A kinematically coupled time-splitting scheme for fluid-structure interaction in blood flow

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Abstract

We present a new time-splitting scheme for the numerical simulation of fluid-structure interaction between blood flow and vascular walls. This scheme deals in a successful way with the problem of the added mass effect. The scheme is modular and it embodies the stability properties of implicit schemes at the low computational cost of loosely coupled ones.

1. Introduction

We study the fluid-structure interaction between blood flow and arterial walls in the flow regime corresponding to blood flow in medium-to-large arteries. Due to the fact that vascular tissue (structure) is "light" relative to blood (fluid), the coupling between fluid and structure at the interface is highly nonlinear. This causes instability and consequent failure of the traditional loosely coupled schemes successfully used in aeroelasticity [1,2,3,4,5,6]. This is due to the *added mass effect* [1] which is most pronounced in problems where fluid and structure have comparable densities.

To get around this difficulty several approaches have been developed. One is based on the *implicit* or *fully-coupled* algorithms, see e.g. [7,1,5,8,6,9], that are quite robust and stable but unfortunately they are quite demanding in terms of computational time. Other, more recent approaches, are based on *semi-implicit schemes* where fluid velocity is decoupled from the strongly coupled pressure-structure system [10,11], and *monolithic-like schemes* where a thin structure is incorporated into the fluid equations

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via a Robin-like boundary condition [12]. Finally, the work in [13] presents a stabilized *explicit* coupling scheme using a penalty term on the fluid force fluctuations acting on the interface.

In this paper, we present a *kinematically coupled time-splitting scheme* which is truly modular and embodies the stability properties of implicit schemes at the computational cost of explicit schemes. The time-splitting is performed on the full problem, written as a first-order system via the kinematic coupling condition, at the differential level. In contrast with classical partitioned schemes, which rely on splitting the fluid from the structure, our strategy is based on splitting the structure in its *hydrodynamic* and *elastic* parts. The hydrodynamic part, consisting of the fluid stress acting on the interface and the viscoelastic terms, is treated together with the fluid. By adding the hydrodynamic part of the structure equation to the fluid equation and by utilizing the kinematic interface condition, we deal with the inertia of both fluid and structure at the same time, thereby getting around the difficulty associated with the added mass effect. The elastic part is treated separately and this enables the use of a wide range of structure models. Our strategy admits extreme flexibility in the choice of solution methods for each sub-problem resulting from the splitting, making this algorithm truly modular. Since no iterations between the two main steps are necessary to achieve stability, the complexity of our method is that of loosely coupled schemes. Because of the crucial role played by the kinematic condition at the interface, we call our numerical scheme a *kinematically-coupled time-splitting scheme*.

2. Mathematical model

Consider the flow of an incompressible, viscous fluid in a two-dimensional channel with thin, deformable walls, and with axial symmetry, see Fig. 1. As in [1] we assume that the structure deformation is small enough so that the fluid domain can be considered fixed. This assumption is not essential for our kinematically coupled scheme since the motion of the fluid domain can be easily incorporated using, e.g., an ALE approach as in [14]. The problem studied here retains the essential difficulties of the full problem while allowing simpler presentation of the numerical scheme.

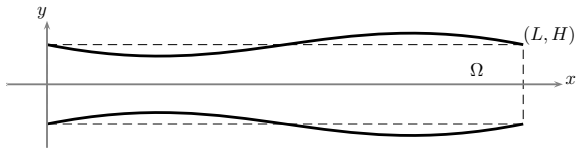

FIG. 1. Sketch of the domain.

