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Abstract

The definition of quasi-Hopf algebra, introduced by Drinfeld in [17], is obtained by relaxing the coassociativity axiom in the definition of Hopf algebras. Their importance lies in their ties to monoidal categories; they are relevant whenever one needs to consider a non-strict category, meaning the associativity of the tensor product is controlled by a non-trivial isomorphism.

Inspired by the work of Radford in [45], for H an arbitrary quasi-Hopf algebra, we describe all the Hopf algebras of dimension 2 within the category of left Yetter-Drinfeld modules over H and determine the biproduct quasi-Hopf algebras uniquely associated to them. Classes of such biproduct quasi-Hopf algebras are obtained by taking H as the Hopf algebra of functions on a group G, endowed with the quasi-Hopf algebra structure provided by a non-trivial 3-cocycle on G, or as a quasi-Hopf algebra with radical of codimension two. Particular attention is dedicated to analyze the case in which G is a finite abelian group. In this way, we discover new classes of quasi-Hopf algebras which can be used as examples to test general theories; see [13] for a paper that uses a class of quasi-Hopf algebras computed here.

We apply the characterization result above to complete the classification of 6-dimensional quasi-Hopf algebras, by proving that any such algebra is semisimple. As byproducts, we provide examples of 6-dimensional quasi-bialgebras that are not semisimple as algebras, as well as the concrete quasi-Hopf structures of the 6-dimensional semisimple quasi-Hopf algebras previously classified by Etingof and Gelaki in [22] in terms of their category of representations.

Per Elisabetta e Eleonora

Declaration

This thesis is based on a joint work with Daniel Bulacu, contained in the papers [10] and [9]. While [10] was subject to peer-review during the publishing process, [9], as a preprint, was not.

The introductory Chapter 1 contains no original results, Chapter 2 is entirely based on [9] and, likewise, Chapter 4 on [10]. With respect to Chapter 3, Section 3.1 contains results proven in [10], while Section 3.2 is based on results from [9]. The content of Appendix A appeared first in [9].

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Contents

Introduction

Quasi-Hopf algebras were introduced in [17] by Drinfeld. Their structure is obtained by relaxing the coassociativity axiom in the definition of Hopf algebras, asking for the coassociativity to hold only up to the conjugation by an invertible element, called the reassociator.

The main motivation to study quasi-Hopf algebra lies in their categorical interpretation, whenever one needs to deal with concrete monoidal categories that are not necessarily strict. This was indeed the reason behind their introduction in [17], motivated by the need to work with monoidal categories in rational conformal field theory. Namely, take $\mathcal{C}$ to be any rigid monoidal category with a field k as unit object and which admits a fiber functor, i.e. an exact and faithful monoidal functor $P : \mathcal{C} \to \mathrm{Vec}_k$; then $\mathcal{C}$ is equivalent to the category of representations of a Hopf algebra H with base field k. Namely, H are the endomorphisms of P, whose coalgebra maps are endowed by the monoidal structure of P.

If instead we ask P to be only a quasi-monoidal functor, meaning we remove the associativity requirement from its monoidal structure, then H becomes a quasi-Hopf algebra; see [24] and [14] for further details. Conversely, if H is a quasi-Hopf algebra, the monoidal structure in its category of left modules ${}_H\mathcal{M}$ is in general non-strict, where the associativity constraint is determined by the reassociator of H. The category of modules is always strict when H is a usual Hopf algebra.

Another notable property of the quasi-Hopf algebra structure is its stability under twist-deformation: if we take H to be a quasi-Hopf algebra and F an arbitrary invertible element in $H \otimes H$, we can construct a new quasi-Hopf algebra, denoted by H_F, on the algebra structure on H, by deforming its coproduct, counit, reassociator and antipode by F; from a categorical point of view, deforming the structure this way is equivalent to changing the quasi-tensor structure on the functor P on a rigid monoidal category $\mathcal{C}$. If, on the other hand, H is a Hopf algebra, deforming H by any twist yields, in general, a quasi-Hopf algebra with non-trivial reassociator; for the deformed structure to be that of a usual Hopf algebra, we need the twist F to satisfy an additional 2-cocycle-like condition.

Coming back to what we stated above, quasi-Hopf algebras where originally introduced in [17] to satisfy the need arising from dealing with non-strictness while studying the representations of Drinfeld-Jimbo algebras. Another application of quasi-Hopf algebras in the setting of mathematical physics comes from the study of some Yang-Baxter equations; two particular classes of solutions, namely the vertex and the face type elliptic solutions, are associated with two types of elliptic quantum groups. It was proven in [26], [25], [18] and [27] that both of them are quasi-Hopf algebras.

One feature of the state of the art in the previously cited areas of mathematical physics is the shortage of examples of non-strict monoidal categories equipped with some extra structures. This is mainly due to the fact that we know only a few "proper" concrete examples of quasi-Hopf algebras, in comparison, for instance, to the many known examples of regular Hopf algebras.

Due to the technical nature of the axioms of quasi-Hopf algebras and their properties, it is usually very difficult to construct examples "by hand" or through computationally easy methods. The alternative, and preferred, way to obtain examples is through general constructions, often categorical in nature (such as the quantum double), that produce quasi-Hopf algebras starting from various kind of data. Furthermore, by classifying quasi-Hopf algebras in low dimension, we obtain some useful ingredients to build, via said general constructions, more complex quasi-Hopf algebras.

As such, this thesis is mainly interested in two problems:

1. classifying finite dimensional quasi-Hopf algebras up to twist-equivalence over an algebraically closed field of characteristic 0,

2. constructing examples of quasi-Hopf algebras that are not just obtained by twisting a known Hopf algebra, thus producing new (tensor) categories.

In [43], Radford proved that any Hopf algebra A with a projection covering the embedding of a sub-Hopf algebra H is isomorphic to a biproduct $B \times H$ between H and a Hopf algebra B in the braided monoidal category of Yetter-Drinfeld modules; a biproduct, in the sense of Radford, is a Hopf algebra structure over the tensor product $B \otimes H$, which generalizes the semidirect product between groups. Note that the category of Yetter-Drinfeld modules over H, introduced by Yetter in [51] and given its current name in [44] by Radford and Towber, is obtained as the center of the left H-modules, a construction that associates a braided monoidal category to any monoidal category.

If we instead take A to be a quasi-Hopf algebra with a projection over a sub-quasi-Hopf, it was proven by Bulacu in [5] and by Bulacu and Nauwelaerts in [11] that the previous results still hold, and we can associate to each braided Hopf algebra B within the category of left Yetter-Drinfeld modules over an arbitrary quasi-Hopf algebra H the biproduct quasi-Hopf algebra $B \times H$, having the same reassociator as its sub-object H. Moreover, it was shown in [14] by Bulacu and Torrecillas that the biproduct structure $B \times H$ can be recovered by reconstruction theory from the monoidal category ${}_B\mathcal{C}$ of representations in $\mathcal{C} = {}_H\mathcal{M}$ of a central Hopf algebra B in $\mathcal{C}$. This thesis focus is precisely the way Radford's biproduct can be used to tackle the problems of classification and construction for quasi-Hopf algebras. We discuss first its role in the classification of (quasi-)Hopf algebras.

Consider any pointed Hopf algebra H; the graded algebra $\mathrm{gr}(H)$, associated with the coradical filtration of H, is a Hopf algebra with a projection, and so it is determined as a biproduct by a braided Hopf algebra R within the category of Yetter-Drinfeld modules over the Hopf algebra of grouplike elements in H. In the context of Hopf algebra classification, the famous lifting method of Andruskiewitsch and Schneider consists in studying the braided Hopf algebra R, transferring the properties found to $\mathrm{gr}(H)$ and, finally, lifting them back to H. This method has proved crucial for classifying pointed Hopf algebras, see the surveys [2, 4] for more details.

When dealing with a quasi-Hopf algebra H, we are unable to approach the technique through the coradical filtration, due to the lack of coassociativity in the coalgebra structure of H. Instead, we can tackle the problem from the dual situation. If H is a basic quasi-Hopf algebra (the dual notion to pointed, meaning that all simple modules are 1-dimensional), the graded algebra $\mathrm{gr}(H)$, this time associated with the filtration by powers of the Jacobson radical of H, is a quasi-Hopf algebra with a projection on $H_0 = \frac{H}{J}$, a semisimple quasi-Hopf algebra. In particular, H_0 is a group Hopf algebra over an abelian group G, considered as a quasi-Hopf algebra via a reassociator determined by a 3-cocycle on G. The lifting method was used in the setting of quasi-Hopf algebras by Etingof and Gelaki in [19, 20, 21], appearing for the first time in [19]. It was also used by Angiono in the paper [3].

The classification of semisimple Hopf algebras in dimension 6 was given by Masuoka in [38], and Ştefan showed in [49] that any Hopf algebra of dimension 6 is semisimple, thus finishing the classification in dimension 6.

In this thesis (see Theorem 4.2.2), we complete, through the lifting method, the classification of 6-dimensional quasi-Hopf algebras, by proving that any such algebra is semisimple and then by using the characterization of semisimple 6-dimensional quasi-Hopf algebras, in terms of their category of representations, given by Etingof and Gelaki in [22]. By translating this result from the categorical language, we provide in Theorem 4.2.5 an explicit list of all 6-dimensional quasi-Hopf algebras. As a byproduct of the classification, we also provide in Remark 4.1.7 examples of 6-dimensional quasi-bialgebras that are not semisimple as algebras, and hence are not quasi-Hopf algebras.

Let us now focus on the role of Radford's biproduct in the construction of new quasi-Hopf algebras. Starting from a well known quasi-Hopf algebra H with a manageable structure (e.g. a group algebra with quasi-Hopf structure endowed by a 3-cocycle), finding braided Hopf algebras in the Yetter-Drinfeld modules over H allows us to construct a new quasi-Hopf algebra, whose structure can be much more complex than that of H. As an example, we can start by a commutative, cocommutative and semisimple H; if there is a braided bialgebra B over H with non-trivial H-action and H-coaction, this leads to a non-commutative, non-cocommutative structure on $B \times H$.

The construction of examples of needed complexity, but simple enough to allow for manual computations was the original motivating desire behind our interest in biproduct quasi-Hopf algebras. Concretely, inspired by the work of Radford in [45], for an arbitrary quasi-Hopf algebra H with invertible antipode over a field k of characteristic different from 2, we describe in Theorem 2.1.8 all the Hopf algebras of dimension 2 within the category of left Yetter-Drinfeld modules over H. As a direct consequence, we determine in Proposition 2.1.9 the biproduct quasi-Hopf algebras uniquely associated to them. All 2-dimensional braided Hopf over H are either trivial, meaning isomorphic to the group Hopf algebra $k[C_2]$ with trivial H-action and coaction, or are isomorphic to $B_{\sigma,v}$, generated by the unit and a primitive element, whose H-action and coaction are respectively controlled by an algebra map $\sigma : H \to k$ and an element $v \in H$ satisfying three relations. Then, computing new examples of quasi-Hopf algebras reduces to finding all couples (σ, v) satisfying the three relations for a given H.

Classes of such biproduct quasi-Hopf algebras are obtained in Section 2.2 by taking H as the Hopf algebra of functions over a group G, endowed with the quasi-Hopf algebra structure provided by a non-trivial 3-cocycle on G, or in Section 2.3 as a quasi-Hopf algebra with radical of codimension two, classified by Etingof and

Gelaki in [19]. Particular attention is dedicated to analyze the case in which G is a finite abelian group; the case in which G is a product of two cyclic groups is treated in great detail in Appendix A.

The structure of the thesis is as follows: Chapter 1 serves as an introduction to some fundamental topics needed to work with quasi-Hopf algebras, and contains no original results.

Chapter 2 is focused on characterizing biproduct quasi-Hopf algebras of rank 2 and producing examples. Section 2.1 contains all the preliminary results needed to prove Theorem 2.1.8. In Section 2.2 the characterization is specialized to function algebras of a group G; Subsection 2.2.1 deals with the case in which G is abelian (see Proposition 2.2.4) and further analyzes the cases when G is the product of one, two or more than three cyclic groups. On the other hand, Subsection 2.2.2 provides an example for G non-abelian, by considering in Proposition 2.2.19 the case in which G is a double dihedral group $\bar{D}_n$. Section 2.3 deals with the classification over well known classes of quasi-Hopf algebras. In particular, Proposition 2.3.3 shows that a finite dimensional quasi-Hopf algebra H with radical of codimension 2 has non-trivial 2 dimensional braided Hopf algebras if and only if it is twist equivalent to a Nichols Hopf algebra H_{2^n}.

Chapter 3 is centered on the study of 4-dimensional quasi-Hopf algebras. In Section 3.1, we provide with Lemma 3.1.1 a characterization of the twist-equivalence classes of the quasi-Hopf structures on the algebras of functions over a finite cyclic group. In Section 3.2 we focus on the classification of 4-dimensional quasi-Hopf algebras; in Subsection 3.2.1, we provide an alternative path to prove that non-semisimple 4-dimensional quasi-Hopf algebras are twist equivalent to the Sweedler Hopf algebra H_4, as originally proven by Etingof and Gelaki in [19]. Moreover, we provide in Proposition 3.2.5 a classification of the isomorphism classes of quasi-Hopf algebras obtained by twisting H_4. In Subsection 3.2.2 we provide a list of all 4-dimensional semisimple quasi-Hopf algebra, up to twist equivalence.

Chapter 4 sole objective is to reach Theorem 4.2.2 and to produce the list of 6-dimensional quasi Hopf algebras contained in Theorem 2.1.8. In Section 4.1, we classify all 3-dimensional YD-Hopf algebras over the two dimensional quasi-Hopf algebra $H(2)$, showing in Proposition 4.1.9 that all of them are associated with semisimple biproducts; this implies that all 6-dimensional biproducts are semisimple, see Corollary 4.1.11. In Section 4.2, we prove Theorem 4.2.2, by showing that all 6-dimensional quasi Hopf algebras are biproducts, the section concludes with Theorem 4.2.5.

Appendix A, after realizing that the relation (2.2.13) in Corollary 2.2.11 reduces to (A.0.1), is entirely devoted to solving the problem of determining the parameters λ, k such that the conic section $\mathcal{C}_{\lambda,k}(x, y) = 0$ has integer solution. The Appendix concludes with a concrete example.

Chapter 1

On quasi-Hopf algebras and their surroundings

This first chapter is intended as a "toolbox", containing all the necessary relation and constructions needed to smoothly introduce the original results that lie at the core of this thesis and are the focus of later chapters. Since most of the more technical results are presented here without proof, we refer the interested reader to the monograph [15] for a comprehensive overview of quasi-Hopf algebras, to [34, 36] for general topics in Hopf algebra theory and to [24] for details on the representation theory of quasi-Hopf algebras. Specific references will be provided when needed.

Unless otherwise specified, k denotes a fixed field and $\otimes$, in the context of vector spaces, denotes $\otimes_k$, the tensor product over k.

1.1 Braided monoidal categories

Monoidal categories provide the correct categorical interpretation of (quasi-)Hopf algebras. Abstractly, monoidal categories generalize the tensor product of vector spaces, allowing the definition of products and coproducts, and therefore of algebras and coalgebras, over objects other than vector spaces.

Definition 1.1.1 A monoidal category is a datum $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$, consisting of a category $\mathcal{C}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product, a distinguished object $\underline{1} \in \mathcal{C}$, called the unit object of $\mathcal{C}$, and the natural isomorphisms

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \quad \text{(the associativity constraint, or associator)},$$
$$l_X : \underline{1} \otimes X \to X \quad \text{(the left unit constraint, or left unitor)},$$
$$r_X : X \otimes \underline{1} \to X \quad \text{(the right unit constraint, or right unitor)},$$

where X, Y, Z are objects in $\mathcal{C}$, satysfing the two following diagrams, respectively called the Pentagon Axiom and the Triangle Axiom, for any $X, Y, Z, T \in \mathcal{C}$:

$$\begin{tikzcd}
& (X \otimes Y) \otimes (Z \otimes T) \arrow[dr, "a_{X,Y,Z\otimes T}"] & \\
((X \otimes Y) \otimes Z) \otimes T \arrow[ur, "a_{X\otimes Y,Z,T}"] \arrow[d, "a_{X,Y,Z}\otimes Id"'] & & X \otimes (Y \otimes (Z \otimes T)) \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "a_{X,Y\otimes Z,T}"'] & & X \otimes ((Y \otimes Z) \otimes T), \arrow[u, "Id\otimes a_{Y,Z,T}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes \underline{1}) \otimes Y \arrow[rr, "aX{,}\underline{1}{,}Y"] \arrow[dr, "r_X\otimes Id"'] & & X \otimes (1 \otimes Y) \arrow[dl, "Id\otimes l_Y"] \\
& X \otimes Y. &
\end{tikzcd}$$

A monoidal category $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$ is strict if a, l and r are defined by identity morphisms in $\mathcal{C}$.

