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Some Inequalities Involving Perimeter and Torsional Rigidity

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Abstract
We consider shape functionals of the form $F_q(\Omega) = P(\Omega)T^q(\Omega)$ on the class of open sets of prescribed Lebesgue measure. Here $q > 0$ is fixed, $P(\Omega)$ denotes the perimeter of Ω and $T(\Omega)$ is the torsional rigidity of Ω. The minimization and maximization of $F_q(\Omega)$ is considered on various classes of admissible domains Ω: in the class $\mathcal{A}_{all}$ of *all domains*, in the class $\mathcal{A}_{convex}$ of *convex domains*, and in the class $\mathcal{A}_{thin}$ of *thin domains*.

Keywords Torsional rigidity · Shape optimization · Perimeter · Convex domains

Mathematics Subject Classification 49Q10 · 49J45 · 49R05 · 35P15 · 35J25

1 Introduction

In this paper, given an open set $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure, we consider the quantities

$$\begin{aligned} P(\Omega) &= \text{perimeter of } \Omega; \\ T(\Omega) &= \text{torsional rigidity of } \Omega. \end{aligned}$$

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The perimeter $P(\Omega)$ is defined according to the De Giorgi formula

$$P(\Omega) = \sup\left\{\int_\Omega \operatorname{div}\phi\, dx \ :\ \phi \in C_c^1(\mathbb{R}^d;\mathbb{R}^d),\ \|\phi\|_{L^\infty(\mathbb{R}^d)} \le 1\right\}.$$

The *scaling property* of the perimeter is

$$P(t\Omega) = t^{d-1}P(\Omega) \qquad \text{for every } t > 0$$

and the relation between $P(\Omega)$ and the Lebesgue measure $|\Omega|$ is the well-known *isoperimetric inequality*:

$$\frac{P(\Omega)}{|\Omega|^{(d-1)/d}} \ge \frac{P(B)}{|B|^{(d-1)/d}} \tag{1.1}$$

where B is any ball in $\mathbb{R}^d$. In addition, the inequality above becomes an equality if and only if Ω is a ball (up to sets of Lebesgue measure zero).

The torsional rigidity $T(\Omega)$ is defined as

$$T(\Omega) = \int_\Omega u\, dx$$

where u is the unique solution of the PDE

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

Equivalently, $T(\Omega)$ can be characterized through the maximization problem

$$T(\Omega) = \max\left\{\Big[\int_\Omega u\, dx\Big]^2 \Big[\int_\Omega |\nabla u|^2\, dx\Big]^{-1} \ :\ u \in H_0^1(\Omega)\setminus\{0\}\right\}.$$

Moreover T is increasing with respect to the set inclusion, that is

$$\Omega_1 \subset \Omega_2 \Longrightarrow T(\Omega_1) \le T(\Omega_2)$$

and T is additive on disjoint families of open sets. The scaling property of the torsional rigidity is

$$T(t\Omega) = t^{d+2}T(\Omega), \qquad \text{for every } t > 0,$$

and the relation between $T(\Omega)$ and the Lebesgue measure $|\Omega|$ is the well-known *Saint-Venant inequality* (see for instance [16,17]):

$$\frac{T(\Omega)}{|\Omega|^{(d+2)/d}} \le \frac{T(B)}{|B|^{(d+2)/d}}. \tag{1.3}$$

Again, the inequality above becomes an equality if and only if Ω is a ball (up to sets of capacity zero). If we denote by B_1 the unitary ball of $\mathbb{R}^d$ and by ω_d its Lebesgue measure, then the solution of (1.2), with $\Omega = B_1$, is

$$u(x) = \frac{1 - |x|^2}{2d}$$

which provides

$$T(B_1) = \frac{\omega_d}{d(d+2)}. \tag{1.4}$$

We are interested in the problem of minimizing or maximizing quantities of the form

$$P^\alpha(\Omega)T^\beta(\Omega)$$

on some given class of open sets $\Omega \subset \mathbb{R}^d$ having a prescribed Lebesgue measure $|\Omega|$, where α, β are two given exponents. Similar problems have been considered for shape functionals involving:

- the torsional rigidity and the first eigenvalue of the Laplacian in [2,3,6,8,11,19,20, 22];
- the torsional rigidity and the Newtonian capacity in [1];
- the perimeter and the first eigenvalue of the Laplacian in [14];
- the perimeter and the Newtonian capacity in [9,13].

