Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education

Editors: Uffe Thomas Jankvist, Marja van den Heuvel-Panhuizen, Michiel Veldhuis
Organised by: Freudenthal Group & Freudenthal Institute, Utrecht University
Year: 2019

erme
european society for research in mathematics education

Publisher

Freudenthal Group & Freudenthal Institute, Utrecht University, Netherlands and ERME

ISBN 978-90-73346-75-8

Recommended citation for the proceedings

Jankvist, U. T., Van den Heuvel-Panhuizen, M., & Veldhuis, M. (Eds.). (2019). *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*. Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Recommended citation for single entries in the proceedings

Authors (2019). Title of paper/poster. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. xxxx-yyyy). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

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National standardized tests database implemented as a research methodology in mathematics education. The case of algebraic powers.

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In this work we show the use of the INVALSI database as an instrument to collect organized data from national standardized tests in order to study students' mathematical thinking on a large scale. It allows us to create appropriate correlations to single out the behavior of the students relative to a specific mathematical topic or a cognitive issue. In this study we have used the INVALSI database to study how grade 10 students work with powers in the domain of algebra. We analyze the quantitative data within a semiotic theoretical framework. Analyzing items taken from the INVALSI tests we are able to single out a general behavior that involves a population of Italian students and that persists in time. We interpret the results within our theoretical framework, thereby giving a quantitative validation to a theoretical perspective that has been proven consistent by qualitative investigations.

Keywords: Standardized assessments, qualitative and quantitative methodologies, algebra, semiotics, powers.

Introduction.

It is more than 10 years that the Italian Ministry of Education (MIUR) has established national standardized tests (INVALSI) in mathematics. In detail, every year since 2008 INVALSI administers Italian and Mathematics (and since 2018, English) large-scale assessment tests to Italian students from Primary School to High Secondary School. The tests are devised according to a robust theoretical framework that has been developed according to mathematics education research. As we will see, a key tool that made it possible to focus our investigation was a database containing data about the INVALSI assessment.

The aim of this paper is to show how the INVALSI database can be effectively used also for research purposes. We study how high school students deal with powers analyzing their answers to items that have been administered throughout several school years. The items have been collected using the INVALSI database GESTINV (Gestinv, 2018) in order to single out consistent data regarding powers.

Our research interest focused on syntactic aspects because INVALSI tests show that Italian high school students have severe difficulties in handling the meaning of algebraic formalism when dealing with powers. This is particularly interesting if we take into account the efforts and attention that both students and teachers devote to algebraic calculations in the first two years of high school.

Our study wants to shed some light on the following issues:

What information can standardized assessments provide about student's learning difficulties regarding powers? What research too ls can we use?

What information can we acquire from standardized assessments about mathematical practices in Italian high schools?

Theoretical framework.

Gestinv is a database which includes all the administered INVALSI tests in Italy. There are now 13,000 users amongst in service and preservice teachers and researchers. As regards mathematics there are almost 1700 items available.

Gestinv contains all the items of the INVALSI tests indexed according to the National Curricula, the results from the statistical point of view, the content, the key words, the percentage of correct, wrong and invalid answers and other characteristics. It is possible to carry out searches correlating these indexes. It has been used as a research tool in several researches to single out difficulties in specific mathematics topics or problems, for example in Ferretti and Gambini (2018), or in Ferretti, Giberti and Lemmo (2018). Gestinv allows both a fine grain and a coarse grain analysis of mathematics didactical phenomena and the data can be organized according to the basic contents and the indications of the Italian national curriculum.

To create a consistent framework for our study, it is necessary to combine the potentialities of Gestinv with a theoretical lens that allows us to look into the complexity of the didactical phenomenon that we want to investigate with the database. In the case of this paper our aim is to use Gestinv to observe the behavior of students in algebra, in particular when working with powers.

The learning of algebra is a key research issue in mathematics education, that involves a leap from procedural to relational and general cognitive functioning. Such a change in cognitive perspective is underpinned by the use of semiotic representations that in algebra implies the use of symbolic language intertwined with other semiotic systems.

