Extended Virtual Element Method for Elliptic Problems with Singularities and Discontinuities in Mechanics

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Abstract. Drawing inspiration from the extended finite element method (X-FEM), we propose for two-dimensional elastic fracture problems, an extended virtual element method (X-VEM). In the X-VEM, we extend the standard virtual element space with the product of vector-valued virtual nodal shape functions and suitable enrichment fields, which reproduce the singularities of the exact solution. We define an extended projection operator that maps functions in the extended virtual element space onto a set spanned by the space of linear polynomials augmented with the enrichment fields. Several numerical examples are adopted to illustrate the convergence and accuracy of the proposed method, for both quadrilateral and general polygonal meshes.

Introduction

Numerical techniques for the solution of problems that admit singular or discontinuous solutions such as fracture propagation in solids have attracted significant attention in the last two decades. Among these, enriched finite element approximations based on the partition-of-unity concept [1] and the eXtended Finite Element Method (X-FEM) [2] have become widely popular. More recently, extended finite element formulations for polygonal meshes have been proposed [3] even though, on polygonal elements, the construction of shape functions is generally cumbersome and additional numerical integration issues must be carefully addressed [4]. The Virtual Element Method (VEM) is a stabilized Galerkin scheme proposed in [5] to solve partial differential equations on general polygonal meshes that overcomes many of the difficulties related to standard polygonal finite element formulations. The VEM can be looked at as a generalization of the Finite Element Method (FEM) in which the explicit knowledge of the basis functions is not needed, since the bilinear form and the continuous linear functional deriving from the variational formulation, are approximated by means of elliptic projections of the basis functions onto suitable polynomial spaces, which turn out to be computable from the degrees of freedom of the method. More recently, taking inspiration from the X-FEM, an eXtended Virtual Element Method (X-VEM) has been proposed in [6,7], for the scalar Laplace problem with singularities and discontinuities, and in [8] for fracture problems in two-dimensional linear elasticity. Here, we summarize the main finding related to the extended virtual element formulation for linear elastic fracture problems proposed in [8], in which the displacement field exhibits both crack-tip singularities and discontinuities. An enriched virtual element space is constructed by resorting to an additional set of virtual basis functions, starting from suitably chosen vectorial enrichment fields which allow to incorporate additional information about the exact solution, improving numerical accuracy in the presence of singularities. The X-VEM for elastic fracture proves to be more flexible with respect to the X-FEM since it is applicable to arbitrary polygonal meshes, while using a very simple one-dimensional quadrature rule for the computation of all the integrals involved.

Two-dimensional elasticity model

Let us consider a linear elastic body occupying the two-dimensional domain $\Omega \in \mathbb{R}^2$, bounded by Γ cut by a traction-free internal crack Γ_c. We denote the displacement field on Ω by $\mathbf{u}(\mathbf{x})$ and assume small strains and displacements. The boundary is such that $\Gamma = \Gamma_u \cup \Gamma_t \cup \Gamma_c$. Prescribed displacements $\mathbf{g} \in C^0(\Gamma_u)$ are imposed on Γ_u, whereas tractions $\bar{\mathbf{t}} \in C^0(\Gamma_t)$ are imposed on Γ_t. Let $\boldsymbol{\sigma}$ be the Cauchy stress tensor. In the absence of body forces, equilibrium equations read

$$\nabla \cdot \boldsymbol{\sigma} = 0 \quad \text{in } \Omega, \tag{1}$$

with the natural boundary conditions

$$\begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{n} &= \bar{\mathbf{t}} \quad \text{on } \Gamma_t, \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \mathbf{0} \quad \text{on } \Gamma_c, \end{aligned} \tag{2}$$

where $\mathbf{n}$ is the unit outward normal, and the essential boundary condition

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_u. \tag{3}$$

The small strain tensor $\boldsymbol{\varepsilon}$ is related to the displacement field $\mathbf{u}$ by the compatibility equation

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla(\mathbf{u}) + \nabla^T(\mathbf{u})\right). \tag{4}$$

Lastly, the isotropic linear elastic constitutive for a homogeneous material reads

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C} : \boldsymbol{\varepsilon}(\mathbf{u}), \tag{5}$$

Where $\mathbf{C}$ is the fourth-order elasticity tensor.

