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Applied Mathematical Sciences, Vol. 17, 2023, no. 9 – 12

Contents

Featured Articles published in Vol. 17, 2023, no. 5 – 8

Applied Mathematical Sciences, Vol. 17, 2023, no. 9, 421 - 428
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A Nonlinear Dynamics for Risk Contagion: Theoretical Remarks

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Abstract

We employ an epidemiological approach to explain how risk may spread in a given economic system. Precisely, the analogy between economic systems and ecosystems is exploited and an original Susceptible-Infected-Recovered model with time delay is adopted to describe risk contagion by a nonlinear dynamics. The economic player population is divided into a set of distinct compartments, which are defined in terms of risk with low and high level. The time delay represents the period of financial immunity that some agents get from recovery after risk infection. Moreover, the contagion phenomenon is modelled by a Holling Type II functional response, which accounts for an incubation time from the contact between susceptible and infected players up to the actual financial distress.

The existence of a unique solution of the proposed delay differential system is stated, moreover the main qualitative features are discussed. Actually, we prove that the dynamics remains positive during the whole time horizon and it admits two different stationary states.

Mathematics Subject Classification: 91B05, 91B55, 34K04

Keywords: Risk contagion, Financial immunity, SIR model, Delay Differential equation

1 Introduction

In the economic and financial literature, the idea of contagion is crucial since both endogenous and exogenous events have the potential to start a relevant chain reaction of crises. Actually, any economic agent entering a crisis or precrisis state will likely cause other economic agents to do the same due to the domino effect. Regarding financial distress as an infection, economic literature deals with several frameworks: infections occurring in the banking system, infections spreading in financial markets, but also infections that occur in terms of credit and debit relationships between companies. As a consequence, an analogy may be established between financial systems and ecosystems; therefore, the so-called Susceptible-Infected-Recovered (SIR) approach can be revisited to model risk contagion among economic players. The SIR epidemiological approach has been adopted in the literature for modelling risk contagion in different contributions among which we cite [1], [4], [5], [6], [10].
In this framework, we focus on an original nonlinear dynamics related to a differential model where economic players with lower risk are susceptible to be infected by other agents with a risk at an extremely high level. Precisely, the economic agents belong to different compartments corresponding to the classes of susceptible, infected and recovered players. The dynamics is described in terms of the densities $S(t)$, $I(t)$ and $R(t)$ of susceptible players, infected and recovered agents, respectively, at any time t. As a further development of the analysis provided in [1], we model the phenomenon of contagion by a Holling Type II functional response as

$$\frac{aS(t)}{1 + haS(t)}, \tag{1}$$

where $a > 0$ is the so-called attack rate, i.e. the average rate at which an infected item gets in contact with a susceptible one, while $h > 0$ is the so-called handling time, i.e. the average time which is spent for infecting a susceptible agent. Under the perspective of the analogy between economic systems and ecosystems, on one hand the infected economic players represent predators, on the other hand the susceptible economic agents may be interpreted as prey species. Thus, the parameter a may measure the degree of association between the agents that transmit each other the risk, while h is an incubation time for the contagion that is the average time passing from the moment of contact up to the moment of actual financial distress. We notice that, as S becomes large, then the term in (1) tends towards the value $1/h$ so that the contagion effect is limited, because when the number of susceptible agents increases then they can gain protection against contagion by means of the increased supervision, self-discipline, and collaboration.
As another original issue, here we suppose that a portion of players indefinitely leaves the economic sector at an elimination rate which depends on the risk level of compartments; thus, we account for γ_L and γ_H which are the death rates of players with low risk and high risk, respectively, and assume $0 < \gamma_L \leq \gamma_H < 1$.