We can relate monoidal structures on categories through monoidal functors.

Definition 1.1.2 Let $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$ and $(\mathcal{D}, \Box, \underline{e}, a', l', r')$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. A monoidal functor is a triple $(F, \varphi_0, \varphi_2)$, where φ_2 is defined by the morphisms

$$\varphi_{2,X,Y} : F(X)\Box F(Y) \to F(X \otimes Y),$$

where $X, Y \in \mathcal{C}$, as a natural morphism, and $\varphi_0 : \underline{e} \to F(\underline{1})$ is a morphism in $\mathcal{D}$ such that the following diagrams are satisfied.

$$\begin{tikzcd}
(F(X)\Box F(Y))\Box F(Z) \arrow[r, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "\varphi_{2,X,Y}\Box Id"'] & F(X)\Box(F(Y)\Box F(Z)) \arrow[d, "Id\Box\varphi_{2,Y,Z}"] \\
F(X \otimes Y)\Box F(Z) \arrow[d, "\varphi_{2,X\otimes Y,Z}"'] & F(X) \otimes F(Y \otimes Z) \arrow[d, "\varphi_{2,X,Y\otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(\alpha_{X,Y,Z})"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{e}\Box F(X) \arrow[r, "\varphi_o\Box Id"] \arrow[d, "l'_{F(X)}"'] & F(\underline{1})\Box F(X) \arrow[d, "\varphi_{2,\underline{1},X}"] \\
F(X) & F(\underline{1} \otimes X) \arrow[l, "F(l_X)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X)\Box\underline{e} \arrow[r, "Id\Box\varphi_o"] \arrow[d, "r'_{F(X)}"'] & F(X)\Box F(\underline{1}) \arrow[d, "\varphi_{2,X,\underline{1}}"] \\
F(X) & F(X \otimes \underline{1}) \arrow[l, "F(r_X)"]
\end{tikzcd}$$

2. The monoidal functor $(F, \varphi_0, \varphi_2)$ is called strong monoidal if φ_0 and φ_2 are defined by isomorphisms in $\mathcal{D}$.

Remark 1.1.3 Let $(\mathcal{C}, \otimes, \underline{e}), (\mathcal{C}', \square, \underline{1})$ and $(\mathcal{D}, \otimes', \underline{e}')$ be monoidal categories; since associator and unitors are not involved in the following construction, we consider them to be strict. The composition of two monoidal functors $(F, \varphi_0, \varphi_2) : \mathcal{C} \to \mathcal{D}$ and $(G, \psi_0, \psi_2) : \mathcal{D} \to \mathcal{C}'$ is still a monoidal functor, namely $(GF, G(\varphi_0) \circ \psi_0, \phi_2)$, where the natural morphism ϕ_2 is defined, for objects $X, Y \in \mathcal{C}$, by

$$GF(X) \otimes' GF(Y) \overset{\psi_{2,F(X),F(Y)}}{\longrightarrow} G(F(X) \square F(Y)) \overset{G(\varphi_{2,X,Y})}{\longrightarrow} GF(X \otimes Y). \tag{1.1.1}$$

Similarly, we can define the monoidal version of natural transformations, by adding compatibility conditions with the monoidal structures of the functors as a requirement.

Definition 1.1.4 Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories (adopting the notation of the previous Remark), and let $(F, \varphi_0, \varphi_2)$, $(G, \psi_0, \psi_2) : \mathcal{C} \to \mathcal{D}$ be monoidal functors. A monoidal natural transformation f (or natural morphism), from $(F, \varphi_0, \varphi_2)$ to (G, ψ_0, ψ_2) is a natural transformation $f : F \to G$ satisfying the relations expressed by the following diagrams for all objects $X, Y \in \mathcal{C}$:

$$\begin{array}{ccc} F(X)\square F(Y) & \xrightarrow{\varphi_{2,X,Y}} & F(X \otimes Y) \\ {\scriptstyle f_X \otimes f_Y}\downarrow & & \downarrow{\scriptstyle f_{X\otimes Y}} \\ G(X)\square G(Y) & \xrightarrow[\psi_{2,X,Y}]{} & G(X \otimes Y) \end{array} \quad \text{and} \quad \begin{tikzcd} G(\underline{1}) & \\ & \underline{e} \arrow[ul, "\psi_0"'] \arrow[dl, "\varphi_0"] \\ F(\underline{1}) \arrow[uu, "f_{\underline{1}}"] & \end{tikzcd} .$$

We can then define the monoidal versions of isomorphism and equivalence of categories, by requiring the functors that realize the isomorphism (or equivalence) to be monoidal.

Definition 1.1.5 Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories.

1. $\mathcal{C}$ and $\mathcal{D}$ are monoidally isomorphic if there exists two monoidal functors $(F, \varphi_0, \varphi_2) : \mathcal{C} \to \mathcal{D}$ and $(G, \psi_0, \psi_2) : \mathcal{D} \to \mathcal{C}$ such that $FG = Id_{\mathcal{D}}$ and $GF = Id_{\mathcal{C}}$ as monoidal functors, where the identities are seen as monoidal functors via trivial isomorphisms. In particular, this implies that F and G are strong monoidal.

2. $\mathcal{C}$ and $\mathcal{D}$ are monoidally equivalent if there exist two monoidal functors F, G (using the same notation as before) and two monoidal natural isomorphisms $GF \to Id_{\mathcal{C}}$ and $FG \to Id_{\mathcal{D}}$, and we call F a monoidal equivalence.

   If F and G both are strong monoidal, we call F a strong monoidal equivalence between $\mathcal{C}$ and $\mathcal{D}$.

Remark 1.1.6 (Mac Lane's Coherence) Mac Lane's strictification theorem, see [35, Section 7.2], states that every monoidal category is strong monoidally equivalent to a strict monoidal category. As a consequence, the commutativity of all diagrams constructed from multiple copies of the unitors and associator is equivalent to the commutativity of the triangle and pentagon axioms.

In particular, whenever we are dealing with the tensor product of multiple objects in a monoidal category, we can move the parenthesis in the order we prefer; as long as the parenthesis are in a fixed order at the start and at the end of the process, any morphism constructed by composing associators, and unitors, with each others will coincide with any other constructed in the same way.

1.1.1 Vector spaces graded by a group

In order to give an illustrative example of monoidal category and introduce notions which will become relevant later in this chapter, we recall some basic notions from group cohomology.

Let G be a group with neutral element e and k a field with unit 1. Denote $k^* = k \setminus \{0\}$ and by $K^n(G,k^*)$ the multiplicative group of maps from G^n to k^* under pointwise multiplication. For any $n \in \mathbb{N}^*$, there exist a map

$$\partial_n : K^n(G,k^*) \to K^{n+1}(G,k^*)$$

defined by

$$\begin{aligned}&\partial_n(f)(g_1,\dots,g_{n+1}) =\\ &\quad = f(g_2,\dots,g_{n+1}) \prod_{i=1}^{n} f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})^{(-1)^i} f(g_1,\dots,g_n)^{(-1)^{n+1}},\end{aligned}$$

for any $g_1,\dots,g_{n+1} \in G$; it is a well known fact that $Im(\partial_{n-1}) \subseteq Ker(\partial_n)$. We call the elements of the group $Z^n(G,k^*) := Ker(\partial_n)$ n-cocycles, while the elements of $B^n(G,k^*) := B^n(G,k^*)$ are referred to as n-coboundaries; the quotient $H^n(G,k^*) := Z^n(G,k^*)/B^n(G,k^*)$ is known as the n-th cohomology group of G over the field k; we say that two cocycle are cohomologus if they are in the same coset. We refer to [48, Section 7.3] for additional details on the context around of the maps ∂_n. For ease of use, we write down ∂_n explicitly for n in the range $\{1,2,3\}$:

$$\begin{aligned}&\partial_1(f)(x,y) = f(x)\ f(xy)^{-1}\ f(y),\\ &\partial_2(g)(x,y,z) = g(y,z)\ g(xy,z)^{-1}\ g(x,yz)\ g(x,y)^{-1},\\ &\partial_3(h)(x,y,z,t) = h(y,z,t)\ h(xy,z,t)^{-1}\ h(x,yz,t)\ h(x,y,zt)^{-1}\ h(x,y,z).\end{aligned}$$

We are now able to explicitly write the definition of n-coboundaries and cocycles, for n equal to 2 or 3.

Definition 1.1.7 2-cocycles and a 3-cocycles on G with coefficients in k^* are maps $\phi : G\times G \to k^*$ and, respectively, $\varphi : G\times G\times G \to k^*$ such that, for all $x,y,z,t \in G$,

$$\phi(xy,z)\ \phi(x,y) = \phi(y,z)\ \phi(x,yz), \tag{1.1.2}$$

$$\varphi(xy,z,t)\ \varphi(x,y,zt) = \varphi(y,z,t)\ \varphi(x,yz,t)\ \varphi(x,y,z). \tag{1.1.3}$$

The cocycles ϕ,φ are 2(respectively 3)-coboundaries if there exist a map $f : G \to k^*$ (respectively $g : G\times G \to k^*$) such that

$$\phi(x,y) = f(x)f(y)f(xy)^{-1}, \quad \varphi(x,y,z) = g(y,z)\ g(xy,z)^{-1}\ g(x,yz)\ g(x,y)^{-1}.$$

Remark 1.1.8 We say that a n-cocycle ϕ is normalized if $\phi(g_1,\dots,g_n) = 1$ whenever $1 \in \{g_1,\dots,g_n\} \subseteq G$. In particular, a 2-cocycle is normalized if and only if $\phi(e,e) = 1$. Indeed, by (1.1.2), we have $\phi(x,z)\ \phi(x,e) = \phi(e,z)\ \phi(x,z)$ and $\phi(x,e)^2 = \phi(e,e)\phi(x,e)$ for all $x,z \in G$, implying $\phi(e,z) = \phi(x,e) = \phi(e,e)$. Moreover, following immediately from the definition, any 2-coboundary defined by a map such that $e \mapsto 1_k$ is a normalized 2-cocycle.

On the other hand, any 3-cocycle φ is normalized if and only if $\varphi(x,e,y) = 1$ for all $x,y \in G$; this condition implies $\varphi(e,x,y) = 1$ and $\phi(x,y,e) = 1$ for all $x,y \in G$, by taking, respectively, either y or z equal to e in (1.1.3). Then, 3-coboundaries are normalized if $\phi(x,e) = \phi(e,y)$ for all $x,y \in G$.

The following lemma explains why we can often consider 3-cocycles to be normalized without loss of generality.

Lemma 1.1.9 *Any* 3*-cocycle is cohomologous to a normalized* 3*-cocycle.*

Proof. In the notation of Definition 1.1.7, by (1.1.3) we obtain the relations $\varphi(x,e,t) = \varphi(e,e,t)\varphi(x,e,e)$ by setting $y,z=e$, and $\varphi(e,e,e)=1$ by taking $x=t=e$. Then, after defining $f : G\times G \to k^* : (x,y) \mapsto \varphi(e,e,y)^{-1}\varphi(x,e,e)$, we get

$$\begin{aligned}\varphi\partial_2(f)(x,e,z) =&\varphi(x,e,z)f(e,z)\ f(x,z)^{-1}\ f(x,z)\ f(x,e)^{-1}\\ &\varphi(x,e,z)\varphi(e,e,z)^{-1}\varphi(e,e,e)\varphi(e,e,z)\\ &\varphi(x,e,e)^{-1}\varphi(e,e,z)^{-1}\varphi(x,e,e)\varphi(e,e,e)\varphi(e,e,x)^{-1}\\ =&\varphi(x,e,z)\varphi(e,e,z)^{-1}\varphi(e,e,x)^{-1}\\ =&\varphi(x,e,z)\varphi(x,e,z)^{-1}=1,\end{aligned}$$

which implies that $\varphi\partial_2(f)$ is normalized. □

Remark 1.1.10 The same property also holds for a 2-cocycle ϕ in a stronger form; it suffices to take $f : G \to k^*$ to be the constant function $x \mapsto \phi(e,e)^{-1}$, then $\phi\partial_1(f) = \phi(e,e)^{-1}\phi$ is normalized. In short, any 2-cocycle is normalized up to a scalar.

Example 1.1.11 Denote by C_n the cyclic group of order n and by g one fixed generator, and suppose k has a primitive n-th roots of the units ξ. It is a well known fact that $H^3(C_n,k^*) \cong C_n$, where a complete system of representatives for the cohomology classes is given, for $0 \le a \le n-1$, by $\varphi_a(x,y,z) = \xi^{ax\lfloor\frac{y+z}{n}\rfloor}$, where $\lfloor m\rfloor$ denotes the integer part of a rational number m; see e.g. [15, Example 1.16].

We say that a k-vector space V is G-graded if there exist a family of sub-vector spaces $\{V_g\}_{g\in G}$ such that $V = \bigoplus_{g\in G} V_g$. We call an element $v \in V$ homogeneous if $v \in V_g$ for some $g \in G$, called the degree of v and denoted by $|v|$. We denote by Vec^G the category whose objects are G-graded vector spaces, with morphisms given by linear maps $f : U \to V$ preserving homogeneous components, meaning $f(U_g) \subseteq V_g$ for all $g \in G$.

We can endow the usual tensor product $\otimes$ on the category of vector spaces Vec_k with extra structure, thus defining a tensor product on the category of G-graded vector spaces Vec^G. Namely, we define the graded components of $U \otimes V$, for two given graded vector spaces $U, V \in Vec^G$, by

$$(U\otimes V)_g := \bigoplus_{\substack{x,y\in G\\ xy=g}} U_x \otimes V_y \qquad \text{for all } g \in G. \tag{1.1.4}$$

Moreover, we denote by l_k and r_k the left and right unitors of Vec_k, defined respectively by the canonical maps $(l_k)_U : k \otimes U \to U$ and $(r_k)_V : V \otimes k \to V$ for any vector spaces U and V.

We are now able to recall the following correspondence of structures, the proof of which is straightforward, if a bit technical. We refer to [15, Proposition 1.13] for all the details.

Proposition 1.1.12 *There is a bijection between monoidal structures on Vec^G of the form $(Vec^G, \otimes, a, k, l_k, r_k)$ and normalized 3-cocycles on G with coefficients in k.*

Namely, the associator a of Vec^G is defined, for any G-graded vector space U, V, W, by a normalized 3-cocycle $\varphi \in Z^3(G, k^*)$ as follows:

$$a_{U,V,W}((u \otimes v) \otimes w) = \varphi(|u|, |v|, |w|) \ \otimes (v \otimes w). \tag{1.1.5}$$

1.1.2 Braided monoidal categories

We define the braiding of a monoidal category, which can be thought of as an extension of the usual flip $\tau : U \otimes V \to V \otimes U : u \otimes v \mapsto v \otimes u$, a morphism in Vec_k, to an arbitrary monoidal category. For any monoidal category $\mathcal{C}$, we can define the flip functor $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by $\tau((A, B)) = (B, A)$ for objects A, B in $\mathcal{C}$ and $\tau((f, g)) = (g, f)$ for morphisms f, g in $\mathcal{C}$.