The fluid motion is governed by the Navier-Stokes equations :

$$\varrho_f\big(\partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}\big)=-\nabla p+\mu\Delta\mathbf{u},\quad \nabla\cdot\mathbf{u}=0,\quad \text{in } \Omega\times(0,T), \tag{1}$$

where $\mathbf{u}=(u_1(x,y;t),u_2(x,y;t))$ is the fluid velocity, $p(x,y;t)$ is the pressure, ϱ_f is the fluid density and μ is the fluid viscosity. The fluid domain Ω is the rectangle $(0,L)\times(0,H)$. At the lower boundary we impose the symmetry condition, while at the inlet and outlet sections we prescribe the generalized Neumann boundary conditions related to a pressure drop, namely :

$$\partial_y u_1|_{y=0}=0,\quad u_2|_{y=0}=0,\quad \big(p\mathbf{n}-\mu(\mathbf{n}\cdot\nabla)\mathbf{u}\big)|_{x=0}=\bar{p}(t)\mathbf{n},\quad \big(p\mathbf{n}-\mu(\mathbf{n}\cdot\nabla)\mathbf{u}\big)|_{x=L}=\mathbf{0}\,. \tag{2}$$

The deformable structure constitutes the upper portion of the domain boundary. In the present article, we will use the generalized viscoelastic string model [15] to describe the structure dynamics. This choice was motivated by the fact that several numerical results are available in literature for this type of structure model, and therefore this constitutes a useful benchmark problem for our algorithm. We remark, though,

that more complicated shell models, such as [16,17,18], could be easily considered. In the generalized viscoelastic string model, the structure undergoes only transverse displacement $\eta = \eta(x,t)$:

$$\varrho_s h_s \partial_t^2 \eta + a\eta - b\partial_x^2 \eta - \gamma \partial_t \partial_x^2 \eta = p|_{y=H} \quad \text{on } (0,L)\times(0,T)\,, \tag{3}$$

where ϱ_s is the structure density, h_s is the structure thickness, and a, b, and γ are elastic constants. More precisely, $a = Eh_s/H^2(1-\sigma^2)$, where E is the Young's modulus and σ is the Poisson ratio, $b = Gh_s$, where G is the shear modulus, and γ is the viscoelastic constant. We remark that, for sake of simplicity, the fluid action on the structure enters only through the pressure, but the influence of the viscous part of the fluid stress could be considered as well. The coupling between the fluid and the structure is both dynamic, through equation (3), and kinematic, through the following kinematic condition :

$$u_1|_{y=H} = 0, \quad \partial_t \eta = u_2|_{y=H}. \tag{4}$$

The following initial and boundary conditions for η and $\mathbf{u}$ are prescribed :

$$\mathbf{u}|_{t=0} = \mathbf{0}, \quad \eta|_{t=0} = 0, \quad \partial_t\eta|_{t=0} = 0, \quad \eta|_{x=0} = 0, \quad \eta|_{x=L} = 0. \tag{5}$$

3. The kinematically-coupled scheme : main ideas

To illustrate the main ideas of the kinematically-coupled scheme, we rewrite (1), (3), and (4) as follows :

$$\begin{cases} \varrho_f \partial_t \mathbf{u} = \Phi(\mathbf{u},p), \quad \nabla\cdot\mathbf{u} = 0, \quad \text{in } \Omega\times(0,T), \\ \varrho_s h_s \partial_t^2 \eta = \Psi(\eta) + \Pi(\partial_t\eta) + \Upsilon(\mathbf{u},p) \quad \text{on } (0,L)\times(0,T), \\ \partial_t\eta = u_2|_{y=H} \quad \text{on } (0,L)\times(0,T), \end{cases} \tag{6}$$

where $\Psi(\eta)$ and $\Pi(\partial_t\eta)$ embody the constitutive equation of the structure, while $\Upsilon(\mathbf{u},p)$ represents the hydrodynamic load on the structure which is due to the fluid stress on the interface. For the particular mathematical model presented in Section 2, $\Phi(\mathbf{u},p) = -\varrho_f \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p + \mu\Delta\mathbf{u}$, $\Psi(\eta) = -a\eta + b\partial_x^2\eta$, $\Pi(\partial_t) = \gamma\partial_x^2\partial_t\eta$, and $\Upsilon(\mathbf{u},p) = p|_{y=H}$.

3.1. *Traditional loosely coupled schemes*

Traditional loosely coupled schemes are based on the simple idea of decoupling the fluid from the structure. This can be achieved by using the structure velocity from the previous time step as Dirichlet data for the fluid velocity on the interface, and then use the computed fluid stress at the interface to force the structure and obtain the new displacement, see, e.g., [1].

