ON SECANT DIMENSIONS AND IDENTIFIABILITY OF FLAG VARIETIES

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Abstract. We investigate the secant dimensions and the identifiablity of flag varieties parametrizing flag of sub vector spaces of a fixed vector space. We give numerical conditions ensuring that secant varieties of flag varieties have the expected dimension, and that a general point on these secant varieties is identifiable.

Contents

1. Introduction

In the most general contest, a flag variety is a projective variety homogeneous under a complex linear algebraic group. Flag varieties play a central role in algebraic geometry, combinatorics, and representation theory [Bri05, BL18].

Fix a vector space $V\cong\mathbb{C}^{n+1}$, over an algebraically closed field K of characteristic zero, and integers $k_1\leq\ldots\leq k_r$. Let $\mathbb{G}(k_i,n)\subset\mathbb{P}^{N_i}$, where $N_i=\binom{n+1}{k_i+1}-1$, be the Grassmannians of k_i-dimensional linear subspace of $\mathbb{P}(V)$ in its Plücker embedding. We have an embedding of the product of these Grassmannians

$$\mathbb{G}(k_1,n)\times\cdots\times\mathbb{G}(k_r,n)\subset\mathbb{P}^{N_1}\times\cdots\times\mathbb{P}^{N_r}\subset\mathbb{P}^N$$

where $N=\binom{n+1}{k_1+1}\cdots\binom{n+1}{k_r+1}-1$.

The flag variety $\mathbb{F}(k_1,\ldots,k_r;n)$ is the set of flags, that is nested subspaces, $V_{k_1}\subset\cdots\subset V_{k_r}\subsetneq V$. This is a subvariety of the product of Grassmannian $\prod_{i=1}^r\mathbb{G}(k_i,n)$. Hence, via a product of Plücker embeddings followed by a Segre embedding we can embed $\mathbb{F}(k_1,\ldots,k_r;n)$

$$\mathbb{F}(k_1,\ldots,k_r;n)\hookrightarrow\mathbb{P}^{N_1}\times\cdots\times\mathbb{P}^{N_r}\hookrightarrow\mathbb{P}^N$$

Consider natural numbers $a_1,\ldots,a_n$ such that $a_{k_1+1}=\cdots=a_{k_r+1}=1$ and $a_i=0$ for all $i\notin\{k_1+1,\ldots,k_r+1\}$. Then, $\mathbb{F}(k_1,\ldots,k_r;n)$ generates the subspace

$$\mathbb{P}(\Gamma_{a_1,\ldots,a_n})\subseteq\mathbb{P}\left(\bigwedge^{k_1+1}V\otimes\cdots\otimes\bigwedge^{k_r+1}V\right)\subseteq\mathbb{P}^N$$

where $\Gamma_{a_1,\ldots,a_n}$ is the irreducible representation of $\mathfrak{sl}_{n+1}\mathbb{C}$ with highest weight $(a_1+\cdots+a_n)L_1+\cdots+a_nL_n$, and $L_1+\cdots+L_k$ is the highest weight of the irreducible representation $\bigwedge^k V$. We will denote $\Gamma_{a_1,\ldots,a_n}$ simply by Γ_a. By the Weyl character formula we have that

$$\dim\mathbb{P}(\Gamma_a)=\prod_{1\leq i<j\leq n+1}\frac{(a_i+\cdots+a_{j-1})+j-i}{j-i}-1$$

Furthermore, $\dim\mathbb{F}(k_1,\ldots,k_r;n)=(k_1+1)(n-k_1)+\sum_{j=2}^i(n-k_j)(k_j-k_{j-1})$ and $\mathbb{F}(k_1,\ldots,k_r;n)=\mathbb{P}(\Gamma_a)\cap\prod_{i=1}^r\mathbb{G}(k_i,n)\subset\mathbb{P}^N$.

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The geometry of these varieties has been investigated mostly from the point of view of Schubert calculus [Bri05] and dual defectivity [Tev05]. Secant varieties of small dimensional flag varieties have been studied in [BD10] by taking advantage of the tropical approach to secant dimensions introduced by J. Draisma in [Dra08].

The *h-secant variety* $\mathbb{S}ec_h(X)$ of a non-degenerate n-dimensional variety $X \subset \mathbb{P}^N$ is the Zariski closure of the union of all linear spaces spanned by collections of h points of X. The *expected dimension* of $\mathbb{S}ec_h(X)$ is $\operatorname{expdim}(\mathbb{S}ec_h(X)) := \min\{nh+h-1, N\}$. In general, the actual dimension of $\mathbb{S}ec_h(X)$ may be smaller than the expected one. In this case, following [CC10, Section 2] we say that X is *h-defective* and the number $\delta_h(X) = \operatorname{expdim}(\mathbb{S}ec_h(X)) - \dim(\mathbb{S}ec_h(X))$ is called the h-secant defect of X.

We investigate secant defectivity of flag varieties following the machinery introduced in [MR19], which we now outline. Given general points $x_1, \dots, x_h \in X \subset \mathbb{P}^N$, consider the linear projection $\tau_{X,h} : X \subseteq \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_h}$, with center $\langle T_{x_1}X, \dots, T_{x_h}X\rangle$, where $N_h := N - 1 - \dim(\langle T_{x_1}X, \dots, T_{x_h}X\rangle)$. [CC02, Proposition 3.5] yields that if $\tau_{X,h}$ is generically finite then X is not $(h+1)$-defective. Given $p_1, \dots, p_l \in X$ general points, we consider the linear projection $\Pi_{T^{k_1,\dots,k_l}_{p_1,\dots,p_l}} : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_{k_1,\dots,k_l}}$ with center the span $\langle T^{k_1}_{p_1}X, \dots, T^{k_l}_{p_l}X\rangle$ of higher order osculating spaces. We can degenerate, under suitable conditions, the linear span of several tangent spaces $T_{x_i}X$ into a subspace contained in a single osculating space T^k_pX. So the tangential projection $\tau_{X,h}$ degenerates to a linear projection with center contained in $\langle T^{k_1}_{p_1}X, \dots, T^{k_l}_{p_l}X\rangle$. If $\Pi_{T^{k_1,\dots,k_l}_{p_1,\dots,p_l}}$ is generically finite, then $\tau_{X,h}$ is generically finite as well, and we conclude that X is not $(h+1)$-defective. In this paper we apply this strategy to flag varieties. We would like to stress that this approach, as the one introduced in [Dra08], depends heavily on an explicit parametrization of X. This method was successfully applied to other classes of homogeneous varieties such as Grassmannians [MR19], Segre-Veronese varieties [AMR19], Lagrangian Grassmannians and Spinor varieties [FMR18]. However, its application to flag varieties involves much more difficult computations compared with the case of the Grassmannians, this is particularly reflected in Section 4 where we introduce submersions of flag varieties into product of Grassmannians in order to study the relation among their higher osculating spaces.

Furthermore, our results on secant defectivity, combined with a recent result in [CM19], allow us to produce a bound for identifiability of flag varieties. Recall that, given a non-degenerated variety $X \subset \mathbb{P}^N$, we say that a point $p \in \mathbb{P}^N$ is h-identifiable if it lies on a unique $(h-1)$-plane in $\mathbb{P}^N$ that is h-secant to X. Especially when $\mathbb{P}^N$ can be interpreted as a tensor space, identifiablity and tensor decomposition algorithms are central in applications for instance in biology, Blind Signal Separation, data compression algorithms, analysis of mixture models psycho-metrics, chemometrics, signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience and graph analysis [DL13a], [DL13b], [DL15], [KAL11], [SB00], [BK09], [CGLM08], [LO15], [MR13]. Our main results in Theorem 4.14 and Corollary 4.15 can be summarized in the following statement.

Theorem 1.1. *Consider a flag variety $\mathbb{F}(k_1, \dots, k_r; n)$. Assume that $n \geq 2k_j + 1$ for some index j and let l be the maximum among these j's. Then, for*

$$h \leq \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^{l} k_j + l - 1)\rfloor}$$

$\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective. Furthermore, under the same bound, the general point of the h-secant variety of $\mathbb{F}(k_1, \dots, k_r; n)$ is h-identifiable.

The paper is organized as follows: in Section 2 we study higher order osculating spaces of products of Grassmannians and the linear projections from them, in Section 3 we apply the method introduced in [MR19] to products of Grassmannians, in Section 4 we get bounds for non-secant defectivity and identifiablity of flag varieties, and in Section 5 we investigate the variety of secant lines of spacial flag varieties of type $\mathbb{F}(0, k; n)$.

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2. Higher osculating behavior of products of Grassmannians

Consider the product $\mathbb{G}(k_1, n) \times \dots \times \mathbb{G}(k_r, n) \subset \mathbb{P}^{N_1} \times \dots \times \mathbb{P}^{N_r} \subset \mathbb{P}^N$, and given a non-negative integer k define

$$\Lambda_k = \{I \subset \{0, \dots, n\} \mid |I| = k+1\}$$

For any $I = \{i_0, \dots, i_k\} \in \Lambda_k$ let $e_I \in \mathbb{G}(k,n)$ be the point corresponding to $e_{i_0} \wedge \cdots \wedge e_{i_k} \in \bigwedge^{k+1} \mathbb{C}^{n+1}$. We will denote by Z_I the Plücker coordinates on $\mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$.

From [MR19] we have a notion of distance in Λ_k given by

$$d(I,J) = |I| - |I \cap J| \tag{2.1}$$

for all $I, J \in \Lambda_k$. More generally, we define

$$\Lambda = \Lambda_{k_1} \times \cdots \times \Lambda_{k_r}$$

Given $I = \{I^1, \dots, I^r\} \in \Lambda$ let $e_I \in \prod_{i=1}^r \mathbb{G}(k_i, n)$ be the point corresponding to $e_{I^1} \otimes \cdots \otimes e_{I^r} \in \mathbb{P}^N$, and by Z_I the corresponding homogeneous coordinate of $\mathbb{P}^N$. Furthermore, for all $I, J \in \Lambda$ with $I = \{I^1, \dots, I^r\}$ and $J = \{J^1, \dots, J^r\}$, we define their distance as

$$d(I,J) = \sum_{i=1}^r d(I^i, J^i)$$

where $d(I^i, J^i)$ is the distance defined in (2.1).

From now on we will assume that $n \geq 2k_r + 1$. Under this assumption Λ has diameter $r + \sum_{i=1}^r k_i$ with respect to this distance.

In the following, we give an explicit description of the osculating spaces of $\prod_{i=1}^r \mathbb{G}(k_i, n)$ at coordinate points.

Proposition 2.2. *For each $s \geq 0$*

$$T^s_{e_I}\left(\prod_{i=1}^r \mathbb{G}(k_i,n)\right) = \langle e_J \,;\, d(I,J) \leq s \rangle = \{Z_J = 0 \,;\, d(I,J) > s\} \subset \mathbb{P}^N$$

In particular, $T^s_{e_I}(\prod_{i=1}^r \mathbb{G}(k_i,n)) = \mathbb{P}^N$ for $s \geq r + \sum_{i=1}^r k_i$.

Proof. Set $I = \{I^1, \dots, I^r\} \in \Lambda$. We may assume that $I^i = \{0, \dots, k_i\}$ for each $1 \leq i \leq r$ and consider the following parametrization of $\prod_{i=1}^r \mathbb{G}(k_i, n)$ in a neighborhood of e_I:

$$\begin{array}{cccc} \varphi & : & \prod_{i=1}^r \mathbb{C}^{(k_i+1)(n-k_i)} & \longrightarrow & \mathbb{P}^N \\ & & \begin{bmatrix} I_{k_i+1} & (x^i_{l,m}) \end{bmatrix}_{i=1,\dots,r} & \longmapsto & \left(\prod_{i=1}^r \det(M_{J^i})\right)_{J=\{J^1,\dots,J^r\}\in\Lambda} \end{array} \tag{2.3}$$

where M_{J^i} is the submatrix obtained from $\begin{bmatrix} I_{k_i+1}, & (x^i_{l,m})_{\substack{0\leq l\leq k_i \\ k_i+1\leq m\leq n}} \end{bmatrix}$ by considering the columns indexed by J^i.

For each $J \in \Lambda$, we will denote $\prod_{i=1}^r \det(M_{J^l})$ simply by $\det(M_J)$. Note that each variable appears in degree at most one in the coordinates of φ. Therefore, deriving two times with respect to the same variable always gives zero. Furthermore, as $\det(M_J)$ has degree at most $r + \sum_{i=1}^r k_i$ all partial derivatives of order greater or equal than $r + \sum_{i=1}^r k_i$ are zero. Thus, it is enough to prove the claim for $s \leq r + \sum_{i=1}^r k_i$.

Given $J = \{J^1, \dots, J^r\} \in \Lambda$, let i, k, k' be integers such that $1 \leq i \leq r$, $k \in \{0 \dots, k_i\}$ and $k' \in \{k_i+1, \dots, n\}$. Then

$$\frac{\partial \det(M_J)}{\partial x^i_{k,k'}} = \begin{cases} \pm \dots \det(M_{J^{i-1}}) \det(M_{J^i,k,k'}) \det(M_{J^{i+1}}) \dots & k' \in J^i \\ 0 & k' \notin J^i \end{cases}$$

where $M_{J^i,k,k'}$ is the submatrix obtained from M_{J^i} by deleting the column indexed by k' and the row indexed by k.

