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The role of convection in the existence of wavefronts for biased movements

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We investigate a model, inspired by Johnston et al. (2017) to describe the movement of a biological population which consists of isolated and grouped organisms. We introduce biases in the movements and then obtain a scalar reaction–diffusion equation that includes a convective term as a consequence of the biases. We focus on the case the diffusivity makes the parabolic equation of forward–backward–forward type and the reaction term models a strong Allee effect, with the Allee parameter lying between the two internal zeros of the diffusion. In such a case, the unbiased equation (i.e., without convection) possesses no smooth traveling-wave solutions; on the contrary, in the presence of convection, we show that traveling-wave solutions do exist for some significant choices of the parameters. We also study the sign of their speeds, which provides information on the long term behavior of the population, namely, its survival or extinction.

KEYWORDS

biased movement, diffusion–convection–reaction equations, population dynamics, sign-changing diffusivity, traveling-wave solutions

MSC CLASSIFICATION

35K65, 35C07, 35K57, 92D25

1 | INTRODUCTION

In this paper, we investigate a model to describe the movement of biological organisms. Its detailed presentation appears in Section 3. Inspired by the recent paper [1], we assume that the population is constituted of isolated and grouped organisms; our discussion is presented in the case of a single spatial dimension but could be extended to the whole space. The first rigorous mathematical deduction of movement for organisms appeared in [2]; since then, several models have been proposed; see for instance [1, 3–6] and references therein. In this context, a common procedure is to start from a discrete framework where the transition probabilities per unit time τ and for a one-step jump-width l are assigned, and then pass to the limit for $\tau, l \to 0$. In the aforementioned papers, the limiting assumptions make the diffusivity totally responsible for the movement, and no convection term appears; see however [5, §5.3] and [7], for instance, for the deduction of a model that also includes a convective effect. Here, we generalize the model in [1] by introducing a possibly *biased* movement, which leads, in general, to a convective term. As a consequence, we show the appearance of a greater variety of dynamics that allow to better investigate the long-term behavior of the population, in particular to predict its survival or extinction.

Our model is described by a reaction–diffusion–convection equation

$$u_t + f(u)_x = (D(u)u_x)_x + g(u), \qquad t \geq 0, x \in \mathbb{R}, \tag{1.1}$$

where the functions f, D, and g satisfy (3.6), (3.7), and (3.8), respectively. The unknown function u denotes the density (or concentration) of the population and then it has bounded range; for simplicity, we assume $u \in [0, 1]$. An interesting feature of Equation (1.1) in this context is that *negative diffusivities* arise for several natural choices of the parameters. As in [1], here we consider a diffusion term that makes Equation (1.1) of forward–backward–forward type. This occurrence was already noticed in other papers; see for instance [8, 9] in the case of a homogeneous population under different assumptions. Notice however that the deduction of the model both in [1] and in the present paper also involves the reaction term, while in [8, 9] it is limited to diffusion. As opposite to *positive* diffusivities, which model the spatial spreading, *negative* diffusivities are usually interpreted to model the "chaotic" movement, which follows from aggregation [8, 9]. In turn, the latter is "a macroscopic effect of the isolated and the grouped motility of the agents, together with competition for space"[10]. At last, we assume that the reaction term g shows the strong Allee effect; that is, it is of the so called *bistable* type (see assumption (g) below).

We focus on the existence of traveling-wave solutions $u(x, t) = \varphi(x - ct)$ to Equation (1.1), for some profiles $\varphi = \varphi(\xi)$ and wave speeds c; see [11] for general information. If the profile is defined in $\mathbb{R}$, it is monotone, nonconstant, and reaches asymptotically the equilibria of (1.1), then the corresponding traveling-wave solution is called a *wavefront*. We consider precisely decreasing profiles, which connect the outer equilibria of g, that is,

$$\varphi(-\infty) = 1 \text{ and } \varphi(\infty) = 0. \tag{1.2}$$

The case when profiles are increasing, and then satisfy $\varphi(-\infty) = 0$, $\varphi(\infty) = 1$, is dealt analogously and leads to a similar discussion. These solutions, even if of a special kind, have several advantages: They are global, they are often in good agreement with experimental data [4], and they can be attractors for more general solutions [12]. Moreover, when u represents the density of a biological species, as in this case, then condition (1.2) means that, for times $t \to \infty$, the species either successfully persists if $c > 0$, or it becomes extinct if $c < 0$. The wavefront profile φ must satisfy the ordinary differential equation

$$\left(D(\varphi)\varphi'\right)' + \left(c - \dot{f}(\varphi)\right)\varphi' + g(\varphi) = 0. \tag{1.3}$$

We used the notation $\cdot := d/du$ and $' := d/d\xi$. Although one can consider the case of discontinuous profiles (see [10, 13] and references therein), in this paper we focus on regular monotone profiles of Equation (1.3). This means that they are continuous and of class C^2 except possibly at points where D vanishes; then solutions to Equation (1.3) are intended in the distribution sense.

The existence of wavefronts is treated here in a quite general framework, which includes, in particular, our biological model. More precisely, we fix three real numbers α, β, γ satisfying

$$0 < \alpha < \gamma < \beta < 1, \tag{1.4}$$

and assume (see Figure 1)

(f) $f \in C^1[0, 1]$;
(D) $D \in C^1[0, 1]$, $D > 0$ in $[0, \alpha) \cup (\beta, 1]$, and $D < 0$ in (α, β);
(g) $g \in C^1[0, 1]$, $g < 0$ in $(0, \gamma)$, $g > 0$ in $(\gamma, 1)$, and $g(0) = g(\gamma) = g(1) = 0$.

Since f in (1.1) is defined up to an additive constant, we can take $f(0) = 0$. The term $\dot{f}(u)$ represents the drift of the total concentration u and prescribes in particular if a concentration wave is moving toward the right ($\dot{f}(u) > 0$) or toward the left ($\dot{f}(u) < 0$). The parabolic equation (1.1) is of *backward* type in the interval (α, β) and of *forward* type elsewhere; moreover, it degenerates at α and β.

The presence of wavefronts to (1.1) satisfying (D) and (g) and with $f = 0$ was first discussed in [14], where it is shown in particular that, if a wavefront exists, then $\gamma \notin [\alpha, \beta]$. Such a situation and many others, again with $f = 0$, was also considered in [1, cases 6.3, 8.3], in the framework of the particular model deduced in that paper. The case with convection

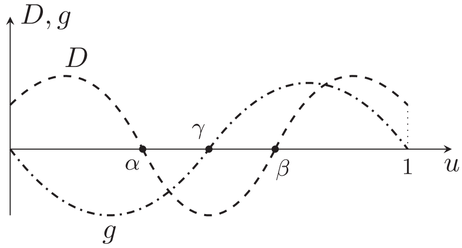

FIGURE 1 Typical plots of the functions D (dashed line) and g (dashdotted line).

is not yet completely understood. Then our issue here is

whether and when the presence of the convective flow allows the existence of wavefronts.

An intuitive argument (see Remark 2.3) shows that the answer is in the affirmative at least for suitable concave f. We now briefly report on the content of this paper.

In Section 2, we investigate the *fine properties* (uniqueness, strict monotonicity, estimates of speed thresholds) of the wavefronts of Equation (1.1) that satisfy (f)-(D) and (g). A similar discussion for a *monostable* reaction term g appeared in [15] and [16], respectively, in the general framework and for the population model with biased movements. We recall that g is called *monostable* if $g > 0$ in $(0, 1)$ and $g(0) = g(1) = 0$. In Section 3, we introduce our biological model, and in Section 4, we give some preliminary results about it. In particular, we point out that the convection in the model can be either concave, or convex or else change convexity–concavity at least once. The following sections separately deal with each of these cases; we provide results for the existence of wavefronts (both sufficient conditions and necessary conditions) and investigate the sign of the propagation speed. Section 7 resumes and compares all the results we obtained. At the same time, we suggest a biological interpretation for them.

As in our aforementioned papers, we exploit here an order-reduction technique. Since we focus on profiles $\varphi = \varphi(\xi)$ that are strictly monotone when $\varphi \in (0, 1)$, we can consider the inverse function $\varphi^{-1}(\varphi)$ of φ and, by denoting $z(\varphi) := D(\varphi)\varphi'\left(\varphi^{-1}(\varphi)\right)$, we reduce the problem (1.3) to a first-order singular boundary-value problem for z in $[0, 1]$. This problem is tackled by the classical techniques of upper and lower solutions. This technique requires lighter assumptions than the phase-plane analysis in [1] and is simpler than the geometric singular perturbation theory exploited in [13]. Then wavefronts satisfying (1.2) are obtained by suitably pasting traveling waves. The results appear in Section 2; they are given for an arbitrary equation (1.1) satisfying conditions (f), (D), and (g), and they are original. About (g), the mere requirement that g is continuous and the product Dg differentiable at 0 would be sufficient for us. Both for (D) and (g), we made slightly stricter assumptions than necessary both for simplicity and because they are satisfied by our biological model with biased movements. The cases when the internal zero of g is before α, i.e. $\gamma \in (0, \alpha)$, or after β, that is $\gamma \in (\beta, 1)$, are not treated here. Equation (1.1) with $f = 0$ admits wavefronts in these cases and we expect that they persist also in the presence of the convective effect f.

The issue of the linear stability of the wavefronts is certainly interesting; we claim that it could be developed as in [13, 17], with a similar discussion.

2 | THEORETICAL RESULTS

In this section, we provide the theoretical results that are needed for the investigation of model (3.5). In the following, we consider Equation (1.1) and we always assume (1.4) and (f), (D), and (g), without any further mention. The existence of a wavefront solution to (1.1), whose profile satisfies (1.2), is obtained by pasting profiles connecting 0 with α, α with γ, γ with β, and β with 1. Each subprofile exists for c larger or smaller than a certain threshold, which varies according to the subinterval: We denote them by

$$c^*_{0,\alpha},\ c^*_{\alpha,\gamma},\ c^*_{\gamma,\beta},\ c^*_{\beta,1},$$

respectively. The expressions of these thresholds are not explicit, but we provide below rather precise estimates for them. We denote

$$c_0 := \min\left\{c^*_{0,\alpha}, c^*_{\alpha,\gamma}\right\} \text{ and } c_1 =: \max\left\{c^*_{\gamma,\beta}, c^*_{\beta,1}\right\}. \tag{2.1}$$

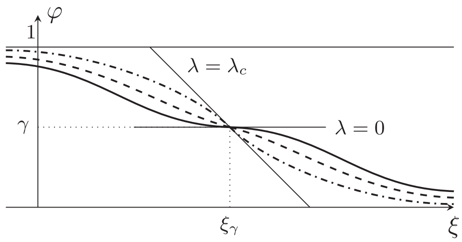

FIGURE 2 Some profiles with the same speed c, in the case $c_1 < c_0$ in Theorem 2.1; profiles have been shifted so that $\varphi(\xi_\gamma) = \gamma$.

In the above pasting framework, c_0 involves the speeds of profiles connecting 0 with α and then α to γ, while c_1 refers to the connections γ to β and then β to 1.

The following main result concerns general necessary and sufficient conditions for the existence of wavefronts.

Theorem 2.1. *If*

$$c_1 < c_0, \tag{2.2}$$

then for every $c \in (c_1, c_0)$, there are wavefronts to Equation (1.1) satisfying (1.2).

Conversely, if $c_1 > c_0$, there exists no wavefronts to Equation (1.1) satisfying (1.2).

We point out that, in the case $c_1 < c_0$, there are *infinitely many* profiles with the *same* speed $c \in (c_1, c_0)$. More precisely, see Figure 2, for every such c there exists $\lambda_c < 0$ and a family of profiles φ_λ, for $\lambda \in [\lambda_c, 0)$, which are characterized by

$$\varphi_\lambda(\xi_\gamma) = \gamma \quad \text{and} \quad \varphi'_\lambda(\xi_\gamma) = \lambda,$$

for some $\xi_\gamma \in \mathbb{R}$. The first condition simply says that all profiles have been shifted so that they reach the value γ at the same $\xi = \xi_\gamma$ (in order to make a comparison possible); the second one states that their slopes at ξ_γ cover the cone centered at (ξ_γ, γ) and opening $[\lambda_c, 0)$. We refer to [18] for more information.

We denote the *difference quotient* of a scalar function of a real variable $F = F(\varphi)$ with respect to a point φ_0 as

$$\delta(F, \varphi_0)(\varphi) := \frac{F(\varphi) - F(\varphi_0)}{\varphi - \varphi_0}, \quad \varphi \neq \varphi_0. \tag{2.3}$$

If F is differentiable in φ_0, then we understand $\delta(F, \varphi_0)(\varphi_0) = \dot{F}(\varphi_0)$.

We also introduce the integral mean of the difference quotient and denote it as

$$\Delta(F, \varphi_0)(\varphi) := \frac{1}{\varphi - \varphi_0} \int_{\varphi_0}^{\varphi} \delta(F, \varphi_0)(\psi)\, d\psi = \frac{1}{\varphi - \varphi_0} \int_{\varphi_0}^{\varphi} \frac{F(\psi) - F(\varphi_0)}{\psi - \varphi_0}\, d\psi. \tag{2.4}$$

Again, if F is differentiable in φ_0, we agree that $\Delta(F, \varphi_0)(\varphi_0) = \dot{F}(\varphi_0)$.