Step 1. (Fluid) Given $\mathbf{u}(t^n) = \mathbf{u}^n$, $\eta(t^n) = \eta^n$, and $\partial_t\eta(t^n) = g^n$, solve :

$$\begin{cases} \varrho_f \partial_t \mathbf{u} = \Phi(\mathbf{u},p), \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(t^n,t^{n+1})\,, \\ \mathbf{u}|_{y=H} = g^n\mathbf{n}\,, \quad \text{on } (0,L)\times(t^n,t^{n+1})\,, \end{cases} \tag{7}$$

and then set $\mathbf{u}(t^{n+1}) = \mathbf{u}^{n+1}$ and $p(t^{n+1}) = p^{n+1}$.

Step 2. (Structure) Given $\eta(t^n) = \eta^n$, and $\partial_t\eta(t^n) = g^n$, solve :

$$\varrho_s h_s \partial_t^2 \eta = \Psi(\eta) + \Pi(\partial_t\eta) + \Upsilon(\mathbf{u}^{n+1}, p^{n+1}) \quad \text{on } (0,L)\times(t^n,t^{n+1})\,, \tag{8}$$

and then set $\eta(t^{n+1}) = \eta^{n+1}$, and $\partial_t\eta(t^{n+1}) = g^{n+1}$.

3.2. *The kinematically-coupled scheme : a new splitting strategy*

As a first fundamental step, we use the kinematic coupling condition to rewrite the structure acceleration in terms of the fluid acceleration at the interface. This allows us to obtain a new formulation of problem (6), involving only first-order differential operators in time :

$$\begin{cases} \varrho_f \partial_t \mathbf{u} = \Phi(\mathbf{u}, p), \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \times (0,T) \,, \\ \varrho_s h_s \partial_t u_2|_{y=H} = \Psi(\eta) + \Pi(u_2|_{y=H}) + \Upsilon(\mathbf{u}, p) \quad \text{on } (0,L) \times (0,T) \\ \partial_t \eta = u_2|_{y=H} \quad \text{on } (0,L) \times (0,T) \,. \end{cases} \tag{9}$$

We now perform the time discretization by operator splitting. Instead of splitting the fluid from the structure, as in the traditional splitting schemes, we split the structure equation in its *hydrodynamic* and *elastic* parts. The hydrodynamic part $\Pi(u_2|_{y=H})+\Upsilon(\mathbf{u},p)$ (viscoelasticity and fluid stress on the interface) will be treated together with the fluid equations. The elastic part, $\Psi(\eta)$, will be treated separately.

Step 1. Given $\mathbf{u}(t^n) = \mathbf{u}^n$, $\eta(t^n) = \eta^n$, and $\partial_t \eta(t^n) = g^n$, solve :

$$\begin{cases} \varrho_f \partial_t \mathbf{u} = \Phi(\mathbf{u}, p), \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \times (t^n, t^{n+1}) \,, \\ \varrho_s h_s \partial_t u_2|_{y=H} = \Pi(u_2|_{y=H}) + \Upsilon(\mathbf{u}, p) \,, \quad \text{on } (0,L) \times (t^n, t^{n+1}) \,, \end{cases} \tag{10}$$

and then set $\mathbf{u}(t^{n+1}) = \mathbf{u}^{n+1/2}$ and $p(t^{n+1}) = p^{n+1}$.

Step 2. Given $\eta(t^n) = \eta^n$, and $\partial_t \eta(t^n) = u_2|_{y=H}^{n+1/2}$, solve :

$$\begin{cases} \partial_t \eta = u_2|_{y=H} \\ \varrho_s h_s \partial_t u_2|_{y=H} = \Psi(\eta) \end{cases} \tag{11}$$

on $(0,L) \times (t^n, t^{n+1})$, and then set $\eta(t^{n+1}) = \eta^{n+1}$ and $\partial_t \eta(t^{n+1}) = u_2|_{y=H}^{n+1}$.