More generally, let $m_1, \dots, m_r$ be non-negative integers such that their sum is bigger than one. For each $i = 1, \dots, r$ consider

$$K_i = \{k^i_1, \dots, k^i_{m_i}\} \subset \{0, \dots, k_i\} \text{ and } K'_i = \{k'^i_1, \dots, k'^i_{m_i}\} \subset \{k_i+1, \dots, n\}$$

with $|K_i| = |K'_i| = m_i$. Now, set $m = m_1 + \cdots + m_r$ and

$$K = \{K_1, \dots, K_r\},\ K' = \{K'_1, \dots, K'_r\}$$

Therefore, denoting $\partial x^1_{k^1_1,k'^1_1} \cdots \partial x^r_{k^r_{m_r},k'^r_{m_r}}$ simply by $\partial^m K, K'$ we have

$$\frac{\partial^m \det(M_J)}{\partial^m K, K'} = \begin{cases} \pm \prod_{i=1}^r \det(M_{J^i,K_i,K'_i}) & \text{if } K' \subset J \text{ and } m \leq d(I,J) = \deg(\det(M_J)) \\ 0 & \text{otherwise} \end{cases}$$

for any $J \in \Lambda$, where $K' \subset J$ means that $\{k'^1_1, \dots, k'^1_{m_1}\} \subset J^1, \dots, \{k'^r_1, \dots, k'^r_{m_r}\} \subset J^r$, and M_{J^i,K_i,K'_i} is the submatrix obtained from M_{J^i} deleting the columns indexed by K'_i and the rows indexed by K_i. Thus,

$$\frac{\partial^m \det(M_J)}{\partial^m K, K'}(0) = \begin{cases} \pm 1 & \text{if } J^i = K'_i \cup (\{I^i \setminus K_i\}) \text{ for each } i = 1, \dots, r \\ 0 & \text{otherwise} \end{cases}$$

Finally, let us denote by $J = K' \cup \{I \setminus K\}$ the element in Λ for which $J^i = K'_i \cup (\{I^i \setminus K_i\})$ for each $i = 1, \dots, r$. Then, we have that

$$\frac{\partial^m \varphi}{\partial^m K, K'}(0) = \pm e_{K' \cup (\{I \setminus K\})}$$

Note that $d(I, K' \cup \{I \setminus K\}) = m$, and any $J \in \Lambda$ with $d(I, J) = m$ may be written as $K' \cup \{I \setminus K\}$. Thus, we get that

$$\left\langle \frac{\partial^m \varphi}{\partial^m K, K'}(0) \mid m \leq s \right\rangle = \langle e_J \mid d(I, J) \leq s \rangle$$

which proves the claim. □

Now, it is immediate to compute the dimension of the osculating spaces of $\prod_{i=1}^r \mathbb{G}(k_i, n)$.

Corollary 2.4. *For any point $p \in \prod_{i=1}^r \mathbb{G}(k_i, n)$ we have*

$$\dim T_p^s(\textstyle\prod_{i=1}^r \mathbb{G}(k_i, n)) = \displaystyle\sum_{\substack{i=1,\dots,r \\ 0 \leq s_i \leq k_i+1, \\ s_1+\dots+s_r \leq s}} \binom{n-k_1}{s_1}\binom{k_1+1}{s_1}\cdots\binom{n-k_r}{s_r}\binom{k_r+1}{s_r}$$

for any $0 \leq s < r + \sum_{i=1}^r k_i$ while $T_p^s(\prod_{i=1}^r \mathbb{G}(k_i, n)) = \mathbb{P}^N$ for any $s \geq r + \sum_{i=1}^r k_i$.

Proof. Since the general linear group $GL(n+1)$ acts transitively on $\prod_{i=1}^r \mathbb{G}(k_i, n)$ the statement follows from Proposition 2.2. □

2.4. **Osculating Projections.** For a general point $p \in \prod_{i=1}^r \mathbb{G}(k_i, n)$, we will denote $T_p^s(\prod_{i=1}^r \mathbb{G}(k_i, n))$ simply by T_p^s. Now, take $0 \leq s \leq r + \sum_{i=1}^r k_i$ and $I \in \Lambda$. By Proposition 2.2 the linear projection of $\prod_{i=1}^r \mathbb{G}(k_i, n)$ from $T_{e_I}^s$ is given by

$$\begin{array}{cccc} \Pi_{T_{e_I}^s} : & \prod_{i=1}^r \mathbb{G}(k_i) & \dashrightarrow & \mathbb{P}^{N'_s} \\ & (Z_J)_{J \in \Lambda} & \longmapsto & (Z_J)_{J \in \Lambda \mid d(I,J) > s} \end{array}.$$

Moreover, given $I' \subset \{0, \dots, n\}$ with $|I'| = m$ we have the linear projection

$$\begin{array}{cccc} \pi_{I'} : & \mathbb{P}^n & \dashrightarrow & \mathbb{P}^{n-m} \\ & (x_i) & \longmapsto & (x_i)_{i \in \{0,\dots,n\} \setminus I'} \end{array}$$

which in turns induces the linear projection

$$\begin{array}{cccc} \Pi_{I'} : & \mathbb{G}(k, n) & \dashrightarrow & \mathbb{G}(k, n-m) \\ & V & \longmapsto & \langle \pi_{I'}(V) \rangle \\ & (Z_J)_{J \in \Lambda_k} & \longmapsto & (Z_J)_{J \in \Lambda_k \mid J \cap I' = \emptyset} \end{array}$$

whenever $k < n - m$.

Finally, let us fix $I = \{I^1, \dots, I^r\} \in \Lambda$ and take $m_1, \dots, m_r$ integers such that $m_i \leq k_i + 1$ for each $i = 1, \dots, r$. Then, given $I'^1 \subset I^1, \dots, I'^r \subset I^r$, with $|I'^i| = m_i$, we have a projection

$$\begin{array}{cccc} \prod_{i=1}^r \Pi_{I'^i} : & \prod_{i=1}^r \mathbb{G}(k_i, n) & \dashrightarrow & \prod_{i=1}^r \mathbb{G}(k_i, n - m_i) \\ & V_1 \times \dots \times V_r & \longmapsto & \Pi_{I'^1}(V_1) \times \dots \times \Pi_{I'^r}(V_r) \end{array}.$$

Note that a general fiber of $\prod_{i=1}^r \Pi_{I'^i}$ is isomorphic to $\prod_{i=1}^r \mathbb{G}(k_i, k_i + m_i)$. Indeed, let $x = \prod_{i=1}^r \Pi_{I'^i}((V_i)_{i=1}^r) \in \prod_{i=1}^r \mathbb{G}(k_i, n - m_l)$ be a general point. Then, we have

$$\overline{\left(\prod_{i=1}^r \Pi_{I'^i}\right)^{-1}(x)} = \left\{(W_i)_{i=1}^r \in \prod_{i=1}^r \mathbb{G}(k_i, n) \mid W_i \subset \langle V_i, e_{j_1^i}, \dots, e_{j_{m_i}^i} \rangle,\ i = 1, \dots, r\right\}.$$

Lemma 2.5. *Let us fix $I = \{I^1, \dots, I^r\} \in \Lambda$. If $0 \leq s \leq r - 2 + \sum_{i=1}^r k_i$ and $I'^i \subset I^i$ with $|I'^i| = m_i$ for each $i = 1, \dots, r$, then the rational map $\Pi_{T_{e_I}^s}$ factors through $\prod_{i=1}^r \Pi_{I'^i}$ whenever $\sum_{i=1}^r m_i = s + 1$.*

Proof. Since the diameter of Λ is $r + \sum k_i$ we have $\{J \in \Lambda \mid d(I, J) \leq s\} \subsetneq \Lambda$ and then $\Pi_{T_{e_I}^s}$ is well-defined.

On the other hand, if $J = \{J^1, \dots, J^r\} \in \Lambda$ is such that $J^i \cap I'^i = \emptyset$ for all $i = 1, \dots, r$, then $d(I, J) \geq \sum_{i=1}^r m_i > s$ which yields that the center of $\Pi_{T_{e_I}^s}$ is contained in the center of $\prod_{i=1}^r \Pi_{I'^i}$. □

Proposition 2.6. *The rational map $\Pi_{T_{e_I}^s}$ is birational for all $0 \leq s \leq r - 2 + \sum_{i=1}^r k_i$.*

Proof. Since $T^s_{e_I}$ contains $T^{s-1}_{e_I}$ it is enough to prove the statement for $s = r-2+\sum_{i=1}^r k_i$. Let us fix $m \in \{1,\dots,r\}$. By Lemma 2.5, for each subset $I'^m \subset I^m$ with $|I'^m| = k_m$ there is a rational map $\pi_{I'^m}$ that makes the following diagram commutative.

$$\begin{array}{ccc} \prod_{i=1}^r \mathbb{G}(k_i,n) & \overset{\Pi_{T^s_{e_I}}}{\dashrightarrow} & \mathbb{P}^{N'_s} \\ & \searrow^{(\prod_{i\neq m}\Pi_{I^i})\times\Pi_{I'^m}} & \downarrow \pi_{I'^m} \\ & & \left(\prod_{i\neq m}\mathbb{G}(k_i,n-k_i-1)\right)\times\mathbb{G}(k_m,n-k_m) \end{array}$$

Let $x = \Pi_{T^s_{e_I}}(\{V_i\}_{i=1}^r)$ be a general point and $X \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ be the fiber of $\Pi_{T^s_{e_I}}$ over x. Set $x_{I'^m} = \pi_{I'^m}(x)$, and denote by $X_{I'^m} \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ the fiber of $\left(\prod_{i\neq m}\Pi_{I^i}\right)\times\Pi_{I'^m}$ over $x_{I'^m}$. Thus,

$$X \subset \bigcap_{I'^m} X_{I'^m}$$

where this intersection runs over all $I'^m \subset I^m$ with $|I'^m| = k_m$ and $m = 1,\dots,r$. Now, if $(W_i)_{i=1}^r$ is a general point in X then

$$W_m \subset \langle e_{j_1},\dots,e_{j_{k_m}},V_m\rangle \text{ for any } I'^m = \{e_{j_1},\dots,e_{j_{k_m}}\} \subset I^m$$

Therefore,

$$W_m \subset \bigcap_{I'^m}\langle e_{j_1},\dots,e_{j_{k_m}},V_m\rangle = V_m$$

This implies $W_m = V_m$ for every $m = 1,\dots,r$. Since we are working in characteristic zero, we conclude that $\Pi_{T^s_{e_I}}$ is birational. □

The next step is to study linear projections from the span of several osculating spaces. In particular, we want to understand when such a projection is birational. First of all, note that the order of osculating spaces can not exceed $r-2+\sum_{i=1}^r k_i$. Furthermore, in order to carry out the computations, we need to consider just the coordinates points of $\prod_{i=1}^r \mathbb{G}(k_i,n)$ such that the corresponding linear subspaces are linearly independent in $\mathbb{C}^{n+1}$, then we can use at most

$$\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$$

of them. Now, let us consider the points $e_{I_1},\dots,e_{I_\alpha} \in \prod_{i=1}^r \mathbb{G}(k_i,n)$, where

$$\begin{array}{l} I_1 = \{I_1^1 = \{0,\dots,k_1\},\dots,I_1^r = \{0,\dots,k_r\}\} \\ I_2 = \{I_2^1 = \{k_r+1,\dots,k_r+k_1+1\},\dots,I_2^r = \{k_r+1,\dots,k_r+k_r+1\}\} \\ \vdots \\ I_\alpha = \{\dots,I_\alpha^i = \{(k_r+1)(\alpha-1),\dots,(k_r+1)(\alpha-1)+k_i\},\dots\} \end{array} \tag{2.7}$$

Let $s_1,\dots,s_\alpha$ be integers such that $0 \leq s_m \leq r-2+\sum_{i=1}^r k_i$. Denote the linear subspace $\langle T^{s_1}_{e_{I_1}},\dots,T^{s_m}_{e_{I_m}}\rangle$ simply by $T^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}$. Then, for $m \leq \alpha$ we have the linear projection

$$\begin{array}{cccc} \Pi_{T^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}} : & \prod_{i=1}^r \mathbb{G}(k_i,n) & \dashrightarrow & \mathbb{P}^{N'_{s_1,\dots,s_m}} \\ & (Z_J)_{J\in\Lambda} & \longmapsto & (Z_J)_{J\in\Lambda \mid d(J,I_1)>s_1,\dots,d(J,I_m)>s_m} \end{array}$$

Now, consider $I_1,\dots,I_\alpha$ as in (2.7), and $I'^i_m \subset I^i_m$ with $|I'^i_m| = s^i_m$ for each $1 \leq m \leq \alpha$ and $i = 1,\dots,r$, where s^i_m are non-negative integers. If I'^i denotes the union $\bigcup_{m=1}^\alpha I'^i_m$, then for each $i = 1,\dots,r$ we have a linear projection of $\mathbb{P}^n$

$$\begin{array}{cccc} \pi_{I'^i} : & \mathbb{P}^n & \dashrightarrow & \mathbb{P}^{n-\sum_{m=1}^\alpha s^i_m} \\ & (x_i)_{0\leq i\leq n} & \longmapsto & (x_i)_{0\leq i\leq n \text{ and } i\notin I'^i} \end{array}$$

which in turns induces the following projection

$$\begin{array}{cccc} \Pi_{I'^i} : & \mathbb{G}(k_i,n) & \dashrightarrow & \mathbb{G}(k_i,n-\sum_{m=1}^\alpha s^i_m) \\ & V & \longmapsto & \langle\pi_{I'^i}(V)\rangle \\ & (Z_J)_{J\in\Lambda_{k_i}} & \longmapsto & (Z_J)_{J\in\Lambda_{k_i} \mid J\cap I'^i=\emptyset} \end{array}$$

whenever $n-\sum_{m=1}^{\alpha}s^i_m\geq k_i$. Finally, if $n-\sum_{m=1}^{\alpha}s^i_m\geq k_i$ for each $i=1,\dots,r$, then the projections above induce a projection

$$\begin{array}{rcccl}\prod_{i=1}^r\Pi_{I'^i} & : & \prod_{i=1}^r\mathbb{G}(k_i,n) & \dashrightarrow & \prod_{i=1}^r\mathbb{G}(k_i,n-\sum_{m=1}^{\alpha}s^i_m)\\ & & (V_1,\dots,V_r) & \longmapsto & (\Pi_{I'^1}(V_1),\dots,\Pi_{I'^r}(V_r))\end{array}$$