Notice that for every $\varphi \in (0, \gamma)$ there is $\varphi_1 \in (0, \gamma)$ such that $\Delta(Dg, \alpha)(\varphi) = \delta(Dg, \alpha)(\varphi_1)$. Then we have

$$\sup_{[0,\gamma]} \Delta(Dg, \alpha) \leq \sup_{[0,\gamma]} \delta(Dg, \alpha), \tag{2.5}$$

and the same estimate holds true in the interval $[\gamma, 1]$ by replacing α with β.

The following results provide necessary and sufficient conditions for the existence of decreasing profiles, in order to make condition (2.2) more explicit in terms of the functions f, D and g. The proofs are deferred to the end of this section.

Corollary 2.1 (Necessary condition)**.** *If there are wavefronts to Equation (1.1) whose profiles satisfy (1.2), then*

$$\min\left\{\inf_{[0,\gamma]} \delta(f, \alpha), \dot{f}(\alpha) - 2\sqrt{\dot{D}(\alpha)g(\alpha)}\right\} \geq \max\left\{\sup_{[\gamma,1]} \delta(f, \beta), \dot{f}(\beta) + 2\sqrt{\dot{D}(\beta)g(\beta)}\right\}. \tag{2.6}$$

In particular, wavefronts exist only if both conditions

$$\dot{f}(\alpha) - \dot{f}(\beta) \geq 2\sqrt{\dot{D}(\alpha)g(\alpha)} + 2\sqrt{\dot{D}(\beta)g(\beta)}, \tag{2.7}$$

$$\inf_{[0,\gamma]} \delta(f,\alpha) - \sup_{[\gamma,1]} \delta(f,\beta) \geq 0, \tag{2.8}$$

are satisfied. Notice that inequality (2.7) separates the behavior of f from that of Dg.

Remark 2.1. When f is strictly concave, we have

$$\inf_{[0,\gamma]} \delta(f,\alpha) = \delta(f,\alpha)(\gamma) = \frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}, \ \sup_{[\gamma,1]} \delta(f,\beta) = \delta(f,\beta)(\gamma) = \frac{f(\gamma)-f(\beta)}{\gamma-\beta}, \tag{2.9}$$

and then

$$\inf_{[0,\gamma]} \delta(f,\alpha) - \sup_{[\gamma,1]} \delta(f,\beta) = \frac{f(\gamma)-f(\alpha)}{\gamma-\alpha} - \frac{f(\gamma)-f(\beta)}{\gamma-\beta} > 0. \tag{2.10}$$

The following result shows, in particular, how far from zero must be the difference in (2.8) in order to have solutions. We denote with $\mathfrak{J}$ the set of *admissible speeds*, that is, the speeds c such that there is a profile with that speed satisfying (1.2).

Corollary 2.2 (Sufficient condition)**.** *We have the following results.*

1. *Assume*

$$\inf_{[0,\gamma]} \delta(f,\alpha) - \sup_{[\gamma,1]} \delta(f,\beta) > 2\sup_{[0,\gamma]} \sqrt{\Delta(Dg,\alpha)} + 2\sup_{[\gamma,1]} \sqrt{\Delta(Dg,\beta)}. \tag{2.11}$$

Then, Equation (1.1) admits wavefronts satisfying (1.2), and $\mathfrak{J}$ *is a bounded interval.*

2. *We have*

(i) *either* $\mathfrak{J} \subset (0,\infty)$ *or* $\mathfrak{J} = \emptyset$*, in the case*

$$\max\left\{\sup_{[\gamma,1]} \delta(f,\beta), \dot{f}(\beta) + 2\sqrt{\dot{D}(\beta)g(\beta)}\right\} > 0; \tag{2.12}$$

(ii) *either* $\mathfrak{J} \subset (-\infty,0)$ *or* $\mathfrak{J} = \emptyset$*, in the case*

$$\min\left\{\inf_{[0,\gamma]} \delta(f,\alpha), \dot{f}(\alpha) - 2\sqrt{\dot{D}(\alpha)g(\alpha)}\right\} < 0. \tag{2.13}$$

We now investigate when the set $\mathfrak{J}$ of admissible speeds contains positive values.

Corollary 2.3. *Assume*

$$\inf_{[0,\gamma]} \delta(f,\alpha) > 2\sup_{[0,\gamma]} \sqrt{\Delta(Dg,\alpha)}. \tag{2.14}$$

Then either $\mathfrak{J} = \emptyset$ *or* $\mathfrak{J} \cap (0,\infty) \neq \emptyset$.

2.1 | The proofs

In the proof of Theorem 2.1, we will reduce the existence of a wavefront to Equation (1.1) satisfying (1.2) to the investigation of a solution z to the following *singular* first-order problem in the interval $[0,1]$:

$$\begin{cases} \dot{z}(\varphi) = \dot{f}(\varphi) - c - \frac{D(\varphi)g(\varphi)}{z(\varphi)} & \text{in } (0,\alpha)\cup(\alpha,\beta)\cup(\beta,1), \\ z < 0 & \text{in } (0,\alpha)\cup(\beta,1), \\ z > 0 & \text{in } (\alpha,\beta), \\ z(0) = z(\alpha) = z(\beta) = z(1) = 0. \end{cases} \tag{2.15}$$

By a solution to (2.15), we mean a function $z(\varphi)$ that is continuous on $[0,1]$ and satisfies Equation $(2.15)_1$ in integral form, that is,

$$z(\varphi)=f(\varphi)-c\varphi-\int_0^{\varphi}\frac{D(\sigma)g(\sigma)}{z(\sigma)}\,d\sigma,\qquad \varphi\in[0,1].$$

Notice that we exploited here the assumption $f(0)=0$. It is clear that such a z belongs to $C^1((0,1)\setminus\{\alpha,\beta\})$. To solve problem (2.15), we divide it into four subproblems, which correspond to the subintervals $[0,\alpha]$, $[\alpha,\gamma]$, $[\gamma,\beta]$, and $[\beta,1]$ of the interval $[0,1]$. In order to have a unified treatment of any of these problems, we now collect results from [18, Lemma 4.1, Corollary 4.1, Remark 4.1] for the problem

$$\begin{cases}\dot z(\varphi)=h(\varphi)-c-\frac{Q(\varphi)}{z(\varphi)}, & \varphi\in(\sigma_1,\sigma_2),\\ z(\varphi)<0, & \varphi\in(\sigma_1,\sigma_2).\end{cases}\tag{2.16}$$

Lemma 2.1. *Let h and Q be continuous functions on $[\sigma_1,\sigma_2]$, with $Q>0$ in (σ_1,σ_2) and $Q(\sigma_1)=Q(\sigma_2)=0$. Then we have:*

- *(a) For any $c\in\mathbb{R}$, there exists a unique $\zeta_c\in C^0[\sigma_1,\sigma_2]\cap C^1(\sigma_1,\sigma_2)$ satisfying (2.16) and $\zeta_c(\sigma_2)=0$.*
- *(b) Denote $c^*(\sigma_1,\sigma_2):=\sup\{c\in\mathbb{R}:\zeta_c(\sigma_1)<0\}\in(-\infty,\infty]$. If $c^*(\sigma_1,\sigma_2)<\infty$, then for every $c>c^*(\sigma_1,\sigma_2)$, there exists $\beta(c)\in(-\infty,0)$ such that there is a unique $z_{c,s}\in C^0[\sigma_1,\sigma_2]\cap C^1(\sigma_1,\sigma_2]$ satisfying (2.16), $z_{c,s}(\sigma_1)=0$, $z_{c,s}(\sigma_2)=s<0$, if and only if $s\geq\beta(c)$. Moreover, we have*

$$\max\left\{\sup_{(\sigma_1,\sigma_2]}\delta(f,\sigma_1),h(\sigma_1)+2\sqrt{\dot Q(\sigma_1)}\right\}\leq c^*(\sigma_1,\sigma_2)\leq\sup_{(\sigma_1,\sigma_2]}\delta(f,\sigma_1)+2\sup_{(\sigma_1,\sigma_2]}\sqrt{\Delta(Q,\sigma_1)},\tag{2.17}$$

  where $f(\varphi):=\int_0^{\varphi}h(\sigma)d\sigma$, $\varphi\in[0,1]$.
- *(c) If $\dot Q(0)$ exists, then $c^*(\sigma_1,\sigma_2)$ is finite.*

Conditions (2.17) also exploit estimates on the threshold speeds recently proposed in [19]. With the help of Lemma 2.1, in the proof of the following proposition we analyze the subproblems we mentioned above.

Proposition 2.1. *Problem (2.15) is solvable if $c_1<c_0$, and it is not solvable if $c_1>c_0$; in the former case, we have $c\in[c_1,c_0]$.*

Estimates for the thresholds $c^*_{0,\alpha}$, $c^*_{\alpha,\gamma}$, $c^*_{\gamma,\beta}$, and $c^*_{\beta,1}$ are provided by (2.21), (2.24), (2.26), and (2.28), respectively.

Proof. The proof analyzes the restriction of problem (2.15) to the four above intervals.

Case $[0,\alpha]$. For $\varphi\in[0,\alpha]$, we define

$$h_1(\varphi)=:-\dot f(-\varphi+\alpha),\ D_1(\varphi)=:D(-\varphi+\alpha),\ g_1(\varphi)=:-g(-\varphi+\alpha).$$

We also define $w(\varphi):=z(-\varphi+\alpha)$ and $\tilde c_1:=-c$. Then, when restricted to the interval $[0,\alpha]$, problem (2.15) is equivalent to

$$\begin{cases}\dot w=h_1-\tilde c_1-D_1g_1/w & \text{in }(0,\alpha),\\ w<0 & \text{in }(0,\alpha),\\ w(0)=w(\alpha)=0.\end{cases}\tag{2.18}$$

Lemma 2.1 applies with $\sigma_1=0$, $\sigma_2=\alpha$ and $Q=D_1g_1$: since Q is differentiable in 0, then Lemma 2.1 provides a threshold $\tilde c^*_{0,\alpha}$ such that (2.18) is solvable iff $\tilde c_1\geq\tilde c^*_{0,\alpha}$, that is, $c\leq-\tilde c^*_{0,\alpha}=:c^*_{0,\alpha}$. By (2.17), we obtain

$$\max\left\{\sup_{(0,\alpha]}\delta(f_1,0),h_1(0)+2\sqrt{\dot D_1(0)g_1(0)}\right\}\leq\tilde c^*_{0,\alpha}\leq\sup_{(0,\alpha]}\delta(f_1,0)+2\sup_{(0,\alpha]}\sqrt{\Delta(D_1g_1,0)},\tag{2.19}$$

with

$$f_1(\varphi)=\int_0^{\varphi}h_1(\sigma)d\sigma=\int_0^{\varphi}-\dot f(-\sigma+\alpha)d\sigma=-\int_{\alpha-\varphi}^{\alpha}\dot f(s)ds=f(\alpha-\varphi)-f(\alpha),$$

whence

$$\sup_{s\in(0,\alpha]}\delta(f_1,0)(s)=\sup_{s\in(0,\alpha]}\frac{f_1(s)}{s}=\sup_{s\in[0,\alpha)}\frac{f(s)-f(\alpha)}{\alpha-s}.$$

Moreover, we have

$$\int_0^s \frac{D_1(\varphi)g_1(\varphi)}{\varphi}\,d\varphi = \int_0^s -\frac{D(-\varphi+\alpha)g(-\varphi+\alpha)}{\varphi}\,d\varphi = \int_{\alpha-s}^{\alpha} -\frac{D(\sigma)g(\sigma)}{\alpha-\sigma}\,d\sigma.$$

Then formula (2.19) can be written as

$$\max\left\{\sup_{[0,\alpha)}[-\delta(f,\alpha)], -\dot f(\alpha) + 2\sqrt{\dot D(\alpha)g(\alpha)}\right\} \le \tilde c^*_{0,\alpha} \le \sup_{[0,\alpha)}[-\delta(f,\alpha)] + 2\sup_{[0,\alpha)}\sqrt{\Delta(Dg,\alpha)}. \tag{2.20}$$

Hence,

$$\inf_{[0,\alpha)}\delta(f,\alpha) - 2\sup_{[0,\alpha)}\sqrt{\Delta(Dg,\alpha)} \le c^*_{0,\alpha} \le \min\left\{\inf_{[0,\alpha)}\delta(f,\alpha), \dot f(\alpha) - 2\sqrt{\dot D(\alpha)g(\alpha)}\right\}. \tag{2.21}$$

Case $[\alpha,\gamma]$. We denote

$$h_2(\varphi) := -\dot f(\varphi),\ D_2(\varphi) := -D(\varphi),\ g_2(\varphi) =: -g(\varphi).$$

We also define $w(\varphi) := -z(\varphi)$ and $c_2 := -c$. Then problem (2.15), when restricted to the interval $[\alpha,\gamma]$, becomes

$$\begin{cases} \dot w = h_2 - c_2 - D_2 g_2/w & \text{in } (\alpha,\gamma), \\ w < 0 & \text{in } (\alpha,\gamma], \\ w(\alpha) = 0. \end{cases} \tag{2.22}$$