Duval's (1995, 2008) semiotic approach highlights a specific cognitive functioning in mathematics, due to the special epistemological nature of its objects that do not allow ostensive references. Thinking and learning in mathematics is identified with the coordination of semiotic registers via treatment and conversion. Treatment is a semiotic transformation from a representation into another within the same register and conversion is a semiotic transformation from a representation in one register into another representation in another register.

The inaccessible nature of mathematical objects leads the student in a cognitive paradox that obliges him to identify the mathematical object with its semiotic representations (Duval, 1993). The cognitive paradox can hinder the student's meaning making processes since he is unable to establish the correct relationship between the mathematical object (the signified) and several representations (its signifiers). Conversion is particularly difficult to carry out because there is no syntactic rule that binds the first representation in one register with the second in the other register. In conversion, the pupil ends up with a series of unrelated representations that he is unable to refer to the same mathematical concept. According to Duval, conversion is the key cognitive function that guarantees the conceptual acquisition of mathematical objects.

Treatments instead rest on the structure of the semiotic register that provides the pupil with the syntactic rules to go from one representation to the other via the semiotic transformation. Therefore, we expect that students are able recognize that different representations involved in treatment transformations refer to the same mathematical object. Within this framework our research questions are: Q1: What precise information, regarding powers, can we acquire from a research that implements Gestinv? Q2: Is it possible to collect information that is coherent with solid research findings? Q3: What methodological tools does Gestinv provide?

Methodology.

Our research methodology intertwines qualitative and quantitative methodologies according to Johnson and Onwuegbuzie (2004) and Iori (2018). In our research design we mixed quantitative

and qualitative methodologies according to the following scheme: QUAL ------> QUAN -----> QUAN+QUAL. From a qualitative point of view our aim was to ascertain higher school students' difficulties in algebra. In particular our focus was on the students' ability to give meaning to the syntactic structure of algebra. Gestinv allowed us to carry out a quantitative analysis based on the INVALSI grade ten mathematics items, selecting the ones with lower scores. Among these we noticed that the management of powers yielded the worst results, in particular the tasks involving treatment operations. At this point we carry out a Quan/Qual analysis that combines our semiotic lens (Qual) with the study of the characteristic curves and the distractor plots (Quan).

In Figure 3 we see an example of our basic tool. On the x-axis are the students' level of competence, measured on the basis of the whole test and on the y-axis are the probabilities of answers to the distractors in function of the level of competence. The solid line represents the curve predicted by the model (characteristic curve) whereas the dotted lines are the empirical results of the test, plotted by deciles. Characteristic curves and distractor plots are an effective quantitative tool to highlight correlations between the students' answers to a specific item according to the basic contents and the indications of the Italian national curriculum, which informs the structure of the INVALSI test.

We point out the fact that we selected items close to the kind of algebra activity usually performed in Italian high schools during the first two years of secondary school. Moreover the items were selected according to the INVALSI theoretical framework, which considers the ability to handle semiotic transformations as one of the basic cognitive processes. We used the functions of Gestinv to choose items that matched our research needs in terms of cognitive processes, mathematical content and learning objectives. We turned out with items that highlight students' difficulties in giving sense to algebraic representations as they undergo treatment transformations. It is important to keep in mind that in Italy students, when learning algebra, are mainly exposed to complicated, heavy, and long algebraic calculations, basically treatments. This is also true for the teaching and learning of powers.

Results.

The following task was administered in 2017 Mathematics Grade 10 INVALSI test.

You can write the expression $\left(a-\frac{1}{a}\right)^2$ like:

A. ☐ $a^2+\frac{1}{a^2}-2$

B. ☐ $\frac{a^4+1}{a^2}$

C. ☐ $a^2-\left(\frac{1}{a}\right)^2$

D. ☐ $\frac{a^2-2a+1}{a^2}$

Figure 1: Task in Mathematics Grade 10 INVALSI Test 2017

This INVALSI test was administered to 540.000 grade 10 students; the results refer to a sample of about 48.000 students representative of the whole population. The task requires to perform a