The case $\beta = 0$ reduces to the isoperimetric inequality, and we have, denoting by Ω_m^* a ball of measure m,

$$\begin{cases} \min\{P(\Omega) \ : \ |\Omega| = m\} = P(\Omega_m^*) \\ \sup\{P(\Omega) \ : \ |\Omega| = m\} = +\infty. \end{cases}$$

Similarly, in the case $\alpha = 0$, the Saint Venant inequality yields

$$\max\{T(\Omega) \ : \ |\Omega| = m\} = T(\Omega_m^*) = \frac{m}{d(d+2)}\Big(\frac{m}{\omega_d}\Big)^{2/d}$$

while

$$\inf\{T(\Omega) \ : \ |\Omega| = m\} = 0.$$

Indeed if we choose $\Omega_n = \cup_{k=1}^n B_{n,k}$ where $B_{n,k}$ are disjoint balls of measure m/n each, we get for every $n \in \mathbb{N}$

$$\inf\{T(\Omega) \ : \ |\Omega| = m\} \le T(\Omega_n) = \frac{m^{(d+2)/d}}{d(d+2)\omega_d^{2/d}} n^{-2/d} \to 0.$$

The case when α and β have a different sign is also immediate; for instance, if $\alpha > 0$ and $\beta < 0$ we have from (1.1) and (1.3)

$$\begin{cases} \min\left\{P^\alpha(\Omega)T^\beta(\Omega)\ :\ |\Omega| = m\right\} = P^\alpha(\Omega_m^*)T^\beta(\Omega_m^*) \\ \sup\left\{P^\alpha(\Omega)T^\beta(\Omega)\ :\ |\Omega| = m\right\} = +\infty, \end{cases}$$

and similarly, if $\alpha < 0$ and $\beta > 0$ we have

$$\begin{cases} \inf\left\{P^\alpha(\Omega)T^\beta(\Omega)\ :\ |\Omega| = m\right\} = 0 \\ \max\left\{P^\alpha(\Omega)T^\beta(\Omega)\ :\ |\Omega| = m\right\} = P^\alpha(\Omega_m^*)T^\beta(\Omega_m^*). \end{cases}$$

The cases we will investigate are the remaining ones; with no loss of generality we may assume $\alpha = 1$, so that the optimization problems we consider are for the quantities

$$P(\Omega)T^q(\Omega), \qquad \text{with } q > 0.$$

In order to remove the Lebesgue measure constraint $|\Omega| = m$ we consider the *scaling free* functionals

$$F_q(\Omega) = \frac{P(\Omega)T^q(\Omega)}{|\Omega|^{\alpha_q}} \qquad \text{with } \alpha_q = 1 + q + \frac{2q-1}{d}.$$

In the following sections we study the minimization and the maximization problems for the shape functionals F_q on various classes of domains. More precisely we consider the cases below.

The class of *all* domains Ω (nonempty)

$$\mathcal{A}_{all} = \left\{\Omega \subset \mathbb{R}^d\ :\ \Omega \neq \emptyset\right\}$$

will be considered in Sect. 2; we show that for every $q > 0$ both the maximization and the minimization problems for F_q on $\mathcal{A}_{all}$ are ill posed.