By Lemma 2.1, we deduce the existence of a threshold $\tilde c^*_{\alpha,\gamma}$ such that (2.22) is solvable iff $c_2 \ge \tilde c^*_{\alpha,\gamma}$, that is, $c \le -\tilde c^*_{\alpha,\gamma} =: c^*_{\alpha,\gamma}$. Moreover, by (2.17), we deduce

$$\max\left\{\sup_{(\alpha,\gamma]}\delta(f_2,\alpha), h_2(\alpha) + 2\sqrt{\dot D_2(\alpha)g_2(\alpha)}\right\} \le \tilde c^*_{\alpha,\gamma} \le \sup_{(\alpha,\gamma]}\delta(f_2,\alpha) + 2\sup_{(\alpha,\gamma]}\sqrt{\Delta(D_2g_2,\alpha)},$$

where

$$f_2(\varphi) := \int_\alpha^\varphi h_2(\sigma)\,d\sigma = -f(\varphi) + f(\alpha),\ \varphi \in [\alpha,\gamma].$$

Whence, by returning to the variables h, D, g, we find

$$\max\left\{\sup_{(\alpha,\gamma]}\{-\delta(f,\alpha)\}, -\dot f(\alpha) + 2\sqrt{\dot D(\alpha)g(\alpha)}\right\} \le \tilde c^*_{\alpha,\gamma} \le \sup_{(\alpha,\gamma]}\{-\delta(f,\alpha)\} + 2\sup_{(\alpha,\gamma]}\sqrt{\Delta(Dg,\alpha)}. \tag{2.23}$$

Hence,

$$\inf_{(\alpha,\gamma]}\delta(f,\alpha) - 2\sup_{(\alpha,\gamma]}\sqrt{\Delta(Dg,\alpha)} \le c^*_{\alpha,\gamma} \le \min\left\{\inf_{(\alpha,\gamma]}\delta(f,\alpha), \dot f(\alpha) - 2\sqrt{\dot D(\alpha)g(\alpha)}\right\}. \tag{2.24}$$

Case $[\gamma,\beta]$. For $\varphi \in [\gamma,\beta]$, we define

$$h_3(\varphi) := \dot f(-\varphi+\gamma+\beta),\ D_3(\varphi) =: -D(-\varphi+\gamma+\beta),\ g_3(\varphi) =: g(-\varphi+\gamma+\beta).$$

We also denote $w(\varphi) := -z(-\varphi+\gamma+\beta)$. Then in the interval $[\gamma,\beta]$, problem (2.15) can be written as

$$\begin{cases} \dot w = h_3 - c - D_3 g_3/w & \text{in } (\gamma,\beta), \\ w < 0 & \text{in } [\gamma,\beta), \\ w(\beta) = 0. \end{cases} \tag{2.25}$$

By Lemma 2.1, problem (2.25) is solvable iff $c \ge c^*_{\gamma,\beta}$, for some threshold $c^*_{\gamma,\beta}$. Upper and lower estimates for $c^*_{\gamma,\beta}$ can be obtained, as in the previous cases, by applying (2.17). In conclusion, we find the estimates

$$\max\left\{\sup_{[\gamma,\beta)}\delta(f,\beta), \dot f(\beta) + 2\sqrt{\dot D(\beta)g(\beta)}\right\} \le c^*_{\gamma,\beta} \le \sup_{[\gamma,\beta)}\delta(f,\beta) + 2\sup_{[\gamma,\beta)}\sqrt{\Delta(Dg,\beta)}. \tag{2.26}$$

Case $[\beta, 1]$. In this case, we directly apply Lemma 2.1: the problem

$$\begin{cases} \dot{z} = \dot{f} - c - Dg/z & \text{in } (\beta, 1), \\ z < 0 & \text{in } (\beta, 1), \\ z(\beta) = z(1) = 0, \end{cases} \tag{2.27}$$

is solvable iff $c \geq c^*_{\beta,1}$, for some $c^*_{\beta,1}$. Again, estimates for $c^*_{\beta,1}$ are deduced by (2.17):

$$\max\left\{\sup_{(\beta,1]} \delta(f,\beta), \dot{f}(\beta) + 2\sqrt{\dot{D}(\beta)g(\beta)}\right\} \leq c^*_{\beta,1} \leq \sup_{(\beta,1]} \delta(f,\beta) + 2\sup_{(\beta,1]} \sqrt{\Delta(Dg,\beta)}. \tag{2.28}$$

This concludes the analysis of the restrictions of problem (2.15) to the four above intervals. Condition $c_1 \leq c_0$ is the requirement that there is a common admissible speed c for the above subproblems. In this case, $c \in [c_1, c_0]$. □

Remark 2.2. Since D and g vanish in the interior of none of the above subintervals, one finds $\varphi' < 0$ if $\varphi \in (0,1) \setminus \{\alpha, \gamma, \beta\}$ (see [18, Proposition 3.1(ii)]). Moreover, by [20, Theorem 2.9 (i)], we deduce that the profile never reaches the value 1 for a finite value of ξ; the same result holds for the value 0, by exploiting again [20, Theorem 2.9 (i)] after the change of variables that led to (2.18). At last, we have $\varphi'(\gamma) < 0$ by the second part of the proof of Proposition 3.1(ii) in [18]. As a consequence, the profile φ is strictly monotone.

Proof of Theorem 2.1. The proof follows an argument based on the reduction of (1.1)–(1.2) to (2.15); see [18].

First, assume $c_1 < c_0$. We argue separately in the four subintervals where $Dg \neq 0$ and then we put together what we found. Thus, let z be the solution of (2.15) associated to some $c \in [c_1, c_0]$. Define $\varphi_{1,\beta}$, $\varphi_{\beta,\gamma}$, $\varphi_{\gamma,\alpha}$, and $\varphi_{\alpha,0}$ as the solutions of

$$\varphi' = \frac{z(\varphi)}{D(\varphi)}, \tag{2.29}$$

with the initial data (respectively)

$$\varphi_{1,\beta}(0) = \frac{1+\beta}{2}, \;\; \varphi_{\beta,\gamma} = \frac{\beta+\gamma}{2}, \;\; \varphi_{\gamma,\alpha}(0) = \frac{\gamma+\alpha}{2}, \;\; \varphi_{\alpha,0}(0) = \frac{\alpha}{2}.$$

Since the right-hand side of (2.29) is locally of class C^1, then $\varphi_{1,\beta}$, $\varphi_{\beta,\gamma}$, $\varphi_{\gamma,\alpha}$, and $\varphi_{\alpha,0}$ exist and are unique in their respective maximal existence intervals.

We focus on the pasting of $\varphi_{1,\beta}$, $\varphi_{\beta,\gamma}$ at β. Let $\varphi_{1,\beta}$, $\varphi_{\beta,\gamma}$ be maximally defined in $(\xi_1, \xi^1_\beta) \subset \mathbb{R}$, $(\xi^2_\beta, \xi^1_\gamma) \subset \mathbb{R}$, with

$$-\infty \leq \xi_1 < 0 < \xi^1_\beta \leq \infty, -\infty \leq \xi^2_\beta < 0 < \xi^1_\gamma \leq \infty,$$

and satisfying

$$\lim_{\xi \to \xi_1^+} \varphi_{1,\beta}(\xi) = 1, \;\; \lim_{\xi \to \{\xi^1_\beta\}^-} \varphi_{1,\beta}(\xi) = \beta, \;\text{ and }\; \lim_{\xi \to \left\{\xi^2_\beta\right\}^+} \varphi_{\beta,\gamma}(\xi) = \beta \;\; \lim_{\xi \to \{\xi^1_\gamma\}^-} \varphi_{\beta,\gamma}(\xi) = \gamma.$$

In order to glue together $\varphi_{1,\beta}$ and $\varphi_{\beta,\gamma}$ (after space shifts), we need to prove $\xi^1_\beta \in \mathbb{R}$ and $\xi^2_\beta \in \mathbb{R}$. We have

$$\lim_{\xi \to \left\{\xi^2_\beta\right\}^+} \varphi'_{\beta,\gamma}(\xi) = \lim_{\xi \to \left\{\xi^2_\beta\right\}^+} \frac{z\left(\varphi_{\beta,\gamma}(\xi)\right)}{D\left(\varphi_{\beta,\gamma}(\xi)\right)} = \lim_{s \to \beta^-} \frac{z(s)}{D(s)} = \lim_{t \to \gamma^+} \frac{w(t)}{D_3(t)},$$

with w and D_3 as in (2.25). The last limit is essentially discussed in the proof of [20, Theorem 2.5]; the only difference is that the interval $[0, 1]$ appearing there is now replaced by $[\gamma, \beta]$. Reasoning as there we obtain that

$$\lim_{t \to \gamma^+} \frac{w(t)}{D_3(t)} \in [-\infty, 0);$$

hence, ξ^2_β is a real value. With a similar reasoning, this time directly applied to $z(\varphi_{1,\beta})$ and $D(\varphi_{1,\beta})$, we can prove that also ξ^2_β is a real value.

The remaining pastings are exactly proved as in the proof of [18, Proposition 3.2], and we refer the reader to that paper for details. To this aim, in particular, we need that $z(\gamma) > 0$, which is satisfied when $c_1 < c_0$ by Proposition 2.1. The proof of the first statement is complete.

We now prove the second statement. Suppose that (1.1)–(1.2) admits a profile φ associated to some speed $c \in \mathbb{R}$. In particular, φ is decreasing and hence it can be decomposed into subprofiles $\varphi_{1,\beta}$, $\varphi_{\beta,\gamma}$, $\varphi_{\gamma,\alpha}$, and $\varphi_{0,\beta}$ connecting, respectively, β to 1, γ to β, and so on. By Remark 2.2, we have $\varphi' < 0$ if $\varphi \in (0,1)\setminus\{\alpha,\gamma,\beta\}$. Therefore, $\varphi_{1,\beta}$ is invertible for $\varphi_{1,\beta} \in (\beta,1)$, $\varphi_{\beta,\gamma}$ is invertible for $\varphi_{\beta,\gamma} \in (\gamma,\beta)$, and so on. Let $\zeta = \zeta(\varphi) : (\beta,1) \to \mathbb{R}$ be the inverse function of $\varphi_{1,\beta}$, and set

$$z(\varphi) := D(\varphi)\varphi'_{1,\beta}(\zeta(\varphi)), \text{ for } \varphi \in (\beta,1).$$

By direct computations, the function z solves $(2.15)_1$ in $(\beta,1)$ (where $z \in C^1$) and, by adapting [18, Lemma 3.1], it can be extended to a function of class $C^0[\beta,1]$, still called z. Also, as in [18, Lemma 3.1], we have $z(1) = 0$. Arguing similarly in the other sub-intervals, one finds that $z \in C^0[0,1]$ is in C^1 in $(0,\alpha)\cup(\alpha,\beta)\cup(\beta,1)$ and it satisfies (2.15). For more details we refer to the similar case presented in the proof of [18, Proposition 3.1 *(ii)*], which applies because g satisfies [18, (2.2)]. According to Proposition 2.1 we obtain $c_1 \le c_0$ and then also the second statement is proved.

Remark 2.3. We now provide a simple argument showing why wavefronts should exist for suitable concave f, in the case the drift $\dot f$ is first positive and then negative. For $\lambda > 0$, let f be defined by λu in $(0,\gamma)$ and $-\lambda(u-2\gamma)$ in $(\gamma,1)$, so that f is Lipschitz continuous with $\dot f = \lambda$ in $(0,\gamma)$ and $\dot f = -\lambda$ in $(\gamma,1)$. In this case, the role of λ is to shift to the right (of magnitude $+\lambda$) the estimates for c_0, as (2.21) and (2.24) show, and to shift to the left (of $-\lambda$) the estimates for c_1 (see (2.26) and (2.28)). Hence, (2.2) holds true for λ large enough.

We denote by $s_{0,\alpha}, s_{\alpha,\gamma}, s_{\gamma,\beta}$, and $s_{\beta,1}$ the lower bounds in (2.21), (2.24), (2.26), and (2.28), respectively, and with $\Sigma_{0,\alpha}, \Sigma_{\alpha,\gamma}$, $\Sigma_{\gamma,\beta}$, and $\Sigma_{\beta,1}$ the corresponding upper bounds. In other words, we rewrite (2.21), (2.24), (2.26), and (2.28) as

$$s_{0,\alpha} \le c^*_{0,\alpha} \le \Sigma_{0,\alpha}, \quad s_{\alpha,\gamma} \le c^*_{\alpha,\gamma} \le \Sigma_{\alpha,\gamma}, \quad s_{\gamma,\beta} \le c^*_{\gamma,\beta} \le \Sigma_{\gamma,\beta}, \quad s_{\beta,1} \le c^*_{\beta,1} \le \Sigma_{\beta,1}. \tag{2.30}$$

Define moreover

$$s_{0,\gamma} := \inf_{[0,\gamma]} \delta(f,\alpha) - 2\sup_{[0,\gamma]}\sqrt{\Delta(Dg,\alpha)} \ \text{ and } \ \Sigma_{\gamma,1} := \sup_{[\gamma,1]}\delta(f,\beta) + 2\sup_{[\gamma,1]}\sqrt{\Delta(Dg,\beta)}.$$

Here above, the arguments of the inf and of the sup's are apparently not defined at α and β, respectively; however, as we wrote below (2.3) and (2.4), since $f, D, g \in C^1$, in those cases we understand them as $-\dot f(\alpha)$, $\dot D(\alpha)g(\alpha)$, $\dot f(\beta)$, and $\dot D(\beta)g(\beta)$, respectively. Under this notation, we immediately deduce the following result.