To state the weak form of the problem we define the test function space as:

$$U_0 = \left\{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v} = 0 \ \text{ on } \ \Gamma_u, \ \mathbf{v} \ \text{ discontinuous on } \ \Gamma_c\right\}. \tag{6}$$

The weak form of the equilibrium equation reads as: Find the admissible displacement field u such that

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{u}) dx = \int_{\Gamma_t} \bar{\mathbf{t}} \cdot \mathbf{v} \, d\Gamma =: b(\mathbf{v}) \quad \forall \mathbf{v} \in U_0. \tag{7}$$

Extended virtual element formulation

We now summarize the formulation of the extended virtual element method for fracture problems in two-dimensional elasticity presented in . Let $\mathrm{T} = \{\Omega_h\}_h$ be a family of decompositions of *Ω* into nonoverlapping polygonal elements E with nonintersecting boundary ∂E , barycenter $\mathbf{x}_E \equiv (x_E, y_E)^T$, area $|E|$, and diameter $h_E = \sup_{\mathbf{x},\mathbf{y}\in E} |\mathbf{x}-\mathbf{y}|$.

Enrichment with singular fields. The main concept of the X-VEM is to enrich the standard virtual element space by means of independent fields carrying information about the singularities affecting the exact solution. For the problem at hand, we choose the enrichment fields $\hat{\mathbf{u}}^I = \mathbf{u}^I / h^{1/2}$ and $\hat{\mathbf{u}}^{II} = \mathbf{u}^{II} / h^{1/2}$, where $\mathbf{u}^I$ and $\mathbf{u}^{II}$ are the exact asymptotic crack-tip displacement fields for mode I and mode II crack opening respectively, and h the maximum elemental diameter of the mesh [8]. We observe that these fields satisfy equilibrium. In order to define the extended virtual element space, we first introduce the local virtual element space $\mathbf{V}^{h,*}(E)$:

$$\mathbf{V}^{h,*}(E) = \left\{ \mathbf{v}^h = (v_x^h, v_y^h)^T \in \mathbf{V}^h(E) : v_x^h = v_y^h \right\}, \tag{8}$$

where $\mathbf{V}^h(E) = \left[V^h(E) \right]^2$ with $V^h(E)$ the standard virtual element space, spanned by the scalar virtual basis functions $\left\{ \varphi_i \right\}_{i=1}^{N_E}$. Hence, the space $\mathbf{V}^{h,*}(E)$ is generated by the linear combination of the basis functions $\left\{ \varphi_i^* = (\varphi_i, \varphi_i)^T \right\}_{i=1}^{N_E}$. Then, we define the matrices ψ^I and ψ^{II} as

$$\psi^I \equiv \begin{bmatrix} \hat{u}_x^I & 0 \\ 0 & \hat{u}_y^I \end{bmatrix}, \; \psi^{II} \equiv \begin{bmatrix} \hat{u}_x^{II} & 0 \\ 0 & \hat{u}_y^{II} \end{bmatrix}, \tag{9}$$

so that the local extended virtual element space $\mathbf{V}_X^h(E)$ reads as

$$\mathbf{V}_X^h(E) \equiv \mathbf{V}^h(E) \oplus \psi^I \mathbf{V}^{h,*}(E) \oplus \psi^{II} \mathbf{V}^{h,*}(E). \tag{10}$$

A basis of this space can be obtained as the union of the basis functions of $\mathbf{V}_X^h(E)$, $\psi^I \mathbf{V}^{h,*}(E)$ and $\psi^{II} \mathbf{V}^{h,*}(E)$. Therefore, at every enriched node the vector-valued field $\mathbf{v}_X^h(\mathbf{x})$ that belongs to the extended virtual element space $\mathbf{V}_X^h(E)$ is characterized by four values and for an element whose nodes are all enriched, we have $4N_E$ degrees of freedom. We denote the basis functions of $\mathbf{V}_X^h(E)$ by the symbol φ_i , $i = 1,\dots,4N_E$, where

$$\varphi_i = \begin{cases} (\varphi_i, 0)^T & \text{for } 1 \le i \le 2N_E, \; i \text{ odd}, \\ (0, \varphi_i)^T & \text{for } 1 \le i \le 2N_E, \; i \text{ even}, \\ (\hat{u}_x^I \varphi_i, \hat{u}_y^I \varphi_i)^T & \text{for } 1 + 2N_E \le i \le 3N_E, \\ (\hat{u}_x^{II} \varphi_i, \hat{u}_y^{II} \varphi_i)^T & \text{for } 1 + 3N_E \le i \le 4N_E. \end{cases} \tag{11}$$

Finally, the extended global virtual element space $\mathbf{V}_X^h$ reads:

$$\mathbf{V}_X^h = \left\{ v_X^h \in \left[H^1(\Omega) \right]^2 : v_{X|E}^h \in \mathbf{V}_X^h(E) \;\; \forall E \in \Omega_h \right\}. \tag{12}$$

Since $\{\varphi_i\}_{i=1}^{4N_E}$ are not known in the interior of the element, we construct a convenient projection operator that allows to compute the approximations $a_X^h(\cdot,\cdot) : \mathbf{V}_X^h(E) \times \mathbf{V}_X^h(E) \to \mathbb{R}$ and