The class of *convex* domains Ω

$$\mathcal{A}_{convex} = \left\{\Omega \subset \mathbb{R}^d\ :\ \Omega \neq \emptyset,\ \Omega \text{ convex}\right\}$$

will be considered in Sect. 3; we show that for $0 < q < 1/2$ the maximization problem for F_q on $\mathcal{A}_{convex}$ is ill posed, whereas the minimization problem is well posed. On the contrary, when $q > 1/2$ the minimization problem for F_q on $\mathcal{A}_{convex}$ is ill posed, whereas the maximization problem is well posed. In the threshold case $q = 1/2$ the precise value of the infimum of $F_{1/2}$ is provided; concerning the precise value of the supremum of $F_{1/2}$ an interesting conjecture is stated. At present, the conjecture has been shown to be true in the case $d = 2$, while the question is open in higher dimensions.

The class of thin domains $\mathcal{A}_{thin}$, suitably defined, will be considered in Sect. 4. If $h(s)$ represents the asymptotical *local thickness* of the thin domain as s varies in a $d-1$ dimensional domain A, the maximization of the functional $F_{1/2}$ on $\mathcal{A}_{thin}$ reduces to the maximization of a functional defined on nonnegative functions h defined on A; this allows us to prove the conjecture for any dimension d on the class of *thin convex* domains.

2 Optimization in the Class of All Domains

In this section we show that the minimization and the maximization problems for the shape functionals F_q are both ill posed, for every $q > 0$.

Theorem 2.1 *There exist two sequences* $(\Omega_{1,n})$ *and* $(\Omega_{2,n})$ *of smooth domains such that for every* $q > 0$ *we have*

$$F_q(\Omega_{1,n}) \to 0 \qquad \textit{and} \qquad F_q(\Omega_{2,n}) \to +\infty.$$

In particular, we have

$$\begin{cases} \inf\left\{F_q(\Omega)\ :\ \Omega \in \mathcal{A}_{all},\ \Omega\ \textit{smooth}\right\} = 0 \\ \sup\left\{F_q(\Omega)\ :\ \Omega \in \mathcal{A}_{all},\ \Omega\ \textit{smooth}\right\} = +\infty. \end{cases}$$

Proof In order to show the sup equality it is enough to take as $\Omega_{2,n}$ a perturbation of the unit ball B_1 such that

$$B_{1/2} \subset \Omega_{2,n} \subset B_2 \qquad \text{and} \qquad P(\Omega_{2,n}) \to +\infty.$$

Then we have

$$|\Omega_{2,n}| \le |B_2|, \qquad T(\Omega_{2,n}) \ge T(B_{1/2}),$$

where we used the monotonicity of the torsional rigidity. Then

$$F_q(\Omega_{2,n}) \ge \frac{P(\Omega_{2,n})T^q(B_{1/2})}{|B_2|^{\alpha_q}} \to +\infty.$$

In order to prove the inf equality we take as $\Omega_{c,\varepsilon}$ the unit ball B_1 from which we remove a periodic array of holes; the centers of two adjacent holes are at distance ε and the radii of the holes are

$$r_{c,\varepsilon} = \begin{cases} e^{-1/(c\varepsilon^2)} & \text{if } d = 2 \\ c\varepsilon^{d/(d-2)} & \text{if } d > 2, \end{cases}$$

where c is a positive constant. It is easy to see that, as $\varepsilon \to 0$, we have

$$|\Omega_{c,\varepsilon}| \to |B_1| \qquad \text{and} \qquad P(\Omega_{c,\varepsilon}) \to P(B_1).$$

Concerning the torsion $T(\Omega_{c,\varepsilon})$, we have (see [10])

$$T(\Omega_{c,\varepsilon}) \to \int_{B_1} u_c \, dx$$

where u_c is the nonnegative function which solves

$$\begin{cases} -\Delta u_c + K_c u_c = 1 & \text{in } B_1 \\ u_c \in H_0^1(B_1), \end{cases}$$

being K_c the constant

$$K_c = \begin{cases} c\pi/2 & \text{if } d = 2 \\ d(d-2)2^{-d}\omega_d c^{d-2} & \text{if } d > 2. \end{cases}$$