treatment operation in the algebraic register, applying a well-known identity. As we can see in the following figure (Figure 2), only 34% of the students gave the correct answer; the majority of students who did not solve the task correctly chose option C. This item disproves Duval's claim that conversion is the key semiotic transformation and that treatment supports students in the construction of the correct meaning of mathematical concepts, in this study, powers. All the incorrect options reveal difficulties in the treatment operations and the mathematical meaning is completely lost. For example, option C formally resembles the original algebraic expression but there is no link to the correct mathematical meaning of powers. It is interesting how the students do not rely on the algebraic transformation rules to give sense to their mathematical activity. This item provides a quantitative confirm of qualitative researches that show how treatment transformations can result in a change or loss of meaning (D'Amore, 2007; Santi, 2011). Such researches show that meaning cannot be reduced to the structure of semiotic systems, but it is rooted in the range of social activity, at a personal and cultural level. In absence of a meaningful personal activity, the student, according to the cognitive paradox, interprets the semiotic representations as if they were unrelated mathematical objects, thereby losing the reference to the common mathematical object, even if he can rely on precise transformation rules. Rules and procedures that Italian students have trained over and over again in their algebra classes.

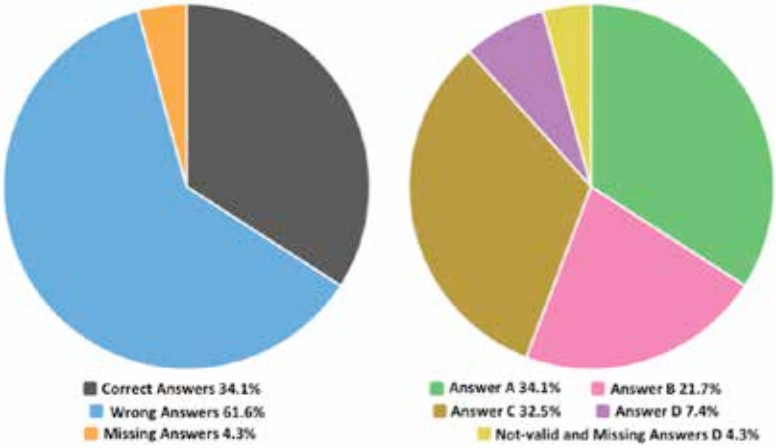

Figure 2: Results referred to the task in Mathematics Grade 10 INVALSI Test 2017

As we can see in the following graph (Figure 3), among the various options of incorrect answers, option C is the most frequently chosen at all levels of competence, up to medium-high skill levels. It testifies that this loss of meaning roots in convictions and beliefs that affect even the most competent students.

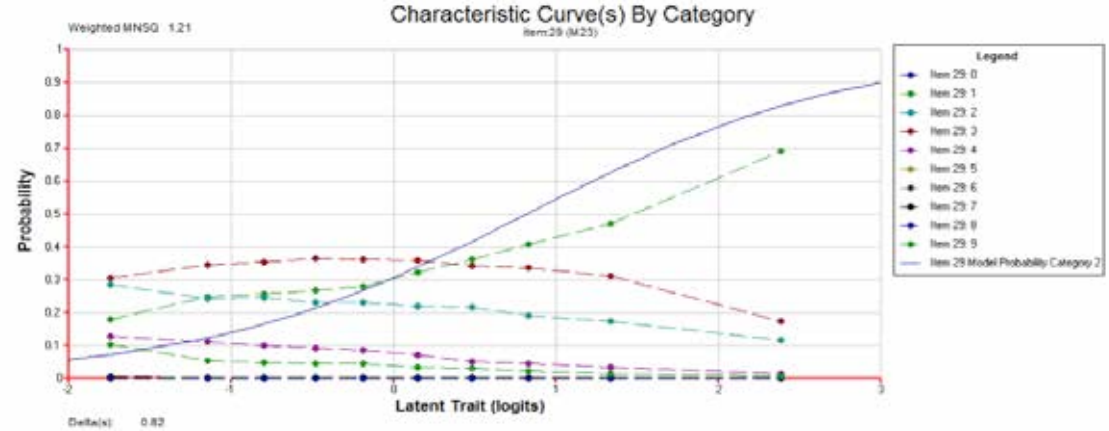

Figure 3: Characteristic Curve referred to the task in Mathematics Grade 10 INVALSI Test 2017

The following task was administered in Mathematics Grade 10 INVALSI Test in the year 2015.