Lemma 2.2. *If $\Sigma_{\gamma,1} < s_{0,\gamma}$, then condition (2.2) is satisfied.*

Proof. According to the right-hand sides of the estimates (2.26) and (2.28), we have $c_1 \le \max\{\Sigma_{\gamma,\beta}, \Sigma_{\beta,1}\} \le \Sigma_{\gamma,1}$. By the assumption $\Sigma_{\gamma,1} < s_{0,\gamma}$, we obtain

$$c_1 \le \Sigma_{\gamma,1} < s_{0,\gamma} \le \min\{s_{0,\alpha}, s_{\alpha,\gamma}\}.$$

Because of (2.21) and (2.24), we deduce $c_1 < \min\{s_{0,\alpha}, s_{\alpha,\gamma}\} \le \min\{c^*_{0,\alpha}, c^*_{\alpha,\gamma}\} = c_0$. □

Proof of Corollary 2.1. If a wavefront exists, then necessarily $c_1 \le c_0$ because of Theorem 2.1. Then, by (2.1) and (2.30), it follows

$$\begin{aligned} &\max\left\{\sup_{[\gamma,1]}\delta(f,\beta), \dot f(\beta) + 2\sqrt{\dot D(\beta)g(\beta)}\right\} = \\ &\max\{s_{\gamma,\beta}, s_{\beta,1}\} \le c_1 \le c_0 \le \min\{\Sigma_{0,\alpha}, \Sigma_{\alpha,\gamma}\} = \\ &\min\left\{\inf_{[0,\gamma]}\delta(f,\alpha), \dot f(\alpha) - 2\sqrt{\dot D(\alpha)g(\alpha)}\right\}, \end{aligned} \tag{2.31}$$

which is (2.6).

Proof of Corollary 2.2. First, notice that (2.11) is exactly $\Sigma_{\gamma,1} < s_{0,\gamma}$ after trivial manipulations. As a consequence, Theorem 2.1 and Lemma 2.2. imply the existence of wavefronts.

To obtain (2.12), we impose $\max\{s_{\gamma,\beta}, s_{\beta,1}\} > 0$ in (2.31), which in turn implies that $c_1 > 0$; notice that the left-hand side in (2.31) is precisely the left-hand side of (2.12). Analogously, to obtain (2.13), we impose $\min\{\Sigma_{0,\alpha}, \Sigma_{\alpha,\gamma}\} < 0$. This implies $c_0 < 0$.

Proof of Corollary 2.3. Assume that $\mathcal{J} \neq \emptyset$; then, according to Theorem 2.1, we have $\mathcal{J} \cap (0,\infty) \neq \emptyset$ if and only if $c_0 > 0$. By (2.21) and (2.24), we have

$$\begin{aligned} c_0 &= \min\{c^*_{0,\alpha}, c^*_{\alpha,\gamma}\} \\ &\geq \min\left\{\inf_{[0,\alpha)} \delta(f,\alpha) - 2\sup_{[0,\alpha)} \sqrt{\Delta(Dg,\alpha)}, \inf_{(\alpha,\gamma]} \delta(f,\alpha) - 2\sup_{(\alpha,\gamma]} \sqrt{\Delta(Dg,\alpha)}\right\} \\ &\geq \inf_{[0,\gamma]} \delta(f,\alpha) - 2\sup_{[0,\gamma]} \sqrt{\Delta(Dg,\alpha)} = s_{0,\gamma}. \end{aligned}$$

If condition (2.14) is satisfied, then $c_0 > 0$.

3 | A BIOLOGICAL MODEL WITH BIASED MOVEMENTS

In this section, we first summarize a model for the movement of organisms recently presented in [1] for populations constituted by two groups of individuals. Then we show how a convective term can appear in the equation because of a biased movement.

The population is divided into *isolated* and *grouped* organisms. Both groups can move, reproduce and die, with possibly different rates. The organisms occupy the sites jl, for $j = 0, \pm1, \pm2, \dots$ and $l > 0$; we denote by c_j the probability of occupancy of the j-th site. Let P^i_m and P^g_m be the movement transitional probabilities for isolated and grouped individuals, respectively; we use the notation $P^{i,g}_m$ to indicate the two sets of parameters together. Analogously, the corresponding probabilities for birth and death are $P^{i,g}_b$ and $P^{i,g}_d$.

Differently from [1], we also introduce the parameters $a^i, b^i \geq 0$ and $a^g, b^g \geq 0$, which characterize a (linearly) biased movement for the isolated and grouped individuals (see Figure 3). For the isolated individuals the bias is toward the left if $a^i - b^i > 0$ and toward the right if $a^i - b^i < 0$; for the grouped individuals, the same occurs when either $a^g - b^g > 0$ or $a^g - b^g < 0$, respectively. In the case of [1], one has $a^i = b^i = a^g = b^g = 1$ and then $a^{i,g} - b^{i,g} = 0$.

Then, the variation δc_j of c_j during a time-step $\tau > 0$ is given by

$$\begin{aligned} \delta c_j = \frac{P^i_m}{2}&\left[a^i c_{j-1}(1-c_j)(1-c_{j-2}) + b^i c_{j+1}(1-c_j)(1-c_{j+2}) - (a^i+b^i)c_j(1-c_{j-1})(1-c_{j+1})\right] \\ &+ \frac{P^g_m}{2}\left[a^g c_{j-1}(1-c_j) + b^g c_{j+1}(1-c_j) - a^g c_j(1-c_{j+1}) - b^g c_j(1-c_{j-1})\right] \\ &- \frac{P^g_m}{2}\left[a^g c_{j-1}(1-c_j)(1-c_{j-2}) + b^g c_{j+1}(1-c_j)(1-c_{j+2})\right. \\ &\left. - (a^g+b^g)c_j(1-c_{j-1})(1-c_{j+1})\right] + \text{reaction terms}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \text{reaction terms} = &\frac{P^i_b}{2}\left[c_{j-1}(1-c_j)(1-c_{j-2}) + c_{j+1}(1-c_j)(1-c_{j+2})\right] + \frac{P^g_b}{2}\left[c_{j-1}(1-c_j) + c_{j+1}(1-c_j)\right] \\ &- \frac{P^g_b}{2}\left[c_{j-1}(1-c_j)(1-c_{j-2}) + c_{j+1}(1-c_j)(1-c_{j+2})\right] - \frac{P^i_d}{2}\left[c_j(1-c_{j-1})(1-c_{j+1})\right] \\ &- \frac{P^g_d}{2}\left[c_j\right] + \frac{P^i_d}{2}\left[c_j(1-c_{j-1})(1-c_{j+1})\right]. \end{aligned}$$

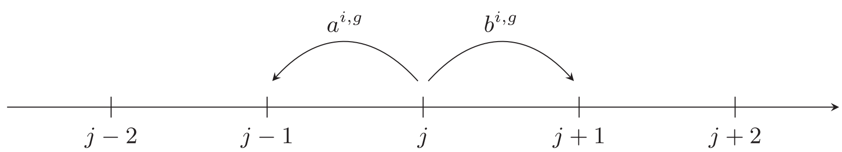

FIGURE 3 Sketch of the meaning of the parameters $a^{i,g}$ and $b^{i,g}$.

By noticing that every bracket is divided by 2, we deduce

$$a^i + b^i = a^g + b^g = 2, \tag{3.2}$$

since in the deduction of (3.1) a bias $a^{i,g}$ implies a converse bias $b^{i,g} = 2 - a^{i,g}$.

The continuum model is obtained by replacing c_j with a smooth function $c = c(x,t)$ and expanding c around $x = jl$ at second order. Then, we divide (3.1) by τ and pass to the limit for $l, \tau \to 0$ while keeping l^2/τ constant; for simplicity, we assume $l^2/\tau = 1$. To perform this step, one makes the following assumptions on the reactive–diffusive terms [1]:

$$\frac{P_m^{i,g} l^2}{2\tau} \sim D_{i,g},\ \frac{P_b^{i,g}}{2\tau} \sim \lambda_{i,g},\ \frac{P_d^{i,g}}{2\tau} \sim k_{i,g},\ P_b^{i,g}, P_d^{i,g} = O(\tau),\ \text{for } l \to 0, \tau \to 0. \tag{3.3}$$

The above limits define the diffusivity parameters $D_{i,g}$, the birth rates $\lambda_{i,g}$, and the death rates $k_{i,g}$; all these parameters are non-negative. About the convection terms, we require

$$a^{i,g}(\tau) \sim 1,\ b^{i,g}(\tau) \sim 1 \text{ and } a^{i,g}(\tau) - b^{i,g}(\tau) \sim C_{i,g}\sqrt{\tau} \text{ for } \tau \to 0, \tag{3.4}$$

for some $C_{i,g} \in \mathbb{R}$. We stress that the parameters $C_{i,g}$ can be either positive or negative according to the values of the bias coefficients $a^{i,g}$ and $b^{i,g}$; in particular, if $C_i > 0$, then we have a bias toward the left of the isolated individuals, and toward the right if $C_i < 0$; the analogous bias for the grouped individuals corresponds to either $C_g > 0$ (left) or $C_g < 0$ (right). If $C_{i,g} = 0$, then the corresponding bias is too weak to pass to Equation (3.5); with a slight abuse of terminology, we say that the corresponding group has no convective movement. At last, assumption $(3.4)_1$ is compatible with (3.2); assumption $(3.4)_2$ is analogous to $(3.3)_4$.

In conclusion, we obtain the equation

$$u_t + f(u)_x = (D(u)u_x)_x + g(u), \tag{3.5}$$

with

$$f(u) = -\left(C_i D_i + C_g D_g\right) u(1-u)^2 - C_g D_g u(1-u), \tag{3.6}$$

$$D(u) = D_i\left(1 - 4u + 3u^2\right) + D_g\left(4u - 3u^2\right), \tag{3.7}$$

$$g(u) = \lambda_g u(1-u) + \left[\lambda_i - \lambda_g - \left(k_i - k_g\right)\right] u(1-u)^2 - k_g u. \tag{3.8}$$

The model (3.5) depends on the eight parameters $C_{i,g}$, $D_{i,g}$, $k_{i,g}$, and $\lambda_{i,g}$. Equation (3.5) coincides with (1.1), but we agree that when we refer to (3.5) we understand f, D, and g as in (3.6)–(3.8). We point out that $f(0) = f(1) = 0$; that is, the convective flow vanishes when the density is either zero or maximum, as physically it should be. When $C_i = 0$, the isolated individuals have no convective movement and the function f is convex in $[0,1]$ if $C_g > 0$ and concave otherwise. Instead, when $C_g = 0$, the grouped individuals have no convective movements and f changes its concavity for $u = 2/3$. The diffusion and reaction terms (3.7)–(3.8) coincide with those in [1, (2)], while f is missing there.

4 | PRELIMINARY RESULTS ON THE MODEL

About the model introduced in Section 3, the case we are interested in is when conditions (D) and (g) are satisfied; the corresponding assumptions on the parameters have already been given in [1, 10].

Lemma 4.1. *The diffusivity D in (3.7) satisfies (D) if and only if* $D_i > 4D_g > 0$. *In this case, we have*

$$\alpha = \frac{2}{3} - \frac{\omega}{3} \text{ and } \beta = \frac{2}{3} + \frac{\omega}{3}, \qquad \text{for } \omega := \sqrt{\frac{D_i - 4D_g}{D_i - D_g}}. \tag{4.1}$$

The reaction term g in (3.8) satisfies (g) if and only if $k_g = 0$, $\lambda_g > 0$ *and* $r_i := k_i - \lambda_i > 0$. *In this case,*

$$\gamma = \frac{r_i}{r_i + \lambda_g}, \tag{4.2}$$

and $\gamma \in (\alpha, \beta)$ *if and only if*

$$\frac{1-\omega}{2+\omega} < \frac{\lambda_g}{r_i} < \frac{1+\omega}{2-\omega}.$$

Proof. The function D in (3.7) is a parabola with $D(0) = D_i$ and $D(1) = D_g$. We have $\dot{D}(u) = -(D_i - D_g)(4-6u)$, which vanishes iff $u = \frac{2}{3}$, and $D\left(\frac{2}{3}\right) = \frac{1}{3}(-D_i + 4D_g)$. Then D is positive–negative–positive if and only if $D_i > 4D_g$; the case $D_g = 0$ is excluded because then D changes sign only once in $(0, 1)$. Then the two zeros α and β of D satisfy $(4.1)_{1,2}$. Moreover, $g(0) = 0$ and $g(1) = 0$ if and only if $k_g = 0$; under this assumption, g also vanishes at γ defined in (4.2). Hence g satisfies condition (g) if and only if $k_g = 0$, $\lambda_g > 0$ and $r_i > 0$. The condition $\gamma \in (\alpha, \beta)$ is then equivalent to the last condition in the statement. □

Notice that $\omega \in (0, 1)$, $\beta - \alpha = 2\omega/3$ and $\alpha + \beta = 4/3$. The condition $k_g = 0$ clearly has no biological sense, and is interpreted in the sense that the life expectancy of grouped individuals is much larger than that of isolated individuals; the condition $r_i > 0$ means that the death rate of isolated individuals is larger than their birth rate. All in all, the scenario favors much more grouped than isolated individuals. We point out that this case was already considered in [1, 10]. Above, γ is the Allee parameter [1, 10].