Since for every $c > 0$ we have that

$$\int_{B_1} |\nabla u_c(x)|^2 + K_c u_c^2(x) \, dx = \int_{B_1} u_c \, dx$$

we get that

$$\int_{B_1} u_c \, dx \le \frac{\omega_d}{K_c}.$$

Therefore, a diagonal argument allows us to construct a sequence $(\Omega_{1,n})$ such that

$$|\Omega_{1,n}| \to |B_1|, \qquad P(\Omega_{1,n}) \to P(B_1), \qquad T(\Omega_{1,n}) \to 0,$$

which concludes the proof. □

3 Optimization in the Class of Convex Domains

In this section we consider only domains Ω which are *convex*. A first remark is in the proposition below and shows that in some cases the optimization problems for the shape functional F_q is still ill posed.

Proposition 3.1 *We have*

$$\begin{cases} \inf\left\{F_q(\Omega) \ : \ \Omega \in \mathcal{A}_{convex}\right\} = 0 & \textit{for every } q > 1/2; \\ \sup\left\{F_q(\Omega) \ : \ \Omega \in \mathcal{A}_{convex}\right\} = +\infty & \textit{for every } q < 1/2. \end{cases}$$

Proof Let A be a smooth convex $d-1$ dimensional set and for every $\varepsilon > 0$ consider the domain $\Omega_\varepsilon \in \mathcal{A}_{convex}$ given by

$$\Omega_\varepsilon = A\times]-\varepsilon/2, \varepsilon/2[.$$

We have (for the torsion asymptotics see for instance [2])

$$\begin{aligned} P(\Omega_\varepsilon) &\approx 2\mathcal{H}^{d-1}(A), \\ T(\Omega_\varepsilon) &\approx \frac{\varepsilon^3}{12}\mathcal{H}^{d-1}(A), \\ |\Omega_\varepsilon| &= \varepsilon\mathcal{H}^{d-1}(A), \end{aligned}$$

so that

$$F_q(\Omega_\varepsilon) \approx \frac{2}{12^q\left(\mathcal{H}^{d-1}(A)\right)^{(2q-1)/d}}\,\varepsilon^{(2q-1)(d-1)/d}. \tag{3.1}$$

Letting $\varepsilon \to 0$ achieves the proof. □

We show now that in some other cases the optimization problems for the shape functional F_q is well posed. Let us begin to consider the case $q = 1/2$.

Proposition 3.2 *We have*

$$\inf\left\{F_{1/2}(\Omega)\ :\ \Omega \in \mathcal{A}_{convex}\right\} = 3^{-1/2} \tag{3.2}$$

and the infimum is asymptotically reached by domains of the form

$$\Omega_\varepsilon = A\times]-\varepsilon/2, \varepsilon/2[$$

as $\varepsilon \to 0$*, where* A *is any* $d-1$ *dimensional convex set.*

Proof Thanks to a classical result by Polya ([21], see also Theorem 5.1 of [11]) it holds

$$T(\Omega) \geq \frac{1}{3}\frac{|\Omega|^3}{(P(\Omega))^2}.$$

Then

$$F_{1/2}(\Omega) = \frac{P(\Omega)(T(\Omega))^{1/2}}{|\Omega|^{3/2}} \geq 3^{-1/2}$$

for any bounded open convex set. Taking into account (3.1), we get (3.2). □

Concerning the supremum of $F_{1/2}(\Omega)$ in the class $\mathcal{A}_{convex}$ we can only show that it is finite.