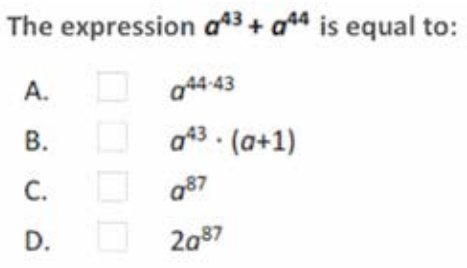

Figure 4: Task in Mathematics Grade 10 INVALSI Test 2015

In 2015 almost 550000 Grade 10 students performed this INVALSI test, and the national results referred to a sample of 48440 students. As we can see in the following graphs, only a third of Italian students provided the correct answer; among the incorrect options, the most chosen option is C. In option C the exponents of the two powers in the text are summed. Again, this protocol shows a loss of meaning, due to a treatment, when the student goes from the original $a^{43} + a^{44}$ to a^{87}. The expression $a^n + a^m$ puzzles the student who is not able to frame it appropriately in the context of powers, thereby he resorts to the well-known identity $a^n \cdot a^m = a^{(n+m)}$ which leads to a loss of the original meaning. This result suggests that factoring out the GCF is meaningless to most students, despite the thorough practice in terms of treatment transformations they are exposed to. Meaningless in the sense that they confuse the algebraic representations (signifiers) with the mathematical object (signified) and they are not able to establish the correct semiotic reference to the mathematical object.

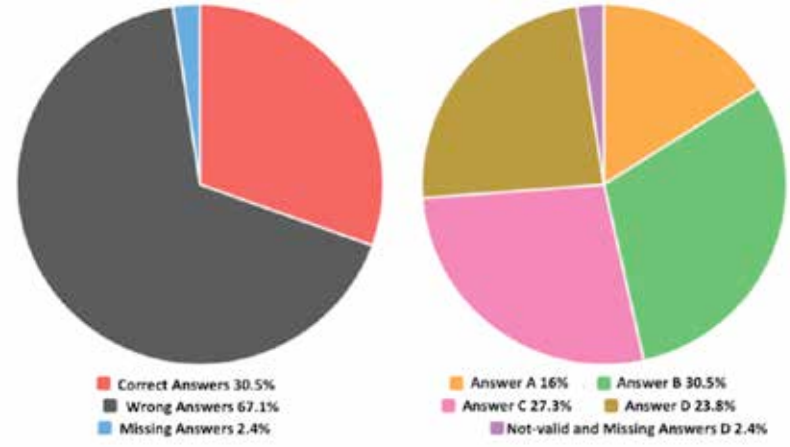

Figure 5: Results referred to the task in Mathematics Grade 10 INVALSI Test 2015

The Characteristic Curve (Figure 6), shows that, among the incorrect options, Option C is the most chosen at all levels of competence.

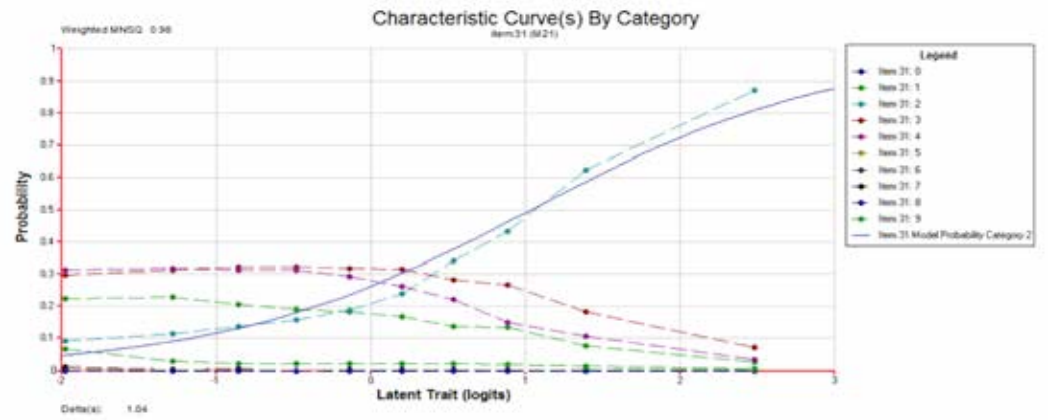

Figure 6: Characteristic Curve referred to the task in Mathematics Grade 10 INVALSI Test 2015

The following question was in the Mathematics INVALSI test that was administered in Italy in 2012 to all Grade 10 students. Almost 533.000 students performed the test and the results referred to a sample of almost 50.000 Grade 10 Italian students. To answer the task correctly, it is necessary to manipulate the exponents of an algebraic expression with two terms.