In the following proofs, we often make use of the notation

$$p := C_i D_i + C_g D_g \text{ and } q := C_g D_g. \tag{4.3}$$

We now rewrite formulas (3.6)–(3.8) by exploiting (4.1), (4.2), and (4.3):

$$f(u) = -pu(1-u)^2 - qu(1-u), \tag{4.4}$$

$$D(u) = 3(D_i - D_g)(u-\alpha)(u-\beta), \tag{4.5}$$

$$g(u) = (r_i + \lambda_g) \cdot u(1-u)(u-\gamma). \tag{4.6}$$

Remark 4.1. We point out that $\dot{f}(0) = -(p+q)$ and $\dot{f}(1) = q$; these quantities can be understood as the drift at very low and maximum concentrations, respectively.

Remark 4.2. The movement velocity $v = v(u)$ is defined by $f(u) =: uv(u)$. Then $v(u) = -pu^2 + (2p+q)u - (p+q) = (1-u)(pu - (p+q))$, and then v vanishes at the maximum density 1; it can also possibly vanish at $u_0 = \frac{p+q}{p}$ (i.e., if $u_0 \in [0, 1)$). This is analogous to similar models in collective movements [21, §3.1].

Assuming $q < 0$ (see Proposition 4.1), it is easy to see that only the following cases may occur (for simplicity we do not include the case $p + q = 0$, when $u_0 = 0$, or $p = 0$, when u_0 is missing, for which slightly different results hold):

1. $q < 0 < p + q$. Then v is concave, it is first negative, then positive; f is convex-concave.
2. $p + q < 0 < p$. Then v is positive and concave; f is concave or convex-concave.
3. $p < 0$, $q < 0$. Then v is positive and convex; f is concave or concave-convex.

Under this notation, for $\varphi, \varphi_0 \in (0, 1)$ we have (see (2.3))

$$\delta(f, \varphi)(\varphi_0) = -(p+q) + (2p+q)(\varphi + \varphi_0) - p\left(\varphi^2 + \varphi\varphi_0 + \varphi_0^2\right). \tag{4.7}$$

Here follows a simple necessary condition for the existence of wavefronts.

Proposition 4.1. *If (3.5) admits wavefronts satisfying condition (1.2), then* $C_g < 0$.

Proof. We apply condition (2.7). Since $\dot{f}(u) = -p(3u^2 - 4u + 1) + q(2u - 1)$, then (2.7) applies if $-p(3\alpha^2 - 4\alpha + 1) + q(2\alpha - 1) > -p(3\beta^2 - 4\beta + 1) + q(2\beta - 1)$, that is,

$$-p\left(3(\alpha^2 - \beta^2) - 4(\alpha - \beta)\right) + 2q(\alpha - \beta) > 0. \tag{4.8}$$

By (4.1), we obtain that (4.8) is equivalent to $2q(\alpha-\beta)=-\frac{4q}{3}\omega>0$, that is, $q<0$. Hence, we deduce $C_g<0$ since $D_g>0$ by Lemma 4.1. □

Notice that the assumption $C_g<0$ becomes $q<0$ under the notation in (4.3). It is easy to see that a necessary condition to have wavefronts satisfying conditions $\varphi(-\infty)=0$ and $\varphi(\infty)=1$, instead of (1.2), is $C_g>0$.

We now summarize the restrictions required on the parameters:

$$C_g<0, \qquad D_i>4D_g>0, \tag{4.9}$$

$$r_i:=k_i-\lambda_i>0, \qquad k_g=0, \qquad \lambda_g>0, \tag{4.10}$$

$$\frac{1-\omega}{2+\omega}<\frac{\lambda_g}{r_i}<\frac{1+\omega}{2-\omega}, \tag{4.11}$$

with ω defined in (4.1). We *always* assume conditions (4.9)–(4.11) in the following, without any further mention.

The results below are preferably stated by referring to the following dimensionless quotients and by lumping the parameters referring to the grouped population into a single dimensionless parameter as follows:

$$s:=\frac{C_i}{|C_g|}, \qquad d:=\frac{D_i}{D_g}>4, \qquad \mu:=\frac{r_i}{\lambda_g}=\frac{k_i-\lambda_i}{\lambda_g}, \qquad E_g:=|C_g|\sqrt{\frac{D_g}{\lambda_g}}. \tag{4.12}$$

Under this notation, we have

$$\omega=\sqrt{\frac{d-4}{d-1}}. \tag{4.13}$$

Notice that E_g gathers the parameters concerning convection, diffusion and reaction of the grouped individuals; the parameter μ is the ratio between the net increasing rate of the isolated and grouped individuals. Notice that condition (4.11) is equivalent to

$$\frac{2-\omega}{1+\omega}<\mu<\frac{2+\omega}{1-\omega}. \tag{4.14}$$

A sufficient condition for the existence of wavefronts to Equation (3.5) is (2.11). The following result provides an upper estimate of the right-hand side of (2.11).

Lemma 4.2. *We have*

$$2\sup_{[0,\gamma]}\sqrt{\Delta(Dg,\alpha)}+2\sup_{[\gamma,1]}\sqrt{\Delta(Dg,\beta)}\le\sqrt{\frac{D_g}{\lambda_g}}\sqrt{d-1}\left(\sqrt{\mu(2+\omega)}+\sqrt{1+\omega}\right).$$

Proof. By (4.5), we have $D(\varphi)g(\varphi)=3(D_i-D_g)(r_i+\lambda_g)\varphi(\varphi-\alpha)(\varphi-\gamma)(\varphi-\beta)(1-\varphi)$. Then, for $\varphi\in[0,\gamma]$, we obtain

$$\begin{aligned}\frac{D(\varphi)g(\varphi)}{\varphi-\alpha}&\le\frac{3}{4}(D_i-D_g)(r_i+\lambda_g)(\varphi-\gamma)(\varphi-\beta)\le\frac{3}{4}(D_i-D_g)(r_i+\lambda_g)\gamma\beta\\&=\frac{1}{4}r_i(D_i-D_g)(2+\omega).\end{aligned} \tag{4.15}$$

By (2.5) and (4.15), we deduce

$$2\sup_{[0,\gamma]}\sqrt{\Delta(Dg,\alpha)}\le\sqrt{D_g}\sqrt{r_i(d-1)(2+\omega)}. \tag{4.16}$$

With a similar reasoning, we have that

$$2\sup_{[\gamma,1]}\sqrt{\Delta(Dg,\beta)}\le\sqrt{D_g}\sqrt{\lambda_g(d-1)(1+\omega)}, \tag{4.17}$$

since we have $(\varphi-\gamma)(\varphi-\alpha) \le (1-\gamma)(1-\alpha)$, for $\varphi \in [\gamma, 1]$, because $(r_i+\lambda_g)(1-\gamma)=\lambda_g$. We complete the proof by combining (4.16) and (4.17). □

The convective term f can change convexity at most once; then, it can be either concave or convex, or else convex–concave or concave–convex. In the following sections, we examine each of these cases; in all of them, we emphasize that s is always multiplied by d. Since the parameter s does not depend on d, we can understand sd as a variable independent from d, which lumps the ratios of the coefficients related to the movement. In this way we shall often deal with the couple (ω, sd) of parameters, where ω depends on d.

5 | THE CONCAVE CASE AND THE CONVEX CASE

In the following sections, we investigate the presence of wavefronts to the biased model (3.5) and prove their main qualitative properties. We make use of the results provided in Section 2 for a general reaction–diffusion–convection process.

The following result characterizes the strict concavity of the function f; see Figure 4.

Lemma 5.1. *The function f in (3.6) is strictly concave if and only if*

$$0 \le sd \le \frac{3}{2}. \tag{5.1}$$

Proof. By (4.4), we compute $\ddot{f}(u)=-6pu+4p+2q$; therefore, $\ddot{f}<0$ in $(0,1)$ if and only if

$$-3pu+2p+q<0, \text{ for any } u \in (0,1). \tag{5.2}$$

The line $-3pu+2p+q=0$ connects the points $(0, 2p+q)$ and $(1, -p+q)$. We remark that $2p+q=-p+q=0$ is not possible since $q<0$ by conditions (4.9)–(4.11) and (4.3). Hence, (5.2) holds if and only if

$$\begin{cases} 2p+q \le 0, \\ -p+q \le 0. \end{cases} \tag{5.3}$$

Conditions (5.3) hold if and only if $q \le p \le -q/2$, which is equivalent to (5.1). □

Remark 5.1. From the proof of Lemma 5.1, we deduce that f is strictly convex iff

$$-\frac{C_gD_g}{2} \le C_iD_i + C_gD_g \le C_gD_g. \tag{5.4}$$

The case when f is convex is then quickly treated in the following theorem: The convexity of f in $[\alpha, \beta]$ prevents the existence of wavefronts.

Theorem 5.1. *If f is convex in the interval $[\alpha, \beta]$, then Equation (3.5) admits no wavefronts satisfying condition (1.2).*

Proof. The function f is strictly convex iff (5.4) is satisfied (see Remark 5.1); however, this condition does not match with the assumption $C_g<0$, which is necessary to have wavefronts to Equation (3.5) satisfying (1.2) by Proposition 4.1.

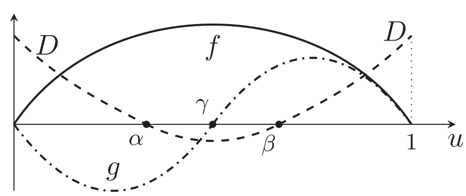

FIGURE 4 Plots of the functions D (dashed line), g (dashdotted line), and f (solid line) in the case f is strictly concave.

Indeed, the bare convexity of f in $[\alpha, \beta]$, as is required in the statement, is sufficient to hinder the existence of such wavefronts, because the right-hand side of (2.7) is strictly positive when D and g are as in (4.5) and (4.6). □

The left-hand side of (2.11) takes a simple form when f is strictly concave (see Remark 2.1). Below, we consider this case.

5.1 | Existence of wavefronts

Theorem 5.2. *If f is strictly concave and*

$$\frac{d-1}{\sqrt{d-4}}\frac{\sqrt{\mu(2+\omega)}+\sqrt{1+\omega}}{2\mu+5+sd(\mu-2)}(\mu+1)<\frac{2}{9}E_g \tag{5.5}$$

holds, then Equation (3.5) admits wavefronts satisfying condition (1.2).

Proof. In order to apply (2.11), we exploit Remark 2.1. Then, by exploiting (4.7), we compute

$$\begin{aligned}\delta(f,\alpha)(\gamma)-\delta(f,\beta)(\gamma)&=(2p+q)(\alpha-\beta)-p(\alpha^2+(\alpha-\beta)\gamma-\beta^2)\\&=(\beta-\alpha)\,(p(\alpha+\beta-2+\gamma)-q)\,,\end{aligned}$$

whence, from $\beta-\alpha=\frac{2}{3}\omega$ and $\alpha+\beta=\frac{4}{3}$, we get

$$\inf_{[0,\gamma]}\delta(f,\alpha)-\sup_{[\gamma,1]}\delta(f,\beta)=\frac{2}{3}\omega\left[p\left(\gamma-\frac{2}{3}\right)-q\right]. \tag{5.6}$$

By (4.3) and (4.2), we can write

$$\begin{aligned}p\left(\gamma-\frac{2}{3}\right)-q&=C_gD_g\left(\frac{r_i}{r_i+\lambda_g}-\frac{5}{3}\right)+C_iD_i\left(\frac{r_i}{r_i+\lambda_g}-\frac{2}{3}\right)\\&=\frac{1}{3(r_i+\lambda_g)}\left(C_gD_g(-2r_i-5\lambda_g)+C_iD_i(r_i-2\lambda_g)\right).\end{aligned}$$

Therefore, when f is strictly concave, we have

$$\inf_{[0,\gamma]}\delta(f,\alpha)-\sup_{[\gamma,1]}\delta(f,\beta)=\frac{2\omega}{9(r_i+\lambda_g)}\left(C_gD_g(-2r_i-5\lambda_g)+C_iD_i(r_i-2\lambda_g)\right).$$

By the above formula, Lemma 4.2, and (4.13), condition (5.5) implies (2.11). □

Corollary 5.1. *Under (5.1), condition (5.5) is satisfied if*

$$\sqrt{\mu(2+\omega)+(1+\omega)}\frac{d-1}{\sqrt{d-4}}<\frac{4}{9\sqrt{2}}E_g. \tag{5.7}$$

Proof. By (5.1), we have $2\mu+5+sd(\mu-2)=(2+sd)\mu+(5-2sd)\geq 2(\mu+1)$; so, condition (5.5) holds if

$$\left(\sqrt{\mu(2+\omega)}+\sqrt{1+\omega}\right)\frac{d-1}{\sqrt{d-4}}<\frac{4}{9}E_g.$$

In turn, this condition is satisfied if (5.7) holds. □

Notice that condition (5.7) can be written as

$$r_i(2+\omega)+\lambda_g(1+\omega)<\frac{8C_g^2D_g}{81}\frac{d-4}{(d-1)^2}. \tag{5.8}$$