Proposition 3.3 *For every $\Omega \in \mathcal{A}_{convex}$ we have*

$$F_{1/2}(\Omega) \le \frac{2^d d^{3d/2}}{\omega_d}\sqrt{\frac{d}{d+2}}\,. \tag{3.3}$$

Proof By the John's ellipsoid Theorem [18], there exists an ellipsoid that, without loss of generality, we may assume centered at the origin,

$$E_a = \left\{x \in \mathbb{R}^d \;:\; \sum_{i=1}^d \frac{x_i^2}{a_i^2} < 1\right\}, \qquad a = (a_1, \dots, a_d),\ \text{with } a_i > 0$$

such that $E_a \subset \Omega \subset dE_a$. Then we have

$$F_{1/2}(\Omega) \le \frac{P(dE_a)\big(T(dE_a)\big)^{1/2}}{|E_a|^{3/2}}. \tag{3.4}$$

Since the solution of (1.2) for E_a is given by

$$u(x) = \frac{1}{2}\Big(\sum_{i=1}^d a_i^{-2}\Big)^{-1}\Big(1 - \sum_{i=1}^d \frac{x_i^2}{a_i^2}\Big),$$

we obtain

$$T(E_a) = \frac{\omega_d}{d+2}\Big(\sum_{i=1}^d a_i^{-2}\Big)^{-1}\prod_{i=1}^d a_i,$$

while

$$|E_a| = \omega_d \prod_{i=1}^d a_i.$$

To estimate $P(E_a)$ we notice that E_a is contained in the cuboid $Q_a = \prod_1^d]-a_i, a_i[$, so that

$$P(E_a) \le P(Q_a) = 2\sum_{i=1}^d \prod_{j\ne i}(2a_j) = 2^d\Big(\sum_{i=1}^d \frac{1}{a_i}\Big)\prod_{i=1}^d a_i.$$

Combining these formulas we have from (3.4)

$$F_{1/2}(\Omega) \le \frac{2^d d^{3d/2}}{\omega_d (d+2)^{1/2}}\Big(\sum_{i=1}^d \frac{1}{a_i}\Big)\Big(\sum_{i=1}^d \frac{1}{a_i^2}\Big)^{-1/2}$$

and finally, by Jensen inequality,

$$F_{1/2}(\Omega) \le \frac{2^d d^{3d/2}}{\omega_d}\sqrt{\frac{d}{d+2}},$$

as required. □

On the precise value of $\sup\{F_{1/2}(\Omega)\ :\ \Omega\in\mathcal{A}_{convex}\}$ we make the following conjecture.

Conjecture 3.4 *We have*

$$\sup\{F_{1/2}(\Omega)\ :\ \Omega\in\mathcal{A}_{convex}\} = d\Big(\frac{2}{(d+1)(d+2)}\Big)^{1/2}$$

and it is asymptotically reached by taking for instance

$$\Omega_\varepsilon = \{(s,t)\ :\ s\in A,\ 0<t<\varepsilon(1-|s|)\}$$

as $\varepsilon\to 0$*, where* A *is the unit ball in* $\mathbb{R}^{d-1}$.

Remark 3.5 We recall that Conjecture 3.4 has been shown to be true in the case $d=2$ (see [21,23], and the more recent paper [12]). In Sect. 4 we prove the conjecture above for every $d\ge 2$ in the class of convex thin domains.

We show now that for F_q in the class $\mathcal{A}_{convex}$ the minimization problem is well posed when $q<1/2$ and the maximization problem is well posed when $q>1/2$. From the bounds obtained in Propositions 3.2 and 3.3 we can prove the following results.

Proposition 3.6 *We have*

$$\begin{cases}\inf\{F_q(\Omega)\ :\ \Omega\in\mathcal{A}_{convex}\}\ge 3^{-1/2}\big(d(d+2)\big)^{1/2-q}\omega_d^{(1-2q)/d} & \text{for every } q\le 1/2\\ \sup\{F_q(\Omega)\ :\ \Omega\in\mathcal{A}_{convex}\}\le \dfrac{2^d d^{3d/2-q+1}}{(d+2)^q\omega_d^{1+(2q-1)/d}} & \text{for every } q\ge 1/2.\end{cases}$$