$b_X^h(\cdot): \mathbf{V}_X^h(E) \to \mathbb{R}$ of the exact bilinear form $a(\cdot,\cdot)$ and the linear functional $b(\cdot)$ appearing in (7). The extended virtual element formulation then reads: Find $\mathbf{u}_X^h \in \mathbf{V}_{X,g}^h$ such that

$$a_X^h(\mathbf{u}_X^h, \mathbf{v}_X^h) = b_X^h(\mathbf{v}_X^h) \quad \forall \mathbf{v}_X^h \in \mathbf{V}_{X,0}^h \tag{13}$$

where the bilinear form $a_X^h(\cdot,\cdot)$ is built element-wise as

$$a_X^h(\mathbf{u}_X^h, \mathbf{v}_X^h) = \sum_{E\in\Omega} a_X^{h,E}(\mathbf{u}_X^h, \mathbf{v}_X^h) \quad \forall \mathbf{u}_X^h, \mathbf{v}_X^h \in \mathbf{V}_X^h, \tag{14}$$

and we set $b_X^h(v_X^h) = b(v_X^h)$. To construct a bilinear form $a_X^{h,E}(\cdot,\cdot)$ which is computable from the degrees of freedom, we extend the vector-valued linear polynomial space $\wp^1(E)$ to a subspace $\wp_X$ of $\mathbf{V}_X^h(E)$ which includes the linear polynomials and the additional enrichment functions $\hat{\mathbf{u}}^I$ and $\hat{\mathbf{u}}^{II}$. Such space is spanned by the eight linearly independent vector fields representing the three fundamental rigid body motions, the three independent deformation modes and the two enrichment fields:

$$\wp_X(E) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \eta \\ -\xi \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \begin{pmatrix} \hat{u}_x^I \\ \hat{u}_y^I \end{pmatrix}, \begin{pmatrix} \hat{u}_x^{II} \\ \hat{u}_y^{II} \end{pmatrix} \right\}. \tag{15}$$

We then define the extended elliptic projection operator $\Pi_X^a : \mathbf{V}_X^h(E) \to \wp_X(E)$ for each element E, which is the solution of the following variational problem:

$$\int_E \boldsymbol{\sigma}(\mathbf{q}_X) : \boldsymbol{\varepsilon}(\Pi_X^a \mathbf{v}_X^h)\, d\mathbf{x} = \int_E \boldsymbol{\sigma}(\mathbf{q}_X) : \boldsymbol{\varepsilon}(\mathbf{v}_X^h)\, dx \quad \forall \mathbf{q}_X \in \wp_X(E), \tag{16}$$

with the additional conditions

$$\begin{aligned} \overline{\Pi_X^a v_X^h} &= \overline{v_X^h}, \\ \overline{(\Pi_X^a v_X^h)_R} &= \overline{(v_X^h)_R}, \end{aligned} \tag{17}$$

where $\overline{(\cdot)}$ and $\overline{(\cdot)}_R$ represent the average translation and rotation. Then, the local extended bilinear form can be computed as:

$$\begin{aligned} a_X^{h,E}(\mathbf{v}_X^h, \mathbf{w}_X^h) &\equiv a^E\left(\Pi_X^a(\mathbf{v}_X^h), \Pi_X^a(\mathbf{w}_X^h)\right) + S_X^E\left(\mathbf{v}_X^h - \Pi_X^a(\mathbf{v}_X^h), \mathbf{w}_X^h - \Pi_X^a(\mathbf{w}_X^h)\right) \\ &= \int_E \boldsymbol{\sigma}(\Pi_X^a(\mathbf{v}_X^h)) : \boldsymbol{\varepsilon}(\Pi_X^a(\mathbf{w}_X^h))\, d\mathbf{x} + S_X^E\left(\mathbf{v}_X^h - \Pi_X^a(\mathbf{v}_X^h), \mathbf{w}_X^h - \Pi_X^a(\mathbf{w}_X^h)\right), \end{aligned} \tag{18}$$

where $S_X^E(\cdot,\cdot)$ is a suitable stabilization term needed to guarantee linear consistency and stability of the method. According to the virtual element methodology, $S_X^E(\cdot,\cdot)$ can be any symmetric, positive definite, continuous bilinear form defined on the kernel of the extended projection operator Π_X^a. In [8], we provide two possible choices of the stabilization term by considering the standard dofi-dofi and D-recipe formulations in our extended setting. Such choices are widely accepted in the VEM literature and in some cases they were theoretically proved to be effective to guarantee stability.