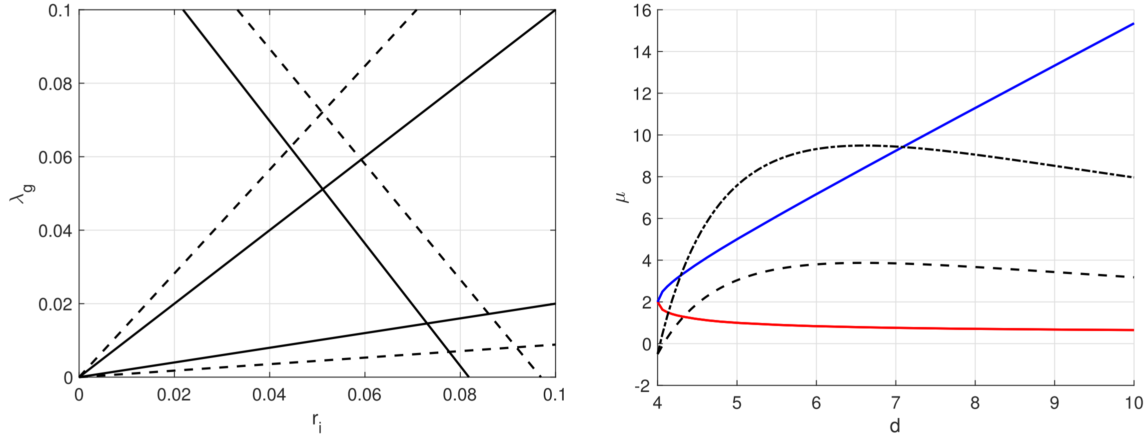

FIGURE 5 On the left: the triangle $\mathcal{T}_g(d)$ is the intersection of three half-planes. Here, $|C_g|\sqrt{D_g} = 6$; dashed lines refer to $d = 5$, solid lines to $d = 8$. On the right: The conditions in (4.14) prescribe that μ must lie between the red (which lies below) and the blue (above) line. Condition (5.7) further prescribes that $\mu > 0$ must belong to the region below the black line: the dashed curve refers to $E_g = 40$, the dash-dotted curve to $E_g = 60$. [Colour figure can be viewed at wileyonlinelibrary.com]

Condition (5.8) contains several parameters; therefore, there are many ways of discussing the results, depending on which parameters are set and which are held constant; in the following remark we focus on two different choices.

Remark 5.2. We consider Equation (5.8) and first focus on the parameters r_i and λ_g. For fixed C_g, D_g, condition (5.8) identifies the triangle (see Figure 5 on the left)

$$\mathcal{T}_g(d) := \left\{(r_i, \lambda_g) \in \mathbb{R}^+ \times \mathbb{R}^+ \,:\, (4.11) \text{ and } (5.8) \text{ hold}\right\}. \tag{5.9}$$

Therefore, under (5.1), if $(r_i, \lambda_g) \in \mathcal{T}_g(d)$, then the assumptions of Theorem 5.2 are satisfied and Equation (3.5) admits wavefronts satisfying (1.2).

Now, we focus instead on the parameters d and μ. Notice that, assuming again (5.1), we can interpret (4.11) and (5.7) (i.e., (5.8)) as relationships between d and μ for fixed E_g; see Figure 5 on the right and Corollary 5.1. In this framework, profiles exist for every couple (d, μ) lying in the region between the red and blue lines and below the black line.

5.2 | Sign of the speed of wavefronts

We now investigate the sign of the speed of wavefronts; this issue is important in the biological framework. We find below conditions in order that wavefronts with *positive* speed exist and conditions assuring that every wavefront has *negative* speed.

About the case of *positive speeds*, by (4.2), (4.3), (4.7), and (4.11), we obtain

$$\begin{aligned}
\inf_{[0,\gamma]} \delta(f,\alpha) &= \frac{f(\gamma)-f(\alpha)}{\gamma-\alpha} = -(p+q) + (2p+q)(\alpha+\gamma) - p(\alpha^2+\alpha\gamma+\gamma^2)\\
&= C_gD_g\left(-\frac{4}{9} - \frac{5}{9}\omega - \frac{\omega^2}{9} + \frac{7+\omega}{3}\gamma - \gamma^2\right) + C_iD_i\left(-\frac{1}{9} - \frac{2}{9}\omega - \frac{\omega^2}{9} + \frac{4+\omega}{3}\gamma - \gamma^2\right)\\
&= \frac{|C_g|D_g}{9}\big((1-sd)(\omega^2+9\gamma^2-3\omega\gamma) + (5-2sd)\omega - 3(7-4sd)\gamma + 4 - sd\big)\\
&=: \frac{|C_g|D_g}{9}\tau(\omega,\gamma,sd).
\end{aligned} \tag{5.10}$$

Denote

$$\mathcal{R} := \left\{(\omega,\gamma) \,:\, \sqrt{3}-1 < \omega < 1 \ \text{ and } \ \frac{2-\omega}{3} < \gamma < 1 - \frac{1}{\sqrt{3}}\right\}. \tag{5.11}$$

Lemma 5.2. *We have $\tau(\omega,\gamma,sd) > 0$ for every $(\omega,\gamma) \in \mathcal{R}$ and $0 \le sd \le 3/2$.*

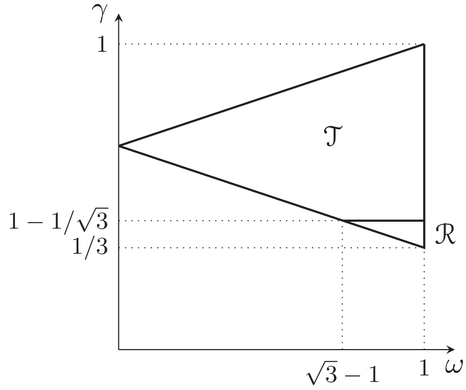

FIGURE 6 The triangles $\mathcal{T}$ and $\mathcal{R}$.

Proof. First, it is easy to show that the function $\partial_{sd}\tau(\omega,\gamma,sd) = -\omega^2 - 9\gamma^2 + 3\omega\gamma - 2\omega + 12\gamma - 1$ has no critical points in the triangle

$$\mathcal{T} := \left\{(\omega,\gamma) \in \mathbb{R}^2 \,:\, 0 < \omega < 1 \text{ and } \frac{2-\omega}{3} < \gamma < \frac{2+\omega}{3}\right\},$$

which contains the set $\mathcal{R}$; see Figure 6. Moreover, on $\partial\mathcal{T}$, we have $\partial_{sd}\tau > 0$ and then

$$\partial_{sd}\tau(\omega,\gamma,sd) > 0 \text{ for } (\omega,\gamma) \in \mathcal{T}. \tag{5.12}$$

Then, by the monotonicity property proved in (5.12), it is sufficient to prove that $\tau(\omega,\gamma,0) > 0$ for every $(\omega,\gamma) \in \mathcal{R}$. We have

$$\tau(\omega,\gamma,0) = \omega^2 + 9\gamma^2 - 3\omega\gamma + 5\omega - 21\gamma + 4 = \left(\omega - \frac{3}{2}\gamma + \frac{5}{2}\right)^2 + \frac{27}{4}(1-\gamma)^2 - 9.$$

This quantity is positive if, in particular,

$$\omega - \frac{3}{2}\gamma + \frac{5}{2} > \frac{3\sqrt{3}}{2} \text{ and } 1 - \gamma > \frac{1}{\sqrt{3}}.$$

The second inequality implies the first one when $\omega > \sqrt{3} - 1$, and then $\tau(\omega,\gamma,0) > 0$ for every $(\omega,\gamma) \in \mathcal{R}$. □

Remark 5.3. We easily see that $(\omega,\gamma) \in \mathcal{R}$ iff $(r_i,\lambda_g) \in \tilde{\mathcal{R}}(d)$ and $\sqrt{3} - 1 < \omega < 1$, where

$$\tilde{\mathcal{R}}(d) = \left\{(r_i,\lambda_g) \in \mathbb{R}^+ \times \mathbb{R}^+ \,:\, \frac{1}{\sqrt{3}-1} < \frac{\lambda_g}{r_i} < \frac{1+\omega}{2-\omega}\right\}.$$

We are now in the position to apply (2.14).

Theorem 5.3. *Assume f is strictly concave, $(r_i,\lambda_g) \in \tilde{\mathcal{R}}(d)$ and $\sqrt{3} - 1 < \omega < 1$. If*

$$\frac{18\sqrt{\mu(d-1)}}{\tau(\omega,\gamma,sd)} < E_g \tag{5.13}$$

is satisfied, then either $\mathcal{J} = \emptyset$ or $\mathcal{J} \cap (0,\infty) \neq \emptyset$.

Proof. By Remark 5.3, Lemma 5.2 applies and then $\tau(\omega,\gamma,sd) > 0$ if $(r_i,\lambda_g) \in \tilde{\mathcal{R}}(d)$, $\sqrt{3} - 1 < \omega < 1$ and $0 \le sd \le \frac{3}{2}$. Now, notice that by (4.16), it follows

$$\sup_{[0,\gamma]} \sqrt{\Delta(Dg,\alpha)} \le \sqrt{r_i D_g(d-1)}. \tag{5.14}$$

Then, condition (5.13) implies (2.14) by (5.10) and (5.14). □

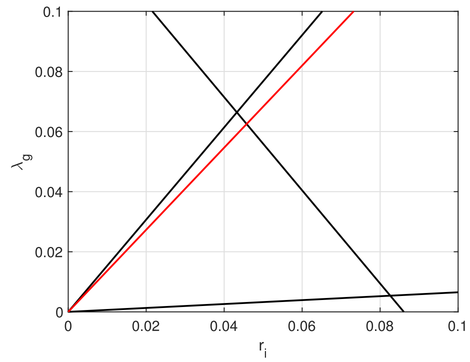

FIGURE 7 The triangle $\mathcal{T}_g(d)$ (thick black lines) and the cone $\tilde{\mathcal{R}}(d)$, which is bounded from below by the red line and from above by a black line. Here, $|C_g|\sqrt{D_g} = 6$ and $d = 10$. [Colour figure can be viewed at wileyonlinelibrary.com]

Remark 5.4. We now show that there is a nonempty intersection between the cone $\tilde{\mathcal{R}}(d)$ in Remark 5.3 and the set of parameters described by Remark 5.2, for $\sqrt{3} - 1 < \omega < 1$, that is, for $d > 4 + 2\sqrt{3} \sim 7.46$. In fact, notice that

$$\frac{1}{\sqrt{3}-1} > \frac{1-\omega}{2+\omega} \text{ for every } 0 < \omega < 1.$$

Then it follows that $\tilde{\mathcal{R}}(d) \cap \mathcal{T}_g(d) \neq \emptyset$ for $d > 4 + 2\sqrt{3}$; see Figure 7. The set $\mathcal{T}_g(d)$ was introduced in (5.9). As a consequence, if $\sqrt{3} - 1 < \omega < 1$, $(r_i, \lambda_g) \in \tilde{\mathcal{R}}(d) \cap \mathcal{T}_g(d)$ and (5.13) are satisfied, then there are wavefronts to Equation (3.5) satisfying (1.2) having positive speeds.

About the case of *negative speeds*, we have the following result, where (2.13) comes into play.

Theorem 5.4. *Assume f is strictly concave and*

$$\frac{\sqrt{(d-1)\omega(1+\omega)(2-\omega)\left((1+\omega)\mu-(2-\omega)\right)}}{(1+\omega)^2 + sd(1-\omega^2) - 3} > E_g. \tag{5.15}$$

Then either $\mathcal{J} \subset (-\infty, 0)$ or $\mathcal{J} = \emptyset$.

Proof. First, we point out that the term $(1+\omega)r_i - (2-\omega)\lambda_g$ under the square root in (5.15) is positive because of (4.11). By (3.6), we compute

$$\begin{aligned}\dot{f}(\alpha) &= -C_g D_g \left(3(\alpha-1)^2 - 1\right) + C_i D_i (1-\alpha)(3\alpha - 1)\\ &= \frac{C_g D_g \left(3 - (1+\omega)^2\right) + C_i D_i (1-\omega^2)}{3}.\end{aligned}$$

By (4.5) and (4.6), we deduce $\dot{D}(\alpha) = 3(\alpha - \beta)(D_i - D_g) = -2\omega(D_i - D_g)$, and, by (4.2),

$$\begin{aligned}g(\alpha) &= (r_i + \lambda_g)\alpha(1-\alpha)(\alpha - \gamma)\\ &= \frac{(r_i + \lambda_g)(2-\omega)(\omega+1)(2-\omega-3\gamma)}{27}\\ &= \frac{(2-\omega)(\omega+1)\left(-(\omega+1)r_i + (2-\omega)\lambda_g\right)}{27}.\end{aligned}$$

Therefore, we have

$$\dot{D}(\alpha)g(\alpha) = \frac{2}{27}(D_i - D_g)\omega(2-\omega)(\omega+1)\left((1+\omega)r_i - (2-\omega)\lambda_g\right).$$

The proof is concluded by applying (2.13) and noticing that $2\sqrt{2/3} \in (1, 2)$. □

FIGURE 8 Thresholds leading to wavefronts with positive or negative speeds when f is strictly concave. The figure is to be interpreted as follows: if $d > 4 + 2\sqrt{3} \sim 7.46$, then there exist wavefronts with positive speed, while, if $4 < d < \left(5 + 2\sqrt{3}\right)/2 \sim 4.23$, then every wavefront has negative speed under the assumptions of Theorem 5.4.