Lemma 2.8. *Let $I_1,\dots,I_\alpha$ be as in (2.7), $m,s_1,\dots,s_m$ integers such that $1<m\leq\alpha$ and $0\leq s_i\leq r-2+\sum_{i=1}^r k_i$. Now, consider $I'^i_1\subset I^i_1,\dots,I'^i_m\subset I^i_m$ with $|I'^i_j|=s^i_j$, where s^i_j is a non-negative integer for each $i=1,\dots,r$ and $1\leq j\leq m$. For $j>m$ and $i=1,\dots,r$ set $I'^i_j=\emptyset\subset I^i_j$. Denote by I'^i the union $\bigcup_{j=1}^{\alpha}I'^i_j$ for each $i=1,\dots,r$ and assume that*

(i) *$n-\sum_{j=1}^m s^i_j\geq k_i$ for each $i=1,\dots,r$;*
(ii) *$\sum_{i=1}^r s^i_j\geq s_j+1$ for each $j=1,\dots,m$.*

Then, the rational maps $\prod_{i=1}^r\Pi_{I'^i}$ and $\Pi_{T^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ are well-defined and the former factors through the latter.

Proof. Note that $\Pi_{T^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ is well-defined if and only if $\{J\in\Lambda\mid d(J,I_1)>s_1,\dots,d(J,I_m)>s_m\}\neq\emptyset$. From (i) we have that for each $1\leq i\leq r$ the set $\{0,\dots,n\}\setminus I'^i$ has at least k_i+1 elements. Therefore, we have a set $J^i\subset\{0,\dots,n\}\setminus I'^i$ of cardinality k_i+1 and taking $J=\{J^1,\dots,J^r\}\in\Lambda$ we have

$$d(I_j,J)=\sum_{i=1}^r d(I^i_j,J^i)\geq\sum_{i=1}^r s^i_j=s_j+1>s_j$$

for each $1\leq j\leq m$. Hence, $\Pi_{T^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ is well-defined. Now, note that (i) yields that $\prod_{i=1}^r\Pi_{I'^i}$ is well-defined. Furthermore, if $J\in\Lambda$ and $J^i\cap I'^i=\emptyset$ for all $i=1,\dots,r$, then $d(J,I_1)>s_1,\dots,d(J,I_m)>s_m$. Thus, the center of $\Pi_{T^{s_1,\dots,s_m}_{e_{I_1},\dots,e_{I_m}}}$ is contained in the center of $\prod_{i=1}^r\Pi_{I'^i}$. □

Proposition 2.9. *Let $I_1,\dots,I_{\alpha-1}$ be as in (2.7) and $s_1,\dots,s_{\alpha-1}$ be integers such that $0\leq s_j\leq s=r-2+\sum_{i=1}^r k_i$. Then, the projection $\Pi_{T^{s_1,\dots,s_{\alpha-1}}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$ is birational.*

Proof. Fix $m\in\{1,\dots,r\}$. For any $j=1,\dots,\alpha-1$ consider $I'^m_j\subset I^m_j$ with $|I'^m_j|=k_m$ and $I'^i_j=I^i_j$ for $i\neq m$. Set $I'^i=\bigcup_{j=1}^{\alpha-1}I'^i_j$, then

$$n-(\alpha-1)(k_i+1)\geq n-(\alpha-1)(k_r+1)\geq n-\frac{(n-k_r)}{k_r+1}(k_r+1)\geq k_r\geq k_i$$

and

$$n-(\alpha-1)k_m\geq n-\frac{(n-k_r)}{k_r+1}k_r=\frac{nk_r+n-nk_r+k_r^2}{k_r+1}\geq\frac{2k_r+1+k_r^2}{k_r+1}\geq k_m+1$$

Thus, our set of subsets I'^i_j satisfies (i) in Lemma 2.8. Furthermore, for each $j=1,\dots,\alpha-1$

$$\sum_{i=1}^r|I'^l_j|=k_m+\sum_{i\neq m}(k_i+1)=r-1+\sum_{i=1}^r k_i=s+1$$

Therefore, by Lemma 2.8 there exists a rational map $\pi_{I'^m}$ that makes the following diagram commutative

$$\begin{array}{ccc}\prod_{i=1}^r\mathbb{G}(k_i,n) & \overset{\Pi_{T_{e_{I_1},\dots,e_{I_{\alpha-1}}}}}{\dashrightarrow} & \mathbb{P}^{N'_{s,\dots,s}}\\ & \searrow^{\prod_{i=1}^r\Pi_{I'^i}} & \downarrow{\scriptstyle\pi_{I'^m}}\\ & & \prod_{i=1}^r\mathbb{G}(k_i,n-\sum_{j=1}^{\alpha-1}|I'^i_j|)\end{array}$$

Now, let $x=\Pi_{T^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}(\{V_i\}_{i=1}^r)$ be a general point in the image of $\Pi_{T^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$, and $X\subset\prod_{i=1}^r\mathbb{G}(k_i,n)$ be the fiber of $\Pi_{T^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$ over x. Set $x_{I'^m}=\pi_{I'^m}(x)$ and denote by $X_{I'^m}\subset\prod_{i=1}^r\mathbb{G}(k_i,n)$ the fiber of $\prod_{i=1}^r\Pi_{I'^i}$ over $x_{I'^m}$. Therefore, $X\subset\bigcap_{I'^m}X_{I'^m}$ where this intersection runs over all subsets $I'^m=\bigcup_{j=1}^{\alpha-1}I'^m_j$ with $I'^m_j\subset I^m_j$ and $|I'^m_j|=k_m$. In particular, if $\{W_i\}_{i=1}^r\in X$ is a general point, then we must have $W_m\subset\langle e_i\mid i\in I'^m\,;\,V_m\rangle$

and hence $W_m \subset \bigcap_{I'^m} \langle e_i \mid i \in I'^m; V_m\rangle$. Now, since $|I'^m_j| = k_m$ we have $\bigcap_{I'^m}\langle e_i \mid i \in I'^m\rangle = \emptyset$ and then $V_m = \bigcap_{I'^m}\langle e_i \mid i \in I'^m\ ;\ V_m\rangle$ which in turn yields $W_m = V_m$, for all $m = 1, \dots, r$. □

Now, we want to understand what is the largest integer s' for which $\Pi_{T^{s,\dots,s,s'}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$ is birational.

Proposition 2.10. *Let $I_1, \dots, I_\alpha$ be as in (2.7) and $s = r-2+\sum_{i=1}^r k_i$. Consider $s'_i = \min\{k_i+1, n-\alpha(k_i+1)\}$ for $i \neq r$, $s'_r = \min\{k_r, n-\alpha k_r - 1\}$, and set $s' = \sum_{i=1}^r s'_i - 1 \leq s$. Then,*

- $\Pi_{T^{s,\dots,s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$ *is birational whenever $\alpha(k_r+1)-1 < n < k_r^2+3k_r+1$;*
- $\Pi_{T^{s,\dots,s}_{e_{I_1},\dots,e_{I_\alpha}}}$ *is birational whenever $n \geq k_r^2+3k_r+1$.*

Proof. First, let us assume that $s'_r < k_r$, that is $n - \alpha k_r - 1 < k_r$, or equivalently

$$n - \alpha k_r < k_r + 1 \ \Leftrightarrow\ n - \frac{(n+1)}{k_r+1}k_r < k_r+1 \ \Leftrightarrow\ n < k_r^2+3k_r+1$$

Now, fix a pair of indexes $(l,m) \in \{1,\dots,\alpha-1\} \times \{1,\dots,r\}$ and consider subsets $I'^i_j \subseteq I^i_j$ with $|I'^i_j| = a_{j,i}$ for each $i \in \{1,\dots,r\}$ and $j \in \{1,\dots\alpha\}$ such that

$$a_{j,i} = \begin{cases} k_i & \text{if} \quad i=m,\ j=l \ \text{ or } \ i=r,\ j\neq l,\alpha; \\ k_i+1 & \text{if} \quad i=r\neq m,\ j=l \ \text{ or } \ i\neq m,r \ \text{ or } \ l\neq m,\alpha; \\ s'_i & \text{if} \quad j=\alpha,\ i\neq m; \\ s'_m - 1 & \text{if} \quad j=\alpha,\ i=m. \end{cases}$$

Note that, since $\alpha(k_r+1)-1 < n$ we have $a_{j,i} \geq 0$ for all $j \in \{1,\dots,\alpha\}$ and $i \in \{1,\dots,r\}$. Moreover, if $m \neq r$ then

$$n - \sum_{j=1}^{\alpha} |I'^m_j| = n-(\alpha-2)(k_m+1)-k_m-|I'^m_\alpha| \geq n-(\alpha-1)(k_m+1)-(n-\alpha(k_m+1)-1) = k_m+2$$

and $n - \sum_{j=1}^{\alpha}|I'^r_j| = n-(\alpha-2)k_r-(k_r+1)-|I'^r_\alpha| \geq n-(\alpha-1)k_r-1-(n-\alpha k_r-1) = k_r$. If $r=m$ we have

$$n - \sum_{j=1}^{\alpha}|I'^r_j| = n-(\alpha-2)k_r-(k_r+1)-|I'^r_\alpha| \geq n-(\alpha-1)k_r-1-(n-\alpha k_r-2) = k_r+1$$

Finally, for $i \neq m, r$ we have

$$n - \sum_{j=1}^{\alpha}|I'^i_j| = n-(\alpha-1)(k_i+1)-|I'^i_\alpha| \geq n-(\alpha-1)(k_i+1)-(n-\alpha(k_i+1)) = k_i+1$$

This yields that (i) in Lemma 2.8 is satisfied by the sets I'^i_j. Moreover, (ii) is satisfied as well. Then, by Lemma 2.8 there exists a rational map $\pi_{I'^m_l, I'^m_\alpha}$ making the following diagram commutative

$$\begin{array}{ccc} \prod_{i=1}^r \mathbb{G}(k_i,n) & \overset{\Pi_{T_{e_{I^1},\dots,e_{I^{\alpha-1}}}}}{\dashrightarrow} & \mathbb{P}^{N'_{s,\dots,s'}} \\ & \searrow^{\prod_{i=1}^r \Pi_{I'^i}} & \downarrow{\scriptstyle \pi_{I'^m_l,I'^m_\alpha}} \\ & & \prod_{i=1}^r \mathbb{G}(k_i, n-\sum_{j=1}^{\alpha}|I'^i_j|) \end{array}$$

where $I'^i = \bigcup_{j=1}^{\alpha} I'^i_j$. Now, let $x = \Pi_{T^{s\ ,\dots,\ s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}(\{V_i\}_{i=1}^r)$ be a general point in the image of $\Pi_{T^{s\ ,\dots,\ s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$, and $X \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ be the fiber of $\Pi_{T^{s\ ,\dots,\ s,s'-1}_{e_{I_1},\dots,e_{I_{\alpha-1}},e_{I_\alpha}}}$ over x. Set $x_{I'^m_l,I'^m_\alpha} = \pi_{I'^m_l,I'^m_\alpha}(x)$ and denote by $X_{I'^m_l,I'^m_\alpha} \subset \prod_{i=1}^r \mathbb{G}(k_i,n)$ the fiber of $\prod_{i=1}^r \Pi_{I'^i}$ over $x_{I'^m_l,I'^m_\alpha}$. Therefore, $X \subset \bigcap_{I'^m_l,I'^m_\alpha} X_{I'^m_l,I'^m_\alpha}$, where the intersection runs over all pairs of sets I'^m_l and I'^m_α with $|I'^m_l| = k_m$ and $|I'^m_\alpha| = s'_m - 1$, and for all pairs of indexes $(l,m) \in \{1,\dots,r\} \times \{1,\dots,\alpha-1\}$. In particular, if $\{W_i\}_{i=1}^r \in X$ is a general point then for every $m \in \{1,\dots,r\}$ we have $W_m \subset \bigcap_{I'^m_l,I'^m_\alpha} \langle e_i \mid i \in I'^m; V_m\rangle$, where the intersection runs over all pair of sets I'^m_l and

I'^m_α with $|I'^m_l| = k_m$ and $|I'^m_\alpha| = s'_m - 1$, and $l \in \{1,\dots,\alpha-1\}$. Now, since $|I'^m_l| = k_m$, $s'_m - 1 \leq k_m$ and $l \in \{1,\dots,\alpha-1\}$, we must have $\bigcap\limits_{I'^m_l, I'^m_\alpha} \langle e_i \mid i \in I'^m\rangle = \emptyset$ and then $V_m = \bigcap\limits_{I'^m}\langle e_i \mid i \in I'^m \ ; \ V_m\rangle$ which yields $W_m = V_m$, for all $m = 1,\dots,r$.