Numerical examples

Patch test. We first conduct an extended patch test, addressing the enrichment with singular fields. The extended patch test ensures that the singular enrichment fields can be exactly reproduced using the X-VEM. To this aim, we consider a square elastic plate that occupies the region $(-1,1)^2$ under plane strain conditions, with a horizontal crack of unit length that extends from $(-1,0)$ to $(0,0)$. A coarse mesh of 64 polygonal elements are considered, where all the nodes in the domain are enriched the near-tip displacement fields are imposed on the boundary of the domain by requiring that all the enriched boundary degrees of freedom are equal to 1 and all the standard boundary degrees of freedom are equal to 0. The relative error in strain energy for the extended patch tests is of the order of 10^{-12}.

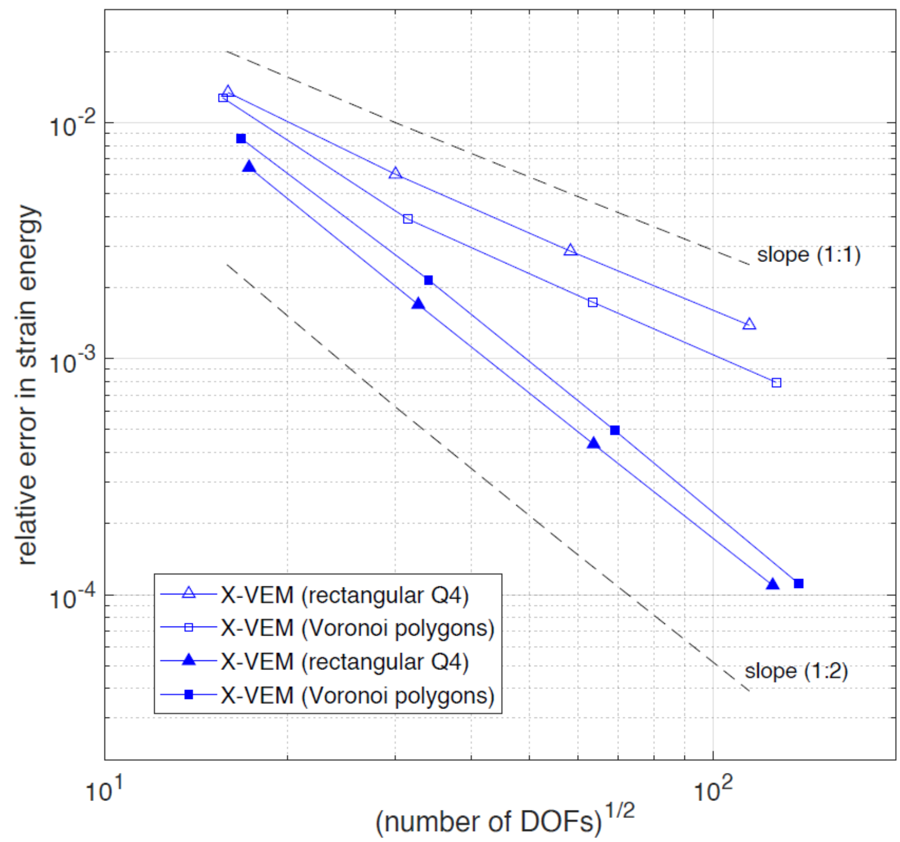

Figure 1: X-VEM convergence in strain energy for the mixed-mode benchmark problem with both topological and enrichment strategies enrichment.

Convergence study. We investigate the convergence of the X-VEM for the problem of a twodimensional square plate under plain strain conditions in the presence of a horizontal crack, extending from the boundary to the center of the specimen. The geometry of the domain is the same adopted as that for the extended patch test. On the boundary of the domain, we apply the exact near-tip mixed mode I and mode II displacement fields, which are also employed as enrichment fields for the X-VEM and represent the exact solution for the problem at hand. Both quadrilateral and general polygonal meshes are considered. To compute the element stiffness matrix, we follow two different strategies: topological enrichment and geometric enrichment. In the topological enrichment, we only enrich the node located at the singularity of the solution whereas in geometric enrichment we enrich all the nodes within a given radius from the origin. As in extended finite element methods, due to the presence of the singularity in the crack tip, the theoretical convergence rate for this problem is R = 1 that is non-optimal. As shown in Fig. 1, both VEM and X-VEM with topological enrichment converge in strain energy with a rate close to 1, in agreement with theory. It turns out that the X-VEM is insensitive to the type of mesh (quadrilaterals or polygons), and the results from the X-VEM are consistently more accurate than those from standard VEM. In order to establish if the proposed X-VEM can deliver the optimal convergence rate R = 2 that is predicted by theory, we enrich all nodes that are located within a ball of radius $r_e = 0.5$ from the origin. As shown in Fig. 1, in the case of geometric enrichment, the convergence rate is close to 2, which is consistent with theory.

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