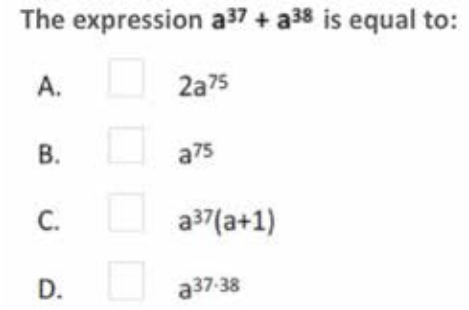

Figure 7: Task in Mathematics Grade 10 INVALSI Test 2012

As we can see in the following graphs (Figure 8), only 35% of the students answered correctly. More than 25% of students chose option B, in which the exponent of the power is the sum of the exponents of the power in the text. Also, in this situation, option B is the most chosen incorrect option at all levels of competences (Figure 9). If we compare this item we the former we can see that the behavior of the students is exactly the same both in terms of scores and of its characteristic curves. The combination of the last two items shows how Gestinv provides quantitative and qualitative information that we can compare across time to gain further insight on a teaching-learning phenomenon

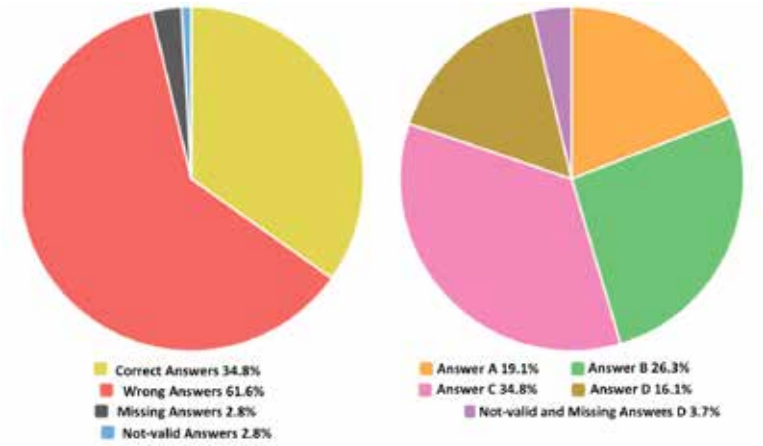

Figure 8: Results referred to the task in Mathematics Grade 10 INVALSI Test 2012

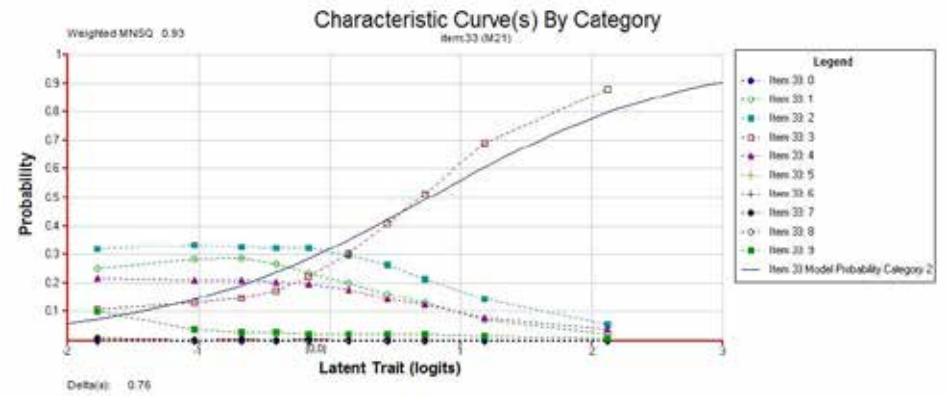

Figure 9: Characteristic Curve referred to the task in Mathematics Grade 10 INVALSI Test 2012

Answer to the research questions.