In addition, the temporal process is affected by a given time delay τ which represents a financial immunity ability got by those players which are recovered after contagion: they remain in a kind of recovery pool for a period of length τ, after that they may revert to come back being susceptible again. The resulting mathematical model describing the risk spreading inside the economic system consists of the following equations

$$\frac{dS(t)}{dt} = b - \gamma_L S(t) - \frac{aS(t)}{1 + haS(t)} I(t) + \delta e^{-\gamma_L \tau} I(t - \tau), \tag{2}$$

$$\frac{dI(t)}{dt} = \frac{aS(t)}{1 + haS(t)} I(t) - (\gamma_H + \delta) I(t), \tag{3}$$

$$\frac{dR(t)}{dt} = \delta I(t) - \gamma_L R(t) - \delta e^{-\gamma_L \tau} I(t - \tau), \tag{4}$$

with parameters $b > 0$ and $0 < \delta < 1$ representing the birth rate of new players and the recovery rate from the high risk, respectively.

In the next Sections, we discuss the well-posedness of the problem. Moreover, we show that the dynamics remains positive for all time when it starts from positive initial data; this issue is worthwhile to be proved because the system describes the temporal evolution of a population of economic agents. Finally, we focus our attention on the existence of two different stationary points at the long run.

2 Existence of a unique risk trajectory

Equation (4) can be integrated once the infected dynamics is known. Indeed, we prescribe an initial condition like $R(0) = \delta \int_{-\tau}^{0} e^{\gamma_L s} I(s)\, ds$ and obtain

$$R(t) = \delta \int_{t-\tau}^{t} e^{\gamma_L s} I(s)\, ds. \tag{5}$$

As a consequence, we omit equation (4) and only focus on system (2)-(3) without loss of generality. The model is completed by initial conditions:

$$S(0) = S_0 > 0, \tag{6}$$

$$I(s) = I_0(s) \geq 0, \quad \text{for all } s \in [-\tau, 0], \quad \text{with } I_0(0) > 0, \tag{7}$$

where $I_0(\cdot)$ is a continuous function and represents the history of the infected class in the whole time lag interval $[-\tau, 0]$.
As a consequence, due to the well-known fundamental theory of functional differential equations (see for instance [7], [9]), there exists a unique solution of problem (2)-(3) which is equipped with initial conditions (6)-(7).

3 Positivity of risk trajectory

As already mentioned, system (2)-(3) describes the evolution of a population of economic players; therefore, it is worthwhile to prove that $S(t)$ and $I(t)$ get positive values at any time when they start from positive initial data as supposed in (6)-(7). With this aim, we provide the next result.

Proposition 3.1 *Let $S(t)$ and $I(t)$ solve problem (2)-(3) which is completed by non-negative initial data (6)-(7). Then it is possible to prove that $S(t)$ and $I(t)$ are positive for all $t > 0$.*

The proof of the previous result can be carried out by an argument which assumes the proposition is false. Under this approach, we set

$$t_1 = \min\{t \;:\; t > 0,\; S(t) \cdot I(t) = 0\}, \tag{8}$$

and, at a first step, suppose that $I(t_1) = 0$. As a consequence, $S(t) \geq 0$ for any $t \in [0, t_1]$. Since $S(t)$ is a differentiable function, then it is continuous and bounded over the interval $[0, t_1]$ and the following minimum value can be defined

$$A_I = \min_{0 \leq t \leq t_1} \left\{ \frac{aS(t)}{1 + haS(t)} - (\gamma_H + \delta) \right\}.$$

This value can be exploited in equation (3) in order to obtain

$$\frac{dI(t)}{dt} \geq A_I I(t),$$

for every time $t \in [0, t_1]$. The previous inequality yields

$$I(t_1) \geq I_0 \, e^{A_I t_1} > 0,$$

which is a contradiction.
As a second step, we suppose that $S(t_1) = 0$. It follows that $I(t) \geq 0$ for any $t \in [-\tau, t_1]$. This inequality can be exploited in equation (2), thus we have

$$\left.\frac{dS(t)}{dt}\right|_{t=t_1} = b + \delta \, e^{-\gamma_L \tau} \, I(t_1 - \tau) > 0. \tag{9}$$

Since $S(0) = S_0 > 0$, defining t_1 as in (8) implies that for $S(t_1) = 0$ we must have

$$\frac{dS(t)}{dt}|_{t=t_1} \leq 0,$$

which is a contradiction with respect to (9). This completes the proof so that Proposition 3.1 is true.

