Applied Mathematical Sciences, Vol. 16, 2022, no. 12, 729 - 736
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2022.917322

Risk Contagion Among Financial Players Modelled by a SIR Model with Time Delay

Mauro Aliano

Department of Economics and Management, University of Ferrara, Italy

Lucianna Cananà

Ionic Department in Legal and Economic Systems of the Mediterranean: society, environment, culture, University of Bari Aldo Moro, Italy

Massimiliano Ferrara

Department of Law, Economics and Human Sciences
University Mediterranea of Reggio Calabria, Italy

Stefania Ragni

Department of Economics and Management, University of Ferrara, Italy

Abstract

We revisit a mathematical approach using epidemiological models to describe risk contagion and propagation among financial players and markets. The link between the financial system and the ecosystem is explained by a SIR model with time delay. We aim to apply this on more complex financial systems, where cryptocurrencies, new participants and traditional financial operators play together. This work represents a starting point for a deep stochastic analysis of the contagion term influence over risk contagion dynamics.

Mathematics Subject Classification: 91B05, 91B55, 34F05

Keywords: Financial distress, Financial immunity, SIR model, Delay Differential equation, Stochastic equation

1 Introduction

In June 2022, a Singapore-based cryptocurrency hedge fund named Three Arrows Capital (3AC) crashed and lost $3.5 billion, according to creditor claims. After a few weeks in July 2022 the cryptocurrency broker Voyager Digital, that promoted the 3AC bankruptcy action, also crashed subsequently to the failure of the so-called margin calls. This domino effect started from the inability of 3AC to pay back a loan from Voyager; this mechanism is well-known in financial markets where the bankruptcy of one financial player represents only a first shock for the financial market. In a more interconnected financial market, as the 3AC experience suggests, cryptocurrencies emerge as a new asset class since they consider specific features and could improve portfolio diversification (cfr. [6]), so that the market has fueled the rise of funds specialized in cryptocurrencies. These funds operate in an asset class corresponding to an unregulated asset with a variety of arbitrage opportunities and risk-taking opportunities (see [4], [11]). Moreover, they limit the competition with more cheap financial instruments that overtake traditional funds, such as Exchange Traded Funds (see [4]). In a complex view, operations among financial companies, hedge funds, banks and stock exchanges create networks where financial distress spreads as an epidemic disease and scholars evaluate the risk of transmission of the so-called default propagation. This classical scheme of contagion can be boosted by many factors, as well as the interconnection among cryptocurrencies, operators and FinTech companies (cfr. [3]). In addition, the linkage between primary broker and hedge funds borrowing is affected by significantly overcollateralized phenomena (see [10]), and often collaterals are represented by cryptocurrencies. The so-called spillover effect could be extended also to other sectors or financial assets (see [2]). As an example, in October 2022 the bankruptcy of FTX (i.e. a Bahamas-based cryptocurrency exchange) was caused by a liquidity crisis of the company token and also led to a collapse in other financial markets and cryptocurrencies.
Our contribution exploits the analogies between financial crises and the spread of a medical disease, since we consider contagion in terms of a transmission mechanism, where the event which sets off the crisis could be caused by liquidity, counterparty, or credit risk. Since the linkage between financial systems and ecosystems has been pointed out by [12], the authors in [7] apply the Susceptible-Infected-Recovered (SIR) epidemic model on company data as one of the first contributions to this topic. The role of contagion is important in the so-called systemic risk, where events could determine a large cascade of crises (cfr. [8]). As a domino effect, if a financial player moves toward a crash, this could lead to a crisis or pre-crisis conditions for other financial players and market instruments. Financial networks, such as the interbank market, play a key role in default propagation (see [13]). According to [5] , the use of

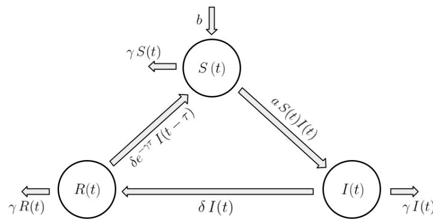

Figure 1: Scheme of risk contagion dynamics.

networks in economic analysis has a long history, and in the last decade it can also be used to explain financial crises. Under a SIR perspective, Susceptible companies are represented by financial players that operate with cryptocurrency markets, while Infected are represented by financial players with financial distress condition, Recovered are players that operate in particular operative conditions due to policy maker activities (laws or economic actions).