Proof We have

$$F_q(\Omega) = F_{1/2}(\Omega)\left(\frac{T(\Omega)}{|\Omega|^{(d+2)/d}}\right)^{q-1/2}.$$

Hence it is enough to apply the bounds (3.2) and (3.3), together with the Saint-Venant inequality (1.3) to get that for every $\Omega\in\mathcal{A}_{convex}$

$$\inf\{F_q(\Omega)\ :\ \Omega\in\mathcal{A}_{convex}\}\ge 3^{-1/2}\left(\frac{T(B)}{B^{(d+2)/d}}\right)^{q-1/2}\qquad \text{if } q\le 1/2$$

$$\sup\{F_q(\Omega)\ :\ \Omega\in\mathcal{A}_{convex}\} < \frac{2^d d^{3d/2}}{\omega_d}\sqrt{\frac{d}{d+2}}\left(\frac{T(B)}{B^{(d+2)/d}}\right)^{q-1/2}\qquad \text{if } q\ge 1/2.$$

By the expression (1.4) for $T(B)$ we conclude the proof. □

We now prove the existence of a convex minimizer when $q < 1/2$ and of a convex maximizer when $q > 1/2$.

Theorem 3.7 *There exists a solution for the following optimization problems:*

$$\begin{cases} \min\left\{F_q(\Omega) \ : \ \Omega \in \mathcal{A}_{convex}\right\} & \textit{for every } q < 1/2; \\ \max\left\{F_q(\Omega) \ : \ \Omega \in \mathcal{A}_{convex}\right\} & \textit{for every } q > 1/2. \end{cases}$$

Proof Suppose $q < 1/2$ and consider (Ω_n) a minimizing sequence for $F_q(\Omega)$. By the John's ellipsoid Theorem we can assume that there exists a sequence of ellipsoids E_{a_n} such that

$$E_{a_n} \subset \Omega_n \subset dE_{a_n}.$$

By rotations, translations and scaling invariance of F_q we can assume without loss of generality that

$$E_{a_n} = \left\{x \in \mathbb{R}^d \ : \ \sum_{i=1}^{d} \frac{x_i^2}{a_{in}^2} < 1\right\}, \quad a_n = (a_{1n}, \dots, a_{dn}),\ 0 < a_{1n} \le \cdots \le a_{dn} = 1.$$

Observe that this implies that the diameter of Ω_n is uniformly bounded in n. We claim that

$$a_{1n} \ge c \qquad \text{for every } n \in \mathbb{N}$$

where c is a positive constant. Then the proof is achieved by extracting a subsequence (Ω_{n_k}) which converges both in the sense of characteristic functions and in the co-Hausdorff metric to some open, non empty, convex, bounded set Ω^- and by using the continuity properties of torsional rigidity, perimeter and volume (see for instance, [7,17]).

To prove the claim we use a strategy similar to the one already used in the proof of Proposition 3.3. Let Q_{a_n} be the cuboid $\prod_{i=1}^d]-a_{in}, a_{in}[$. Since

$$d^{-1/2} Q_{a_n} \subset E_{a_n}$$

we have, for n large enough,

$$F_q(B_1) \ge F_q(\Omega_n) \ge \frac{1}{d^{(d-1)/2} d^{d\alpha_q}} \frac{T^q(E_{a_n}) P(Q_{a_n})}{|E_{a_n}|^{\alpha_q}}. \tag{3.5}$$

An explicit computation shows

$$\frac{T^q(E_{a_n}) P(Q_{a_n})}{|E_{a_n}|^{\alpha_q}} = \frac{2^d \omega_d^{q-\alpha_q}}{(d+2)^q} \left(\frac{\sum_{i=1}^d a_{in}^{-1}}{\left(\sum_{i=1}^d a_{in}^{-2}\right)^{1/2}}\right) \left(\frac{\left(\sum_{i=1}^d a_{in}^{-2}\right)^{1/2}}{(\prod_{i=1}^d a_{in}^{-1})^{1/d}}\right)^{1-2q}.$$