As a consequence of Proposition 3.1, we remark that $R(t)$ is positive for all $t > 0$ since it is evaluated by (5) where function $I(s) > 0$ is integrated in in the interval $[t - \tau, t]$.

As another remark, we focus on the total density $N(t) = S(t) + I(t) + R(t)$ at any time t and notice that

$$\frac{dN}{dt} \leq b - \gamma_L N(t).$$

By integrating the previous relationship in time, we get

$$N(t) \leq N(0)e^{-\gamma_L t} + \frac{b}{\gamma_L}(1 - e^{-\gamma_L t}).$$

As a consequence, since all the variables are non-negative valued, then the previous inequality implies that $S(t)$, $I(t)$ and $R(t)$ are upper bounded by the term $N(0)e^{-\gamma_L t} + b(1 - e^{-\gamma_L t})/\gamma_L$.

4 Equilibria at the long run

An equilibrium point $E^* = (S^*, I^*)$ for the dynamics in (2)-(3) satisfies the following system of equations

$$b - \gamma_L S^* - \frac{aS^*}{1 + haS^*} I^* + \delta e^{-\gamma_L \tau} I^* = 0, \tag{10}$$

$$\frac{aS^*}{1 + haS^*} I^* - (\gamma_H + \delta) I^* = 0. \tag{11}$$

Therefore, model (2)-(3) admits the risk-free equilibrium

$$E_0^* = \left(\frac{b}{\gamma_L}, 0 \right),$$

and one more non-zero steady state $E_\tau^* = (S_\tau^*, I_\tau^*)$ such that

$$S_\tau^* = \frac{\gamma_H + \delta}{a(1 - h(\gamma_H + \delta))},$$

and

$$I_\tau^* = \frac{b - \gamma_L S_\tau^*}{\gamma_H + \delta(1 - e^{-\gamma_L \tau})} = \frac{\gamma_L(\gamma_H + \delta)}{(\gamma_H + \delta(1 - e^{-\gamma_L \tau}))\, a(1 - h(\gamma_H + \delta))}(\rho_0 - 1),$$

where

$$\rho_0 = \frac{ba(1 - h(\gamma_H + \delta))}{\gamma_L(\gamma_H + \delta)},$$

is the so-called *basic reproduction number* of the risk infection. We point out that this non-trivial point $E_\tau^* = (S_\tau^*, I_\tau^*)$ represents an endemic equilibrium which is feasible under the assumption that $\rho_0 > 1$.

5 Further development

The model we propose here and discuss in the framework of risk contagion is inspired by the analysis developed in [8] for describing a disease transmission in the context of an epidemic, in a different framework of application. Our approach differs in the employment of a Holling Type II functional response depending on the susceptible density. This kind of functional response is exploited as a tool for describing predator-prey dynamics, where the susceptible agents play the role of prey species and the infected players are the predators. Holling Type II functional response allows for investigating risk infection dynamics by taking into account the so-called incubation time together with the financial immunity period. As another difference, we assume that a portion of economic agents are eliminated from the market at a rate which is dependent on the risk level of the compartments they are a part of.
In this respect, we would like to point out that the SIR idea is known in the literature, anyway here the paradigm is original and its application is worthy of attention in the economic and financial context.

In this paper, the well-posedness of the proposed model has been stated and the existence of two different steady states has been pointed out. As a further development, the analysis of the problem is completed in [2] and [3] where the risk trajectory behaviour is studied at the long run. Indeed, the stability is investigated for both risk-free equilibrium and endemic steady state in

[2] and [3], respectively. They highlight the crucial role of the incubation period and the period of financial immunity on the nature of long-term equilibria.

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