Now, assume that $n \geq k_r^2 + k_r + 1$. In this case we have that

$$n - \alpha k_r - 1 \geq n - \frac{(n+1)}{k_r+1}k_r - 1 = \frac{n(k_r+1)-(n+1)k_r-(k_r+1)}{k_r+1} \geq k_r$$

and for $i < r$ we have

$$n - \alpha(k_i+1) \geq n - \alpha(k_r) \geq n - \frac{(n+1)}{k_r+1}k_r = \frac{n-k_r}{k_r+1} \geq \frac{k_r^2+3k_r+1-k_r}{k_r+1} = k_r + 1 > k_i + 1$$

Now, for each pair of indexes $(l,m) \in \{1,\dots,\alpha\} \times \{1,\dots,r\}$ we can consider subsets $I'^i_j \subseteq I^i_j$ with $|I'^i_j| = a_{j,i}$ for each $i \in \{1,\dots,r\}$ and $j \in \{1,\dots\alpha\}$ such that

$$a_{j,i} = \begin{cases} k_i & \text{if} \quad i = m,\ j = l \ \text{ or } \ i = r,\ j \neq l; \\ k_i + 1 & \text{if} \quad i = r \neq m,\ j = l \ \text{ or } \ i \neq m, r \ \text{ or } \ l \neq m. \end{cases}$$

Therefore, arguing as in the proof of the first claim we conclude that $\Pi_{T^{s,\dots,s}_{e_{I_1},\dots,e_{I_\alpha}}}$ is birational. □

2.10. **Degenerating tangential projections to osculating projections.** In this section we study how the notion of osculating regularity introduced in [MR19] behaves for products of Grassmannians. Let us recall [MR19, Definition 5.5, Assumption 5.2] and [AMR19, Definition 4.4].

Definition 2.11. Let $X \subset \mathbb{P}^N$ be a projective variety. We say that X has *m-osculating regularity* if the following property holds: given general points $p_1,\dots,p_m \in X$ and an integer $s \geq 0$, there exists a smooth curve C and morphisms $\gamma_j : C \to X$, $j = 2,\dots,m$, such that $\gamma_j(t_0) = p_1$, $\gamma_j(t_\infty) = p_j$, and the flat limit T_0 in the Grassmannian of the family of linear spaces

$$T_t = \left\langle T^s_{p_1}, T^s_{\gamma_2(t)}, \dots, T^s_{\gamma_m(t)} \right\rangle,\ t \in C\backslash\{t_0\}$$

is contained in $T^{2s+1}_{p_1}$. We say that $\gamma_2,\dots,\gamma_m$ realize the m-osculating regularity of X for $p_1,\dots,p_m$.

We say that X has *strong 2-osculating regularity* if the following property holds: given general points $p,q \in X$ and integers $s_1, s_2 \geq 0$, there exists a smooth curve $\gamma : C \to X$ such that $\gamma(t_0) = p$, $\gamma(t_\infty) = q$ and the flat limit T_0 in the Grassmannian of the family of linear spaces

$$T_t = \left\langle T^{s_1}_p, T^{s_2}_{\gamma(t)} \right\rangle,\ t \in C\backslash\{t_0\}$$

is contained in $T^{s_1+s_2+1}_p$.

For a discussion on the notions of m-osculating regularity and strong 2-osculating regularity we refer to [MR19, Section 5] and [AMR19, Section 4].

Proposition 2.12. *The variety $\prod_{i=1}^r \mathbb{G}(k_i,n)$ has strong 2-osculating regularity.*

Proof. Let $p,q \in \prod_{i=1}^r \mathbb{G}(k_i,n)$ be general points. We may assume that $p = e_{I_1}$ and $q = e_{I_2}$ with I_1, I_2 as in (2.7) and consider the degree $r + \sum_{i=1}^r k_i$ rational normal curve given by

$$\gamma(s:t) = \prod_{i=1}^r (se_0 + te_{k_r+1}) \wedge \dots \wedge (se_{k_i} + te_{k_r+k_i+1})$$

We work on the affine chart $s = 1$ and set $t = (1:t) \in \mathbb{P}^1$. Now, consider the points

$$e_0,\dots,e_n, e_0^t = e_0 + te_{k_r+1}, \dots, e^t_{k_r} = e_{k_r} + te_{2k_r+1}, e^t_{k_r+1} = e_{k_r+1}, \dots, e^t_n = e_n$$

and, for each $I = \{I^1,\dots,I^r\} \in \Lambda$, the corresponding points in $e^t_I = e^t_{I^1} \otimes e^t_{I^2} \otimes \dots \otimes e^t_{I^r} \in \prod_{i=1}^r \mathbb{G}(k_i,n)$ where, setting $I^j = \{i^j_1,\dots,i^j_{k_j}\}$, $e^t_{I^j} = e^t_{i^j_1} \wedge \dots \wedge e^t_{i^j_{k_j}}$.

Given integers $s_1, s_2 \geq 0$, let us consider the family of linear spaces

$$T_t = \langle T^{s_1}_{e_{I_1}}, T^{s_2}_{e_{\gamma(t)}} \rangle,\ \ t \in \mathbb{P}^1 \setminus \{0\}$$

By Proposition 2.2 we have

$$T_t = \langle e_J \mid d(I_1, J) \leq s_1 \ ; \ e^t_J \mid d(I_2, J) \leq s_2 \rangle,\ t \neq 0$$

and

$$T^{s_1+s_2+1}_{e_{I_1}} = \langle e_J \mid d(I_1,J) \leq s_1+s_2+1\rangle = \{Z_J = 0 \mid d(I_1,J) > s_1+s_2+1\}$$

Now, let T_0 be the flat limit of $\{T_t\}_{t\in\mathbb{P}^1\setminus\{0\}}$, we want to show that $T_0 \subset T_p^{s_1+s_2+1}$. In order to do this it is enough to exhibit, for each index $I \in \Lambda$ with $d(I_1,I) > s_1+s_2+1$, a hyperplane H_I of type $Z_I + t\left(\sum_{J\neq I} f_J(t)Z_J\right) = 0$ where $f_J(t) \in \mathbb{C}[t]$ for every J. We define, for each $l \geq 0$ and $I = \{I^1,\dots,I^r\} \in \Lambda$,

$$\Delta(I,l) = \left\{\{(I^j \setminus J^j)\cup(J^j+k_r+1)\}_{1\leq j\leq r} \mid J^j \subset I^j \cap I_1^j \text{ and } \sum |J^j| = l\right\} \subset \Lambda$$

Furthermore, for each $l > 0$ we define

$$\Delta(I,-l) = \{J \mid I \in \Delta(J,l)\};$$
$$s_I^+ = \max_{l\geq 0}\{\Delta(I,l) \neq \emptyset\} \in \{0,\dots,\sum k_j + r\}$$
$$s_I^- = \max_{l>0}\{\Delta(I,-l) \neq \emptyset\} \in \{0,\dots,\sum k_j + r\}$$
$$\Delta(I)^+ = \bigcup_{0\leq l}\Delta(I,l) = \bigcup_{0\leq l\leq s_I^+}\Delta(I,l)$$
$$\Delta(I)^- = \bigcup_{0\leq l}\Delta(I,-l) = \bigcup_{0\leq l\leq s_I^-}\Delta(I,-l)$$

Now, let us write e_I^t with $d(I_1,I) \leq s_2$, in the basis e_J with $J \in \Lambda$. For any $I \in \Lambda$ we have

$$\begin{aligned} e_I^t &= e_I + t\sum_{J\in\Delta(I,1)} \operatorname{sign}(J)e_J + \dots + t^{s_I^+}\sum_{J\in\Delta(I,s_I^+)}\operatorname{sign}(J)e_J \\ &= \sum_{l=0}^{s_I^+}\left(t^l \sum_{J\in\Delta(I,l)}\operatorname{sign}(J)e_J\right) = \sum_{J\in\Delta(I)^+} t^{d(J,I)}\operatorname{sign}(J)e_J. \end{aligned}$$

where $\operatorname{sign}(J) = \pm 1$. Note that $\operatorname{sign}(J)$ depends on J but not on I, then we can write $e_I^t = \sum_{J\in\Delta(I)^+} t^{d(J,I)}e_J$. Therefore, we have

$$T_t = \left\langle e_I \mid d(I_1,I) \leq s_1\,;\ \sum_{J\in\Delta(I)^+} t^{d(J,I)}e_J \mid d(I_1,I)\leq s_2\right\rangle$$

Finally, we define

$$\Delta := \{I\ :\ d(I_1,I)\leq s_1\}\bigcup\left(\bigcup_{d(I_1,I)\leq s_2}\Delta(I)^+\right) \subset \Lambda$$

Let $I \in \Lambda$ be an index such that $d(I_1,I) > s_1+s_2+1$. If $I \notin \Delta$ then $T_t \subset \{Z_I = 0\}$ for any $t \neq 0$ and we are done.

Assume that $I \in \Delta$. For any e_K^t with non-zero coordinate Z_I we have $I \in \Delta(K)^+$, that is $K \in \Delta(I)^-$. Now, it is enough to find a hyperplane H_I of type

$$F_I = \sum_{J\in\Delta(I)^-} t^{d(J,I)}c_J Z_J = 0$$

with $c_J \in \mathbb{C}$ and $c_I \neq 0$, and such that $T_t \subset H_I$ for each $t \neq 0$. In the following, let us write $s_{i,I}^- = s$. Now, let us check what conditions we get by requiring $T_t \subset \{F_I = 0\}$ for $t \neq 0$. Given $K \in \Delta(I)^-$ we have that $d(I,K) \leq s_K^+$ and

$$\begin{aligned} F_I(e_K^t) &= F_I\left(\sum_{J\in\Delta(K)^+} t^{d(J,K)}e_J\right) = \sum_{J\in\Delta(I)^-} t^{d(J,I)}c_J\left(\sum_{J\in\Delta(K)^+} t^{d(J,K)}e_J\right) \\ &= \sum_{J\in\Delta(I)^-\cap\Delta(K)^+} t^{d(J,I)+d(J,K)}c_J = t^{d(I,K)}\left[\sum_{J\in\Delta(I)^-\cap\Delta(K)^+} c_J\right] \end{aligned}$$

Therefore,

$$F_I(e_K^t) = 0 \ \forall\, t \neq 0 \Leftrightarrow \sum_{J\in\Delta(I)^-\cap\Delta(K)^+} c_J = 0$$

Note that this is a linear condition on the coefficients c_J, with $J \in \Delta(I)^-$. Hence

$$\begin{array}{rcl} T_t \subset \{F_I = 0\} \text{ for } t \neq 0 & \Leftrightarrow & \left\{\begin{array}{ll} F_I(e_K) = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_1] \\ F_I(e_K^t) = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_2] \end{array}\right. \\ & \Leftrightarrow & \left\{\begin{array}{ll} c_K = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_1] \\ \displaystyle\sum_{J\in\Delta(I)^-\cap\Delta(K)^+} c_J = 0 & \forall K \in \Delta(I)^- \cap B[I_1, s_2] \end{array}\right. \end{array} \tag{2.13}$$

where $B[J,l] := \{K \in \Lambda \mid d(J,K) \leq l\}$. The number of conditions on the c_J is then $c := |\Delta(I)^- \cap B[I_1, s_1]| + |\Delta(I)^- \cap B[I_1, s_2]|$.

The problem is now reduced to find a solution of the linear system given by the c equations (2.13) in the $|\Delta(I)^-|$ variables c_J, $J \in \Delta(I)^-$ such that $c_I \neq 0$. Therefore, it is enough to find $s+1$ complex numbers $c_I = c_0 \neq 0, c_1, \dots, c_s$ satisfying the following conditions

$$\left\{\begin{array}{ll} c_j = 0 & \forall\ j = s, \dots, d - s_1 \\ \displaystyle\sum_{m=0}^{d(I,K)} |\Delta(I)^- \cap \Delta(K, l)| c_{d(I,K)-m} = 0 & \forall\ K \in \Delta(I)^- \cap B[I_1, s_2] \end{array}\right. \tag{2.14}$$

where $d = d(I_1, I) > s_1 + s_2 + 1$. Note that (2.14) can be written as

$$\left\{\begin{array}{ll} c_j = 0 & \forall\ j = s, \dots, d - s_1 \\ \displaystyle\sum_{m=0}^{j} \binom{j}{j-m} c_m = 0 & \forall\ j = s, \dots, d - s_2 \end{array}\right.$$

that is

$$\left\{\begin{array}{l} c_s = 0 \\ \vdots \\ c_{d-s_1} = 0 \end{array}\right. \quad \left\{\begin{array}{l} \binom{s}{0}c_s + \binom{s}{1}c_{s-1} + \cdots + \binom{s}{s-1}c_1 + \binom{s}{s}c_0 = 0 \\ \vdots \\ \binom{d-s_2}{0}c_{d-s_2} + \binom{d-s_2}{1}c_{d-s_2-1} + \cdots + \binom{d-s_2}{d-s_2-1}c_1 + \binom{d-s_2}{d-s_2}c_0 = 0 \end{array}\right. \tag{2.15}$$

Now, it is enough to show that the linear system (2.15) admits a solution with $c_0 \neq 0$. If, $s < d - s_2$ then the system (2.15) reduces to $c_s = \cdots = c_{d-s_1} = 0$ and then we can take $c_0 = 1$ and $c_1 = \cdots = c_s = 0$, since $d - s_1 > s_2 + 1 > 1$.