Corollary 5.2. *Under (5.1), condition (5.15) is satisfied if* $d < \left(5 + 2\sqrt{3}\right)/2$.

Proof. For $A(\omega) := \omega(1+\omega)(2-\omega)$ and $B(\omega) := (1+\omega)\mu - (2-\omega)$, condition (5.15) is

$$\frac{\sqrt{d-1}}{E_g}\sqrt{A(\omega)B(\omega)} > (1+\omega)^2 + sd(1-\omega^2) - 3 =: E(\omega, sd). \tag{5.16}$$

Condition (5.15) is satisfied if the right-hand side of (5.16) is negative. If $C_i = s = 0$, then this happens if $\omega < \sqrt{3} - 1$. In the general case, we notice that

$$E(\omega, sd) \le E\left(\omega, \frac{3}{2}\right) = -\frac{1}{2}\omega^2 + 2\omega - \frac{1}{2} := \varphi(\omega).$$

We have $\varphi(0) = -\frac{1}{2}$, $\varphi(1) = 1$, φ is an increasing function when $\omega \in (0,1)$, and $\varphi(\omega) = 0$ for $\omega \in (0,1)$ iff $\omega = 2 - \sqrt{3}$. Then condition (5.15) is satisfied if $\omega < 2 - \sqrt{3}$, i.e., for $d < \left(5 + 2\sqrt{3}\right)/2 \sim 4.23$. □

Remark 5.5. Let us fix C_g, D_g. By Remark 5.2, for every $s > 0$ and $d > 4$ satisfying (5.1) the existence of wavefronts to (3.5) satisfying (1.2) holds for (r_i, λ_g) in the triangle $\mathcal{T}_g(d)$. For $d \in \left(4, \left(5 + 2\sqrt{3}\right)/2\right)$, every pair $(r_i, \lambda_g) \in \mathcal{T}_g(d)$ provides profiles, and all of them have *negative* speeds.

Remark 5.6. We now briefly resume the results we obtained about the sign of the propagation speed. For $(r_i, \lambda_g) \in \mathcal{T}_g(d)$, we have (see Figure 8):

- if $d > 4+2\sqrt{3}$ and (5.13) is satisfied, then (3.5) admits wavefronts satisfying (1.2) with *positive* speeds (see Remark 5.4);
- if $d \in \left(4, \left(5 + 2\sqrt{3}\right)/2\right)$, then *every* pair $(r_i, \lambda_g) \in \mathcal{T}_g(d)$ provides profiles and all of them have *negative* speeds (see Remark 5.5).

Remark 5.7. When $\gamma \to \alpha$, that is, when $\gamma \to (2-\omega)/3$, we get $\tau(\omega, sd, \gamma) \to 3E(\omega, sd)$, for E as in (5.16). Hence, if $\gamma \sim \alpha$, the condition $\tau < 0$ implies that only wavefronts with negative speeds can agree with (3.5)–(1.2) (from (5.16)). This implies that the model only supports extinction.

6 | THE CASE WHEN f CHANGES CONVEXITY

We now consider a convective term f as in (3.6) (see also (4.4)), which changes its concavity in $[0,1]$ and show that also in this case the model (3.5) can support wavefronts satisfying condition (1.2). Due to the definition of f, a concavity change occurs iff $p \neq 0$, and in this case only once, namely, at $\frac{2}{3} + \frac{q}{3p}$. Moreover, when this occurs, then concavity and convexity are strict.

Lemma 6.1. *Assume that* f *has an inflection point in* $(0,1)$*. Then:*

(i) f *is first convex and then concave if and only if* $sd > \frac{3}{2}$.
(ii) f *is first concave and then convex if and only if* $s < 0$.

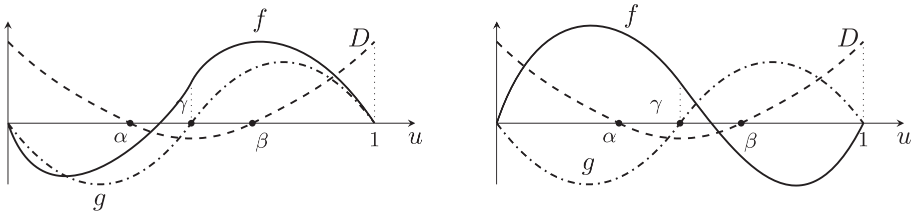

FIGURE 9 Plots of the functions D (dashed line), g (dashdotted line), and f (solid line) in the case f is convex–concave (on the left) and concave–convex (on the right), with γ as inflection point of f.

Proof. We argue as in the proof of Lemma 5.1. About *(i)*, the statement is equivalent to $2p + q > 0$ and $-p + q < 0$, that is, $-2p < q < p$; hence, $p > 0$, and we conclude by (4.3) and $(4.9)_1$.

About *(ii)*, the statement is equivalent to $2p + q < 0$ and $-p + q > 0$, that is, $p < q < -2p$; hence, $p < 0$ and $s < 0$ by $(4.9)_1$. □

To simplify calculations, in the following *we only consider the case when* γ*, which is the inner zero of g and is given by* (4.2), *coincides with the inflection point of* f; that is, we assume in the current section (without further mention)

$$\gamma = \frac{2}{3} + \frac{C_g D_g}{3(C_g D_g + C_i D_i)} = \frac{3 - 2sd}{3(1 - sd)}. \tag{6.17}$$

Notice that the assumptions $p \neq 0$ and $r_i \neq 0$ are equivalent to $sd \neq 1$ (because of $(4.9)_1$) and $sd \neq \frac{3}{2}$, respectively. Then

$$r_i = \left(2 - \frac{3}{s} d\right) \lambda_g. \tag{6.18}$$

We now briefly comment on the biological meaning of assumption (6.17). Recall that γ represents the Allee parameter [1], which describes the threshold separating a decrease of concentration (if $u < \gamma$) from an increase of concentration (if $u > \gamma$). The assumption that f has an inflection point at γ means that the maximum drift $\dot{f}$ (if f is convex-concave) or the minimum drift (if f is concave-convex) is precisely reached at γ. We refer to Figure 9 for an illustration of both cases.

6.1 | The convex–concave case

We consider a function f that is first convex and then concave, with γ as inflection point; see Figure 9 on the left.

We recall that we are assuming $\gamma \in (\alpha, \beta)$; see (4.11). We now check the implications of this condition on sd, because γ also satisfies (6.17). By Lemma 6.1*(i)* and $(4.9)_1$, we obtain $C_i D_i + C_g D_g > 0$ and hence $\gamma < \frac{2}{3} < \beta$ by (6.17). On the other hand, the condition $\gamma > \alpha$ is equivalent to $sd > 1 + \frac{1}{\omega} > 2$ because of (4.1) and (6.17), which strengthens the previous requirement $sd > \frac{3}{2}$. Summing up, under the assumptions of the current case, the parameters sd and γ must satisfy the conditions

$$sd > 1 + \frac{1}{\omega} \qquad \text{and} \qquad \gamma \in \left(\frac{1}{3}, \frac{2}{3}\right). \tag{6.19}$$

We now consider the issue of the existence of profiles. By making use of (4.7), the left-hand side of (2.11) becomes

$$\begin{aligned} \inf_{[0,\gamma]} \delta(f, \alpha) - \sup_{[\gamma,1]} \delta(f, \beta) &= \frac{f(\alpha)}{\alpha} - \frac{f(\gamma) - f(\beta)}{\gamma - \beta} \\ &= (2p + q)(\alpha - \beta - \gamma) + p(\beta^2 - \alpha^2 + \gamma\beta + \gamma^2) \\ &= C_g D_g H_1(\omega, \gamma) + C_i D_i H_2(\omega, \gamma) \\ &= |C_g| D_g \left(H_1(\omega, \gamma) + sd H_2(\omega, \gamma)\right), \end{aligned} \tag{6.20}$$

where

$$H_1(\omega, \gamma) := -\gamma^2 - \gamma\left(\frac{\omega - 7}{3}\right) + \frac{10}{9}\omega \text{ and } H_2(\omega, \gamma) := \gamma^2 + \gamma\left(\frac{\omega - 4}{3}\right) - \frac{4}{9}\omega.$$

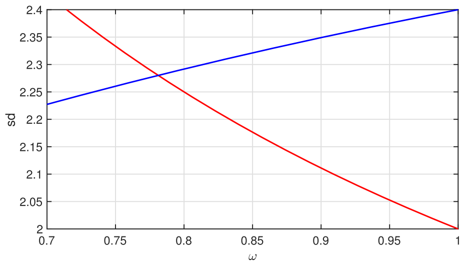

FIGURE 10 The set $\mathcal{S}$ is the plane region bounded from above by the blue curve and from below by the red curve. [Colour figure can be viewed at wileyonlinelibrary.com]

We now investigate the sign of (6.20): its positivity is necessary for (2.11) to hold. First, we introduce the set (see Figure 10)

$$\mathcal{S} := \left\{ (\omega, sd) \, : \, 1 + \frac{1}{\omega} < sd < \frac{12(2+3\omega)}{(4+\omega)^2} \right\}. \tag{6.21}$$

Proposition 6.1. *The quantity in (6.20) is positive for every* $(\omega, sd) \in \mathcal{S}$.

Proof. We know that γ, provided by (6.17), is entirely determined by sd and that it varies in $\left(\frac{1}{3}, \frac{2}{3}\right)$ by (6.19). However, to simplify computations, we treat γ in the current proof as an independent variable ranging in $\left(\frac{1}{3}, \frac{2}{3}\right)$.

First, we claim that for $\omega \in (0, 1)$ and $\gamma \in \left(\frac{1}{3}, \frac{2}{3}\right)$, we have

$$H_1(\omega, \gamma) > \frac{2}{3} + \omega \text{ and } H_2(\omega, \gamma) > -\left(\frac{4+\omega}{6}\right)^2. \tag{6.22}$$

In fact, estimate $(6.22)_1$ follows because the function $\gamma \mapsto H_1(\omega, \gamma)$ is increasing for $\gamma \in \left(\frac{1}{3}, \frac{2}{3}\right)$. Concerning $(6.22)_2$, we have $\min_{\gamma \in \left(\frac{1}{3}, \frac{2}{3}\right)} H_2(\omega, \gamma) = H_2\left(\omega, \frac{4-\omega}{6}\right) = -\left(\frac{4+\omega}{6}\right)^2$.

Next, according to (6.22) and since $sd > 0$, we have, for all $\gamma \in \left(\frac{1}{3}, \frac{2}{3}\right)$,

$$H_1(\omega, \gamma) + sdH_2(\omega, \gamma) > \frac{2+3\omega}{3} - sd\left(\frac{4+\omega}{6}\right)^2. \tag{6.23}$$

The latter quantity is positive iff $sd < \frac{12(2+3\omega)}{(4+\omega)^2}$. By $(6.19)_1$, we need $\frac{12(2+3\omega)}{(4+\omega)^2} > 1 + \frac{1}{\omega}$, and this is equivalent to require $\omega > \omega_0$, where $\omega_0 \sim 0.78$ is the only root of $\omega^3 - 27\omega^2 + 16$ in the interval $(0, 1)$. □

Theorem 6.1. *Assume that f is convex in $[0, \gamma]$, concave in $[\gamma, 1]$, and $(\omega, sd) \in \mathcal{S}$. If*

$$\frac{\sqrt{(d-1)}}{H_1(\omega, \gamma) + sdH_2(\omega, \gamma)} < \frac{E_g}{4} \tag{6.24}$$

holds, then Equation (3.5) has wavefronts satisfying condition (1.2).

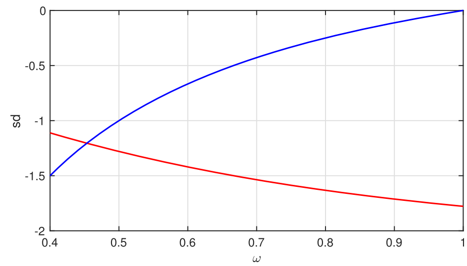

FIGURE 11 The set $\tilde{\mathcal{S}}$ is the plane region bounded from above by the blue curve and from below by the red curve. [Colour figure can be viewed at wileyonlinelibrary.com]

Proof. According to (6.18), Lemma 4.2 and the fact that $sd > 0$, we have

$$2\sup_{[0,\gamma]}\sqrt{\Delta(Dg,\alpha)} + 2\sup_{[\gamma,1]}\sqrt{\Delta(Dg,\beta)} \le \sqrt{D_g}\sqrt{d-1}\left(\sqrt{r_i(2+\omega)} + \sqrt{\lambda_g(1+\omega)}\right)$$
$$= \sqrt{D_g}\sqrt{\lambda_g(d-1)}\left(\sqrt{\left(2-\frac{3}{s}d\right)(2+\omega)} + \sqrt{1+\omega}\right)$$
$$\le \sqrt{D_g}\sqrt{\lambda_g(d-1)}\left(\sqrt{\left(6-\frac{9}{s}d\right)} + \sqrt{2}\right) \le 4\sqrt{D_g}\sqrt{\lambda_g(d-1)}. \tag{6.25}$$

Now, we assumed $(\omega, sd) \in \mathcal{S}$ and then $H_1(\omega,\gamma) + sdH_2(\omega,\gamma) > 0$ by Proposition 6.1. Hence, if (6.24) is satisfied, then condition (2.11) holds true by (6.20) and then (3.5) has wavefronts satisfying (1.2). □

6.2 | The concave–convex case

We now assume that f is concave in $[0,\gamma]$ and convex in $[\gamma,1]$, with γ as inflection point; see Figure 9. Again, we show that (3.5) admits wavefronts satisfying (1.2) under some conditions.