2 Risk spread among financial players

The evolution of financial contagion can be modelled in terms of risk with low and high level. Actually, some financial players with lower risk are susceptible (S) to infection by others which are characterized by an extremely high risk and considered as infected (I). After infection, some players are recovered (R) and get the capability of risk control, thus they become able to keep their risk at a low level and are no longer infectious for a period of time but not life long. Therefore, we describe a SIR dynamics starting at $t = 0$ when $N_0 > 0$ is the initial total number of financial players. We normalize the number of items in each category to the initial value N_0, thus we define the densities $S(t)$, $I(t)$ and $R(t)$ for any class at every time $t \geq 0$. The total number of players is evaluated as $N(t) = S(t) + I(t) + R(t)$ at each t, with $N(0) = 1$. The whole dynamics is represented in Figure 1 and is similar to the one proposed in [9] in a different application framework. We assume that new financial players come into the susceptible class so that it increases by a given growth rate $b > 0$. Meanwhile, a portion of low risk players come into contact with some infected and leave the susceptible class according to the bilinear incidence term $a\,S(t)I(t)$, where $a > 0$ represents the contagion removal rate. On the other hand, a portion of infected related to the term $\delta I(t)$ leaves the same class of infectives and is recovered; parameter δ is the rate at which contagious players take the corresponding measures to get financial immunity. Thus, δ may measure the intervention of laws or economic actions aimed to overcome financial distress; it is assumed $0 < \delta < 1$. After curing, the immunity ability of recovered players is temporary but not lifelong: the financial immunity period τ represents the time lag of the model. Actually, recovered items remain in the recovery pool for the time period between $t-\tau$ and t; after that, the portion $e^{-\gamma\tau}\delta I(t-\tau)$ reverts to the susceptible class assuming that γ models the regulatory authorities final elimination rate of the players from the market. We suppose $0 < \gamma < 1$, moreover we notice that the choice of term $e^{-\gamma\tau}\delta I(t-\tau)$ is common in the literature (see [9] and [14]). The resulting differential problem consists of the following equations

$$\frac{dS(t)}{dt} = b - \gamma S(t) - aS(t)\,I(t) + \delta e^{-\gamma\tau} I(t-\tau), \tag{1}$$

$$\frac{dI(t)}{dt} = aS(t)\,I(t) - (\gamma+\delta)I(t), \tag{2}$$

completed by suitable initial conditions:

$$S(0) = S_0 > 0, \tag{3}$$

$$I(s) = I_0(s) \geq 0, \quad \text{for all } s \in [-\tau, 0], \quad \text{with } I_0(0) > 0. \tag{4}$$

Once $I(t)$ is evaluated, then the recovered class evolution is obtained by the following rule

$$\frac{dR(t)}{dt} = \delta I(t) - \gamma R(t) - \delta e^{-\gamma\tau} I(t-\tau), \tag{5}$$

which can be integrated in order to have $R(t) = \delta \int_{t-\tau}^{t} e^{-\gamma(t-s)} I(s)\,ds$, provided that $R(0) = \delta \int_{-\tau}^{0} e^{\gamma s} I_0(s)\,ds$. The following result states the existence of the unique positive solution of the model.

Theorem 2.1 *Assume that $I_0(\cdot)$ is a continuous function in $[-\tau, 0]$. Then there exists a unique solution of problem (1)-(2) which is equipped with initial conditions (3)-(4). In addition, it holds that $S(t) > 0$ and $I(t) > 0$ for all $t \geq 0$.*

The proof of existence and uniqueness arises from the continuity feature of the initial condition together with the fact that the forcing terms are Lipschitz continuous with respect to S and I; moreover, the approach for proving solution positivity is similar to the one developed in [9] and [1]. As a consequence of Theorem 2.1, since $R(t)$ is obtained by integrating $I(s) > 0$, then we also get that $R(t)$ is positive for all $t \geq 0$.

We remark that the dynamics of the total number of players is described by $dN/dt = b - \gamma N$, with $N(0) = 1$. By integrating this relationship, it is not so difficult to verify that $N(t) \leq 1 + b/\gamma$ for each t, thus both $S(t) \leq 1 + b/\gamma$ and $I(t) \leq 1 + b/\gamma$ over the whole time horizon.