So, let us assume that $s \geq d - s_2$. Since $c_s = \cdots = c_{d-s_1} = 0$ our problem is translated into checking that the system (2.15) admits a solution involving the variables $c_{d-s_1-1}, \dots, c_0$ with $c_0 \neq 0$. First of all, note that the system (2.15) can be rewritten as follows

$$\left\{\begin{array}{l} \binom{s}{s-(d-s_1-1)}c_{d-s_1-1} + \binom{s}{s-(d-s_2-2)}c_{d-s_1-2} + \cdots + \binom{s}{s-1}c_1 + \binom{s}{s}c_0 = 0 \\ \vdots \\ \binom{d-s_2}{d-s_2-(d-s_1-1)}c_{d-s_1-1} + \binom{d-s_2}{d-s_2-(d-s_1-2)}c_{d-s_1-2} + \cdots + \binom{d-s_2}{d-s_2-1}c_1 + \binom{d-s_2}{d-s_2}c_0 = 0 \end{array}\right.$$

Thus, it is enough to check that the $(s - d + s_2 + 1) \times (d - s_1 - 1)$ matrix

$$M = \begin{pmatrix} \binom{s}{s-(d-s_1-1)} & \binom{s}{s-(d-s_1-2)} & \cdots & \binom{s}{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{d-s_2}{d-s_2-(d-s_1-1)} & \binom{d-s_2}{d-s_2-(d-s_1-2)} & \cdots & \binom{d-s_2}{d-s_2-1} \end{pmatrix}$$

has maximal rank. Now, note that $s \leq d$ and $d > s_1 + s_2 + 1$ yield $s - d + s_2 + 1 < s - s_1 \leq d - s_1$ and then $s - d + s_2 + 1 \leq d - s_1 - 1$. Therefore, we have to show that the $(s - d + s_2 + 1) \times (s - d + s_2 + 1)$ submatrix

$$M' = \begin{pmatrix} \binom{s}{s-d+s_2+1} & \binom{s}{s-d+s_2} & \cdots & \binom{s}{1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{d-s_2}{s-d+s_2+1} & \binom{d-s_2}{s-d+s_2} & \cdots & \binom{d-s_2}{1} \end{pmatrix}$$

has non-zero determinant. Finally, since $d - s_2 > s_1 + 1 \geq 1$ [GV85, Corollary 2] yields that $\det(M') \neq 0$. $\square$

Proposition 2.16. *Set $\alpha = \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$. Then, the variety $\prod_{i=1}^{r} \mathbb{G}(k_i, n)$ has α-osculating regularity.*

Proof. First of all, note that if $\alpha = 2$ then the statement follows form Proposition 2.12. Then we may assume $\alpha \geq 3$.

Let $p_1, \dots, p_\alpha \in \prod_{i=1}^r \mathbb{G}(k_i, n)$ be general points. We may assume that $p_j = e_{I_j}$ for $j = 1, \dots, \alpha$. Each e_{I_j}, $j \geq 2$, is connected to e_{I_1} by the degree $r + \sum_{i=1}^r k_i$ rational normal curve defined by

$$\gamma_j(s:t) = \prod_{i=1}^{r}(se_0 + te_{(k_r+1)(j-1)}) \wedge \dots \wedge (se_{k_i} + te_{(k_r+1)(j-1)+k_i})$$

We work on the affine chart $s = 1$ and set $t = (1:t)$. Now, given $s \geq 0$ we consider the family of linear subspaces

$$T_t = \langle T^s_{e_{I_1}}, T^s_{\gamma_2(t)}, \dots, T^s_{\gamma_\alpha(t)} \rangle,\ t \in \mathbb{P}^1 \setminus \{0\}$$

Our goal is to show that the flat limit T_0 of $\{T_t\}_{t\in\mathbb{P}^1\setminus\{0\}}$ in $\mathbb{G}(\dim(T_t), N)$ is contained in $T^{2s+1}_{e_{I_1}}$. In order to do this, let us consider the points

$$e_0, \dots, e_n, e_0^{j,t} = e_0 + te_{(k_r+1)(j-1)}, \dots, e_{k_r}^{j,t} = e_{k_r} + te_{(k_r+1)j-1}, e_{k_r+1}^{j,t} = e_{k_r+1}, \dots, e_n^{j,t} = e_n$$

and, for each $I = \{I^1, \dots, I^r\} \in \Lambda$ and $j = 2, \dots, \alpha$, the corresponding points in $e_I^{j,t} = e_{I^1}^{j,t} \otimes e_{I^2}^{j,t} \otimes \dots \otimes e_{I^r}^{j,t} \in \mathbb{P}^N$. By Proposition 2.2 we have

$$T_t = \langle e_I \mid d(I_1, I) \leq s; e_I^{j,t} \mid d(I_j, I) \leq s,\ j = 2, \dots, \alpha \rangle,\ t \neq 0$$

and

$$T^{2s+1}_{e_{I_1}} = \langle e_J \mid d(I_1, J) \leq 2s+1 \rangle = \{Z_J = 0 \mid d(I_1, J) > 2s+1\}$$

In order to show that $T_0 \subset T_p^{2s+1}$, it is enough to exhibit, for each index $I \in \Lambda$ with $d(I_1, I) > 2s+1$, an hyperplane H_I of type $Z_I + t\left(\sum_{J\neq I} f_J(t) Z_J\right) = 0$ such that $T_t \subset \{H_i = 0\}$ for $t \neq 0$.

For each $l \geq 0$, $j = 2, \dots, \alpha$ and $I = \{I^1, \dots, I^r\} \in \Lambda$ we define

$$\Delta(I,l)_j = \left\{ \{(I^k \setminus J^k) \cup (J^k + (j-1)(k_r+1)\}_{1\leq k\leq r} \mid J^k \subset I^k \cap I_1^k \text{ and } \sum |J^k| = l \right\} \subset \Lambda$$

where $L + \lambda = \{i + \lambda \mid i \in L\}$ is the translation of the set L by the integer λ. For any $l > 0$ we define

$$\Delta(I,-l)_j = \{J \mid I \in \Delta(J,l)_j\}$$
$$s^+_{I,j} = \max_{l\geq 0}\{\Delta(I,l)_j \neq \emptyset\} \in \{0, \dots, \sum k_j + r\}$$
$$s^-_{I,j} = \max_{l>0}\{\Delta(I,-l)_j \neq \emptyset\} \in \{0, \dots, \sum k_j + r\}$$
$$\Delta(I)^+_j = \bigcup_{0\leq l} \Delta(I,l)_j = \bigcup_{0\leq l\leq s^+_{I,j}} \Delta(I,l)_j$$
$$\Delta(I)^-_j = \bigcup_{0\leq l} \Delta(I,-l)_j = \bigcup_{0\leq l\leq s^-_{I,j}} \Delta(I,-l)_j$$

Note that for any l we have

$$J \in \Delta(I,l)_j \Rightarrow d(J,I) = |l| \text{ and } d(J,I_1) = d(I,I_1) + l \tag{2.17}$$

We will write e_I^t with $d(I_1, I) \leq s$, in the basis e_J with $J \in \Lambda$. For any $I \in \Lambda$ we have

$$\begin{aligned} e_I^{j,t} &= e_I + t \sum_{J\in\Delta(I,1)_j} \mathrm{sign}(J) e_J + \dots + t^{s^+_{I,j}} \sum_{J\in\Delta(I,s^+_{I,j})} \mathrm{sign}(J) e_J \\ &= \sum_{l=0}^{s^+_{I,j}} \left(t^l \sum_{J\in\Delta(I,l)_j} \mathrm{sign}(J) e_J \right) = \sum_{J\in\Delta(I)^+_j} t^{d(J,I)} \mathrm{sign}(J) e_J. \end{aligned}$$

where $\mathrm{sign}(J) = \pm 1$. Note that $\mathrm{sign}(J)$ depends on J but not on I, then we can write $e_I^{j,t} = \sum_{J\in\Delta(I)^+_j} t^{d(J,I)} e_J$.

Therefore, we have

$$T_t = \left\langle e_I \mid d(I_1, I) \leq s\,;\ \sum_{J\in\Delta(I)^+_j} t^{d(J,I)} e_J \mid d(I_1, I) \leq s,\ 2 \leq j \leq \alpha \right\rangle$$

Finally we define

$$\Delta := \{I \,:\, d(I_1,I)\leq s\}\bigcup\left(\bigcup_{\substack{d(I_1,I)\leq s\\ 2\leq j\leq\alpha}}\Delta(I)_j^+\right)\subset\Lambda$$

Let $I\in\Lambda$ be an index such that $d(I_1,I)>2s+1$. If $I\notin\Delta$, then $T_t\subset\{Z_I=0\}$ for any $t\neq 0$ and we are done.

Now, assume that $I\in\Delta$. We will show that $\Delta(K_1)^+_{j_1}\cap\Delta(K_2)^+_{j_2}=\emptyset$ whenever $K_1,K_2\in\Lambda$ with $d(K_1,I_1),d(K_2,I_2)\leq s$ and $2\leq j_1,j_2\leq\alpha$ with $j_1\neq j_2$.

In fact, suppose that $\Delta(K_1)^+_{j_1}\cap\Delta(K_2)^+_{j_2}\neq\emptyset$, that is there exists $I\in\Lambda$ such that

$$I\in\Delta(K_1,l_1)_{j_1}\cap\Delta(K_2,l_2)_{j_2}\text{ for some } l_1 \text{ and } l_2$$

Now, consider the following sets

$$\begin{array}{rcl} I^0 & := & I\cap I_1\\ I^1 & := & I\cap\{K_1+(j_1-1)(k_r+1)\}\\ I^2 & := & I\cap\{K_2+(j_2-1)(k_2+1)\}\\ I^3 & := & I\setminus(I^0\cup I^1\cup I^2)\end{array}$$

Since $I\in\Delta(K_1,l_1)_{j_1}\cap\Delta(K_2,l_2)_{j_2}$ we have $|I^1|=l_1$ and $|I^2|=l_2$. Set $|I^3|=u$, then

$$d(I,I_1)=l_1+l_2+u\leq l_1+l_2+2u\overset{(2.17)}{=}d(K_1,I_1)+d(K_1,I_1)\leq 2s$$

contradicting $d(I_1,I)>2s+1$. Therefore we conclude that there is a unique j_I for which

$$I\in\bigcup_{d(I_1,I)\leq s}\Delta(I)^+_{j_I}$$

Now, let $J\in\Lambda$ such that $d(J,I_1)\leq s$ and $I\in\Delta(J)^+_{j_I}$. Note that

$$d(I,I_1)-s(I)^-_{j_I}\leq d(I,I_1)-d(I,J)=d(J,I_1)\leq s\ \Rightarrow\ s+1-D+s(I)^-_{j_I}>0$$

where $D=d(I,I_1)>2s+1$. We define

$$\Gamma(I)=\sum_{0\leq l\leq s+1-D+s(I)^-_{j_I}}\Delta(I,-l)_{j_I}\subset\Gamma$$

Our aim now is to find a hyperplane of the form

$$H_I=\left\{\sum_{J\in\Gamma(I)}t^{d(J,I)}c_JZ_J=0\right\}\tag{2.18}$$

such that $T_t\subset H_I$ and $c_I\neq 0$. First, note that

$$J\in\Gamma(I)\ \Rightarrow\ J\notin\bigcup_{\substack{d(I_1,K)\leq s\\ 2\leq j\leq\alpha\,;\,j\neq j_I}}\Delta(K)^+_j\tag{2.19}$$

In fact, suppose that $J\in\Delta(I,-l)_{j_I}\cap\Delta(K,m)_j$, for some $K\in\Lambda$ with $d(K,I_1)\leq s$, and $0\leq j\leq s+1-D+s(I)^-_{j_I}$ with $j\neq j_I$. Then, since $J\in\Delta(I,-l)_{j_I}$ we have

$$|J\cap I_{j_I}|=|I\cap I_{j_I}|-l\geq s(I)^-_{j_I}-l\geq D-k-1>s$$

On the other hand, since $J\in\Delta(K,m)_j$ with $j\neq j_I$ we have

$$|J\cap I_{j_I}|=|K\cap I_{j_I}|\leq d(K,I_1)\leq s$$

which is a contradiction. Now, note that if $K\in\Lambda$ is such that $d(K,I_1)\leq s$ and $K\in\Gamma(I)$, then

$$d(K,I_1)=d(I,I_1)-d(I,K)>2s+1-(s+1-D+s(I)^-_{j_I})>s+D-s(I)^-_{j_I}>s$$

Thus (2.19) yields that the hyperplane H_I given by (2.18) is such that

$$\left\langle e_K\mid d(I_1,K)\leq s\,;\ \sum_{J\in\Delta(K)^+_j}t^{d(J,K)}e_J\mid d(I_1,K)\leq s,\ 2\leq j\leq\alpha\,;\,j\neq j_I\right\rangle\subset H_I,\ t\neq 0$$

Therefore

$$T_t \subset H_I,\ t \neq 0 \ \Leftrightarrow \ \left\langle \sum_{J\in\Delta(K)^+_{j_I}} t^{d(J,K)} e_J \mid d(I_1,K) \leq s \right\rangle \subset H_I,\ t\neq 0$$

Now, arguing as in the proof of Proposition 2.12 we get

$$T_t \subset H_I,\ t \neq 0 \ \Leftrightarrow \sum_{J\in\Delta(K)^+_{j_I}\cap\Gamma(I)} c_J = 0, \ \ \forall\, K \in \Delta(I)^-_{j_I} \cap B[I_1, s] \tag{2.20}$$

So, the problem is reduced to find a solution $(c_J)_{J\in\Gamma(I)}$ for the linear system (2.20) such that $c_I \neq 0$. We set $c_J = c_{d(I,J)}$ and reduce, as in the proof of Proposition 2.12, to the linear system

$$\sum_{l=0}^{s+1+D-s(I)^-_{j_I}} \binom{D-i}{D-i-l} c_l, \ \ D - S(I)^-_{j_I} \leq i \leq k \tag{2.21}$$

We have $s+2+D-s(I)^-_{j_I}$ variables $c_0,\dots,c_{s+1+D-s(I)^-_{j_I}}$ and $s+1+D-s(I)^-_{j_I}$ equations. Finally, the argument used in the last part of the proof of Proposition 2.12 shows that the linear system (2.21) admits a solution with $c_0 \neq 0$. □

3. On secant defectivity of products of Grassmannians

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension n and let

$$\Gamma_h(X) \subset X \times \cdots \times X \times \mathbb{G}(h-1, N)$$

where $h \leq N$, be the closure of the graph of the rational map $\alpha : X \times \cdots \times X \dashrightarrow \mathbb{G}(h-1,N)$ taking h general points to their linear span $\langle x_1,\dots,x_h\rangle$. Observe that $\Gamma_h(X)$ is irreducible and reduced of dimension hn.