This scheme has the following three appealing features :

(i) *Stability properties of implicit schemes with the complexity of traditional loosely coupled schemes.*

(ii) *The non-dissipative sub-problems can be treated with non-dissipative solvers.*

(iii) *Modularity : existing fluid and structure solvers can be used as "black boxes".*

4. Numerical experiments

The kinematically coupled scheme was applied to the model described in Section 2, and compared with the analogous problem solved by Formaggia et al. in [15] using an implicit scheme. Geometry, fluid and structure parameters are $L = 6$ cm, $H = 1$ cm, μ=0.035 poise, ρ_f=1 g/cm^3, $E = 0.75 \times 10^6$ dynes/cm^3, $\sigma = 0.5$, ρ_s=1.1 g/cm^2, h_s=0.1 cm, $G = E/2(1+\sigma)$, γ=0.01 poise·cm. The inlet pressure is the pulse given by $\bar{p}(t) = p_{max}$ for $t \le t_{max}$, and $\bar{p}(t) = 0$ for $t > t_{max}$, where $p_{max} = 2 \cdot 10^4$ dynes/cm^2 and $t_{max} = 0.005$ s. Problem (10) is solved in three substeps : (1) the generalized Stokes problem is solved with half the viscosity, then (2) the fluid advection, and (3) the generalized Stokes problem with the remaining half of the viscosity. This approach enforces the divergence free condition and smooths out the solution. The two generalized Stokes problems are solved using a conjugate gradient method in combination with a Backward Euler scheme [19]. The fluid advection and problem (11) are solved using a wave-like method [19,20,14]. The time step is mainly dictated by a CFL condition for the proper resolution of the wave-propagation in the structure. Numerical experiments showed that our algorithm is stable for a wide range of Δt. We

tested values of Δt between 10^{-6} and 10^{-3}, which is the same range as for implicit schemes [1,15] ; our method was stable for all the values. Results shown in Fig. 2 have been obtained with $\Delta t = 5 \cdot 10^{-5}$. A smaller time substep of $\Delta t/5$ is used in non-dissipative sub-problems. Space discretization is obtained using a $P1-iso-P2$ and $P1$ finite element approximation, where the pressure is approximated on a mesh which is twice coarser than that used for the velocity field. Results shown in Fig. 2 have been obtained using a uniform mesh, where the size of the related pressure mesh was $h_p = H/8$. The numerical solution obtained with the kinematically-coupled scheme (10)-(11) is shown in Fig. 2 at six different snap-shots. Each plot contains the information on pressure (colormap), velocity field (streamlines), and structure displacement. Fig. 2 shows a pressure wave moving forward and reflecting once it reaches the end of the domain. The reflected wave is characterized by the negative values of pressure and positive flow rates [15,21]. The results obtained with our method are in excellent agreement with those obtained in [15] using an implicit scheme.

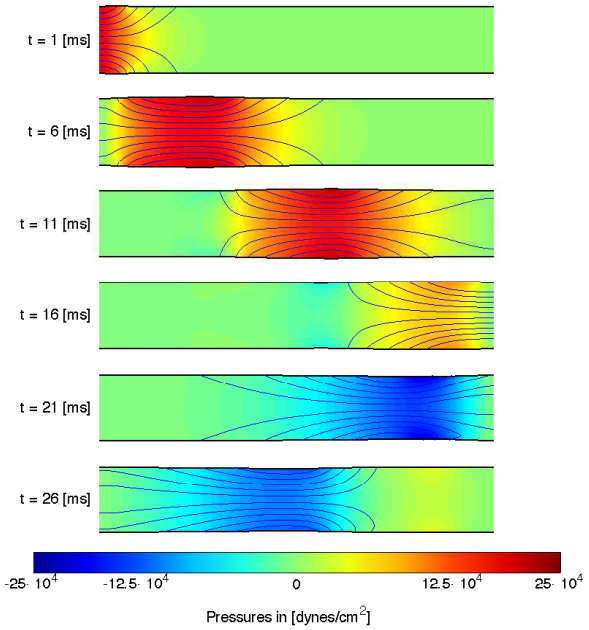

FIG. 2. Pressure maps, membrane displacements, black lines, and stream lines at different simulations times.

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