In addition, model (1)-(2) has the risk-free equilibrium $E_0^* = (b/\gamma, 0)$ and one more non-zero steady state $E_\tau^* = (S_\tau^*, I_\tau^*)$ with

$$S_\tau^* = (\gamma+\delta)/a, \qquad I_\tau^* = \frac{\gamma(\gamma+\delta)}{a(\gamma+\delta-\delta e^{-\gamma\tau})}(\rho_0 - 1),$$

where

$$\rho_0 = \frac{ba}{\gamma(\gamma+\delta)},$$

represents the basic reproduction number. It is known that E_τ^* represents an endemic equilibrium and is feasible under the assumption that $\rho_0 > 1$. In order to understand whether the risk will continue to exist in or it will be eliminated from the market, steady state stability needs investigation as in the following proposition.

Theorem 2.2 *The following propositions hold:*

- *if $\rho_0 < 1$, then the risk-free equilibrium E_0^* is locally asymptotically stable and no other equilibrium is feasible;*
- *in the opposite case when $\rho_0 > 1$, then E_0^* is unstable while E_τ^* becomes both feasible and locally asymptotically stable.*

The proof can be carried out by linearising system (1)-(2) near the equilibrium points. Concerning the risk-free steady state, the results arise from noting that the eigenvalues of the linearisation are evaluated as $\lambda_1 = -\gamma$ and $\lambda_2 = (\gamma+\delta)(\rho_0-1)$ and they are both negative in the case when $\rho_0 < 1$. On the other hand, with the aim of understanding the non-trivial equilibrium stability, we remark that the linearisation matrix of system (1)-(2) near E_τ^* has the following characteristic equation in λ

$$\lambda^2 + (\gamma + aI_\tau^*)\lambda + aI_\tau^*(\gamma + \delta - \delta e^{-\gamma\tau}) = 0.$$

In the case when $\rho_0 > 1$, the coefficients in the previous equation are positive. This yields that both the eigenvalues of the linearisation matrix cannot be purely imaginary and have negative real parts. It follows that both the feasibility of E_τ^* and its local stability are proved.

3 Fluctuation of the contact coefficient

The contagion parameter a can be perturbed by a white noise with the aim of accounting for the uncertainty of contacts between players in the financial market. Contact rate a in model (1)-(2) is replaced by $a + \sigma(dB/dt)$, where $B(t)$ is a Brownian motion; thus, the resulting stochastic system is

$$dS(t) = \left[b - \gamma S(t) - aS(t)I(t) + \delta e^{-\gamma\tau} I(t-\tau)\right] dt - \sigma S(t)I(t)dB(t), \quad (6)$$
$$dI(t) = \left[aS(t)I(t) - (\gamma+\delta)I(t)\right] dt + \sigma S(t)I(t)dB(t), \quad (7)$$

coupled with (5) which remains a deterministic equation.
The study of well-posedness starts from considering a complete probability

space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ which is right continuous and increasing, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. In this framework, the following result states the conditions of existence of the unique positive solution of model (6)-(7).

Theorem 3.1 *Assume that initial conditions (3)-(4) are imposed and function $I_0(\cdot)$ is continuous over $[-\tau, 0]$. Then there exists a unique solution of problem (6)-(7) and, for any time $t \geq 0$, it holds that $S(t) > 0$, $I(t) > 0$ with probability one.*

The proof can be carried out by applying the Itô's lemma and exploiting the Lyapunov function $V(S, I) = -\log(\gamma S/(\gamma + b)) - \log(\gamma I/(\gamma + b))$.

4 Concluding remarks and future development

This work is concerned with the relevant problem of modelling how risk may spread across financial markets. As there exists a link between financial systems and ecosystems, risk contagion is modelled by the mathematical approach of epidemiological models describing infectious disease spread that begins as outbreak. The analysis of these epidemiological problems is not a novelty, anyway the paradigm and the application to players in financial markets is an interesting topic which is worthy of consideration.