Definition 1.1.13 Let $\mathcal{C}$ be a monoidal category. A pre-brading on $\mathcal{C}$ is a natural transformation $\sigma : \otimes \to \otimes \circ \tau$ defined by morphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ satisfying, for any object A, B, C in $\mathcal{C}$, the Hexagon Axioms, meaning, in the notations of Definition 1.1.1, the commutativity of the following diagrams:

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\sigma_{A,B\otimes C}} & (B \otimes C) \otimes A \\
{\scriptstyle \sigma_{A,B} \otimes 1_C} \downarrow & & & & \downarrow {\scriptstyle \alpha_{B,C,A}} \\
(B \otimes A) \otimes C & \xrightarrow[\alpha_{B,A,C}]{} & B \otimes (A \otimes C) & \xrightarrow[1_B \otimes \sigma_{A,C}]{} & B \otimes (C \otimes A),
\end{array} \tag{1.1.6}$$

$$\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{\alpha^{-1}_{A,B,C}} & (A \otimes B) \otimes C & \xrightarrow{\sigma_{A\otimes B,C}} & C \otimes (A \otimes B) \\
{\scriptstyle 1_A \otimes \sigma_{B,C}} \downarrow & & & & \downarrow {\scriptstyle \alpha^{-1}_{C,A,B}} \\
A \otimes (C \otimes B) & \xrightarrow[\alpha^{-1}_{A,C,B}]{} & (A \otimes C) \otimes B & \xrightarrow[\sigma_{A,C} \otimes 1_B]{} & (C \otimes A) \otimes B.
\end{array} \tag{1.1.7}$$

We call σ a braiding if it is a natural isomorphism. Moreover, the couple $(\mathcal{C}, \sigma)$ is called a (pre-)braided monoidal category if σ is a (pre-)braiding on $\mathcal{C}$.

Example 1.1.14

- The flips $\tau_{U,V} : U \otimes V \to V \otimes U : u \otimes v \mapsto v \otimes u$, for any vector spaces U and V, define a braiding on Vec_k.
- The braiding of a monoidal category is generally not unique; if $(\mathcal{C}, c)$ is a braided monoidal category, we can define another braiding on $\mathcal{C}$ through the inverse of the braiding c. Namely, $\mathcal{C}^{in} := (\mathcal{C}, \underline{c})$, where $\underline{c}$ is defined by $\underline{c}_{X,Y} = \underline{c}^{-1}_{Y,X}$.

Definition 1.1.15 Let $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \sigma')$ be (pre-)braided monoidal categories. A strong monoidal functor $(F, \varphi_0, \varphi_2)$ is said to be (pre-)braided monoidal if it satisfies the following diagram for all $X, Y \in \mathcal{C}$, where $\otimes$ and $\boxtimes$ denote, respectively, the tensor products in $\mathcal{C}$ and $\mathcal{D}$.

$$\begin{array}{ccc}
F(X) \boxtimes F(Y) & \xrightarrow{\varphi_{2,X,Y}} & F(X \otimes Y) \\
{\scriptstyle \sigma'_{F(X),F(Y)}} \downarrow & & \downarrow {\scriptstyle F(\sigma_{X,Y})} \\
F(Y) \boxtimes F(X) & \xrightarrow[\varphi_{2,Y,X}]{} & F(Y \otimes X).
\end{array} \tag{1.1.8}$$

Moreover,

- $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \sigma')$ are isomorphic as (pre-)braided monoidal categories if there exists (pre-)braided monoidal functors $(F, \varphi_0, \varphi_2) : \mathcal{C} \to \mathcal{D}$ and $(G, \phi_0, \phi_2) : \mathcal{D} \to \mathcal{C}$ such that $GF = id_{\mathcal{C}}$ and $FG = id_{\mathcal{D}}$, where $id_{\mathcal{C}}$ and $id_{\mathcal{D}}$ are seen as (pre-)braided monoidal functors;

- similarly, $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \sigma')$ are equivalent as (pre-)braided monoidal categories if they are equivalent as monoidal categories through two (pre-)braided monoidal functors F and G.

1.1.3 Braided bialgebras

Algebraic structures in certain braided monoidal categories play a central role in our constructions; as intuition suggests, one can define algebraic structures on an object in a braided monoidal category by requiring the structure maps to be morphism in the category, and by dealing with parenthesis and the unit object (playing the role of the base field) through the associator and unitors.

Definitions 1.1.16 Let $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$ be a monoidal category.

- A monoid in $\mathcal{C}$ is a triple (M, μ, u), where $\mu : M \otimes M \to M$ and $u : \underline{1} \to M$ are morphisms in $\mathcal{C}$ such that

$$\begin{tikzcd} & M \otimes (M \otimes M) \arrow[dr, "id\otimes\mu"] & \\ (M \otimes M) \otimes M \arrow[ur, "a_{M,M,M}"] \arrow[d, "\mu\otimes id"'] & & M \otimes M \arrow[d, "\mu"] \\ M \otimes M \arrow[rr, "\mu"'] & & M, \end{tikzcd} \tag{1.1.9}$$

$$\begin{tikzcd} \underline{1} \otimes M \arrow[r, "u\otimes id"] \arrow[d, "l"'] & M \otimes M \arrow[dl, "\mu"] \\ M & M \otimes \underline{1}. \arrow[l, "r"] \arrow[u, "id\otimes u"'] \end{tikzcd} \tag{1.1.10}$$

- A comonoid in $\mathcal{C}$ is a triple (C, δ, ϵ), where $\delta : C \to C \otimes C$ and $\epsilon : C \to \underline{1}$ are morphisms in $\mathcal{C}$ such that

$$\begin{tikzcd} & C \otimes (C \otimes C) & \\ (C \otimes C) \otimes C \arrow[ur, "a_{C,C,C}"] & & C \otimes C \arrow[ul, "id\otimes\delta"'] \\ C \otimes C \arrow[u, "\delta\otimes id"] & & C, \arrow[ll, "\delta"] \arrow[u, "\delta"'] \end{tikzcd} \tag{1.1.11}$$

$$\begin{tikzcd} \underline{1} \otimes C \arrow[d, "l"'] & C \otimes C \arrow[l, "\epsilon\otimes id"'] \arrow[d, "id\otimes\epsilon"] \\ C \arrow[ur, "\delta"] & C \otimes \underline{1}. \arrow[l, "r"] \end{tikzcd} \tag{1.1.12}$$

Example 1.1.17 A monoid in the category Vec^G of vector spaces graded by a group G is a G-graded algebra, assuming Vec^G is strict as a monoidal category.

In what follows, we will call monoids and comonoids in the category of left modules ${}_H\mathcal{M}$ over a (quasi-)Hopf algebra H, respectively, algebras and coalgebras in ${}_H\mathcal{M}$. Equivalently, they are known as, respectively, H-module algebras and H-module coalgebras. We define a quasi-Hopf algebra in the next section.

Consider now two monoids (M, μ_M, u_M) and (N, μ_N, u_N) in a braided monoidal category $(\mathcal{C}, \otimes, \underline{1}, a, l, r)$ with braiding σ. We can endow the tensor product $M \otimes N$ with a monoid structure, namely $M \underline{\otimes} N = (M \otimes N, \mu_{M\underline{\otimes}N}, u_{M\underline{\otimes}N})$, where $\mu_{M\underline{\otimes}N}$ and $u_{M\underline{\otimes}N}$ are morphisms defined through composition as described by the following diagrams.

$$
\begin{tikzcd}
(M\otimes N)\otimes(M\otimes N) \arrow[rr, "\mu_{M\underline{\otimes}N}"] \arrow[d, "a_{M,N,M\otimes N}"'] & & M\otimes N \\
M\otimes(N\otimes(M\otimes N)) \arrow[d, "id_M\otimes a^{-1}_{N,M,N}"'] & & (M\otimes M)\otimes(N\otimes N) \arrow[u, "\mu_M\otimes\mu_N"'] \\
M\otimes((N\otimes M)\otimes N) \arrow[dr, "id_M\otimes(c_{N,M}\otimes id_N)"'] & & M\otimes(M\otimes(N\otimes N)) \arrow[u, "a^{-1}_{M,M,N\otimes N}"'] \\
& M\otimes((M\otimes N)\otimes N), \arrow[ur, "id_M\otimes a_{M,N,N}"'] &
\end{tikzcd}
\tag{1.1.13}
$$

$$
\begin{tikzcd}
M\otimes N \arrow[rr, "u_{M\underline{\otimes}N}"] \arrow[dr, "u_M\otimes u_N"'] & & \underline{1} \\
& \underline{1}\otimes\underline{1}. \arrow[ur, "l_{\underline{1}}=r_{\underline{1}}"'] &
\end{tikzcd}
\tag{1.1.14}
$$

Note that we can define $\mu_{M\underline{\otimes}N}$ using different compositions of associators in order to move the parenthesis; by coherence, see Remark 1.1.6, all such definitions coincide.

Definitions 1.1.18 Let $\mathcal{C}$ be a braided monoidal category.

- A bialgebra in $\mathcal{C}$ is a collection $(B, \mu, u, \Delta, \epsilon)$ such that (B, μ, u) is an monoid in $\mathcal{C}$, (B, Δ, ϵ) is a comonoid in $\mathcal{C}$ and $\Delta : B \to B \underline{\otimes} B$ is multiplicative, in the sense that the following diagram commutes.

$$
\begin{tikzcd}
B\otimes B \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "\mu"'] & (B\underline{\otimes}B)\otimes(B\underline{\otimes}B) \arrow[d, "\mu_{B\underline{\otimes}B}"] \\
B \arrow[r, "\Delta"'] & B\underline{\otimes}B.
\end{tikzcd}
\tag{1.1.15}
$$

- A Hopf algebra in $\mathcal{C}$ is a bialgebra B in $\mathcal{C}$ such that the identity id_B has an inverse $\underline{S}$ in $Hom_{\mathcal{C}}(B, B)$ with respect to the convolution product $*$, meaning a morphism $\underline{S} : B \to B$ such that $\underline{S} * id_B = id_B * \underline{S} = u \circ \epsilon$. We refer to [15, Lemma 2.57] for further details on the convolution product, defined by $f * g = \mu \circ (f \otimes g) \circ \delta$, for all $f, g \in Hom_{\mathcal{C}}(B, B)$.

We call bialgebras and Hopf algebras in braided monoidal categories braided bialgebras and braided Hopf algebras, respectively.

1.2 Quasi-Hopf algebras

Quasi-Hopf algebras, introduced by V.G. Drinfeld in [17], extend the class of Hopf algebras, by relaxing the coassociativity axiom.

Definition 1.2.1 A quasi-bialgebra is a quadruple $(H, \Delta, \varepsilon, \Phi)$ consisting of a k-associative algebra H with unit 1, two algebra maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ and an invertible element Φ in $H \otimes H \otimes H$, called the reassociator, such that the following relations hold

$$(\mathrm{Id}_H \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes \mathrm{Id}_H)(\Delta(h))\Phi^{-1}, \tag{1.2.16}$$
$$(\mathrm{Id}_H \otimes \varepsilon)(\Delta(h)) = h, \quad (\varepsilon \otimes \mathrm{Id}_H)(\Delta(h)) = h, \tag{1.2.17}$$

for all $h \in H$. Moreover, Φ is a normalized 3-cocycle, in the sense that

$$\begin{aligned}(1 \otimes \Phi)(\mathrm{Id}_H \otimes \Delta \otimes \mathrm{Id}_H)(\Phi)(\Phi \otimes 1)\\ = (\mathrm{Id}_H \otimes \mathrm{Id}_H \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{Id}_H \otimes \mathrm{Id}_H)(\Phi),\end{aligned} \tag{1.2.18}$$
$$(\mathrm{Id} \otimes \varepsilon \otimes \mathrm{Id}_H)(\Phi) = 1 \otimes 1. \tag{1.2.19}$$

We use a Sweedler-like notation for both the reassociator and the coproduct map Δ. More precisely, we omit the summation and summarize the components of the tensor product, for all h in H, as $\Delta(h) = h_1 \otimes h_2$ for the coproduct and as

$$\Phi = X^1 \otimes X^2 \otimes X^3 = Y^1 \otimes Y^2 \otimes Y^3 = Z^1 \otimes Z^2 \otimes Z^3 = \cdots$$
$$\Phi^{-1} = x^1 \otimes x^2 \otimes x^3 = y^1 \otimes y^2 \otimes y^3 = z^1 \otimes z^2 \otimes z^3 = \cdots$$

for the reassociator, using upper case letters to denote Φ and lower case letters to denote its multiplicative inverse.

Example 1.2.2 Let k be a field of characteristic different from 2 and let $p_\pm = (1 \pm g)/2$. The first nontrivial example of quasi-bialgebra is the quadruple $(k[C_2], \Delta, \varepsilon, \Phi)$, where $k[C_2]$ is the group algebra associated with the cyclic group of order 2, generated by an element g, Δ is determined by $\Delta(g) = g \otimes g$, ε is determined by $\varepsilon(g) = 1$ and $\Phi = 1 \otimes 1 \otimes 1 - 2p_- \otimes p_- \otimes p_-$.

Definition 1.2.3 A quasi-bialgebra H is called a quasi-Hopf algebra if there exists an anti-algebra morphism $S : H \to H$ and elements $\alpha, \beta \in H$ such that, for all $h \in H$, we have:

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha, \quad h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{1.2.20}$$
$$X^1\beta S(X^2)\alpha X^3 = 1, \quad S(x^1)\alpha x^2\beta S(x^3) = 1. \tag{1.2.21}$$

In this case, we call (S, α, β) the antipode of H.

Remarks 1.2.4 We recall some useful properties of quasi-bialgebras and quasi-Hopf algebras, which will be of use in the later chapters.

1. We say that a quasi-bialgebra (quasi-Hopf algebra) is genuine when it is not a bialgebra (respectively Hopf algebra), meaning its reassociator Φ is different from $1 \otimes 1 \otimes 1$.

2. Deviating from Drinfeld's definition in [17], we do not require a quasi-Hopf algebra H to have a bijective antipode. However, whenever H is finite dimensional or quasi-triangular, H has a bijective antipode. We refer to [6] and [12] for all the details.

3. Differently from Hopf algebras, an antipode (S, α, β) of a quasi-Hopf algebra H is not unique. Namely, for any invertible element u in H, $(uSu^{-1}, u\alpha, \beta u^{-1})$ is an antipode of H. Moreover, it is possible to prove that all antipodes of H are related to (S, α, β) by conjugation as seen above; see [15, Proposition 3.17].

4. Consider two finitely dimensional k-vector spaces U, V; we can assume, without loss of generality, that the left (or, equivalently, right) tensor components of each element in $U \otimes V$ are linearly independent. Indeed, if we take $u_1 \otimes v_1 + \ldots + u_n \otimes v_n \in U \otimes V$, $(e_1, \ldots, e_m)$ to be the basis on U and $u_i = u_1^i e_1 + \ldots + u_m^i e_m$, we can compute

$$\begin{aligned} u_1 \otimes v_1 + \ldots + u_n \otimes v_n =& (u_1^1 e_1 + \ldots + u_m^1 e_m) \otimes v_1 + \ldots \\ & + (u_1^n e_1 + \ldots + u_m^n e_m) \otimes v_n \\ =& e_1 \otimes (u_1^1 v_1 + \ldots + u_1^n v_n) + \ldots \\ & + e_m \otimes (u_m^1 v_1 + \ldots + u_m^n v_n). \end{aligned}$$

   As an immediate consequence, we can consider all elements in the image of a coproduct Δ to have either left or right component linearly independent, without loss of generality. This will often prove useful during computations.

5. For a quasi-Hopf algebra H, we define a left integral in H as an element $t \in H$ such that $ht = \varepsilon(h)t$ for all $h \in H$. As far as we are concerned, the importance of integrals in H resides in the fact that they characterize the semisimplicity of H; more precisely, H is semisimple as an algebra if and only if there exists a left integral t in H such that $\varepsilon(t) = 1$. We refer to [40] for all the details.

   Owing to [6, Theorem 2.5], if H has bijective antipode, the space of left integrals in H is non-zero if and only if H is finite dimensional. Moreover, if it is nonzero, the space of left integrals in H has dimension one.

Definition 1.2.5 Let $(H, \Delta, \epsilon, \Phi)$ and $(H, \Delta', \epsilon', \Phi')$ be quasi-bialgebras. A quasi-bialgebra morphism $f : (H, \Delta, \epsilon, \Phi) \to (H, \Delta', \epsilon', \Phi')$ is an algebra map $f : H \to H'$ such that

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \quad \epsilon' \circ f = \epsilon \quad \text{and} \quad (f \otimes f \otimes f)(\Phi) = \Phi'.$$

When f is invertible, we say that it is a quasi-bialgebra isomorphism.