Let $\pi_2 : \Gamma_h(X) \to \mathbb{G}(h-1,N)$ be the natural projection, and $\mathcal{S}_h(X) := \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h-1,N)$. Again $\mathcal{S}_h(X)$ is irreducible and reduced of dimension hn. Finally, consider

$$\mathcal{I}_h = \{(x,\Lambda) \mid x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h-1,N)$$

with natural projections π_h and ψ_h onto the factors.

The *abstract h-secant variety* is the irreducible variety $\mathrm{Sec}_h(X) := (\psi_h)^{-1}(\mathcal{S}_h(X)) \subset \mathcal{I}_h$. The *h-secant variety* is $\mathbb{S}ec_h(X) := \pi_h(Sec_h(X)) \subset \mathbb{P}^N$. Then $\mathrm{Sec}_h(X)$ is an $(hn+h-1)$-dimensional variety.

The number $\delta_h(X) = \min\{hn+h-1, N\} - \dim \mathbb{S}ec_h(X)$ is called the *h-secant defect* of X. We say that X is *h-defective* if $\delta_h(X) > 0$. We refer to [Rus03] for a comprehensive survey on the subject.

Determining secant defectivity is a classical problem in algebraic geometry. A new strategy to determine the non secant defectivity was introduced in [MR19, Theorem 5.3], the method is based on degenerating the span of several tangent spaces $T_{x_i}X$ in a single osculating space T^s_xX.

To state the criterion for non secant defectivity in [MR19] we introduce a function $h_m : \mathbb{N}_{\geq 0} \longrightarrow \mathbb{N}_{\geq 0}$ counting how many tangent spaces can be degenerated into a higher order osculating space.

Definition 3.1. Given an integer $m \geq 0$ we define a function

$$h_m : \mathbb{N}_{\geq 0} \longrightarrow \mathbb{N}_{\geq 0}$$

as follows: for $h_m(0) = 0$ and for any $k > 0$ write

$$k+1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geq 1$ and $\varepsilon \in \{0,1\}$, then

$$h_m(k) = m^{\lambda_1-1} + m^{\lambda_2-1} + \cdots + m^{\lambda_l-1}$$

Theorem 3.2. [MR19, Theorem 5.3] *Let $X \subset \mathbb{P}^N$ be a projective variety having m-osculating regularity and strong 2-osculating regularity. Let $s_1,\dots,s_l \geq 1$ integers such that the general osculating projection $\Pi^{s_1,\dots,s_l}_{p_1,\dots,p_l}$ is generically finite. If*

$$h \leq \sum_{j=1}^{l} h_m(s_j)$$

then X is not $(h+1)$-defective.

Now, we are ready to prove our main result on non-defectivity of product of Grassmannians. We follow the notation introduced in the previous sections.

Theorem 3.3. *Assume that $n \geq 2k_r + 1$. Set*

$$\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$$

and let h_α be as in Definition 3.1. Assume that

- *either $n \geq k_r^2 + 3k_r + 1$ and $h \leq \alpha h_\alpha(\sum_{i=1}^r k_i + r - 2)$ or*
- *$\alpha(k_r+1) - 1 < n < k_r^2 + 3k_r + 1$ and $h \leq (\alpha-1)h_\alpha(\sum_{i=1}^r k_i + r - 2) + h_\alpha(s')$*

where $s' = \sum_{i=1}^r s_i - 2$ with $s_i' = \min\{k_i + 1, n - \alpha(k_i+1)\}$ for $i \neq r$ and $s_r' = \min\{k_r, n - \alpha k_r - 1\}$. Then $\prod_{i=1}^r \mathbb{G}(k_i, n)$ is not $(h+1)$-defective.

Proof. We have shown in Propositions 2.16, 2.12 that $\prod_{i=1}^r \mathbb{G}(k_i, n)$ has respectively α-osculating regularity for $\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$, and strong 2-osculating regularity. The statement then follows immediately from Proposition 2.10 and Theorem 3.2. □

Corollary 3.4. *The variety $\prod_{i=1}^r \mathbb{G}(k_i, n)$ is not $(h+1)$-defective for $h \leq \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_j + r - 1) \rfloor}$.*

Proof. We may write

$$\sum_{i=1}^r k_i + r - 1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon \tag{3.5}$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geq 1$ and $\varepsilon \in \{0, 1\}$. Then $h_\alpha(\sum_{i=1}^r k_i + r - 2) = \alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1}$.

The first bound in Theorem 3.3 gives $h \leq \alpha^{\lambda_1} + \cdots + \alpha^{\lambda_l}$. Furthermore, considering just the first summand in the second bound in Theorem 3.3 we get that $\prod_{i=1}^r \mathbb{G}(k_i, n)$ is not $(h+1)$-defective for $h \leq (\alpha - 1)(\alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1})$.

Finally, from (3.5) we get that $\lambda_1 = \lfloor \log_2(r - 1 + \sum k_i) \rfloor$. Hence, asymptotically we have $h_\alpha(\sum k_j + r - 2) \sim \alpha^{\lfloor \log_2(r-1+\sum k_i) \rfloor - 1}$, and by Theorem 3.3 if $h \leq \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_j + r - 1) \rfloor}$ then the variety $\prod_{i=1}^r \mathbb{G}(k_i, n)$ is not $(h+1)$-defective. □

4. On secant defectivity of flag varieties

Our goal is to compute the higher osculating spaces of $\mathbb{F}(k_1, \dots, k_r; n)$. In order to do this, we will use the following notion introduced in [FMR18, Definition 3.2].

Definition 4.1. Let $X \subset \mathbb{P}^N$ be an irreducible variety and $Y = \mathbb{P}^k \cap X$ be a linear section of X. We say that Y is *osculating well-behaved* if for each smooth point $p \in Y$ we have

$$T_p^s Y = \mathbb{P}^k \cap T_p^s X$$

for every $s \geq 0$.

Let us denote by M_i the following $(k_i + 1) \times (n+1)$ matrix

$$M_i = \begin{bmatrix} I_{k_1+1} & \cdots & \cdots & (x_{l,m}^1)_{\substack{0 \leq l \leq k_1 \\ k_1+1 \leq m \leq n}} \\ 0 & I_{k_2 - k_1} & \cdots & (x_{l,m}^2)_{\substack{k_1+1 \leq l \leq k_2 \\ k_2+1 \leq m \leq n}} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I_{k_i - k_{i-1}} & (x_{l,m}^i)_{\substack{k_{i-1}+1 \leq l \leq k_i \\ k_i+1 \leq m \leq n}} \end{bmatrix}$$

and consider the map

$$\begin{array}{cccc} \varphi' : & \prod_{i=1}^r \mathbb{C}^{(k_1+1)(n-k_1) + \sum_{j=2}^i (n-k_j)(k_j - k_{j-1})} & \longrightarrow & \mathbb{P}^N \\ & (M_1, \dots, M_r) & \longmapsto & (\prod_{i=1}^r \det(M_{J^i}))_{J = \{J^1, \dots, J^r\} \in \Lambda} \end{array}$$

where M_{J^i} is the submatrix obtained from M_i by considering only the columns indexed by J^i.

For each $2 \leq i \leq r$ and $m \leq k_l$, let us take $x_{l,m}^i = 0$ in M_i. Then φ' becomes the parametrization φ of $\prod_{i=1}^r \mathbb{G}(k_i, n)$ in (2.3).

Now, set $x^i_{l,m} = x^r_{l,m}$ in M_i for each $i = 1, \dots, r-1$ and $1 \le l < m \le n$. Hence φ becomes the parametrization of $\mathbb{F}(k_1, \dots, k_r; n)$ given by

$$\begin{array}{rccc} \overline{\varphi}: & \mathbb{C}^{(k_1+1)(n-k_1)+\sum_{j=2}^r (n-k_j)(k_j-k_{j-1})} & \longrightarrow & \mathbb{P}(\Gamma_a) \subset \mathbb{P}^N \\ & M_r & \longmapsto & \varphi\left(\overline{M}_1, \dots, \overline{M}_r\right) \end{array}$$

where $\overline{M}_i$ is the submatrix obtained from M_r by considering only the first $k_i + 1$ rows.

Lemma 4.2. *Let* $T^s_{\varphi'}\left(\prod_{i=1}^r \mathbb{G}(k_i, n)\right) := \left\langle \frac{\partial^{|I|}\varphi'}{\partial x_{|I|}}(0) \mid |I| \le s \right\rangle$ *be the s-osculating space of* $\prod_{i=1}^r \mathbb{G}(k_i, n)$ *with respect to* φ'. *Then* $T^s_{\varphi'}\left(\prod_{i=1}^r \mathbb{G}(k_i, n)\right) = T^s\left(\prod_{i=1}^r \mathbb{G}(k_i, n)\right)$ *for every* $s \le r + \sum_{i=1}^r k_i$. *In particular,*

$$\frac{\partial^s \varphi'}{\partial x_{|I|}}(0) = \frac{\partial^{|J|}\varphi}{\partial x_{|J|}}(0)$$

for some J with $|J| \le |I|$.

Proof. First, note that if for any $x^i_{l,m} \in x_{|I|}$ we have $m > k_i$, then $\frac{\partial^s \varphi'}{\partial x_{|I|}}(0) = \frac{\partial^{|I|}\varphi}{\partial x_{|I|}}(0)$ and we are done.

Now, let $2 \le i \le r$ and consider a derivative $\frac{\partial^{|I|}\varphi'}{\partial x_{|I|}}(0)$ such that $x^i_{l,m} \in x_{|I|}$ with $m \le k_i$. Therefore, to prove the statement it is enough to show that this partial derivative can be written in terms of another partial derivative $\frac{\partial^{|J|}\varphi'}{\partial x_{|J|}}(0)$ with $x^i_{l,m} \notin x_{|J|}$, $m \le k_i$ and $|J| < |I|$.

Fix $2 \le i \le r$ and let $x^i_{l_1,m_1}, \dots, x^i_{l_h,m_h}, x^i_{l_{h+1},m_{h+1}}, \dots, x^i_{l_b,m_b} \in x_{|I|}$ such that $m_a \le k_i$ for every $a = 1, \dots, h$ and $b \le k_i + 1$.

If $\frac{\partial^b \varphi'}{\partial x^i_{l_1,m_1} \cdots \partial x^i_{l_h,m_h} \partial x^i_{l_{h+1},m_{h+1}} \cdots \partial x^i_{l_b,m_b}}(0) \ne 0$ consider the minor M_{J^i} of M_i such that the monomial

$$x^i_{l_1,m_1} \dots x^i_{l_h,m_h} x^i_{l_{h+1},m_{h+1}} \dots x^i_{l_b,m_b}$$

appears in the expression of $\det(M_{J^i})$. Then, there exist variables $x^i_{\sigma_{J^i}(l_{h+1}),\sigma_{J^i}(m_{h+1})}, \dots, x^i_{\sigma_{J^i}(l_b),\sigma_{J^i}(m_b)}$ such that $x^i_{\sigma_{J^i}(l_{h+1}),\sigma_{J^i}(m_{h+1})} \cdots x^i_{\sigma_{J^i}(l_b),\sigma_{J^i}(m_b)}$ is also a monomial in $\det(M_J)$, where σ_{J^i} is a permutation on the indexes such that $\sigma_{J^i}(m_a) > k_i$ for all $h + 1 \le a \le b$.