Observe that, by Cauchy–Schwarz inequality,

$$1 \le \frac{\sum_{i=1}^d a_{in}^{-1}}{\left(\sum_{i=1}^d a_{in}^{-2}\right)^{1/2}} \le \sqrt{d}, \tag{3.6}$$

while for the last term it holds

$$\frac{\left(\sum_{i=1}^d a_{in}^{-2}\right)^{1/2}}{\left(\prod_{i=1}^d a_{in}^{-1}\right)^{1/d}} = \frac{\left(\sum_{i=1}^d a_{in}^{-2}\right)^{1/2}}{\left(\prod_{i=1}^{d-1} a_{in}^{-1}\right)^{1/d}} \ge \frac{a_{1n}^{-1}}{\left(a_{1n}^{-1}\right)^{(d-1)/d}} = \left(\frac{1}{a_{1n}}\right)^{1/d} \tag{3.7}$$

Therefore, putting together (3.5)–(3.7) and using the fact that $q < 1/2$ we obtain that, if n is large enough, the sequence a_{1n} must be greater than some positive constant c, which proves the claim.

The case $q > 1/2$ can be proved in a similar way. If (Ω_n) is a maximizing sequence for $F_q(\Omega)$ and E_{a_n} are ellipsoids such that $E_{a_n} \subset \Omega_n \subset dE_{a_n}$, we have

$$F_q(B_1) \le F_q(\Omega_n) \le \frac{P(dE_{a_n})T^q(dE_{a_n})}{|E_{a_n}|^{\alpha_q}} = d^{d-1+q(d+2)}\frac{P(E_{a_n})T^q(E_{a_n})}{|E_{a_n}|^{\alpha_q}} . \tag{3.8}$$

If Q_{a_n} is the cuboid $\prod_{i=1}^d]-a_{in}, a_{in}[$ we have $E_{a_n} \subset Q_{a_n}$, so that

$$P(E_{a_n}) \le P(Q_{a_n}) = 2^d \left(\sum_{i=1}^d a_{in}^{-1}\right) \prod_{i=1}^d a_{in} .$$

Hence (3.8) implies, for a suitable constant $C_{q,d}$ depending only on q and on d,

$$F_q(B_1) \le C_{q,d} \frac{\sum_{i=1}^d a_{in}^{-1}}{\left(\sum_{i=1}^d a_{in}^{-2}\right)^q \left(\prod_{i=1}^d a_{in}\right)^{(2q-1)/d}} \le d^q C_{q,d} \left(\frac{\left(\prod_{i=1}^d a_{in}^{-1}\right)^{1/d}}{\sum_{i=1}^d a_{in}^{-1}}\right)^{2q-1},$$

where in the last inequality we used the Cauchy–Schwarz inequality (3.6). Finally, since $a_{in} \le a_{dn} = 1$, we obtain

$$F_q(B_1) \le d^q C_{q,d} \left(a_{in}^{-1}\right)^{(2q-1)/d}$$

and, since $q > 1/2$, the conclusion follows as in the previous case. □

4 Optimization in the Class of Thin Domains

In this section we consider the class of *thin domains*, that we define below through the families of domains

$$\Omega_\varepsilon = \Big\{(s,t)\ :\ s \in A,\ \varepsilon h_-(s) < t < \varepsilon h_+(s)\Big\} \tag{4.1}$$

where ε is a small positive parameter, A is a (smooth) domain of $\mathbb{R}^{d-1}$, and h_-, h_+ are two given (smooth) functions. We denote by $h(s)$ the *local thickness*

$$h(s) = h_+(s) - h_-(s),$$

and we assume that $h(s) \geq 0$. More precisely, we call *thin domain* a family $(\Omega_\varepsilon)_{\varepsilon>0}$ as above; in other words a thin domain is characterized by the $d-1$ dimensional domain A and by the local thickness function h.