Definition 1.2.6 Let $(H, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ and $(H', \Delta', \epsilon', \Phi', S', \alpha', \beta')$ be quasi-Hopf algebras. A quasi-Hopf algebra morphism

$$f : (H, \Delta, \epsilon, \Phi, S, \alpha, \beta) \to (H, \Delta', \epsilon', \Phi', S', \alpha', \beta')$$

is a quasi-bialgebra morphism $f : (H, \Delta, \epsilon, \Phi) \to (H, \Delta', \epsilon', \Phi')$ such that

$$S' \circ f = f \circ S, \quad f(\alpha) = \alpha \quad \text{and} \quad f(\beta) = \beta.$$

When f is invertible, we say that it is a quasi-Hopf algebra isomorphism.

One important property of the structure of a quasi-Hopf algebra H is the stability under deformation by an arbitrary invertible element F in $H \otimes H$; for technical reasons, we require here F to also be "normalized" with respect to the counit ϵ. This is a remarkable departure from the behavior of Hopf algebras, whose structure is stable only under twist deformation by an invertible element in $H \otimes H$ satisfying 2-cocycle-like properties. Indeed, twisting a Hopf algebra structure by a general invertible element results in a quasi-Hopf algebra.

Proposition 1.2.7 *Let H be a quasi-bialgebra and let $F = F^1 \otimes F^2$ be an invertible element in $H \otimes H$, with inverse $F^{-1} = G^1 \otimes G^2$, such that $\epsilon(F^1)F^2 = 1 = \epsilon(F^2)F^1$. Define*

$$\Delta_F : H \to H \otimes H : h \mapsto F\Delta(h)F^{-1}, \tag{1.2.22}$$
$$\Phi_F := (1 \otimes F)(id_H \otimes \Delta)(F)\Phi(\Delta \otimes id_H)(F^{-1})(F^{-1} \otimes 1). \tag{1.2.23}$$

Then $H_F = (H, \Delta_F, \epsilon, \Phi_F)$ is a quasi-bialgebra.

Proof. It is easy to check that H_F satisfies all quasi-bialgebra axioms; we refer to [15, Proposition 3.5] for all the computational details. $\square$

Corollary 1.2.8 *In the notations of Proposition 1.2.7, if H is a quasi-Hopf algebra, then H_F is also a quasi-Hopf algebra.*

Proof. One can directly check that $(H, \Delta_F, \epsilon, \Phi_F)$ admits the antipode (S_F, α_F, β_F) defined by

$$S_F = S, \quad \alpha_F = S(G^1)\alpha G^2 \text{ and } \beta_F = F^1\beta S(F^2). \tag{1.2.24}$$

$\square$

Definition 1.2.9 We call two quasi-bialgebras or quasi-Hopf algebras H and H' twist equivalent if there exist an invertible element F in $H \otimes H$ such that H' is isomorphic to H_F.

We present some important identities holding for a quasi-Hopf algebra H with invertible antipode, relating the following elements

$$p_R = p^1 \otimes p^2 = x^1 \otimes x^2\beta S(x^3), \tag{1.2.25}$$
$$q_R = q^1 \otimes q^2 = X^1 \otimes S^{-1}(\alpha X^3)X^2, \tag{1.2.26}$$
$$p_L = \tilde{p}^1 \otimes \tilde{p}^2 = X^2 S^{-1}(X^1\beta) \otimes X^3, \tag{1.2.27}$$
$$q_L = \tilde{q}^1 \otimes \tilde{q}^2 = S(x^1)\alpha x^2 \otimes x^3. \tag{1.2.28}$$

The following relations will be used extensively in Chapter 2.

Proposition 1.2.10 *Let H be a quasi-Hopf algebra; whenever S^{-1} appears, we assume H to have invertible antipode. For all $h \in H$, we then have*

$$\Delta(h_1)p_R(1 \otimes S(h_2)) = p_R(h \otimes 1), \tag{1.2.29}$$
$$(1 \otimes S^{-1}(h_2))q_R\Delta(h_1) = (h \otimes 1)q_R, \tag{1.2.30}$$
$$\Delta(h_2)p_L(S^{-1}(h_1) \otimes 1_H) = p_L(1 \otimes h), \tag{1.2.31}$$
$$(S(h_1) \otimes 1)q_L\Delta(h_2) = (1 \otimes h)q_L, \tag{1.2.32}$$

Furthermore, the following relations hold:

$$(1 \otimes S^{-1}(p^2))q_R\Delta(p^1) = 1 \otimes 1, \tag{1.2.33}$$
$$\Delta(q^1)p_R(1 \otimes S(q^2)) = 1 \otimes 1, \tag{1.2.34}$$
$$(S(\tilde{p}^1) \otimes 1)q_L\Delta(\tilde{p}^2) = 1 \otimes 1, \tag{1.2.35}$$
$$\Delta(\tilde{q}^2)p_L(S^{-1}(\tilde{q}^1) \otimes 1) = 1 \otimes 1. \tag{1.2.36}$$

1.2.1 A few examples

Next, we provide some useful examples of quasi-Hopf algebras, each of which will play a role in a later chapter.

A (trivial) example of quasi-Hopf algebra is given by any Hopf algebra with unit 1, by taking $1 \otimes 1 \otimes 1$ as reassociator and the unit as both the distinguished elements of the antipode; the rest of the structure is induced by the structure maps of the Hopf algebra. In particular, the first example of a non-commutative and non-cocommutative Hopf algebra is the Sweedler's Hopf algebra H_4.

Definition 1.2.11 The Sweedler's Hopf algebra H_4 is a unital 4-dimensional algebra generated by g, x with relations $g^2 = 1$, $x^2 = 0$, $xg = -gx$, whose Hopf algebra structure is determined by

$$\begin{gathered}\Delta(g) = g \otimes g,\ \Delta(x) = g \otimes x + x \otimes 1,\\ \varepsilon(g) = 1,\ \varepsilon(x) = 0,\\ S(g) = g \text{ and } S(x) = -gx.\end{gathered}$$

The study of H_4 in the context of twist-deformation will be the focus of Section 3.2 in Chapter 3.

Let G be a finite group and k^G the Hopf algebra of functions on G, meaning the set of algebra maps $k[G] \to k$ under pointwise multiplication, where $k[G]$ denotes the group algebra of G in k and the unit of k^G is the constant map 1_k. For $\omega \in Z^3(G, k^*)$ a normalized 3-cocycle of G with values in k^*, define $\Phi_\omega \in k^G \otimes k^G \otimes k^G$ by

$$\Phi_\omega = \sum_{g_1, g_2, g_3 \in G} \omega(g_1, g_2, g_3)\ P_{g_1} \otimes P_{g_2} \otimes P_{g_3}, \tag{1.2.37}$$

where $\{P_g\}_{g \in G}$ is the basis of k^G dual to the basis $\{g\}_{g \in G}$ of the group Hopf algebra $k[G]$. Then k^G, endowed with the (co)multiplication and (co)unit coming from the (co)algebra structure of k^G, can also be seen as a quasi-bialgebra with reassociator Φ_ω. From here on, we denote this quasi-bialgebra structure on k^G by k^G_ω; it is easy to check that k^G_ω is, moreover, a quasi-Hopf algebra, with antipode determined by the triple (S, α, β) consisting of $S(P_g) = P_{g^{-1}}$, for all $g \in G$, $\alpha = \varepsilon$, the counit of $k[G]$, and β given by $\beta(g) = \omega(g, g^{-1}, g)^{-1}$, for all $g \in G$. Sometimes, in order to avoid an excessively heavy notation, we will write (k^G, Φ_ω) to address k^G_ω.

Take $G = C_n$, where C_n denotes the cyclic group of order $n \geq 2$, and denote by g its generator. Suppose that there exist a n-th root of the unit ξ in k. We can identify k^{C_n} with the group algebra $k[C_n]$, via the correspondence $P_g^j \mapsto 1_j$, for $j \in \{0, \dots, n-1\}$, where $1_j := \frac{1}{n} \sum_{t=0}^{n-1} \xi^{-tj} g^t$ defines a complete system of orthogonal idempotents in $k[C_n]$; see [15, Proposition 3.57]. Under this identification, a normalized 3-cocycle ω endows $k[C_n]$ with a quasi Hopf algebra structure, from here on denoted by $k_\omega[C_n]$. In general, this same construction works for an arbitrary finite abelian group G. We will see this in detail at the start of Section 2.2.1.

Consider now $n = 2$; it is a well known fact that, up to cohomological equivalence, the only non-trivial normalized 3-cocycle in $Z^3(C_2, k^*)$ is $\overline{\omega} : (g^a, g^b, g^c) \mapsto (-1)^{a\lfloor \frac{b+c}{2} \rfloor}$, where $\lfloor r \rfloor$ denotes the integer part of a rational number r. Thus, we recover the simplest non-trivial example of quasi-Hopf algebra, $H(2) := k_{\overline{\omega}}[C_2]$, described in Example 1.2.2.

1.3 Yetter-Drinfeld modules

The categorical setting of large part of the results in this thesis is the category of left Yetter-Drinfeld modules over a quasi-bialgebra H, denoted by ${}_H^H\mathcal{Y}D$, introduced for both the Hopf and quasi-Hopf case by S. Majid in [37].

The Yetter-Drinfeld modules form a subcategory of the left modules over H, and can be realized as the categorical center of such category; we refer to [15, Section 8] for all the details on this construction. In actuality, there are 4 types of Yetter-Drinfeld modules, depending on whether we use the left or right center construction of the left or right modules over H. We will focus for now only on the left-left version.

Definition 1.3.1 Let H be a quasi-bialgebra. The objects of ${}_H^H\mathcal{Y}D$ are left H-modules M endowed with a left H-coaction, denoted by $\lambda_M : M \to H \otimes M$, with $\lambda_M(m) = m_{[-1]} \otimes m_{[0]}$, such that $\varepsilon(m_{[-1]})m_{[0]} = m$ and, for all $m \in M$,

$$\begin{aligned} X^1 m_{[-1]} \otimes (X^2 \cdot m_{[0]})_{[-1]} X^3 \otimes (X^2 \cdot m_{[0]})_{[0]} \\ = X^1 (Y^1 \cdot m)_{[-1]_1} Y^2 \otimes X^2 (Y^1 \cdot m)_{[-1]_2} Y^3 \otimes X^3 \cdot (Y^1 \cdot m)_{[0]}. \end{aligned} \tag{1.3.38}$$

It is compatible with the left H-module structure on M, in the sense that

$$h_1 m_{[-1]} \otimes h_2 \cdot m_{[0]} = (h_1 \cdot m)_{[-1]} h_2 \otimes (h_1 \cdot m)_{[0]}, \tag{1.3.39}$$

for all $h \in H$ and $m \in M$.

Remark 1.3.2 We recall a few basic facts on Yetter-Drinfeld modules over quasi-bialgebras.

1. ${}_H^H\mathcal{Y}D$ is a pre-braided monoidal category, with pre-braiding:

   $$M \otimes N \to N \otimes M : m \otimes n \mapsto m_{(-1)} \cdot n \otimes m_{(0)};$$

   moreover, when H is a quasi-Hopf algebra with bijective antipode, ${}_H^H\mathcal{Y}D$ is a braided monoidal category; see [15, Proposition 8.12].

2. The monoidal structure on ${}_H^H\mathcal{Y}D$, induced by the left center construction on ${}_H\mathcal{M}$, is such that the forgetful functor ${}_H^H\mathcal{Y}D \to {}_H\mathcal{M}$ is strong monoidal; in particular, given two modules M, N in ${}_H^H\mathcal{Y}D$, the H-coaction $\lambda_{M\otimes N}$ on M, N is defined, for each $m \in M$ and $n \in N$, by

   $$\begin{aligned} \lambda_{M\otimes N}(m \otimes n) = X^1 (x^1 Y^1 \cdot m)_{[-1]} x^2 (Y^2 \cdot n)_{[-1]} Y^3 \\ \otimes X^2 \cdot (x^1 Y^1 \cdot m)_{[0]} \otimes X^3 x^3 \cdot (Y^2 \cdot n)_{[0]}. \end{aligned} \tag{1.3.40}$$

3. The main difference in the monoidal structure of ${}_H^H\mathcal{Y}D$ between the cases in which H is a Hopf or a quasi-Hopf algebra lies in the non-strictness of the category. When H is a quasi-Hopf algebra with reassociator $\Phi = X^1 \otimes X^2 \otimes X^3$, the Yetter-Drinfeld modules ${}_H^H\mathcal{Y}D$ inherits the associator $a_{U,V,W}$, where U, V, W are left (Yetter-Drinfeld) H-modules, from the left modules ${}_H\mathcal{M}$; namely, for all u, v, w in U, V and W respectively, $a_{U,V,W}$ is determinde by Φ as follows:

   $$a_{M,N,V}((u \otimes v) \otimes w) \mapsto \Phi \cdot (u \otimes (v \otimes w)) = X^1 \cdot u \otimes (X^2 \cdot v \otimes X^3 \cdot w).$$

   The left and right unitors are trivial in both ${}_H\mathcal{M}$ and ${}_H^H\mathcal{Y}D$.

In this and the following chapters, we will often call an algebra, coalgebra, and so on, in ${}^H_H\mathcal{Y}D$ a Yetter-Drinfeld algebra, coalgebra and so on. This terminology will sometimes be abbreviated to YD-algebra, coalgebra, et cetera.

Proposition 1.3.3 *Let B be a Yetter-Drinfeld module on a quasi-Hopf algebra H. We can characterize braided structures on ${}^H_H\mathcal{Y}D$ as follows:*

1. *B is a YD-algebra when equipped with a unital multiplication $\underline{m}_B : B\otimes B \to B : b\otimes b' \mapsto bb'$ such that*

$$\begin{aligned} h\cdot(\mathfrak{b}\mathfrak{b}') = (h_1\cdot\mathfrak{b})(h_2\cdot\mathfrak{b}'), \quad h\cdot 1_B = \varepsilon(h)1_B\ , \\ 1_{B[-1]}\otimes 1_{B[0]} = 1_H\otimes 1_B, \end{aligned} \tag{1.3.41}$$

$$(\mathfrak{b}\mathfrak{b}')\mathfrak{b}'' = (X^1\cdot\mathfrak{b})[(X^2\cdot\mathfrak{b}')(X^3\cdot\mathfrak{b}'')], \tag{1.3.42}$$

$$\begin{aligned} (\mathfrak{b}\mathfrak{b}')_{[-1]}\otimes(\mathfrak{b}\mathfrak{b}')_{[0]} = X^1(x^1Y^1\cdot\mathfrak{b})_{[-1]}x^2(Y^2\cdot\mathfrak{b}')_{[-1]}Y^3 \\ \otimes(X^2\cdot(x^1Y^1\cdot\mathfrak{b})_{[0]})(X^3x^3\cdot(Y^2\cdot\mathfrak{b}')_{[0]}), \end{aligned} \tag{1.3.43}$$

   for all $h\in H$ and $\mathfrak{b},\mathfrak{b}',\mathfrak{b}''\in B$, where $1_B\in B$ denotes the unit of B.