A1) At a coarse-grained level, Gestinv highlights rooted difficulties in students facing treatments regarding powers. This is a quantitatively relevant macro phenomenon that persists with the same features across time.

A2) At a fine-grained level, Gestinv provides data that are coherent with solid research findings. In particular, it unveils at a quantitative level the phenomenon of change/loss of meaning due to treatment semiotic transformations. Several studies (D'Amore, 2007; D'Amore & Fandiño Pinilla, 2007; Santi, 2011) have shown that at all school levels, including prospective teachers, also treatment bewilders students who experience a loss or a change of meaning in treatment transformations. The loss and/or change of meaning due to treatment transformations implies that mathematical cognition in general and in particular the algebraic one cannot be reduced to a complex transformation of signs. Meaning is beyond the mere relation sign-object and it is necessary to take into account other basic features that characterize sense-making processes in mathematics. Moreover, data easily available in Gestinv show that this phenomenon also affects students with medium-high levels of competences. Thus teacher's didactical awareness is not only aimed at helping weak students but also the so-called stronger ones, devising an effective didactical transposition and didactical engineering that encompass the complexity of mathematical thinking and learning.

A3) Gestinv is an effective tool that entangles quantitative and qualitative research methodologies. As regards the quantitative approaches, they are based on a statistically significant population. It allows us to provide a quantitative validation of theoretical results, confirmed at a qualitative level. Furthermore, the characteristic curves are a powerful tool to intertwine quantitative and qualitative analyses.

Conclusions.

Gestinv implements a new research method that combines quantitative and qualitative approaches. In this study, it allowed us to highlight a macro didactical phenomenon that is quantitatively relevant. Thanks to the combination of the quantitative and qualitative approaches we were able both to frame the students' behavior within an appropriate theoretical framework and provide a statistically based quantitative validation of such a theoretical result.

From an educational point of view, Gestinv shows how the school mathematical practices confirm the research results of mathematics education.

Further research is required to understand if Gestinv, implemented as a research method, is able to single out new didactical phenomena that cannot be framed in the current perspectives and therefore prompts new theoretical research.

References

D'Amore, B. (2007). Mathematical objects and sense: how semiotic transformations change the sense of mathematical objects. *Acta Didactica Universitatis Comenianae*. 7, 23-45.

D'Amore, B. & Fandiño Pinilla, M.I. (2007). Change of the meaning of mathematical objects due to the passage between their different representations. How other disciplines can be useful to the analysis of this phenomenon. *Rome, Symposium on the occasion of the 100th anniversary of ICMI, March 2008. WG5: The evolution of theoretical framework in mathematics education*, organizers: Gilah Leder and Luis Radford. www.unige.ch/math/EnsMath/Rome2008.

Duval, R. (1993). *Registres de représentations sémiotique et fonctionnement cognitif de la pensée. Annales de Didactique et de Sciences Cognitives*, ULP, IREM Strasbourg. 5, 37-65.

Duval, R. (1995). *Sémiosis et pensée humaine: Registres sémiotiques et apprentissages intellectuels.* Berne: Peter Lang.

Duval, R. (2008). Eight problems for a semiotic approach in mathematics education. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 39–61). Rotterdam: Sense Publishers.

Ferretti, F., & Gambini, A. (2017). A vertical analysis of difficulties in mathematics from secondary school to university level; some evidence stems from standardized assessment. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education*. Dublin, Ireland: DCU Institute of Education & ERME.

Ferretti, F., Giberti, C., & Lemmo, A. (2018). The Didactic Contract to Interpret Some Statistical Evidence in Mathematics Standardized Assessment Tests. *EURASIA Journal of Mathematics, Science and Technology Education*, *14*(7), 2895-2906. https://doi.org/10.29333/ejmste/90988

Gestinv 2.0. (07.01.2018). *Archivio interattivo delle prove Invalsi.* Retrieved from http://www.gestinv.it.

Iori, M. (2018). Teachers' awareness of the semio-cognitive dimension of learning mathematics. *Educational Studies in Mathematics, 98(1),* 95-113.

Johnson, R. B., & Onwuegbuzie, A. J. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Researcher, 33(7),* 14-26.

Santi, G. (2011). Objectification and semiotic function. *Educational Studies in Mathematics, 66(77),* 285–311.