The following asymptotics hold for the quantities we are interested to (for the torsional rigidity we refer to [5]):

$$\begin{aligned} P(\Omega_\varepsilon) &\approx 2\mathcal{H}^{d-1}(A),\\ T(\Omega_\varepsilon) &\approx \frac{\varepsilon^3}{12}\int_A h^3(s)\,ds,\\ |\Omega_\varepsilon| &= \varepsilon\int_A h(s)\,ds, \end{aligned}$$

which together give the asymptotic formula when $q = 1/2$

$$\begin{aligned} F_{1/2}(\Omega_\varepsilon) &\approx 3^{-1/2}\mathcal{H}^{d-1}(A)\Big[\int_A h^3(s)\,ds\Big]^{1/2}\Big[\int_A h(s)\,ds\Big]^{-3/2}\\ &= 3^{-1/2}\Big[\Big[\fint_A h^3(s)\,ds\Big]\Big[\fint_A h(s)\,ds\Big]^{-3}\Big]^{1/2} \end{aligned} \tag{4.2}$$

where we use the notation

$$\fint_A f(s)\,ds = \frac{1}{\mathcal{H}^{d-1}(A)}\int_A f(s)\,ds.$$

We then define the functional $F_{1/2}$ on the thin domain $(\Omega_\varepsilon)_{\varepsilon>0}$ associated with the $d-1$ dimensional domain A and the local thickness function h by

$$F_{1/2}(A,h) = 3^{-1/2}\Big[\Big[\fint_A h^3(s)\,ds\Big]\Big[\fint_A h(s)\,ds\Big]^{-3}\Big]^{1/2}.$$

By Hölder inequality we have

$$F_{1/2}(A,h) \geq 3^{-1/2}$$

and the value $3^{-1/2}$ is actually reached by taking the local thickness function h constant, which corresponds to Ω_ε a thin *slab*.

A sharp inequality from above is also possible for $F_{1/2}(A, h)$, if we restrict the analysis to *convex* domains, that is to local thickness functions h which are *concave*. The following result will be used, for which we refer to [4,15].

Theorem 4.1 *Let $1 \le p \le q$. Then for every convex set A of $\mathbb{R}^N$ ($N \ge 1$) and every nonnegative concave function f on A we have*

$$\Big[\fint_A f^q \, dx \Big]^{1/q} \le C_{p,q} \Big[\fint_A f^p \, dx \Big]^{1/p}$$

where the constant $C_{p,q}$ is given by

$$C_{p,q} = \binom{N+p}{N}^{1/p} \binom{N+q}{N}^{-1/q}.$$

In addition, the inequality above becomes an equality when A is a ball of radius 1 and $f(x) = 1 - |x|$.

We are now in a position to prove the Conjecture 3.4 for convex thin domains.

Theorem 4.2 *If $(\Omega_\varepsilon)_{\varepsilon>0}$ is a thin convex domains given by (4.1), we have*

$$F_{1/2}(A, h) \le d \Big(\frac{2}{(d+1)(d+2)} \Big)^{1/2}. \tag{4.3}$$

In addition, the inequality above becomes an equality taking for instance as A the unit ball of $\mathbb{R}^{d-1}$ and as the local thickness $h(s)$ the function $1 - |s|$.

Proof Since the local thickness function h is concave, by Theorem 4.1 with $N = d-1$, $q = 3$, $p = 1$, we obtain

$$\fint_A h^3 \, dx \le C_{1,3}^3 \Big[\fint_A h \, dx \Big]^3,$$

so that

$$F_{1/2}(A, h) \le 3^{-1/2} C_{1,3}^{3/2} = d \Big(\frac{2}{(d+1)(d+2)} \Big)^{1/2}$$

as required. Finally, an easy computation shows that in (4.3) the inequality becomes an equality if A is the unit ball of $\mathbb{R}^{d-1}$ and $h(s) = 1 - |s|$. □

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