2. *A YD-coalgebra on B is obtained by endowing it with a comultiplication $\underline{\Delta}_B : B\to B\otimes B : b\mapsto b_{\underline{1}}\otimes b_{\underline{2}}$ and counit $\underline{\varepsilon}_B : B\to k$ such that, for all $b\in B$,*

$$\begin{aligned} \underline{\Delta}_B(h\cdot\mathfrak{b}) = h_1\cdot\mathfrak{b}_{\underline{1}}\otimes h_2\cdot\mathfrak{b}_{\underline{2}}, \quad \underline{\varepsilon}_B(h\cdot\mathfrak{b}) = \varepsilon(h)\underline{\varepsilon}_B(\mathfrak{b}), \\ \underline{\varepsilon}_B(\mathfrak{b}_{\underline{1}})\mathfrak{b}_{\underline{2}} = \mathfrak{b} = \underline{\varepsilon}_B(\mathfrak{b}_{\underline{2}})\mathfrak{b}_{\underline{1}}, \end{aligned} \tag{1.3.44}$$

$$X^1\cdot\mathfrak{b}_{\underline{11}}\otimes X^2\cdot\mathfrak{b}_{\underline{12}}\otimes X^3\cdot\mathfrak{b}_{\underline{2}} = \mathfrak{b}_{\underline{1}}\otimes\mathfrak{b}_{\underline{21}}\otimes\mathfrak{b}_{\underline{22}}, \tag{1.3.45}$$

$$\begin{aligned} \mathfrak{b}_{[-1]}\otimes\mathfrak{b}_{[0]_{\underline{1}}}\otimes\mathfrak{b}_{[0]_{\underline{2}}} = X^1(x^1Y^1\cdot\mathfrak{b}_{\underline{1}})_{[-1]}x^2(Y^2\cdot\mathfrak{b}_{\underline{2}})_{[-1]}Y^3 \\ \otimes X^2\cdot(x^1Y^1\cdot\mathfrak{b}_{\underline{1}})_{[0]}\otimes X^3x^3\cdot(Y^2\cdot\mathfrak{b}_{\underline{2}})_{[0]}, \\ \underline{\varepsilon}_B(\mathfrak{b}_{[0]})\mathfrak{b}_{[-1]} = \underline{\varepsilon}_B(\mathfrak{b})1, \end{aligned} \tag{1.3.46}$$

   where we used the notation

$$\begin{aligned} \mathfrak{b}_{\underline{11}}\otimes\mathfrak{b}_{\underline{12}}\otimes\mathfrak{b}_{\underline{2}} := (\underline{\Delta}_B\otimes\mathrm{Id}_B)(\underline{\Delta}_B(\mathfrak{b})), \\ \mathfrak{b}_{\underline{1}}\otimes\mathfrak{b}_{\underline{21}}\otimes\mathfrak{b}_{\underline{22}} := (\mathrm{Id}_B\otimes\underline{\Delta}_B)(\underline{\Delta}_B(\mathfrak{b})). \end{aligned}$$

3. *A bialgebra in ${}^H_H\mathcal{Y}D$ on B is a YD-algebra on the module B that is at the same time a YD-coalgebra such that the comultiplication $\underline{\Delta}_B$ and the counit $\underline{\varepsilon}_B$ are algebra morphisms in ${}^H_H\mathcal{Y}D$, where $B\otimes B$ is considered an algebra in ${}^H_H\mathcal{Y}D$ via the tensor product algebra structure defined in (1.1.13). Explicitly, for all $b,b'\in B$, the following relations must hold:*

$$\underline{\Delta}_B(1_B) = 1_B\otimes 1_B, \quad \underline{\varepsilon}_B(1_B) = 1_k, \quad \underline{\varepsilon}_B(\mathfrak{b}\mathfrak{b}') = \underline{\varepsilon}_B(\mathfrak{b})\underline{\varepsilon}_B(\mathfrak{b}'), \tag{1.3.47}$$

$$\begin{aligned} \underline{\Delta}_B(\mathfrak{b}\mathfrak{b}') = (y^1X^1\cdot\mathfrak{b}_{\underline{1}})(y^2Y^1(x^1X^2\cdot\mathfrak{b}_{\underline{2}})_{[-1]}x^2X^3_1\cdot\mathfrak{b}'_{\underline{1}}) \\ \otimes(y^3_1Y^2\cdot(x^1X^2\cdot\mathfrak{b}_{\underline{2}})_{[0]})(y^3_2Y^3x^3X^3_2\cdot\mathfrak{b}'_{\underline{2}}). \end{aligned} \tag{1.3.48}$$

Proof. Follows directly from Section 1.1.3, by computing the relations displayed by the diagram in the section using the monoidal structure of ${}^H_H\mathcal{Y}D$, as described in Remark 1.3.2.

Regarding the YD-algebra case, note that (1.3.41) expresses the H-linearity of the multiplication and unit morphisms of B, (1.3.42) is just the associativity of $\underline{m}_B$

in ${}_H\mathcal{M}$ and ${}^H_H\mathcal{Y}D$ and (1.3.43) is the H-colinearity of $\underline{m}_B$ in ${}^H_H\mathcal{Y}D$; see (1.3.40) for the coaction of the tensor product. Note that (1.3.41, 1.3.42) are the required condition on B for it to be an algebra in ${}_H\mathcal{M}$ or, equivalently, an H-module algebra.

Likewise, in the braided coalgebra case, observe that (1.3.44) expresses the fact that $\underline{\Delta}_B$ and $\underline{\varepsilon}_B$ are left H-linear and $\underline{\varepsilon}_B$ is counit for $\underline{\Delta}_B$. In other words, B is a left H-module coalgebra, that is, a coalgebra in ${}_H\mathcal{M}$. The relation (1.3.45) corresponds to the coassociativity of $\underline{\Delta}_B$ in ${}_H\mathcal{M}$ or, equivalently, in ${}^H_H\mathcal{Y}D$, while (1.3.46) expresses the left H-colinearity of $\underline{\Delta}_B$ and $\underline{\varepsilon}_B$ in ${}^H_H\mathcal{Y}D$.

The last point follow similarly, by computing (1.3.48) from (1.1.13). □

In conclusion, we recall by Definition 1.1.18, translated to the setting of Proposition 1.3.3, that a Hopf algebra in ${}^H_H\mathcal{Y}D$ is a YD-bialgebra B for which there exists a morphism $\underline{S}_B : B \to B$ in ${}^H_H\mathcal{Y}D$, called antipode, such that $\underline{S}_B(b_{\underline{1}})b_{\underline{2}} = \underline{\varepsilon}_B(b)1_B = b_{\underline{1}}\underline{S}(b_{\underline{2}})$, for all $b \in B$.

1.4 Biproduct quasi-Hopf algebras

While the research contained in this thesis mainly focuses on classifying YD-Hopf algebras, we are interested in constructing regular quasi-Hopf algebras, for the sake of low-dimensional classification and to be used as examples. The bridge covering the gap between these two structures is the biproduct construction.

It was proven by D. E. Radford in [43] that any bialgebra with a projection over one of its sub-bialgebras can be characterized as a biproduct between a braided bialgebra and the sub-bialgebra; as we are about to see, a biproduct, in the sense of Radford, is a bialgebra structure over the tensor product between the sub-bialgebra and the braided bialgebra.

The same result also holds for quasi-bialgebras and quasi-Hopf algebras; owing to [5, 41, 11], we recall the biproduct construction, piece by piece.

Proposition 1.4.1 *Let B be a module over a quasi-Hopf algebra H.*

- *If B is a H-module algebra (or, in particular, a YD-algebra over H), we can associate to it a k-algebra $B\#H$, called the smash product of B and H. As a vector space $B\#H$ equals $B \otimes H$, while its multiplication is defined by*

$$(\mathfrak{b}\#h)(\mathfrak{b}'\#h') = (x^1 \cdot \mathfrak{b})(x^2 h_1 \cdot \mathfrak{b}')\#x^3 h_2 h',$$

  for all $\mathfrak{b}, \mathfrak{b}' \in B$ and $h, h' \in H$. This multiplication is unital, with unit $1_B \times 1$.

- *If B is a YD-coalgebra (so, in particular, it is a YD-module), we can associate to it the k-coalgebra $B \rtimes H$, called the smash product coalgebra of B and H, built on the k-vector space $B \otimes H$ as follows, where we write $\mathfrak{b} \rtimes h$ in place of $\mathfrak{b} \otimes h$ to distinguish this coalgebra structure from $B \otimes H$.*

$$\begin{aligned}\underline{\Delta}(\mathfrak{b} \rtimes h) = y^1X^1 \cdot \mathfrak{b}_{\underline{1}} \rtimes y^2Y^1(x^1X^2 \cdot \mathfrak{b}_{\underline{2}})_{[-1]}x^2X^3_1h_1 \\ \otimes y^3_1Y^2 \cdot (x^1X^2 \cdot \mathfrak{b}_{\underline{2}})_{[0]} \rtimes y^3_2Y^3x^3X^3_2h_2\end{aligned} \tag{1.4.49}$$

  and $\underline{\varepsilon}(\mathfrak{b} \rtimes h) = \underline{\varepsilon}_B(\mathfrak{b})\varepsilon(h)$, for all $\mathfrak{b} \in B$ and $h \in H$.

- *Last but not least, if B is a YD-bialgebra, $B\#H$ and $B \bowtie H$ determine a quasi-bialgebra structure on $B \otimes H$, with reassociator*

$$(1_B \otimes X^1) \otimes (1_B \otimes X^2) \otimes (1_B \otimes X^3).$$

The structure is denoted in what follows by $B \times H$ and called the biproduct quasi-bialgebra of B and H. Moreover, if B is a YD-Hopf algebra with antipode $\underline{S}_B$, $B \times H$ is a quasi-Hopf algebra with antipode defined by

$$\begin{aligned}\mathcal{S}(\mathfrak{b} \times h) = (1_B \times S(X^1 x^1_1 \mathfrak{b}_{[-1]} h)\alpha)& \\ (X^2 x^1_2 \cdot \underline{S}_B(\mathfrak{b}_{[0]}) \times X^3 x^2 \beta S(x^3))&\end{aligned} \tag{1.4.50}$$

and by the distinguished elements $1_B \times \alpha$ and $1_B \times \beta$.

Proof. We refer to [5, 41] for the proof of the first two points, and to [11] for the proof of the last. □

As peviously stated and similarly to the Hopf case, according to [11] biproduct quasi-Hopf algebras characterize quasi-Hopf algebras with a projection. This means that, if there exist quasi-Hopf algebra morphisms $H \underset{\pi}{\overset{i}{\rightleftarrows}} A$ such that $\pi i = \mathrm{Id}_H$, then there exists a Hopf algebra B in ${}^H_H\mathcal{YD}$ such that A is isomorphic to $B \times H$ as a quasi-Hopf algebra. Conversely, it is clear that any biproduct quasi-Hopf algebra $B \times H$ has a projection on H, namely with $i : H \to B \times H : h \mapsto 1 \times h$ and $\pi : B \times H \to H : b \times h \mapsto \underline{\epsilon}_B(b)h$.

1.4.1 The graded algebra

We conclude this section by recalling the construction of the graded algebra associated to a quasi-Hopf algebra; we will see how, in some cases, the graded algebra can be characterized as a biproduct.

Let A be a finite dimensional nonsemisimple quasi-Hopf algebra, meaning that the underlying algebra structure is nonsemisimple, and denote by J the Jacobson radical of A. Then $J \neq 0$ is a nilpotent ideal of A and the quotient algebra A/J is semisimple. For a positive integer m, denote $A_m := J^m/J^{m+1}$ and, by convention, $J^0 = A$. Denote by $\mathrm{gr}(A) = \bigoplus_{m\geq 0} A_m$ the graded algebra associated to the filtration of A given by the powers of J.

We say that a quasi-Hopf algebra A is basic if all the irreducible representations of A are 1-dimensional or, equivalently, $A/J \simeq k \times \cdots \times k$ as an algebra.

Definition 1.4.2 Let H be a quasi-Hopf algebra. An ideal I of the underlying algebra structure of H is called a quasi-Hopf ideal if $\Delta(I) \subseteq I \otimes H + H \otimes I$, $\varepsilon(I) = 0$ and $S(I) \subseteq I$.

As the intuition suggests, the definition of a quasi-Hopf ideal is such that the quotient H/I admits a unique quasi-Hopf algebra structure such that the canonical projection $\pi : A \to A_0$ turns into a quasi-Hopf algebra morphism; we refer to [15, Proposition 3.29] for further details. The following has an identical proof as [28, Lemma 1.1], its Hopf version.

Proposition 1.4.3 *If a quasi-Hopf algebra H is a basic algebra, its Jacobson radical J is a quasi-Hopf ideal of H.*

Let A be a finite dimensional nonsemisimple quasi-Hopf algebra which is also basic; we want to describe A_0. Since J is a quasi-Hopf ideal of A, the quotient $A_0 = A/J$ is a quasi-Hopf algebra. Since A is basic, and so $A_0 \simeq k \times \cdots \times k$ as algebras, it follows that $A_0 \simeq k^G \simeq k[G]$, for some finite abelian group G, and therefore A_0 is a usual group Hopf algebra viewed as a quasi-Hopf algebra via a normalized 3-cocycle ϕ of G. In other words, $A_0 = k_\phi[G]$ and the reassociator Φ of A projects through $\pi^{\otimes 3}$ onto ϕ.

Coming back to the graded algebra, $\mathrm{gr}(A)$ admits a quasi-Hopf algebra structure such that the canonical linear maps $A_0 \underset{p}{\overset{i}{\rightleftarrows}} \mathrm{gr}(A)$ are quasi-Hopf algebra morphisms obeying $pi = \mathrm{Id}_{A_0}$.

Namely, by denoting as $p_m : J^m \to A_m$ the canonical k-linear projection, so that $p_0 = p$, the quasi-coalgebra structure of $\mathrm{gr}(A)$ is defined, for all $x \in J^m$, $m \geq 0$, by

$$\Delta_m : A_m \to \sum_{s+t=m} A_s \otimes A_t : p_m(x) \mapsto \sum_{s+t=m} (p_s \otimes p_t)\Delta(x);$$

note that Δ_m is well defined since one can show by induction that

$$\Delta(J^m) \subseteq \sum_{s+t=m} J^s \otimes J^t, \quad J^{m+1} \subseteq \mathrm{Ker}(\sum_{s+t=m} (p_s \otimes p_t)\Delta).$$

Likewise, the antipode of $\mathrm{gr}(A)$ is given, for all $x \in J^m$, $m \geq 0$, by

$$S_m : A_m \to A_m : p_m(x) \mapsto p_m(S(x)).$$

The other structures of $\mathrm{gr}(A)$ are induced by the quasi-Hopf algebra structure of A_0, as a substructure of $\mathrm{gr}(A)$.

In conclusion, since $gr(A)$ is a quasi-Hopf algebra with a projection on A_0, we know that there is a braided Hopf algebra B in ${}_{A_0}^{A_0}\mathcal{YD}$ such that $\mathrm{gr}(A) \simeq B \times A_0$, as quasi-Hopf algebras; in particular, we get that $\dim_k A_0$ divides $\dim_k \mathrm{gr}(A)$; this fact will be used extensively while proving the main classification results in chapters 3 and 4.

Chapter 2

Biproduct quasi-Hopf algebras of rank 2

In this chapter, we classify all rank 2 biproducts over a quasi-Hopf algebra H with invertible antipode, over a field k of characteristic different from 2. Thus, we provide a way to produce examples of quasi-Hopf algebras from previously known ones, to be used to test general theories (see for instance [13]). Computing such concrete structures is the focus of Section 2.2 and Section 2.3.

2.1 Two-dimensional braided Hopf algebras

Inspired by the Hopf case [45], we will characterize the 2-dimensional Hopf algebras B in ${}_H^H\mathcal{Y}D$. To reach this objective, we start by describing the YD-algebras of dimension 2 in ${}_H^H\mathcal{Y}D$.

First, let B be a 2-dimensional vector space with basis $\{1, n\}$ that admits a left H-module algebra structure. Since $n^2 + rn + s1 = 0$ for some $r, s \in k$, as in [45], one can assume without loss of generality that $n^2 = \omega 1$ for a certain $\omega \in k$, by replacing n with $m := n + \frac{r}{2}1$; then $\{1, m\}$ is a basis of B such that $m^2 = \omega 1$, where $\omega = \frac{r^2}{4} - s$.

If $\cdot : H \otimes B \to B$ provides the left H-module structure on B, it follows that the H-action is determined by two k-linear maps $\sigma, \psi : H \to k$, this means elements in the dual space H^* of H, in the sense that, for all $h \in H$, the action is trivial on the unit, meaning $h \cdot 1 = \varepsilon(h)1$, and

$$h \cdot n = \psi(h)1 + \sigma(h)n. \tag{2.1.1}$$

Remark 2.1.1 We endow H^* with a possibly non-associative multiplication given by the convolution product $(\sigma\psi)(h) = \sigma(h_1)\psi(h_2)$, for all $\sigma, \psi \in H^*$; it is unital with unit ε, the counit of H.