We argue as in the convex–concave case. Lemma 6.1*(ii)* implies $\gamma \in (2/3, 1)$. Moreover, the condition $\gamma < \beta$ is equivalent to $sd < 1 - \frac{1}{\omega} < 0$ by (4.1) and (6.17). Summing up, the parameters sd and γ must now satisfy the conditions

$$sd < 1 - \frac{1}{\omega} \text{ and } \gamma \in \left(\frac{2}{3}, 1\right). \tag{6.26}$$

We now compute the left-hand side of (2.11). According to (4.7), we have

$$\inf_{[0,\gamma]}\delta(f,\alpha) - \sup_{[\gamma,1]}\delta(f,\beta) = \frac{f(\gamma)-f(\alpha)}{\gamma-\alpha} - \frac{f(1)-f(\beta)}{1-\beta}$$
$$= (2p+q)(\alpha-\beta+\gamma-1) + p(\beta^2-\alpha^2-\alpha\gamma-\gamma^2+\beta+1),$$
$$= C_gD_g\tilde{H}_1(\omega,\gamma) + C_iD_i\tilde{H}_2(\omega,\gamma)$$
$$= |C_g|D_g\left(\tilde{H}_1(\omega,\gamma) + sd\tilde{H}_2(\omega,\gamma)\right), \tag{6.27}$$

for

$$\tilde{H}_1(\omega,\gamma) := \gamma^2 - \gamma\left(\frac{\omega+7}{3}\right) + \frac{7}{9}\omega + \frac{4}{3} \text{ and } \tilde{H}_2(\omega,\gamma) := -\gamma^2 + \gamma\left(\frac{\omega+4}{3}\right) - \frac{\omega}{9} - \frac{1}{3}.$$

We now discuss the sign of (6.27); to this aim, we define the set (see Figure 11)

$$\tilde{\mathcal{S}} := \left\{(\omega, sd) : -\frac{16\omega}{(\omega+2)^2} < sd < 1 - \frac{1}{\omega}\right\}. \tag{6.28}$$

If $(\omega, sd) \in \tilde{\mathcal{S}}$ and estimate (6.31) holds, then Equation (3.5) has wavefronts satisfying condition (1.2).

Proposition 6.2. *The quantity in (6.27) is positive for $(\omega, sd) \in \tilde{\mathcal{S}}$.*

Proof. Notice that, according to (6.17), γ depends on sd; however, as in the proof of Proposition 6.1, we treat γ as an independent variable ranging in $(0, 1)$.

First, we claim that for all $\omega, \gamma \in (0, 1)$, we have

$$\tilde{H}_1(\omega,\gamma) > \frac{4}{9}\omega \text{ and } \tilde{H}_2(\omega,\gamma) < \left(\frac{\omega+2}{6}\right)^2. \tag{6.29}$$

About $(6.29)_1$, since $\tilde{H}_1(\omega,\gamma)$ is a decreasing function for $\gamma \in \left(-\infty, \frac{\omega+7}{6}\right)$ and $\frac{\omega+7}{6} > 1$, we obtain $\tilde{H}_1(\omega,\gamma) > \tilde{H}_1(\omega,1) = \frac{4}{9}\omega$ for $\gamma \leq 1$. About $(6.29)_2$, we have $\max_{\gamma\in\mathbb{R}}\tilde{H}_2(\omega,\gamma) = \tilde{H}_2\left(\omega, \frac{\omega+4}{6}\right) = \left(\frac{\omega+2}{6}\right)^2$.

Next, according to (6.29) and since $sd < 0$, we have

$$\tilde{H}_1(\gamma) + sd\tilde{H}_2(\gamma) > \frac{4}{9}\omega + sd\left(\frac{\omega+2}{6}\right)^2, \tag{6.30}$$

which is positive when $sd > -16\omega/(\omega+2)^2$. Since we have $sd < 1 - \frac{1}{\omega}$, we need that

$$1 - \frac{1}{\omega} > -\frac{16\omega}{(\omega+2)^2},$$

which is true when $\omega > \tilde{\omega}_0$, where $\tilde{\omega}_0 \sim 0.45$ is the only root of $\omega^3 + 19\omega^2 - 4$ in $(0, 1)$. This completes the proof. □

Theorem 6.2. *Assume that f is concave in $[0, \gamma]$, convex in $[\gamma, 1]$ and $(\omega, sd) \in \tilde{\mathbb{S}}$. If*

$$\frac{5\sqrt{d-1}}{(1-\omega)\left(\tilde{H}_1(\omega,\gamma) + sd\tilde{H}_2(\omega,\gamma)\right)} < E_g \tag{6.31}$$

holds, then Equation (3.5) has wavefronts satisfying (1.2).

Proof. As in the proof of Theorem 6.1, by taking $sd < 1 - \frac{1}{\omega}$ into account, we have

$$\begin{aligned} 2\sup_{[0,\gamma]}\sqrt{\Delta(Dg,\alpha)} + 2\sup_{[\gamma,1]}\sqrt{\Delta(Dg,\beta)} &\leq \sqrt{\lambda_g D_g}\sqrt{d-1}\left(\sqrt{\left(6 - \frac{9}{s}d\right)} + \sqrt{2}\right) \\ &\leq \sqrt{\lambda_g D_g}\sqrt{d-1}\left(\sqrt{\frac{6+3\omega}{1-\omega}} + \sqrt{2}\right) \leq \sqrt{\lambda_g D_g}\sqrt{d-1}\left(\frac{3}{\sqrt{1-\omega}} + \sqrt{2}\right) \\ &\leq \sqrt{\lambda_g D_g}\sqrt{d-1}\frac{3+\sqrt{2}}{\sqrt{1-\omega}} \leq 5\sqrt{\lambda_g(d-1)}\frac{\sqrt{D_g}}{1-\omega}. \end{aligned} \tag{6.32}$$

Since we assumed $(\omega, sd) \in \tilde{\mathbb{S}}$, then, by (6.27), conditions (6.31) and (6.32) imply (2.11); according to Corollary 2.2, the model (3.5) has wavefronts satisfying (1.2). □

7 | CONCLUSIONS

We investigate a model for the movement of biological organisms that includes a convective term f. The population is split into isolated and grouped individuals. We focus on the existence of decreasing wavefront solutions. This model is inspired by [1], where it was first proposed in the case $f = 0$. We consider a diffusivity D, which makes the equation of forward–backward–forward type and assume that the reaction term g has the strong Allee effect with its inner zero between the two inner zeros of the diffusivity. In this case, there are no wavefronts if $f = 0$.

The convection f has four possibilities: It can be either concave, or convex, or convex–concave, or else concave–convex. A key role in our discussion is played by the *adimensional* term $E_g = |C_g|\sqrt{D_g/\lambda_g}$, which *only* depends on behavior of the *grouped* population and lumps all their significative parameters.

- When f is convex, wavefronts do not exist.
- When f is concave, wavefronts exists if condition (5.5) is satisfied; this is an inequality, whose left-hand side depends on the mutual behavior of grouped and isolated individuals and involves many parameters. In Remark 5.2, we vary *pairs* of them while fixing the others and show plane regions where (5.5) holds. Its right-hand side is just E_g. As a consequence, (5.5) is certainly satisfied when E_g is large, that is, if either the convective and diffusivity coefficients of the grouped population are sufficiently large or else its birth rate is small.

  Assume now wavefronts exist. If condition (5.13) holds, then some of them have positive speed; this suggests the persistence of the species in the long period. On the contrary, when (5.15) holds, all the wavefronts have negative speed suggesting the extinction of this species in the long term. Again, both conditions are inequalities with E_g on the right-hand side; in particular, (5.13) is satisfied if E_g is large, while (5.15) is valid if E_g is small. A detailed discussion appears in Remark 5.6.
- When f changes convexity, we focus for simplicity on the case its inflection point coincides with the inner zero of g. In both the convex–concave and concave–convex cases, we give sufficient conditions for the existence of wavefronts: They exist if E_g is large enough.

AUTHOR CONTRIBUTIONS

Diego Berti: Conceptualization; Investigation; methodology; writing—original draft; writing—review & editing. **Andrea Corli:** Conceptualization; Investigation; writing—original draft; methodology; writing—review & editing. **Luisa Malaguti:** Conceptualization; Investigation; methodology; writing—review & editing; writing—original draft.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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REFERENCES

1. S. T. Johnston, R. E. Baker, S. D. L. McElwain, and M. J. Simpson, *Co-operation, competition and crowding: A discrete framework linking Allee kinetic, nonlinear diffusion, shocks and sharp-fronted travelling waves*, Sci. Rep. **7** (2017), 42134.
2. C. S. Patlak, *Random walk with persistence and external bias*, Bull. Math. Biophys. **15** (1953), 311–338, DOI 10.1007/bf02476407.
3. D. Horstmann, K. J. Painter, and H. G. Othmer, *Aggregation under local reinforcement: From lattice to continuum*, European J. Appl. Math. **15** (2004), no. 5, 546–576, DOI 10.1017/S0956792504005571.
4. J. D. Murray, *Mathematical biology. II*, 3rd ed., Springer-Verlag, New York, 2003.
5. A. Okubo and S. A. Levin, *Diffusion and ecological problems: Modern perspectives*, 2nd ed., Springer-Verlag, New York, 2001, DOI 10.1007/978-1-4757-4978-6.
6. H. G. Othmer and A. Stevens, *Aggregation, blowup, and collapse: The ABCs of taxis in reinforced random walks*, SIAM J. Appl. Math. **57** (1997), no. 4, 1044–1081, DOI 10.1137/S0036139995288976.
7. V. Padrón, *Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation*, Trans. Amer. Math. Soc. **356** (2004), no. 7, 2739–2756, DOI 10.1090/S0002-9947-03-03340-3.
8. F. Sánchez-Garduño, P. K. Maini, and J. Pérez-Velázquez, *A non-linear degenerate equation for direct aggregation and traveling wave dynamics*, Discrete Contin. Dyn. Syst. Ser. B **13** (2010), no. 2, 455–487, DOI 10.3934/dcdsb.2010.13.455.
9. P. Turchin, *Population consequences of aggregative movements*, J. Anim. Ecol. **58** (1989), 75–100.
10. Y. Li, P. van Heijster, M. J. Simpson, and M. Wechselberger, *Shock-fronted travelling waves in a reaction–diffusion model with nonlinear forward–backward–forward diffusion*, Phys. D. **423** (2021), 132916.

11. B. H. Gilding and R. Kersner, *Travelling waves in nonlinear diffusion-convection reaction*, Birkhäuser Verlag, Basel, 2004, DOI 10.1007/978-3-0348-7964-4.
12. P. C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomathematics, Vol. **28**, Springer-Verlag, Berlin-New York, 1979.
13. Y. Li, P. van Heijster, R. Marangell, and M. J. Simpson, *Travelling wave solutions in a negative nonlinear diffusion-reaction model*, J. Math. Biol. **81** (2020), no. 6-7, 1495–1522, DOI 10.1007/s00285-020-01547-1.
14. M. Kuzmin and S. Ruggerini, *Front propagation in diffusion-aggregation models with bi-stable reaction*, Discrete Contin. Dyn. Syst. Ser. B. **16** (2011), no. 3, 819–833, DOI 10.3934/dcdsb.2011.16.819.
15. D. Berti, A. Corli, and L. Malaguti, *Traveling waves for degenerate diffusion-convection reaction equations with sign-changing diffusivity*, Discrete Contin. Dyn. Syst. **41** (2021), no. 12, 6023–6048.
16. D. Berti, A. Corli, and L. Malaguti, *Analysis, Applications, and Computations*, U. Kähler, M. Reissig, I. Sabadini, and J. Vindas (eds.), Trends in Mathematics, Birkhäuser, Cham, 2022.
17. D. H. Sattinger, *Weighted norms for the stability of traveling waves*, J. Differ. Equ. **25** (1977), no. 1, 130–144, DOI 10.1016/0022-0396(77)90185-1.
18. D. Berti, A. Corli, and L. Malaguti, *Diffusion-convection reaction equations with sign-changing diffusivity and bistable reaction term*, Nonlinear Anal. Real World Appl. **67** (2022), 103579, 29, DOI 10.1016/j.nonrwa.2022.103579.
19. C. Marcelli and F. Papalini, *A new estimate of the minimal wave speed for travelling fronts in reaction-diffusion-convection equations*, Electron. J. Qual. Theory Differ. Equ. **10** (2018), 1–13, DOI 10.14232/ejqtde.2018.1.10.
20. A. Corli and L. Malaguti, *Semi-wavefront solutions in models of collective movements with density-dependent diffusivity*, Dyn. Partial Differ. Equ. **13** (2016), no. 4, 297–331, DOI 10.4310/DPDE.2016.v13.n4.a2.
21. G. B. Whitham, *Linear and nonlinear waves*, Wiley-Interscience, New York-London-Sydney, 1974.

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