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A Nonlinear Dynamics for Risk Contagion: Analyzing the Not Risk-Free Equilibrium

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Abstract

In this paper we carry out the stability analysis of the not risk-free steady state which is involved in a financial contagion dynamics. Starting from an analogy between economic sectors and ecosystems, the Susceptible-Infected-Recovered (SIR) approach is employed to describe the risk dynamics by a nonlinear differential system with time delay. A

main assumption is that contagion phenomenon is modelled by a Holling Type II functional responce so that an incubation time for risk infection is accounted for; moreover, after contagion, some agents may be recovered from high risk and get a temporary immunity for a temporal period represented by the time delay characterizing the dynamics.
We perform the analysis around the not risk-free equilibrium in terms of asymptotic stability and point out the crucial role of the incubation time and the financial immunity period in establishing whether risk crisis continues to exists in the economic sector at the long run or it can be eliminated.

Mathematics Subject Classification: 91B05, 91B55, 34K04

Keywords: Risk contagion, Financial immunity, SIR model, Delay Differential equation, Stability analysis

1 Introduction

This study serves as the conclusion to a thorough investigation done on the differential problem suggested by [2]. The model is based on a compartmental approach describing the nonlinear dynamics associated with the spread of risk contagion in a certain economic sector. Precisely, it consists of a SIR differential system which is revisited according to an analogy existing between economic systems and ecosystems. An economic player population is divided into a set of distinct compartments or classes, which are defined in terms of risk with low and high level: the players with lower risk are susceptible to be infected by other agents with a risk at an extremely high level. The dynamics is affected by a time delay τ representing a financial immunity ability reached by those players which are recovered since they get a capability of risk control after infection: they remain in a kind of recovery pool for a period of length τ, after that they may revert to come back being susceptible again.
We suppose that $b > 0$ and $0 < \delta < 1$ are the birth rate of new players and the recovery rate from the high risk, respectively. Moreover, we also assume that some players indefinitely leave the economic sector at an elimination rate depending on the risk level of compartments: $\gamma_L > 0$ and $\gamma_H > 0$ represent the death rates of players with low risk and high risk, respectively. In addition, the phenomenon of contagion is modelled by a Holling Type II functional response with attack rate $a > 0$ and incubation time $h > 0$.
At any time t we denote by $S(t)$, $I(t)$ and $R(t)$ the densities of susceptible players, infected and recovered agents, respectively; then, according to the model in [2], the recovered density $R(t)$ can be explicitly integrated once $I(t)$ is known. Therefore, it is possible to focus only on investigating the dynamics

in terms of $S(t)$ and $I(t)$ defined by the following system

$$\frac{dS(t)}{dt} = b - \gamma_L S(t) - \frac{aS(t)}{1+haS(t)} I(t) + \delta e^{-\gamma_L \tau} I(t-\tau), \tag{1}$$

$$\frac{dI(t)}{dt} = \frac{aS(t)}{1+haS(t)} I(t) - (\gamma_H + \delta) I(t),, \tag{2}$$

equipped with suitable initial conditions

$$S(0) = S_0 > 0, \tag{3}$$

$$I(s) = I_0(s) \geq 0, \quad \text{for all } s \in [-\tau, 0], \quad \text{with } I_0(0) > 0, \tag{4}$$

where $I_0(\cdot)$ is a continuous function and represents the history of the infected class in the whole time lag interval $[-\tau, 0]$. In [2] we show that problem (1)-(2) admits a unique positive solution. Moreover, it is characterized by the risk-free equilibrium $E_0^* = (b/\gamma_L, 0)$ and one more non-trivial steady state $E_\tau^* = (S_\tau^*, I_\tau^*)$ for certain values of the parameters. Precisely, we have

$$S_\tau^* = \frac{\gamma_H + \delta}{a(1 - h(\gamma_H + \delta))}, \qquad I_\tau^* = \frac{\gamma_L(\gamma_H + \delta)}{(\gamma_H + \delta(1 - e^{-\gamma_L \tau}))\, a(1 - h(\gamma_H + \delta))}(\rho_0 - 1),$$

where

$$\rho_0 = \frac{ba(1 - h(\gamma_H + \delta))}{\gamma_L(\gamma_H + \delta)},$$

is the basic reproduction number of the risk infection. In the following Sections, we assume that

$$\rho_0 > 1, \tag{5}$$

which assures that E_τ^* is a feasible steady state. According to the analysis developed in [3], condition (5) implies that the risk-free equilibrium E_0^* is unstable.

The main focus of this paper consists of discussing both local and global asymptotic stability related to the not risk-free equilibrium point E_τ^* under the assumption (5).