Lemma 2.1.2 *There is a* 1*-to-*1 *correspondence between H-module algebra structures on a* 2*-dimensional vector space B with basis $\{1, n\}$ as above ($n^2 = \omega 1$ for a certain $\omega \in k$) and couples $(\sigma, \psi) \in H^* \times H^*$, such that σ is a k-algebra map and,* *for all $h, h' \in H$, the following relations are satisfied:*

$$\psi(hh') = \varepsilon(h)\psi(h') + \psi(h)\sigma(h') \ , \ \psi\sigma = -\sigma\psi \ , \ \psi^2 = \omega(\varepsilon - \sigma^2), \quad (2.1.2)$$

$$\begin{aligned} &\psi(X^1)\psi(X^2)\psi(X^3) + \omega[\psi(X^1)\sigma(X^2X^3) + \\ &\qquad + \sigma(X^1X^2)\psi(X^3) + \sigma(X^1X^3)\psi(X^2)] = 0, \end{aligned} \quad (2.1.3)$$

$$\begin{aligned} \omega[1 - \sigma(X^1X^2X^3)] &= \psi(X^1)\psi(X^2)\sigma(X^3) + \\ &+ \sigma(X^1)\psi(X^2)\psi(X^3) + \psi(X^1)\sigma(X^2)\psi(X^3). \end{aligned} \quad (2.1.4)$$

Proof. Recall (2.1.1); since $h\cdot(h'\cdot n) = (hh')\cdot n$, for all $h, h' \in H$, and $\{1, n\}$ is a basis of B, it follows that σ is an algebra map and ψ is an (ε, σ)-derivation (meaning the first equality in (2.1.2) holds). These two conditions on (σ, ψ) characterize H-module structures on B.

We see now when B is, moreover, an H-module algebra. Looking at the associativity of the multiplication $\underline{m}_B$ of B, by counitality of H, one can show that (1.3.42) holds if and only if it holds for $\mathfrak{b} = \mathfrak{b}' = \mathfrak{b}'' = n$; the latter is equivalent to

$$\omega n = [\psi(X^1)1 + \sigma(X^1)n][\psi(X^2)1 + \sigma(X^2)n][\psi(X^3)1 + \sigma(X^3)n],$$

and a simple computation ensures us that this happens only if (2.1.3, 2.1.4) are satisfied.

Finally, similar to the computation performed in [45, Proposition 1], the action-product compatibility condition (1.3.41) is satisfied if and only if it is verified, for all $h \in H$, by $\mathfrak{b} = \mathfrak{b}' = n$; in turn, in this case, the condition is equivalent to $\psi\sigma + \sigma\psi = 0$ and $\psi^2 + \omega\sigma^2 = \omega\varepsilon$. □

We now move to analize the coalgebra case. Assume from now on that B is a 2-dimensional H-module algebra, using the same notation as in Lemma 2.1.2.

Lemma 2.1.3 *Let B be as in the above and assume that B admits a left H-module coalgebra structure given by $(\underline{\Delta}_B, \underline{\varepsilon}_B)$ such that $\underline{\Delta}_B(1) = 1 \otimes 1$ and $\underline{\varepsilon}_B(1) = 1_k$. Then $\psi = 0$, $\omega(\sigma^2 - \varepsilon) = 0$, $\omega[1 - \sigma(X^1X^2X^3)] = 0$, and the coalgebra structure of B is completely determined by two elements $b, b' \in B$ obeying the following conditions:*

$$\underline{\Delta}_B(b) = b \otimes 1 + b' \otimes b, \ \underline{\Delta}_B(b') = b' \otimes b', \ \underline{\varepsilon}_B(b) = 0, \ \underline{\varepsilon}_B(b') = 1_k, \quad (2.1.5)$$

$$h \cdot b = \sigma(h)b, \ h \cdot b' = \varepsilon(h)b', \quad (2.1.6)$$

for all $h \in H$. In addition, $n = b + \underline{\varepsilon}_B(n)b'$ and $\underline{\Delta}_B(n) = b \otimes 1 + b' \otimes n$.

Proof. The proof is similar to the one given in [45, Corollary 1]; the major difference lies in the additional complications that arise in proving $\psi = 0$.

We write $\underline{\Delta}_B(n) = b \otimes 1 + b' \otimes n$ for some $b, b' \in B$. From the counit properties, we get that $\underline{\varepsilon}_B(b) = 0$, $\underline{\varepsilon}_B(b') = 1_k$ and $n = b + \underline{\varepsilon}_B(n)b'$. Now, the coassociativity of $\underline{\Delta}_B$ is fulfilled if and only if (1.3.45) is verified for $\mathfrak{b} = n$, which is the same as

$$\begin{aligned} &\underline{\Delta}_B(b) = b \otimes 1 + b' \otimes b + \psi(x^3)x^1 \cdot b' \otimes x^2 \cdot b', \\ &\underline{\Delta}_B(b') = \sigma(x^3)x^1 \cdot b' \otimes x^2 \cdot b'. \end{aligned}$$

Likewise, (1.3.44) holds if and only if it holds for $\mathfrak{b} = n$ or, equivalently, if

$$\begin{aligned} \psi(h)1 + \sigma(h)b &= h \cdot b + \psi(h_2)h_1 \cdot b' \\ \text{and} \ \ \sigma(h)b' &= \sigma(h_2)h_1 \cdot b', \end{aligned} \quad (2.1.7)$$

for all $h \in H$. We claim that the two conditions above are equivalent to the fact that $h \cdot b' = \varepsilon(h)b'$, $h \cdot b = \sigma(h)b$, for all $h \in H$, and $\psi = 0$. Indeed, for q_R as in (1.2.26), we compute that, for all $h \in H$,

$$\begin{aligned}\sigma(S^{-1}(h_2))\sigma(q^2h_{12})q^1h_{11}\cdot b' &= \sigma(S^{-1}(h_2)q^2h_{12})q^1h_{11}\cdot b'\\ &\overset{(1.2.30)}{=} \sigma(q^2)hq^1\cdot b'.\end{aligned}$$

On the other hand, by using the fact that σ is an algebra morphism and the second relation in (2.1.7), we get that

$$\begin{aligned}\sigma(S^{-1}(h_2))\sigma(q^2h_{12})q^1h_{11}\cdot b' &= \sigma(S^{-1}(h_2))\sigma(q^2)q^1\cdot(\sigma(h_{12})h_{11}\cdot b')\\ &\overset{(2.1.7)}{=} \sigma(S^{-1}(h_2))\sigma(q^2)\sigma(h_1)q^1\cdot b'\\ &\overset{(1.2.26)}{=} \sigma(S^{-1}(S(h_1)\alpha h_2))\sigma(S^{-1}(X^3)X^2)X^1\cdot b'\\ &\overset{(1.2.20)}{=} \varepsilon(h)\sigma(q^2)q^1\cdot b',\end{aligned}$$

for all $h \in H$. Thus, we have shown, for all $h \in H$,

$$\sigma(q^2)hq^1\cdot b' = \varepsilon(h)\sigma(q^2)q^1\cdot b'. \tag{2.1.8}$$

Consequently,

$$\begin{aligned}\sigma(S^{-1}(\alpha))h\cdot b' &= \sigma(S^{-1}(\alpha x^3X^3)x^2X^2)hx^1X^1\cdot b'\\ &\overset{(1.2.26)}{=} \sigma(S^{-1}(x^3)x^2)\sigma(q^2)hx^1q^1\cdot b'\\ &\overset{(2.1.8)}{=} \varepsilon(hx^1)\sigma(S^{-1}(x^3)x^2)\sigma(q^2)q^1\cdot b' = \varepsilon(h)\sigma(q^2)q^1\cdot b',\end{aligned}$$

for all $h \in H$. Together with (1.2.33) and the second equality in (2.1.7), this implies that, for all $h \in H$,

$$\begin{aligned}\varepsilon(h)b' &\overset{(1.2.33)}{=} \varepsilon(h)\sigma(S^{-1}(p^2)q^2p^1_2)q^1p^1_1\cdot b'\\ &\overset{(2.1.7)}{=} \varepsilon(h)\sigma(S^{-1}(p^2)p^1)\sigma(q^2)q^1\cdot b'\\ &\overset{(1.2.25)}{=} \sigma(S^{-1}(\alpha p^2)p^1)h\cdot b' = h\cdot b',\end{aligned}$$

as stated. Now, by considering $b = n - \underline{\varepsilon}_B(n)b'$ in (2.1.7) we get that $\psi(h_2)h_1 \cdot b' - \underline{\varepsilon}_B(n)h\cdot b' = -\underline{\varepsilon}_B(n)\sigma(h)b'$, for all $h \in H$, and so $(\psi(h) - \varepsilon(h)\underline{\varepsilon}_B(n))b' = -\underline{\varepsilon}_B(n)\sigma(h)b'$, for all $h \in H$. As $\underline{\varepsilon}_B(b') = 1_k$, the last equality coincides with $\psi = \underline{\varepsilon}_B(n)(\varepsilon - \sigma)$; in particular, σ and ψ commute and anti-commute and the same time. Since we work over a field of characteristic different from 2, it follows that $\psi\sigma = \sigma\psi = 0$ in H^*. Thus, with the help of (1.2.34) we see that

$$\begin{aligned}\psi(h) &\overset{(1.2.34)}{=} \psi(q^1_1p^1h)\sigma(q^1_2p^2S(q^2))\\ &\overset{(1.2.29)}{=} \psi((q^1h_1)_1p^1)\sigma((q^1h_1)_2p^2S(q^2h_2))\\ &\overset{\psi\ der.}{=} \psi(p^1)\sigma(q^1h_1p^2S(q^2h_2)) + \sigma(p^1)\psi((q^1h_1)_1)\sigma((q^1h_1)_2)\sigma(p^2S(q^2h_2))\\ &\overset{\psi\sigma=0,(1.2.20)}{=} \varepsilon(h)\psi(x^1)\sigma(q^1\beta S(q^2))\sigma(x^2S(x^3)) \overset{(1.2.26)}{=} \varepsilon(h)\kappa,\end{aligned}$$

where we use in the third equality the fact that ψ is an (ε,σ)-derivation; here $\kappa := \psi(x^1)\sigma(x^2S(x^3))$. Then, the (ε,σ)-derivation condition on ψ reads as $\sigma(h')\kappa = 0$, for all $h' \in H$, and therefore, since κ is invertible, $\psi = 0$ and $h \cdot b = \sigma(h)b$, for all $h \in H$. The remaining details follow by Lemma 2.1.2 from straightforward computations. □

Summing up, we have proved so far that a 2-dimensional H-module algebra and coalgebra B is determined by an algebra morphism $\sigma \in H^*$, a scalar ω and two elements $b, b' \in B$. Next we analyze what happens when we require B to be a (co)algebra in ${}_H^H\mathcal{Y}D$; we denote by $\rho : B \ni \mathfrak{b} \mapsto \mathfrak{b}_{[-1]} \otimes \mathfrak{b}_{[0]} \in H \otimes B$ a left coaction of H on B and assume that $\rho(1) = 1 \otimes 1$.

Lemma 2.1.4 *For B as above, the left Yetter-Drinfeld module structures on B are determined by two elements $u, v \in H$ satisfying $\varepsilon(u) = 0$, $\varepsilon(v) = 1$ and*

$$\begin{aligned}
\Delta(u) &= \sigma(x^1)ux^2 \otimes x^3 + \sigma(x^1X^2)X^1vx^2 \otimes uX^3x^3, && (2.1.9)\\
\Delta(v) &= \sigma(x^3X^2y^1)x^1X^1vy^2 \otimes x^2vX^3y^3, && (2.1.10)\\
hu &= \sigma(h_1)uh_2, && (2.1.11)\\
\sigma(h_2)h_1v &= \sigma(h_1)vh_2 && (2.1.12)
\end{aligned}$$

for all $h \in H$, defining the H-coaction by $\rho(n) = u \otimes 1 + v \otimes n$.

Furthermore,

- *B is a coalgebra in ${}_H^H\mathcal{Y}D$ if and only if $u = \underline{\varepsilon}_B(n)(1 - v)$ and*

$$\begin{aligned}
&u \otimes 1 + v \otimes b = b_{[-1]} \otimes b_{[0]} + b'_{[-1]}u \otimes b'_{[0]}, && (2.1.13)\\
&v \otimes b' = \sigma(X^3)X^1b'_{[-1]}v \otimes X^2 \cdot b'_{[0]}; && (2.1.14)
\end{aligned}$$

- *B is an algebra in ${}_H^H\mathcal{Y}D$ if and only if*

$$\begin{aligned}
&u[1 + \sigma(x^1x^3)x^2] = 0, && (2.1.15)\\
&u^2 = \omega[\sigma(x^1x^3X^2X^3)X^1vx^2v - \sigma(y^1y^2)y^3]. && (2.1.16)
\end{aligned}$$

Proof. It is clear that ρ is defined by $\rho(n)$. As ρ is counital, we have that $\varepsilon(u) = 0$ and $\varepsilon(v) = 1$. Moreover, the coassociativity of ρ in (1.3.38), which reduces to the case when $\mathfrak{b} = n$, is equivalent to

$$\begin{aligned}
&u \otimes 1 \otimes 1 + \sigma(X^2)X^1v \otimes uX^3 \otimes 1 + \sigma(X^2)X^1v \otimes vX^3 \otimes n\\
&\qquad = \sigma(Y^1)\Delta(u)(Y^2 \otimes Y^3) \otimes 1 + \sigma(X^3Y^1)(X^1 \otimes X^2)\Delta(v)(Y^2 \otimes Y^3) \otimes n.
\end{aligned}$$

Since $\{1, n\}$ is a basis of B, the above relation is equivalent to the two conditions (2.1.9) and (2.1.10). In a similar manner one can prove that (1.3.39) reduces in our case to the two conditions (2.1.11) and (2.1.12). Indeed, it suffices for the Yetter-Drinfeld condition (1.3.39) to be verified for $b = n$, which is equivalent to

$$hu \otimes 1 + \sigma(h_2)h_1v \otimes n = \sigma(h_1)uh_2 \otimes 1 + \sigma(h_2)vh_2 \otimes n,$$

for all $h \in H$. Then, by a simple direct computation we obtain that the last condition holds if and only if (2.1.11) and (2.1.12) are satisfied. We now check when B is a coalgebra in ${}_H^H\mathcal{Y}D$.

From the third equality in (1.3.44), i.e. the counitality of B, we get $\underline{\varepsilon}_B(1)u + \underline{\varepsilon}_B(n)v = \underline{\varepsilon}_B(n)1$, which is equivalent to $u = \underline{\varepsilon}_B(n)(1 - v)$; in particular, u and v commute. Similarly, the comodule coassociativity condition (1.3.46) holds for any $\mathfrak{b} \in B$ if and only if it holds for n; by using the fact that $\underline{\Delta}_B(n) = b \otimes 1 + b' \otimes n$, one can easily see that (1.3.46) holds for $\mathfrak{b} = n$ if and only if

$$\begin{aligned}
&u \otimes 1 \otimes 1 + v \otimes b \otimes 1 + v \otimes b' \otimes n\\
&\qquad = b_{[-1]} \otimes b_{[0]} \otimes 1 + b'_{[-1]}u \otimes b'_{[0]} \otimes 1 + \sigma(X^3)X^1b'_{[-1]}v \otimes X^2 \cdot b'_{[0]} \otimes n.
\end{aligned}$$

By using again the fact that $\{1, n\}$ is a basis for B, it follows immediately that the above equality coincides with the relations (2.1.13, 2.1.14).

In our context, B being an algebra in ${}^H_H\mathcal{Y}D$ reduces to the condition (1.3.43). As in the coalgebra case, it can be directly verified that in our context the coproduct multiplicativity condition on the coaction (1.3.43) holds if and only if

$$\begin{aligned}\sigma(x^1x^3Y^1Y^2)ux^2vY^3 &+ \sigma(Y^1Y^2)vuY^3 = 0, &(2.1.17)\\ \sigma(x^1x^3Y^1Y^2)ux^2uY^3 &+ \omega\sigma(x^1x^3Y^1Y^2X^2X^3)X^1vx^2vY^3 = \omega 1. &(2.1.18)\end{aligned}$$

As in the Hopf case, see the proof of [45, Corollary 1], we show that v has a right inverse, implying that (2.1.17, 2.1.18) and (2.1.15, 2.1.16) are equivalent.