This shows that

$$\frac{\partial^m \varphi'}{\partial x^i_{l_1,m_1} \cdots \partial x^i_{l_h,m_h} \partial x^i_{l_{h+1},m_{h+1}} \cdots \partial x^i_{l_b,m_b}}(0) = \frac{\partial^m \varphi'}{\partial x^i_{\sigma_{J^i}(l_{h+1}),\sigma_{J^i}(m_{h+1})}, \dots, \partial x^i_{\sigma_{J^i}(l_b),\sigma_{J^i}(m_b)}}(0)$$

We have thus decreased the number of variables with respect we differentiate and thus lowered the order of the derivatives. Finally, since $\frac{\partial \varphi}{\partial x^i_{l,m}}(0) = \frac{\partial \varphi'}{\partial x^i_{l,m}}(0)$ for $m > k_i$ we are done. $\square$

Lemma 4.3. *Since $\overline{\varphi}$ is a sub-parametrization of φ' by the chain rule we have*

$$\frac{\partial^s \overline{\varphi}}{\partial x_{|I|}}(0) = \sum_{|K|} \frac{\partial^s \varphi'}{\partial x_{|K|}}(0) = \sum_{|J|} \frac{\partial^s \varphi}{\partial x_{|J|}}(0)$$

where $|K| = |I| = s$ and $|J| \le |I|$. Let $\frac{\partial^s \overline{\varphi}}{\partial x_{|I|}}(0) \ne 0$ *with $|I| = s$ such that for each $x^i_{l,m} \in x_{|I|}$ we have that $m > k_i$. Then, in the above decomposition there is at least a vector* $\frac{\partial^s \varphi}{\partial x_{|J|}}(0)$ *with $|J| = s$.*

Proof. For any $x^i_{l,m} \in x_{|I|}$ let $h(m)$ be the maximum index in $\{1, \dots, r\}$ such that $m > k_{h(m)}$. Since for each $x^i_{l,m} \in x_{|I|}$ we have that $m > k_i$ and $\frac{\partial^s \overline{\varphi}}{\partial x_{|I|}}(0) \ne 0$, we get that any $x^i_{l,m} \in x_{|I|}$ appears at most $h(m)$ times in $x_{|I|}$.

Now, for any $s \le h(m)$, the chain rule expression of $\frac{\partial^s \overline{\varphi}}{(\partial x^i_{l,m})^s}(0)$ contains the factor

$$\frac{\partial^s \varphi'}{\partial x^1_{l,m} \partial x^2_{l,m} \dots \partial x^{h(m)}_{l,m}}(0) = \frac{\partial^s \varphi}{\partial x^1_{l,m} \partial x^2_{l,m} \dots \partial x^{h(m)}_{l,m}}(0)$$

Repeating this argument for all indexes $x^i_{l,m} \in x_{|I|}$ we conclude. $\square$

Proposition 4.4. *The flag variety is osculating well-behaved, that is*

$$T_p^s\mathbb{F}(k_1,\dots,k_r;n) = T_p^s\prod_{i=1}^r\mathbb{G}(k_i,n)\cap\mathbb{P}(\Gamma_a)$$

for any $p\in\mathbb{F}(k_1,\dots,k_r;n)$ *and non-negative integer* s.

Proof. We may assume that $p=e_I$ where $I=\{I^1,\dots,I^r\}$ and $I^l=\{0,\dots,k_l\}$ for each $1\le l\le r$. Let us first assume that $s=r+\sum_{i=1}^r k_i$. Note that s is the smallest integer for which $T_p^s\mathbb{F}(k_1,\dots,k_r;n)=\mathbb{P}(\Gamma_a)$ and $T_p^s\prod_{i=1}^s\mathbb{G}(k_i,n)=\mathbb{P}^N$, in this case $T_p^s\mathbb{F}(k_1,\dots,k_r;n)=\mathbb{P}(\Gamma_a)=\mathbb{P}(\Gamma_a)\cap\mathbb{P}^N=\mathbb{P}(\Gamma_a)\cap T_p^s\prod_{i=1}^s\mathbb{G}(k_i,n)$ and we are done. Now, assume $s<r+\sum_{i=1}^r k_i$. Let

$$v=\sum_{|I|\le s-1}\alpha_{|I|}\frac{\partial^{|I|}\varphi}{\partial x_{|I|}}(0) \tag{4.5}$$

be a general vector in $T_p^{s-1}\prod_{i=1}^r\mathbb{G}(k_i,n)$, and assume that

$$v\in T_p^{s-1}\prod_{i=1}^r\mathbb{G}(k_i,n)\cap\mathbb{P}(\Gamma_a)\subset T_p^s\prod_{i=1}^r\mathbb{G}(k_i,n)\cap\mathbb{P}(\Gamma_a)=T_p^s\mathbb{F}(k_1,\dots,k_r;n)$$

this yields that v can be written as

$$v=\sum_{|I|\le s-1}\beta_{|I|}\frac{\partial^{|I|}\overline{\varphi}}{\partial^{|I|}x_{|I|}}(0)+\sum_{|I|=s}\beta_{|I|}\frac{\partial^{|I|}\overline{\varphi}}{\partial^{|I|}x_{|I|}}(0) \tag{4.6}$$

Now, recall that for any I such that there are variables $x_{l,m}^i\in x_{|I|}$ with $m\le k_i$ we can find another set J for which $|J|<|I|$ and

$$\frac{\partial^s\varphi'}{\partial x_{|I|}}(0)=\frac{\partial^{|J|}\varphi}{\partial x_{|J|}}(0)$$

Therefore, we can assume that any set I in the second summand of (4.6) is such that $m>k_i$ for any $x_{l,m}^i\in x_{|I|}$. Thus, by Lemma 4.3, we will have an equality in (4.5) and (4.6) if and only if $\beta_{|I|}=0$ for any set I such that $m>k_i$ for all $x_{l,m}^i\in x_{|I|}$. Hence $v\in T_p^{s-1}\mathbb{F}(k_1,\dots,k_r;n)$. □

4.6. **Osculating Projections.** Let $s_1,\dots,s_\alpha$ be integers such that $0\le s_m\le r-2+\sum_{i=1}^r k_i$. Denote $T_p^s\mathbb{F}(k_1,\dots,k_r;n)$ simply by $T_p^s\mathbb{F}$ and the linear subspace $\langle T_{e_{I_1}}^{s_1}\mathbb{F},\dots,T_{e_{I_m}}^{s_m}\mathbb{F}\rangle$ by $T_{e_{I_1},\dots,e_{I_m}}^{s_1,\dots,s_m}\mathbb{F}$. Then, for $m\le\alpha$ we have the linear projection

$$\Pi_{T_{e_{I_1},\dots,e_{I_m}}^{s_1,\dots,s_m}\mathbb{F}}:\mathbb{F}(k_1,\dots,k_r;n)\dashrightarrow\mathbb{P}^{N_{s_1,\dots,s_m}}$$

Proposition 4.7. *Let* $I_1,\dots,I_\alpha$ *be as in (2.7) and* $s=r-2+\sum_{i=1}^r k_i$. *Then,*

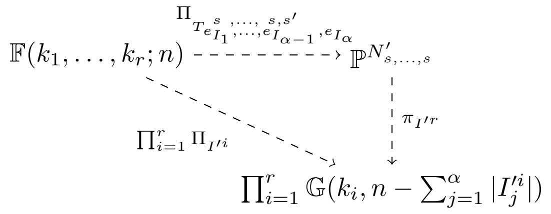
- $\Pi_{T_{e_{I_1},\dots,e_{I_{\alpha-1}}}^{s,\dots,s}\mathbb{F}}$ *is birational;*
- $\Pi_{T_{e_{I_1},\dots,e_{I_\alpha}}^{s,\dots,s}\mathbb{F}}$ *is birational whenever* $n\ge k_r^2+3k_r+1$.

Proof. Since $\Pi_{T_{e_{I_1},\dots,e_{I_{\alpha-1}}}^{s,\dots,s}}$ factors trough $\Pi_{T_{e_{I_1},\dots,e_{I_{\alpha-1}}}^{s,\dots,s}\mathbb{F}}$, it is enough to show that the restriction of $\Pi_{T_{e_{I_1},\dots,e_{I_{\alpha-1}}}^{s,\dots,s}}$ to $\mathbb{F}(k_1,\dots,k_r)$ is birational.

For any $i\ne r$ and $1\le j\le\alpha-1$ consider $I_j'^i=I_j^i$ and $I_j'^r\subset I_j^r$ of cardinality k_r. Since $n\ge 2k_r+1$ and $k_r\ge k_i$ we must have

$$n-\sum_{j=1}^{\alpha-1}|I_j'^i| \;=\; n-(\alpha-1)(k_i+1)\ge n-(\alpha-1)k_r\ge k_r+1\le k_i+1$$

Now, let us denote by I'^i the union $\bigcup_{j=1}^{\alpha-1}I_j'^i$. Then, by Lemma 2.8 there exists a rational map $\pi_{I'^r}$ making the following diagram commutative



Now, let $x = (\{V_i\}_{i=1}^r)$ be a general point in the image of $\Pi_{T^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$ and $X \subset \mathbb{F}(k_1,\dots,k_r;n)$ be the fiber of $\Pi_{T^{s,\dots,s}_{e_{I_1},\dots,e_{I_{\alpha-1}}}}$ over x. Set $x_{I'^r} = \pi_{I'^r}(x)$ and denote by $X_{I'^r} \subset \mathbb{F}(k_1,\dots,k_r;n)$ the fiber of $\prod_{i=1}^r \Pi_{I'^i}$ over $x_{I'^r}$.

Therefore, $X \subset \bigcap_{I'^r} X_{I'^r}$, where the intersection runs over all sets $I'^r = \bigcup_{j=1}^{\alpha-1} I'^r_j$ with $I'^r_j \subset I^r_j$ and $|I'^r_j| = k_r$ for $1 \leq j \leq \alpha - 1$.

Now, note that if $\{W_i\}_{i=1}^r \in X$ is a general point, then we must have $W_i \subset \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^i_j \,;\, V_i\rangle$ for any choice of $\bigcup_{j=1}^{\alpha} I'^i_j$. Hence,

$$W_i \subset \bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^i_j \,;\, V_i\rangle \tag{4.8}$$

In particular, $W_r \subset \bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^r_j ; V_r\rangle$. Now, since $|I'^r_j| \leq k_r$ we must have $\bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^r_j\rangle = \emptyset$ and then $V_r = \bigcap_{I'^r} \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I'^r_j \,;\, V_r\rangle$ which yields $W_r = V_r$.

Now, set $i \leq r-1$. Since $\{V_i\}_{i\in K}$ is general in $\mathbb{F}(k_1,\dots,k_r;n)$ and $n - \sum_{j=1}^{\alpha-1} |I'^i_j| \geq k_r + 1$ we have $V_r \cap \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I^i_j\rangle = \emptyset$. On the other hand $W_i \subset W_r = V_r$ for all $i \leq r-1$, then $W_i \cap \langle e_m \mid m \in \bigcup_{j=1}^{\alpha} I^i_j\rangle = \emptyset$. Hence, by (4.8) we must have $W_i = V_i$ for any $i \leq r-1$.

Now, assume that $n \geq k_r^2 + 3k_r + 1$ then

$$\begin{aligned} n - \alpha(k_i+1) \geq n - \alpha k_r &\geq n - \frac{(n+1)}{k_r+1}k_r = \frac{n(k_r+1)-(n+1)k_r}{k_r+1} \\ &= \frac{n-k_r}{k_r+1} \geq \frac{k_r^2+3k_r+1-k_r}{k_r+1} = k_r+1 \end{aligned}$$

Then, arguing as in the proof of the first case, for any choice of subsets $I'^i_j \subset I^i_j$, $I'^i_j = I^i_j$ with $i \neq r$ and $1 \leq j \leq \alpha - 1$, $I'^r_j \subsetneq I^r_j$ of cardinality k_r we get, by Lemma 2.8, a rational map $\pi_{I'^r}$ making the following diagram commutative

$$\begin{array}{ccc} \mathbb{F}(k_1,\dots,k_r;n) & \overset{\Pi_{T_{e_{I_1},\dots,e_{I_\alpha}}}}{\dashrightarrow} & \mathbb{P}^{N'_{s,\dots,s}} \\ & {\scriptstyle \prod_{i=1}^r \Pi_{I'^i}} \searrow & \downarrow {\scriptstyle \pi_{I'^r}} \\ & & \prod_{i=1}^r \mathbb{G}(k_i, n - \sum_{j=1}^{\alpha} |I'^i_j|) \end{array}$$

where $I'^i = \bigcup_{j=1}^{\alpha} I'^i_j$, $i = 1,\dots,r$. Now, to conclude it is enough to follow the same argument used in the end of the proof of the first claim. □

4.8. **Non-Secant defectivity of flag varieties.** We recall [FMR18, Proposition 4.4] which describes how the notion of osculating regularity behaves under linear sections.

Proposition 4.9. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and $Y = \mathbb{P}^k \cap X$ a linear section of X that is osculating well-behaved. Assume that given general points $p_1,\dots,p_m \in Y$ one can find smooth curves $\gamma_j : C \to X, j = 2,\dots,m$, realizing the m-osculating regularity of X for $p_1,\dots,p_m$ such that $\gamma_j(C) \subset Y$. Then Y has m-osculating regularity as well. Furthermore, the analogous statement for strong 2-osculating regularity holds as well.*

Proposition 4.10. *The flag variety $\mathbb{F}(k_1,\dots,k_r;n)$ has strong 2-osculating regularity and α-osculating regularity, where $\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$.*

Proof. The statement follows immediately from Propositions 2.12, 2.16, 4.9. □

Now, we are ready to prove our main result on non-defectivity of flags varieties.

Theorem 4.11. *Assume that $n \geq 2k_r + 1$. Set*

$$\alpha := \left\lfloor \frac{n+1}{k_r+1} \right\rfloor$$

and let h_α be as in Definition 3.1. If either

- $n \geq k_r^2 + 3k_r + 1$ *and* $h \leq \alpha h_\alpha(\sum k_j + r - 2)$ *or*
- $n < k_r^2 + 3k_r + 1$ *and* $h \leq (\alpha - 1)h_\alpha(\sum k_j + r - 2)$.