We also remark that the present paper represents a starting point for a deeper investigation about the influence of a stochastic treatment of the contagion term over the whole risk spread dynamics. In this respect, it is worth noting that stochastic model (6)-(7) has the same risk-free steady state E_0^* as the corresponding deterministic system. In that regard, we remark that the assumption in Theorem 2.1 can be strengthened to state a sufficient condition for assuring the global asymptotic stability of E_0^* in the deterministic framework. More precisely, starting from the assumption

$$\rho_0 < 1 - \frac{a}{\gamma + \delta},$$

which is more restrictive than the requirement $\rho_0 < 1$ in Theorem 2.1, it is possible to get $I(t) \leq I_0(0)e^{ct}$ with $c = a + (\gamma + \delta)(\rho_0 - 1) < 0$, by integrating equation (2). This result can be exploited to prove that any solution of the deterministic model (1)-(2) converges to the risk-free equilibrium E_0^* at the long run.

Inspired by this property, our future purpose will consist of providing a complete study of the stochastic stability of the same equilibrium point E_0^* concerning the stochastic process defined in (6)-(7). This analysis will be developed by choosing suitable Lyapunov functions and applying the classical techniques of stochastic analysis.

As a final remark, we notice that the exact solution of the stochastic model (6)-(7) is not available in closed form, due to the fact that the model is nonlinear. Therefore, it is necessary to approximate risk contagion dynamics by numerical integration. It will be interesting to combine the numerical algorithms for delay equations together with ad-hoc techniques for approximating stochastic processes. The issue of investigating the employment of suitable numerical schemes will be another important topic to be developed in our future research.

References

[1] M. Aliano, L. Cananá, G. Cestari, S. Ragni, A dynamical model with time delay for risk contagion, *Submitted,* (2022).

[2] C. Boido, M. Aliano, Digital art and non-fungible-token: Bubble or revolution?, *Finance Research Letters,* (2022), in press.

[3] W. Bazan-Palomino, Interdependence, contagion and speculative bubbles in cryptocurrency markets, *Finance Research Letters,* **49** (2022), 103132. https://doi.org/10.1016/j.frl.2022.103132

[4] D. Bianchi, M. Babiak, *On the performance of cryptocurrency funds* (2020), Available at SSRN 3559092.

[5] M. Callon, Techno-economic Networks and Irreversibility, *The Sociological Review*, **1** suppl (1990), 132-161. https://doi.org/10.1111/j.1467-954x.1990.tb03351.x

[6] L. Charfeddine, N. Benlagha, Y. Maouchi, Investigating the dynamic relationship between cryptocurrencies and conventional assets: Implications for financial investors, *Economic Modelling,* **85** (2020), 198-217. https://doi.org/10.1016/j.econmod.2019.05.016

[7] A. Garas, P. Argyrakis, C. Rozenblat, Worldwide spreading of economic crisis, *New J. Phys.* **12** (2), (2012), 185-188. https://doi.org/10.1088/1367-2630/12/11/113043

[8] A.G. Haldane, R.M. May, Systemic risk in banking ecosystems, *Nature,* **469** (2011), 351-355. https://doi.org/10.1038/nature09659

[9] Y.N. Kyrychko, K.B. Blyuss, Global properties of a delayed SIR model with temporary immunity and nonlinear incidence rate, *Nonl. Anal. RWA,* **6** (3) (2005), 495-507. https://doi.org/10.1016/j.nonrwa.2004.10.001

[10] M.S. Kruttli, F. Monin Sumudu, W. Watugala, The life of the counterparty: Shock propagation in hedge fund-prime broker credit networks, *Journal of Financial Economics,* **146** (3) (2022), 965-988. https://doi.org/10.1016/j.jfineco.2022.02.002

[11] I. Makarov, A. Schoar, Trading and arbitrage in cryptocurrency markets, *Journal of Financial Economics,* **135** (2) (2020), 293-319. https://doi.org/10.1016/j.jfineco.2019.07.001

[12] R.M. May, S.A. Levin, G. Sugihara, Complex systems: ecology for bankers, *Nature,* **451** (7181) (2008), 893-895. https://doi.org/10.1038/451893a

[13] T. Roukny, H. Bersini, H. Pirotte, G. Caldarelli, S. Battiston, Default cascades in complex networks: topology and systemic risk, *Sci. Rep.,* **3** (2013), 2759. https://doi.org/10.1038/srep02759

[14] C. Zhao, M. Li, J. Wang, S. Ma, S. The mechanism of credit risk contagion among internet P2P lending platforms based on a SEIR model with time-lag, *Research in International Business and Finance,* **57** (2021), 101407. https://doi.org/10.1016/j.ribaf.2021.101407

Received: December 1, 2022; Published: December 29, 2022