Recall that u and v commute. For $h \in H$, we have that

$$\begin{aligned} hv &\overset{(1.2.33)}{=} \sigma(S^{-1}(p^2)q^2p^1_2)hg^1p^1_1v \\ &\overset{(1.2.30)}{=} \sigma(S^{-1}(h_2p^2)q^2(h_1p^1)_2)q^1(h_1p^1)_1v \\ &\overset{(2.1.12)}{=} \sigma(S^{-1}(h_2p^2)q^2)\sigma(h_{11}p^1_1)q^1vh_{12}p^1_2 = \sigma(q^2)q^1v\underline{h}, \end{aligned} \tag{2.1.19}$$

where, for $h \in H$, we denote $\underline{h} := \sigma(S^{-1}(h_2p^2))\sigma(h_{11}p^1_1)h_{12}p^1_2$. Now, from the formula of $\Delta(v)$ in (2.1.10), since $\varepsilon(v) = 1$, we deduce that

$$\begin{aligned} 1 &\overset{(1.2.26)}{=} q^1\beta S(q^2) \\ &\overset{(1.2.20)}{=} q^1v_1\beta S(q^2v_2) \\ &\overset{(2.1.10)}{=} \sigma(y^1X^2x^3)q^1x^1X^1vy^2\beta S(q^2x^2vX^3y^3) \\ &\overset{(2.1.19)}{=} \sigma(Q^2)Q^1v\hbar, \end{aligned}$$

where $Q^1 \otimes Q^2$ is a second copy of q_R and $\hbar := \sigma(y^1X^2x^3)\underline{q^1x^1X^1y^2}\beta S(q^2x^2vX^3y^3)$. So $\sigma(Q^2)Q^1v$ has a right inverse. Since σ is an algebra morphism, (1.2.21) implies that $\sigma(Q^2)Q^1$ is invertible in H with inverse given by $\sigma(S^{-1}(x^3)x^2)\sigma(S^{-1}(\alpha))^{-1}x^1$, where $\sigma(S^{-1}(\alpha))^{-1} = \sigma(S^{-1}(S(x^1)x^2\beta S(x^3)) = \sigma(S^{-1}(X^1\beta S(X^2)X^3))$. Hence v has a right inverse, as stated. □

At last, we need one final preliminary result.

Proposition 2.1.5 *Let B be a 2-dimensional bialgebra in ${}^H_H\mathcal{Y}D$ with basis $\{1, n\}$, determined by $\sigma \in H^*$, $\omega \in k$, $u, v \in H$ and $b, b' \in B$ as in the above. Then*

$$\begin{aligned} b^2 + \sigma(u)b'b + \omega b'^2 &= \omega 1, &(2.1.20)\\ \sigma(v)b'b + bb' &= 0. &(2.1.21)\end{aligned}$$

Consequently, if $b' = 1$ then $u = 0$.

Proof. We have studied all the conditions required for B to be an algebra and a coalgebra in ${}^H_H\mathcal{Y}D$. The only one left in order to completely describe the bialgebra structure of B in ${}^H_H\mathcal{Y}D$ is the product-coproduct compatibility condition (1.3.48), which holds in our case if and only if it holds for $\mathfrak{b} = \mathfrak{b}' = n$. By using the relations obtained so far, one can show that $\underline{\Delta}_B$ is multiplicative if and only if

$$b^2 \otimes 1 + bb' \otimes n + \sigma(u)b'b \otimes 1 + \sigma(v)b'b \otimes n + \omega b'^2 \otimes 1 \quad = \quad \omega 1 \otimes 1.$$

It is clear that the above relation is equivalent to the relations (2.1.20) and (2.1.21), thus completing the description of B as a braided bialgebra.

Assume now $b' = 1$, so $n = b + \underline{\varepsilon}_B(n)1$. As $\{1, n\}$ is a basis in B, we cannot have $b = 0$, and we can not have $b = 1$ since $\underline{\varepsilon}_B(b) = 0$. Therefore $\{1, b\}$ is another basis for B. In addition, (2.1.20) yields $b^2 = -\sigma(u)b$; thus $n^2 = \omega 1$ leads us to $\sigma(u) = 2\underline{\varepsilon}_B(n)$ and $\omega = \underline{\varepsilon}_B(n)^2$.

By way of contradiction, assume that $u \neq 0$. Then $u = \underline{\varepsilon}_B(n)(1 - v)$ implies $\underline{\varepsilon}_B(n) \neq 0$, and so $\omega \neq 0$ and $\sigma(u) \neq 0$. Hence (2.1.2) entails to $\sigma^2 = \varepsilon$. It follows that $v = 1 - 2\sigma(u)^{-1}u$, and the equality (2.1.12) reads as

$$\sigma(h_2)h_1(1 - 2\sigma(u)^{-1}u) \;\;=\;\; \sigma(h_1)(1 - 2\sigma(u)^{-1}u)h_2,$$

for all $h \in H$, which, owing to (2.1.11), it is equivalent to

$$\sigma(h_2)h_1 - \sigma(h_1)h_2 \;\;=\;\; 2\sigma(u)^{-1}(\sigma(h_2)h_1 - h)u,$$

for all $h \in H$. By applying σ to the both sides of the above equality, since $\sigma^2 = \varepsilon$, we get that $2\sigma(u)^{-1}(\varepsilon - \sigma) = 0$, hence $\sigma = \varepsilon$. But this yields $0 = \varepsilon(u) = \sigma(u) \neq 0$, a contradiction. □

We now have all the ingredients needed to describe the 2-dimensional braided Hopf algebras B within ${}_H^H\mathcal{YD}$, provided that H is a quasi-Hopf algebra with bijective antipode. To simplify the presentation, we introduce the following family of braided Hopf algebras.

Definition 2.1.6 Let H be a quasi-Hopf algebra and (σ, v) a couple consisting of an algebra map $\sigma : H \to k$ and an element $v \in H$ verifying $\sigma(v) = -1$, $\varepsilon(v) = 1$, (2.1.10) and (2.1.12). Then $B_{\sigma,v}$ is the 2-dimensional vector space with basis $\{1, n\}$ endowed with the following braided Hopf algebra structure:

- $B_{\sigma,v}$ belongs to ${}_H^H\mathcal{YD}$ via the H-action determined by $h \cdot 1 = \varepsilon(h)1$, $h \cdot n = \sigma(h)n$, for all $h \in H$, and the H-coaction given by $1 \mapsto 1 \otimes 1$, $n \mapsto v \otimes n$;
- as an algebra, $B_{\sigma,v}$ is unital with unit 1 and is generated by 1 and n under the relation $n^2 = 0$;
- the coalgebra structure of $B_{\sigma,v}$ is defined by $\underline{\Delta}(1) = 1 \otimes 1$, $\underline{\Delta}(n) = n \otimes 1 + 1 \otimes n$ and $\underline{\varepsilon}(1) = 1$, $\underline{\varepsilon}(n) = 0$;
- the braided antipode $\underline{S}$ is determined by $\underline{S}(1) = 1$, $\underline{S}(n) = -n$.

The fact that $B_{\sigma,v}$ is indeed a braided Hopf algebra will follow from the proof of Theorem 2.1.8.

Remark 2.1.7 It is easy to see that the trivial H-(co)module structure (given by the counit, respectively by the unit, of H) turns $k[C_2]$ into an object of ${}_H^H\mathcal{YD}$ and, moreover, into a braided Hopf algebra in ${}_H^H\mathcal{YD}$ (the braided Hopf structure equals the one considered in the Hopf case); in what follows, we will refer to this braided Hopf algebra structure of $k[C_2]$ as being the trivial one.

The next result can be viewed as the quasi-Hopf version of [45, Theorem 1].

Theorem 2.1.8 *Let H be a quasi-Hopf algebra with bijective antipode over a field k of characteristic different from 2. If B is a braided Hopf algebra in ${}_H^H\mathcal{YD}$ then B is isomorphic to a braided Hopf algebra of type $B_{\sigma,v}$ or to the group Hopf algebra $k[C_2]$ considered in ${}_H^H\mathcal{YD}$ via the trivial structure.*

Proof. As before, we consider B determined by $\sigma \in H^*$, $\omega \in k$, $u, v \in H$ and $b, b' \in B$ satisfying the relations mentioned above. Parallel to the Hopf case in [45, Theorem 1], we distinguish two cases.

Case 1. If $b' = 1$, by Proposition 2.1.5 we get $u = 0$; clearly, $b \neq 0$. Thus (2.1.21) is equivalent to $\sigma(v) = -1$; consequently, $v \neq 1$ and so $u = \underline{\varepsilon}_B(n)(1 - v)$ is equivalent to $\underline{\varepsilon}_B(n) = 0$. Therefore, $n = b$ and $\omega 1 = n^2 = b^2 = -\sigma(u)b = 0$; see the last part of the proof of Proposition 2.1.5. We conclude that $\omega = 0$ and $n^2 = 0$. Now, a simple inspection tells us that the braided Hopf algebra B is nothing but $B_{\sigma,v}$, as presented in Definition 2.1.6; the details are left to the reader. In particular, this shows that $B_{\sigma,v}$ is indeed a Hopf algebra in ${}_H^H\mathcal{Y}D$, as previously claimed.

Case 2. Assume that $b' \neq 1$. Since $\underline{\varepsilon}_B(b') = 1$, it follows that $\{1, b'\}$ is a basis of B for which $h \cdot 1 = \varepsilon(h)1$ and $h \cdot b' = \varepsilon(h)b'$, for all $h \in H$. We deduce from here that $h \cdot \mathfrak{b} = \varepsilon(h)\mathfrak{b}$, for all $\mathfrak{b} \in B$. But $h \cdot n = \sigma(h)n$, for all $h \in H$, and therefore $\sigma = \varepsilon$ in this case. We obtain from (2.1.15) that $u = 0$, and from (2.1.14) that $b'_{[-1]}v \otimes b'_{[0]} = v \otimes b'$. We know from the proof of Lemma 2.1.4 that, in general, v is right invertible, hence $b'_{[-1]} \otimes b'_{[0]} = 1 \otimes b'$. Keeping in mind that $\{1, b'\}$ is a basis of B and the fact that $\rho(1) = 1 \otimes 1$, it follows that $\rho(\mathfrak{b}) = 1 \otimes \mathfrak{b}$, for all $\mathfrak{b} \in B$. By taking $\mathfrak{b} = n$, with the help of $\rho(n) = u \otimes 1 + v \otimes n = v \otimes n$, we conclude that $v = 1$. Summing up, $u = 0$, $v = 1$ and $\sigma = \varepsilon$; similarly to the previous case, a simple inspection guarantees us that B is in this case the group Hopf algebra $k[C_2]$, equipped with the trivial braided Hopf algebra structure. For this, note that

$$\begin{aligned}(\underline{\Delta}_B(n))^2 &= (b \otimes 1 + b' \otimes n)^2 = b^2 \otimes 1 + b'^2 \otimes n^2 \\ &= (b^2 \otimes \omega b'^2) \otimes 1 \overset{(2.1.21)}{=} \omega 1 \otimes 1 = \underline{\Delta}_B(n^2),\end{aligned}$$

since (2.1.21) reduces to $b'b = -bb'$; but $\{1, b'\}$ is a basis for B, so B is a commutative algebra, and therefore $bb' = b'b = 0$. The above equality tells us that $\underline{\Delta}_B$ is multiplicative in the usual way, hence $\underline{\Delta}_B(b'^2) = (\underline{\Delta}_B(b'))^2 = b'^2 \otimes b'^2$. As $\underline{\varepsilon}_B(b') = 1$, it follows that b'^2 is a grouplike element of the (ordinary) Hopf algebra B, hence $1, b', b'^2$ are linearly independent elements of B, provided that they are pairwise distinct. Since B is 2-dimensional it follows that $b'^2 \in \{1, b'\}$. But we required for B to have an antipode, and this occurs only in the case when $b'^2 = 1$; for more details we refer to [16, Exercise 4.3.7]. □

For further use, we present the quasi-Hopf algebra structures of $B \times H$ coming from 2-dimensional braided Hopf algebras B. As before, we work over a field of characteristic different from 2. Also, remark that, in general, $B \times H$ is a free right H-module of rank equals $\dim_k(B)$.

Proposition 2.1.9 *Let A be a quasi-Hopf algebra with a projection $H \underset{\pi}{\overset{i}{\rightleftarrows}} A$, $\pi i = \mathrm{Id}_H$, such that A is a free right H-module of rank 2. The algebra A identifies either to H_g or $H(\theta)_{\zeta,v}$, where:*

- *H_g is the unital k-algebra generated by g and H with relations $g^2 = 1$ and $gh = hg$, for all $h \in H$. H is a quasi-Hopf subalgebra of H_g, and $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g$ complete the quasi-Hopf algebra structure of H_g. In other words, H_g is the tensor product of the quasi-Hopf algebras $k[C_2]$ and H, the former being seen as the group Hopf algebra with trivial reassociator.*

- *Let (σ, v) be a couple consisting of an algebra morphism $\sigma : H \to k$ and an element $v \in H$ subject to relations*

$$\begin{aligned}
\sigma(v) &= -1, && (2.1.22)\\
\varepsilon(v) &= 1, && (2.1.23)\\
\Delta(v) &= \sigma(y^1x^3X^2)x^1X^1vy^2 \otimes x^2vX^3y^3, && (2.1.24)\\
\sigma(h_2)h_1v &= \sigma(h_1)vh_2, && (2.1.25)
\end{aligned}$$

for all $h \in H$. Then $H(\theta)_{\sigma,v}$ is the unital k-algebra generated by θ and H with relations $\theta^2 = 0$ and $h\theta = \sigma(h_1)\theta h_2$, for all $h \in H$. The quasi-Hopf algebra structure of $H(\theta)_{\sigma,v}$ is defined in such a way that H is a quasi-Hopf subalgebra of it and

$$\Delta(\theta) = \sigma(X^2x^1)vX^1x^2 \otimes \theta X^3x^3 + \sigma(x^1)\theta x^2 \otimes x^3, \qquad (2.1.26)$$
$$S(\theta) = -\sigma(X^2x^1_2)S(X^1x^1_1v)\alpha\theta X^3x^2\beta S(x^3). \qquad (2.1.27)$$

Proof. By [11], A identifies to $B \times H$ as a quasi-Hopf algebra, for some 2-dimensional braided Hopf algebra B. If $B = k[C_2]$, A identifies to H_g. When $B = B_{\sigma,v}$, A identifies to $H(\theta)_{\sigma,v}$. Note that $\theta \equiv n \times 1$, where $\{1, n\}$ is the basis of B as in the Case 1 of the proof of Theorem 2.1.8; the formulas for $\Delta(\theta)$ and $S(\theta)$ follow from (1.4.49) and (1.4.50), respectively. $\square$

To conclude the section, we present a Corollary that can be used to apply Theorem 2.1.8 to the classification of small-dimensional quasi-Hopf algebras.

Corollary 2.1.10 *Let A, H be quasi-Hopf algebras over a field k of characteristic different from 2 and suppose that there are quasi-Hopf algebra morphisms $H \underset{\pi}{\overset{i}{\rightleftarrows}} A$ such that $\pi i = \mathrm{Id}_H$. If A is a free right H-module of rank 2 and H is finite dimensional, the following assertions hold.*

1. *If A is semisimple then so is H and $A \simeq H_g$ as a quasi-Hopf algebra.*
2. *If A is not semisimple and H is semisimple then $A \simeq H(\theta)_{\sigma,v}$ as a quasi-Hopf algebra, for some σ, v, θ as in Proposition 2.1.9;*
3. *If both A, H are not semisimple then $A \simeq H_g$ or $A \simeq H(\theta)_{\sigma,v}$ as a quasi-Hopf algebra.*

Proof. We have $A \simeq B \times H$, for some 2-dimensional braided Hopf algebra B; so Theorem 2.1.8 applies.