Then, $\mathbb{F}(k_1, \dots, k_r; n)$ *is not* $(h+1)$*-defective. In particular, if*

$$h \leq \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_j + r - 1)\rfloor}$$

then $\mathbb{F}(k_1, \dots, k_r; n)$ *is not* $(h+1)$*-defective.*

Proof. The first part is an immediately consequence of Propositions 4.9, 4.7 and Theorem 3.2. For the last claim note that if we write

$$\sum k_j + r - 1 = 2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_l} + \varepsilon \tag{4.12}$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_l \geq 1$ and $\varepsilon \in \{0, 1\}$. Then

$$h_\alpha(\sum k_j + r - 2) = \alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1}$$

Therefore, the first bound in Theorem 4.11 yields

$$h \leq \alpha^{\lambda_1} + \alpha^{\lambda_2} + \cdots + \alpha^{\lambda_l}$$

Furthermore, by the second bound in Theorem 4.11 we get that $\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective for

$$h \leq (\alpha - 1)(\alpha^{\lambda_1 - 1} + \alpha^{\lambda_2 - 1} + \cdots + \alpha^{\lambda_l - 1})$$

Finally, by (4.12) we get that $\lambda_1 = \lfloor \log_2(\sum k_j + r - 1)\rfloor$. Hence, asymptotically we have $h_\alpha(\sum k_j + r - 2) \sim \alpha^{\lfloor \log_2(\sum k_j + r - 1)\rfloor}$, and by Theorem 4.11 for $h \leq \alpha^{\lfloor \log_2(\sum k_j + r - 1)\rfloor} \leq \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_j + r - 1)\rfloor}$ the flag variety $\mathbb{F}(k_1, \dots, k_r; n)$ is not $(h+1)$-defective. □

Remark 4.13. Now, given a flag $\mathbb{F}(k_1, \dots, k_r; n)$ with $n < 2k_r + 1$. Assume that $n \geq 2k_j + 1$ for some index j and let l be the maximum among these j's. Then we have a natural projection

$$\begin{array}{rccc} \pi & : & \mathbb{F}(k_1, \dots, k_r; n) & \longrightarrow & \mathbb{F}(k_1, \dots, k_l; n) \\ & & \{V_i\}_{i=1,\dots,r} & \longmapsto & \{V_i\}_{i=1,\dots,l} \end{array}$$

The fiber of π over a general point in $\mathbb{F}(k_1, \dots, k_l; n)$ is isomorphic to $\mathbb{F}(k_{l+1} - k_l - 1, \dots, k_r - k_l - 1; n - k_l - 1)$. Now let $p_1, \dots, p_h \in \mathbb{F}(k_1, \dots, k_l; n)$ be general points, and $T_{p_i}\mathbb{F}(k_1, \dots, k_l; n)$ be the tangent space at p_i. Then, we have

$$T_{\pi^{-1}(p_i)}\mathbb{F}(k_1, \dots, k_r; n) = \left\langle T_{p_i}\mathbb{F}(k_1, \dots, k_l; n), T_{\pi^{-1}(p_i)}\mathbb{F}(k_{l+1} - k_l, \dots, k_r - k_l; n - k_l)\right\rangle$$

and $T_{p_i}\mathbb{F}(k_1, \dots, k_l; n) \cap T_{\pi^{-1}(p_i)}\mathbb{F}(k_{l+1} - k_l, \dots, k_r - k_l; n - k_l) = \emptyset$.

Now, observe that if $T_{\pi^{-1}(p_i)}\mathbb{F}(k_1, \dots, k_r; n) \cap T_{\pi^{-1}(p_j)}\mathbb{F}(k_1, \dots, k_r; n) \neq \emptyset$ then

$$\dim \left\langle T_{\pi^{-1}(p_j)}\mathbb{F}(k_1, \dots, k_r; n)\ ;\ j = 1, \dots, h\right\rangle \leq h \dim \mathbb{F}(k_1, \dots, k_l; n) + h - 2$$

Since $T_{\pi^{-1}(p_i)}\mathbb{F}(k_{l+1} - k_l - 1, \dots, k_r - k_l - 1; n - k_l - 1)$ is contracted by π for any $j = 1, \dots, h$ we have that

$$\begin{array}{rcl} \dim \pi(T) & \leq & h \dim \mathbb{F}(k_1, \dots, k_r; n) + h - 2 - h \dim \mathbb{F}(k_{l+1} - k_l, \dots, k_r - k_l; n - k_l) \\ & = & h \dim \mathbb{F}(k_1, \dots, k_l; n) + h - 2 \end{array}$$

where $T = \left\langle T_{\pi^{-1}(p_i)}\mathbb{F}(k_1, \dots, k_r; n)\ ;\ i = 1, \dots, h\right\rangle$.

In particular, by Terracini's lemma [Ter12] we have that if $\mathbb{F}(k_1, \dots, k_l; n)$ is not h-defective, then $\mathbb{F}(k_1, \dots, k_r; n)$ is not h-defective.

Theorem 4.14. *Consider a flag variety* $\mathbb{F}(k_1, \dots, k_r; n)$ *with* $n < 2k_r + 1$*. Assume that* $n \geq 2k_j + 1$ *for some index* j *and let* l *be the maximum among these* j*'s. Then, for*

$$h \leq \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^{l} k_j + l - 1)\rfloor}$$

$\mathbb{F}(k_1, \dots, k_r; n)$ *is not* $(h+1)$*-defective.*

Proof. It is an immediate consequence of Theorem 4.11 and Remark 4.13. □

4.14. **On identifiability of products of Grassmannians and flag varieties.** Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerated variety. A point $p \in \mathbb{P}^N$ is said to be h-identifiable, with respect to X, if it lies on a unique $(h-1)$-plane h-secant to X. Furthermore, X is said to be h-identifiable if a general point of $\mathbb{S}ec_h(X)$ is h-identifiable.

Now, we combine our bounds on non-secant defectivity of products of Grassmannians and flag varieties and [CM19, Theorem 3] to get the following.

Corollary 4.15. *Consider the product of Grassmannians* $\prod_{i=1}^r \mathbb{G}(k_i, n)$. *Assume that* $2\prod_{i=1}^r (k_i+1)(n-k_i) - 1 \leq \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_i + r - 1)\rfloor}$. *Then,* $\prod_{i=1}^r \mathbb{G}(k_i, n)$ *is h-identifiable for* $h \leq \left(\frac{n+1}{k_r+1}\right)^{\lfloor \log_2(\sum k_i + r - 1)\rfloor}$.

Furthermore, let us suppose that $n \geq 2k_j + 1$ *for some index* j *and consider* l *the maximum among these* j*'s. Assume that* $2((k_1+1)(n-k_1) + \sum_{j=2}^i (n-k_j)(k_j - k_{j-1})) - 1 \leq \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^l k_j + l - 1)\rfloor}$. *Then* $\mathbb{F}(k_1, \dots, k_r; n)$ *is h-identifiable for* $h \leq \left(\frac{n+1}{k_l+1}\right)^{\lfloor \log_2(\sum_{j=1}^l k_j + l - 1)\rfloor}$.

Proof. It is enough to apply Corollary 3.4, Theorem 4.14 and [CM19, Theorem 3]. □

5. On the chordal variety of $\mathbb{F}(0, k; n)$

In this section we consider particularly flag varieties parametrizing chains of type $p \in H^k \subset \mathbb{P}^n$.

Proposition 5.1. *Let us consider the flag variety* $\mathbb{F}(0, k; n) \subset \mathbb{P}(\Gamma) \subset \mathbb{P}^N$, *where* $0 < k < n$. *Then,* $\mathbb{S}ec_2\mathbb{F}(0, k; n)$ *has always the expected dimension except when* $k = n-1$, *in this case* $\mathbb{F}(0, n-1; n)$ *is 2-defective with 2-defect* $\delta_2(\mathbb{F}(0, n-1; n)) = 1$.

Proof. Let $p, q \in \mathbb{F}(0, k; n)$ be two general points, without lose the generality we can assume that $p = e_{0,\{0,\dots,k\}} = e_{0,I_0}$ and $q = e_{n,\{n-k,\dots,n\}} = e_{n,I_1}$.

Now, Proposition 4.4 yields that

$$T_{e_{0,I_0}}\mathbb{F}(0, k; n) \quad = \quad \langle e_{i,I} \mid d((i, I), (0, I_0)) \leq 1\rangle \cap \mathbb{P}(\Gamma)$$

and

$$T_{e_{n,I_1}}\mathbb{F}(0, k; n) \quad = \quad \langle e_{i,I} \mid d((i, I), (n, I_n)) \leq 1\rangle \cap \mathbb{P}(\Gamma)$$

Note that $d((i, I), (0, I_0)) = 1$ if and only if either $i \neq 0$ and $I = I_0$ or $i = 0$ and $|I \cap I_0| = k$. Similarly, $d((i, I), (n, I_1)) = 1$ if and only if either $i \neq n$ and $I = I_1$ or $i = n$ and $|I \cap I_1| = k$. Therefore, since $n \neq 0$ and $I_1 \neq I_0$ we have that $e_{i,I} \in \{e_{i,I} \mid d((i, I), (0, I_0)) \leq 1\} \cap \{e_{i,I} \mid d((i, I), (n, I_1)) \leq 1\}$ if and only if either $I = I_0$ and $i = n$ or $I = I_1$ and $i = 0$.

Now, assume that $I = I_0$ and $i = n$, this is $e_{i,I} \in T_{e_{0,I_0}}\mathbb{F}(0, k; n) \cap T_{e_{n,I_1}}\mathbb{F}(0, k; n)$, in particular we have $|I \cap I_1| = |I_0 \cap I_1| = k$ and hence $\{1, \dots, k\} \subset I_1$ once $0 \notin I_1$. So we must have $k = n-1$. Similarly, if $I = I_1$ and $i = 0$ we conclude that $k = n-1$.

Therefore, if $k < n-1$, we get

$$\{e_{i,I} \mid d((i, I), (0, I_0)) \leq 1\} \cap \{e_{i,I} \mid d((i, I), (n, I_1)) \leq 1\} = \emptyset$$

and hence

$$\{e_{i,I} \mid d((i, I), (0, I_0)) \leq 1\} \cap \{e_{i,I} \mid d((i, I), (n, I_1)) \leq 1\} \cap \mathbb{P}(\Gamma) = \emptyset$$

which implies that

$$\dim \left\langle T_{e_{0,I_0}}\mathbb{F}(0, k; n), T_{e_{n,I_1}}\mathbb{F}(0, k; n)\right\rangle = 2\dim \mathbb{F}(0, k; n) + 1$$

So, Terracini's lemma [Ter12] yields that $\mathbb{S}ec_2\mathbb{F}(0, k; n)$ has the expected dimension whenever $k < n-1$.

Now, assume that $k = n-1$. In this case we have

$$\{e_{i,I} \mid d((i, I), (0, I_0)) \leq 1\} \cap \{e_{i,I} \mid d((i, I), (n, I_1)) \leq 1\} = \{e_{0,\{1,\dots,n\}}, e_{n,\{0,\dots,n-1\}}\}$$

Furthermore, $\mathbb{F}(0, n-1; n)$ is the hypersurface cutting out in $\mathbb{P}^n \times \mathbb{P}^{n*}$ by

$$\sum_{i=0}^n (-1)^i Z_{i, I_n \setminus \{i\}} = 0$$

where $I_n = \{0, \dots, n\}$.

Therefore, we get that $T_{e_{0,I_0}}\mathbb{F}(0, n-1; n) = \langle e_{i,I} \mid d((i, I), (0, I_0)) \leq 1\rangle \cap \mathbb{P}(\Gamma)$ is given by

$$\left\langle e_{0,\{1,\dots,n\}} + (-1)^{n+1} e_{n,\{0,\dots,n-1\}}\ ;\ e_{i,I} \mid d((i, I), (0, I_0)) \leq 1 \text{ and } i, I \neq \left\{ \begin{array}{l} 0, \{1, \dots, n\} \\ n, \{1, \dots, n-1\} \end{array} \right. \right\rangle$$

and $T_{e_{n,I_1}}\mathbb{F}(0,n-1;n) = \langle e_{i,I} \mid d((i,I),(n,I_1)) \leq 1\rangle \cap \mathbb{P}(\Gamma)$ is given by

$$\left\langle e_{0,\{1,\dots,n\}} + (-1)^{n+1} e_{n,\{0,\dots,n-1\}} \ ; \ e_{i,I} \mid d((i,I),(n,I_1)) \leq 1 \text{ and } i,I \neq \left\{ \begin{array}{l} 0,\{1,\dots,n\} \\ n,\{1,\dots,n-1\} \end{array} \right. \right\rangle$$

Therefore,

$$\dim \left\langle T_{e_{0,I_0}}\mathbb{F}(0,n-1;n), T_{e_{n,I_1}}\mathbb{F}(0,n-1;n)\right\rangle = 2\dim \mathbb{F}(0,n-1;n) < \operatorname{expdim} \mathbb{S}ec_2\mathbb{F}(0,n-1;n)$$

Finally, since $\operatorname{expdim} \mathbb{S}ec_2\mathbb{F}(0,k;n) = 2\dim \mathbb{F}(0,n-1;n)+1$ we have that $\mathbb{F}(0,n-1,n)$ is 2-defective with 2-defect $\delta_2(\mathbb{F}(0,n-1;n)) = 1$. □

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