2 Local stability of the not risk-free equilibrium

With the aim of discussing the local stability of the not risk-free equilibrium, we focus on the characteristic equation at E_τ^*:

$$\lambda^2 + \left(\gamma_L + \frac{aI_\tau^*}{(1+haS_\tau^*)^2}\right)\lambda + \frac{aI_\tau^*}{(1+haS_\tau^*)^2}(\gamma_H + \delta - \delta e^{-(\gamma_L+\lambda)\tau}) = 0. \tag{6}$$

When $\tau = 0$, it reduces to the following equation with positive coefficients

$$\lambda^2 + \left(\gamma_L + \frac{aI_\tau^*}{(1+haS_\tau^*)^2}\right)\lambda + \frac{aI_\tau^*}{(1+haS_\tau^*)^2}\gamma_H = 0,$$

which has roots with negative real parts. Therefore, under condition (5) equilibrium E_τ^* is locally stable when $\tau = 0$.
In the other case when $\tau > 0$, we suppose that $\lambda = iv$ $(v > 0)$ is a solution of equation (6) and obtain

$$-v^2 + \left(\gamma_L + \frac{aI_\tau^*}{(1+haS_\tau^*)^2}\right)iv + \frac{aI_\tau^*}{(1+haS_\tau^*)^2}(\gamma_H + \delta - \delta e^{-\gamma_L\tau}e^{-iv\tau}) = 0.$$

After separating real and imaginary parts, we have

$$\begin{aligned} -v^2 + (\gamma_H + \delta)\tfrac{aI_\tau^*}{(1+haS_\tau^*)^2} &= \tfrac{aI_\tau^*}{(1+haS_\tau^*)^2}\delta e^{-\gamma_L\tau}\cos(v\tau), \\ \left(\gamma_L + \tfrac{aI_\tau^*}{(1+haS_\tau^*)^2}\right)v &= -\tfrac{aI_\tau^*}{(1+haS_\tau^*)^2}\delta e^{-\gamma_L\tau}\sin(v\tau), \end{aligned}$$

which yields

$$v^4 + p_0 v^2 + \left(\frac{aI_\tau^*}{(1+haS_\tau^*)^2}\right)^2 (\gamma_H^2 + 2\gamma_H\delta + \delta^2(1 - e^{-2\gamma_L\tau})) = 0, \tag{7}$$

where we set $p_0 = p(\frac{aI_\tau^*}{(1+haS_\tau^*)^2})$ with $p(\cdot)$ corresponding to the following polynomial function

$$p(\omega) = \omega^2 - 2(\gamma_H + \delta - \gamma_L)\omega + \gamma_L^2.$$

Condition $p_0 \geq 0$ implies that equation (7) cannot be solved; in this respect, it represents a sufficient condition to garantee that E_τ^* is locally asymptotically stable. The previous argument proves the following Proposition.

Proposition 2.1 *Under condition (5), the not free-risk equilibrium E_τ^* is locally asymptotically stable for any delay $\tau \geq 0$ such that the following assumption holds*

$$p\left(\frac{aI_\tau^*}{(1+haS_\tau^*)^2}\right) = \left(\frac{aI_\tau^*}{(1+haS_\tau^*)^2}\right)^2 - 2(\gamma_H + \delta - \gamma_L)\frac{aI_\tau^*}{(1+haS_\tau^*)^2} + \gamma_L^2 \geq 0.$$

3 Global stability for not trivial equilibrium

The analysis is completed by investigating the global stability at E_τ^*. With this aim, we introduce new variables defined as

$$u_1(t) = S(t) - S_\tau^*, \qquad u_2(t) = I(t) - I_\tau^*;$$

and rewrite the problem in the following form

$$\begin{aligned}\frac{du_1(t)}{dt} &= -\gamma_L u_1(t) - \frac{aS(t)}{1+haS(t)}(u_2(t)+I^*_\tau) + \delta e^{-\gamma_L\tau}u_2(t-\tau) + (\gamma_H+\delta)I^*_\tau,\\ \frac{du_2(t)}{dt} &= \left(\frac{aS(t)}{1+haS(t)} - (\gamma_H+\delta)\right)(u_2(t)+I^*_\tau).\end{aligned}$$

We set

$$\omega = \frac{\gamma_L+\gamma_H+\delta}{(1-h(\gamma_H+\delta))\,aI^*_\tau} > 0, \tag{8}$$

and consider the following functional

$$V(u(t)) = \frac{1}{2}(u_1(t)+u_2(t))^2 + \frac{1}{2}\omega u_2^2(t),$$

in $u(t) = (u_1(t), u_2(t))$, whose derivative corresponds to

$$\begin{aligned}\frac{dV}{dt} &= (u_1(t)+u_2(t))\left(-\gamma_L u_1(t) - (\gamma_H+\delta)u_2(t) + \delta e^{-\gamma_L\tau}u_2(t-\tau)\right)\\ &\quad +\omega\left(\frac{aS(t)}{1+haS(t)} - (\gamma_H+\delta)\right)(u_2(t)+I^*_\tau)u_2(t),\end{aligned}$$

According to the results proved in [2], we notice that $S(t) > 0$ for any t; therefore, in the previous derivative we may employ the bound $0 < \frac{aS(t)}{1+haS(t)} \le \frac{1}{h}$ and alternatively $\frac{aS(t)}{1+haS(t)} - (\gamma_H+\delta) = \frac{au_1(1-h(\gamma_H+\delta))}{1+ha(u_1+S^*_\tau)} \le au_1(1-h(\gamma_H+\delta))$. Furthermore, the CauchySchwartz inequality is applied to the term $(u_1(t)+u_2(t))\cdot\delta e^{-\gamma_L\tau}u_2(t-\tau)$, so that we get

$$\frac{dV}{dt} \le -\frac{\gamma_L}{2}u_1^2(t) + C_1u_2^2(t-\tau) - (C_2-\omega C_3)\,u_2^2(t),$$

where

$$C_1 = \frac{\delta^2(\gamma_L+\gamma_H+\delta)}{2\gamma_L(\gamma_H+\delta)}, \quad C_2 = \frac{\gamma_H+\delta}{2}, \quad C_3 = \frac{1-h(\gamma_H+\delta)}{h}.$$

Then we consider the Lyapunov functional of the form

$$U(u(t)) = V(u(t)) + C_1\int_{t-\tau}^{t} u_2^2(\theta)d\theta,$$

in order to obtain

$$\frac{dU}{dt} = \frac{dV}{dt} + C_1u_2^2(t) - C_1u_2^2(t-\tau) \le -\frac{\gamma_L}{2}u_1^2(t) - (C_2-\omega C_3-C_1)\,u_2^2(t).$$

We aim to get $C_2 - \omega C_3 - C_1 > 0$ which is equivalent to $\omega < (C_2 - C_1)/C_3$ that is

$$(\gamma_H + \delta)\left(1 - \frac{\gamma_L(\gamma_H + \delta)}{\gamma_L + \gamma_H + \delta} \cdot \frac{C_2 - C_1}{C_3}(\rho_0 - 1)\right) < \delta e^{-\gamma_L \tau}, \qquad (9)$$

due to (8). Inequality (9) assures that U is negative definite. Then, a direct application of the LyapunovLaSalle type theorem (see Theorem 2.5.3 in [4], p. 30) implies that both $u_1(t) \to 0$ and $u_2(t) \to 0$ as $t \to +\infty$ when (9) holds. As a consequence, (9) represents a sufficient condition to guarantee that equilibrium E^*_τ is globally asymptotically stable.
In this respect, a specific case when (9) is satisfied for any time delay $\tau > 0$ corresponds to have the basic reproduction number being sufficiently large such that

$$\rho_0 > 1 + \frac{\gamma_L + \gamma_H + \delta}{\gamma_L(\gamma_H + \delta)};$$

as a consequence, in that case the not risk-free equilibrium is globally asymptotically stable.

4 Concluding remarks

As already pointed out, the not risk-free steady state is feasible under assumption (5) on the basic reproduction number ρ_0. The length of h affects the feasibility of E^*_τ, indeed requirement (5) results in h small enough to satisfy $h < (ba - \gamma_L(\gamma_H + \delta))/(ba(\gamma_H + \delta))$. Furthermore, we remark that time delay τ also affects the global stability behaviour according to condition (9). In this respect, both incubation time and immunity period play a role in the dynamics at the long run near the not risk-free steady state.

As the last remark, we notice that the dynamics at the long run may represent a tool to understand whether risk disappears or continues to exists at the steady state in specific real economic systems. As an example on a toy simulation problem, the model can be applied to simulate a risk spreading by exploiting the same data provided in [1] concerning the sector of food industries in Emilia-Romagna Italian region, which plays a key economic role in the entire territory of northern Italy. In this respect, as in [1] some parameters are estimated as $b = 0.04$, $\delta = 0.05$, $a = 0.12$. Moreover, we set the elimination rates at $\gamma_L = 0.009$ and $\gamma_H = 0.03$ in a hypothetical scenario. The differential system starts from the same initial conditions estimated in [1]. As incubation time and financial immunity period are not estimated, then it is possible to choose hypothetical values such as $\tau = 1$ and $h = 0.5$. The dynamics is simulated by the built-in function `dde23` in MATLAB environment, which integrates DDEs with constant delays: the result is plot in Figure 1. SIR

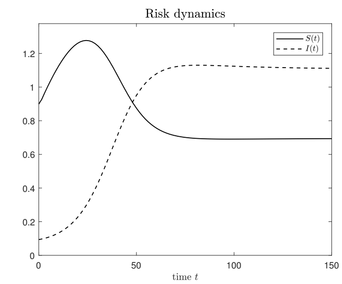

Figure 1: Dynamics of risk contagion. Solid line: susceptible dynamics. Dashed line: infected dynamics.

trajectory approaches the not free-risk equilibrium, thus risk infection remains in the economy in the long run.

In this framework, the mathematical model may be a useful tool for predicting and simulating different risk contagion scenarios to support policy makers on their financial actions to contain a financial distress.

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