As in [43, §2 Proposition 3], one can show that a left integral t in $B \times H$ is of the form $t_B \times t_H$ with $t_B \in B$ and $t_H \in H$ such that $\mathfrak{b}t_B = \underline{\varepsilon}_B(\mathfrak{b})t_B$, for all $\mathfrak{b} \in B$, and $h \cdot t_B = \varepsilon(h)t_B$, $ht_H = \varepsilon(h)t_H$, for all $h \in H$. Consequently, $B \times H$ is semisimple if and only if H is semisimple and t_B satisfies, moreover, $\underline{\varepsilon}_B(t_B) = 1_k$. Note that, for $B = B_{\sigma,v}$ such an element t_B does not exist. On the other hand, for $B = k[C_2]$ one can take $t_B = \frac{1}{2}(1 + g)$, where g is the generator of C_2.

1. Since A is finite dimensional and semisimple and H can be identified to a quasi-Hopf subalgebra of A, by [15, Corollary 7.82] we obtain that H is semisimple. By the above comments it follows that $A \simeq H_g \equiv k[C_2] \otimes H$ as a quasi-Hopf algebra.

2., 3. If A is not semisimple then either H is not semisimple and B is arbitrary or H is semisimple and $B = B_{\sigma,v}$. $\square$

2.2 The algebra of functions on a finite group

Let G be a finite group, ω a normalized 3-cocycle on G and k_ω^G the Hopf algebra of functions on G, seen as a quasi-Hopf algebra with reassociator Φ_ω as in (4.2.2). In this section, with the help of Theorem 2.1.8, we characterize 2-dimensional braided Hopf algebras B in ${}_{k_\omega^G}^{k_\omega^G}\mathcal{YD}$. We will see that the structure of B is determined by a pair $(\rho, \mathfrak{g})$ consisting of a map $\rho : G \to k^*$ and an element $\mathfrak{g} \in G$ satisfying certain conditions.

The next lemma can be seen as the non-commutative version of the result stated at the beginning of subsection 2.2 in [33].

Lemma 2.2.1 *Let ω be a normalized 3-cocycle on G and $\mathfrak{g} \in G$. Define $\flat_{\mathfrak{g}}\omega : G \times G \to k^*$, for any $x, y \in G$, by*

$$\flat_{\mathfrak{g}}\omega(x,y) = \frac{\omega(\mathfrak{g},x,y)\omega(x,y,\mathfrak{g})}{\omega(x,\mathfrak{g},y)}.$$

Then $\flat_{\mathfrak{g}}\omega$ is a normalized 2-cocycle on G if and only if

$$\omega(\mathfrak{g}x,y,z)\omega(x,\mathfrak{g}y,z)\omega(x,y,\mathfrak{g}z) = \omega(x\mathfrak{g},y,z)\omega(x,y\mathfrak{g},z)\omega(x,y,z\mathfrak{g}), \tag{2.2.1}$$

For all $x, y, z \in G$. In particular, $\flat_{\mathfrak{g}}\omega$ is a normalized 2-cocycle on G, provided that $\mathfrak{g}$ is in the center of G.

Proof. $\flat_{\mathfrak{g}}\omega$ is normalized since ω is so. By applying the 3-cocycle condition (1.1.3) for several times, we compute that

$$\begin{aligned}
\flat_{\mathfrak{g}}\omega(x,y)\flat_{\mathfrak{g}}\omega(xy,z) &= \frac{\omega(\mathfrak{g},x,y)\omega(x,y,\mathfrak{g})\omega(\mathfrak{g},xy,z)\omega(xy,z,\mathfrak{g})}{\omega(x,\mathfrak{g},y)\omega(xy,\mathfrak{g},z)} \\
&= \frac{\omega(\mathfrak{g}x,y,z)\omega(\mathfrak{g},x,yz)\omega(x,y,\mathfrak{g})\omega(xy,z,\mathfrak{g})}{\omega(x,\mathfrak{g},y)\omega(xy,\mathfrak{g},z)\omega(x,y,z)} \\
&= \frac{\omega(\mathfrak{g}x,y,z)\omega(\mathfrak{g},x,yz)\omega(\mathfrak{g},x,yz)\omega(x,y,\mathfrak{g}z)\omega(xy,z,\mathfrak{g})}{\omega(x,\mathfrak{g},y)\omega(y,\mathfrak{g},z)\omega(x,y\mathfrak{g},z)\omega(x,y,z)} \\
&= \frac{\omega(\mathfrak{g}x,y,z)\omega(\mathfrak{g},x,yz)\omega(x,y,\mathfrak{g}z)\omega(y,z,\mathfrak{g})\omega(x,yz,\mathfrak{g})}{\omega(x,\mathfrak{g},y)\omega(y,\mathfrak{g},z)\omega(x,y\mathfrak{g},z)\omega(x,y,z\mathfrak{g})} \\
&= \flat_{\mathfrak{g}}\omega(x,yz)\flat_{\mathfrak{g}}(y,z)\frac{\omega(\mathfrak{g}x,y,z)\omega(x,\mathfrak{g}y,z)\omega(x,y,\mathfrak{g}z)}{\omega(x\mathfrak{g},y,z)\omega(x,y\mathfrak{g},z)\omega(x,y,z\mathfrak{g})},
\end{aligned}$$

for all $x, y, z \in G$, and so our assertions follow. □

To simplify the notation, we denote the map ∂_2, defined in Subsection 1.1.1, by ∂ for the rest of this section.

Lemma 2.2.2 *Let G be a finite group and ω a normalized 3-cocycle on G. Giving a pair (σ, v) for k_ω^G satisfying the conditions in Definition 2.1.6 is equivalent to giving a pair $(\mathfrak{g}, \rho)$ consisting of an element $\mathfrak{g}$ in the center of G and a map $\rho : G \to k^*$ obeying the relations*

$$\rho(e) = 1, \quad \rho(\mathfrak{g}) = -1 \quad \text{and} \quad \flat_{\mathfrak{g}}\omega = \partial\rho.$$

Proof. Keeping in mind the quasi-Hopf algebra structure of k^G, an algebra map $\sigma : k^G \to k$ is completely determined by an element $\mathfrak{g} \in k[G]$ satisfying $\Delta(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}$ and $\varepsilon(\mathfrak{g}) = 1$. Thus, $\mathfrak{g} \in G$ and $\sigma(\varphi) = \varphi(\mathfrak{g})$, for all $\varphi \in k^G$.

An element $v \in k^G$ is completely determined by a map $\rho : G \to k$, in the sense that $v = \sum\limits_{x \in G} \rho(x) P_x$, where, as before, $\{P_g\}_{g \in G}$ is the basis of k^G dual to the basis $\{g\}_{g \in G}$ of $k[G]$. We have $\varepsilon(v) = 1$ and $\sigma(v) = -1$ if and only if $\rho(e) = 1$ and $\rho(\mathfrak{g}) = -1$, respectively.

We now look at the condition $\sigma(\varphi_2)\varphi_1 v = \sigma(\varphi_1) v \varphi_2$, which must be valid for all $\varphi \in k^G_\omega$ or, equivalently, for any P_h, $h \in G$. Since $\Delta(P_h) = \sum_{xy=h} P_x \otimes P_y$, we get that $\sum_{xy=h} \sigma(P_y) P_x v = \sum_{xy=h} \sigma(P_x) v P_y$, for all $h \in H$. Thus $P_{h\mathfrak{g}^{-1}} v = v P_{\mathfrak{g}^{-1}h}$, for all $h \in H$, and this fact holds if and only if $\mathfrak{g}h = h\mathfrak{g}$, for all $h \in G$; in other words, if and only if $\mathfrak{g}$ is an element in the center of G.

Finally, we claim that $\Delta(v) = \sigma(y^1 x^3 X^2) x^1 X^1 v y^2 \otimes x^2 v X^3 y^3$ is equivalent to $\flat_{\mathfrak{g}} \omega = \partial \rho$. Indeed, since $\Phi_\omega^{-1} = \sum_{g_1, g_2, g_3 \in G} \omega(g_1, g_2, g_3)^{-1} P_{g_1} \otimes P_{g_2} \otimes P_{g_3}$, we compute that

$$\begin{aligned} \sigma(X^2) X^1 v \otimes v X^3 &= \sum_{g_1, g_3 \in G} \omega(g_1, \mathfrak{g}, g_3) P_{g_1} v \otimes v P_{g_3} \\ &= \sum_{x, y \in G} \omega(x, \mathfrak{g}, y) \rho(x) \rho(y) P_x \otimes P_y \end{aligned}$$

and, likewise,

$$\sigma(y^1 X^2) X^1 v y^2 \otimes v X^3 y^3 = \sum_{x, y \in G} \frac{\omega(x, \mathfrak{g}, y)}{\omega(\mathfrak{g}, x, y)} \rho(x) \rho(y) P_x \otimes P_y.$$

We then have

$$\begin{aligned} \sigma(y^1 x^3 X^2) x^1 X^1 v y^2 \otimes x^2 v X^3 y^3 &= \sum_{x, y \in G} \frac{\omega(x, \mathfrak{g}, y)}{\omega(\mathfrak{g}, x, y) \omega(x, y, \mathfrak{g})} \rho(x) \rho(y) P_x \otimes P_y \\ &= \sum_{x, y \in G} \frac{\rho(x) \rho(y)}{\flat_{\mathfrak{g}} \omega(x, y)} P_x \otimes P_y. \end{aligned}$$

On the other hand, $\Delta(v) = \sum\limits_{g, y \in G} \rho(g) P_{gy^{-1}} \otimes P_y$, and so

$$\Delta(v) = \sigma(y^1 x^3 X^2) x^1 X^1 v y^2 \otimes x^2 v X^3 y^3$$

holds if and only if $\sum_{g \in G} \rho(g) P_{gy^{-1}} = \sum_{x \in G} \frac{\rho(x)\rho(y)}{\flat_{\mathfrak{g}} \omega(x, y)} P_x$, for all $y \in G$. The latest assertion is then clearly equivalent to $\rho(xy) \flat_{\mathfrak{g}} \omega(x, y) = \rho(x) \rho(y)$, for all $x, y \in G$. If there exist some $x \in G$ such that $\rho(x) = 0$, then $\rho(xy) = 0$, for all $y \in G$, meaning that $\rho(g) = 0$, for all $g \in G$, which is false. So we can consider $\rho : G \to k^*$, and we conclude that $\flat_{\mathfrak{g}} \omega = \partial \rho$. □

The result below is an immediate consequence of Lemma 2.2.2 and Theorem 2.1.8.

Proposition 2.2.3 *Non-trivial 2-dimensional braided Hopf algebras in* ${}^{k^G_\omega}_{k^G_\omega}\mathcal{YD}$ *are in a one-to-one correspondence with couples* $(\mathfrak{g}, \rho)$ *consisting of an element* $\mathfrak{g}$ *in the center of* G *and a map* $\rho : G \to k^*$ *obeying* $\rho(e) = 1$, $\rho(\mathfrak{g}) = -1$ *and* $\flat_{\mathfrak{g}} \omega = \partial \rho$.

In the following, we apply Proposition 2.2.3 to various finite groups G.

2.2.1 Finite abelian groups

Let $G = C_1 \times \ldots \times C_n$ be a finite abelian group; C_j stands for the cyclic group of order m_j, generated by g_j, $1 \le j \le n$. In general, for m a non-zero positive integer, denote by ξ_m a primitive m-th root of unity in k. Then $1_{j_1,\ldots,j_n}$ ($0 \le j_l \le m_l - 1$, for all $1 \le l \le n$) given by

$$1_{j_1,\ldots,j_n} := \frac{1}{m_1 \ldots m_n} \prod_{l=1}^{n} \left(\sum_{t=0}^{m_l-1} \xi_{m_l}^{-tj_l} g_l^t \right) \tag{2.2.2}$$

define a family of orthogonal idempotents of $k[G]$. It is well known that k^G and $k[G]$ identify as Hopf algebras through the correspondence $P_{g_1^{j_1}\ldots g_n^{j_n}} \mapsto 1_{j_1,\ldots,j_n}$. The inverse of this correspondence maps $g_1^{j_1}\ldots g_n^{j_n}$ to $\prod_{l=1}^{n}\left(\sum_{t=0}^{m_l-1}\xi_{m_l}^{tj_l}\widetilde{P}_{g_l^t}\right)$, where $\widetilde{P}_{g_l^t} \in k^G$ is determined by $\widetilde{P}_{g_l^t}(g_1^{a_1}\ldots g_n^{a_n}) = \delta_{t,a_l}$, for all $g_1^{a_1}\ldots g_n^{a_n} \in G$.

A set of representative 3-cocycles on G was given in [33]; namely, denote by $\mathcal{A}$ the set of sequences $\underline{a}$ with integer elements, having the form

$$(c_1, \ldots, c_l, \ldots, c_n, c_{12}, \ldots, c_{ij}, \ldots, c_{n-1,n}, c_{123}, \ldots, c_{rst}, \ldots, c_{n-2,n-1,n}) \tag{2.2.3}$$

with $0 \le c_l \le m_l - 1$, $0 \le c_{ij} \le (m_i, m_j)$ and $0 \le c_{rst} \le (m_r, m_s, m_t)$, for all $1 \le l \le n$, $1 \le i < j \le n$ and $1 \le r < s < t \le n$; the c_{ij}'s and the c_{rst}'s are ordered by the lexicographical order and (u, v) stands for the greatest common divisor between the integers u, v, and an analogous notation is used for (u, v, w).

For $\underline{a}$ in $\mathcal{A}$, define $\omega_{\underline{a}} : G \times G \times G \to k^*$ by

$$\begin{aligned}\omega_{\underline{a}}(g_1^{i_1}\ldots g_n^{i_n}, g_1^{j_1}\ldots g_n^{j_n}, g_1^{k_1}\ldots g_n^{k_n}) :=\\ = \prod_{l=1}^{n} \xi_{m_l}^{c_l i_l \lfloor \frac{j_l+k_l}{m_l} \rfloor} \prod_{1\le s<t\le n} \xi_{m_t}^{c_{st} i_t \lfloor \frac{j_s+k_s}{m_s} \rfloor} \prod_{1\le r<s<t\le n} \xi_{(m_r,m_s,m_t)}^{-c_{rst}k_r j_s i_t}.\end{aligned} \tag{2.2.4}$$

Owing to [32, Proposition 3.8], $\{\omega_{\underline{a}}\}_{\underline{a}\in\mathcal{A}}$ is a set of representative normalized 3-cocycles for G. Furthermore, $\omega_{\underline{a}}$ is an abelian 3-cocycle if and only if $c_{rst} = 0$ for all $1 \le r < s < t \le n$ (for the definition of an abelian 3-cocycle we refer to [32]).

In the computations we are about to tackle, sometimes we need a common root of unity, one from which we can derive all the others roots of unity. This is why we set $M := LCM\{(m_r, m_s, m_t) | 1 \le r < s < t \le n\}$ and fix ξ_M a primitive M-th root of unity in k; then, without loss of generality, we assume that $\xi_{(m_r,m_s,m_t)} = \xi_M^{M_{rst}}$, where, for all $1 \le r < s < t \le n$, $M_{rst} := \frac{M}{(m_r,m_s,m_t)}$. In addition, we define $M_{rst}^c = M_{rst}c_{rst}$, for all $1 \le r < s < t \le n$.

Proposition 2.2.4 *For $\mathfrak{g} = g_1^{f_1}\ldots g_n^{f_n} \in C_1 \times \ldots \times C_n$ and $\underline{a} = (c_1, \ldots, c_{n-2,n-1,n})$ as in (2.2.3), $\flat_{\mathfrak{g}}\omega_{\underline{a}}$ is a coboundary 2-cocycle if and only if $n = 2$ or $n \ge 3$ and for any $1 \le r < s \le n$ the following divisibility hold:*

$$M | \sum_{u=1}^{r-1} M_{urs}^c f_u - \sum_{v=r+1}^{s-1} M_{rvs}^c f_v + \sum_{w=s+1}^{n} M_{rsw}^c f_w; \tag{2.2.5}$$

as a convention, a sum for which its summands do not exist is equal to zero.

Moreover, if (2.2.5) holds, then $\flat_{\mathfrak{g}}\omega_{\underline{a}} = \flat_{\mathfrak{g}}\omega_{\bar{a}}$, where $\bar{a} = (\bar{c}_1, \ldots, \bar{c}_{n-2,n-1,n})$ defined by $\bar{c}_i = c_i$, $\bar{c}_{ij} = c_{ij}$ and $\bar{c}_{rst} = 0$ induces the abelian 3-cocycle $\omega_{\bar{a}}$ on G associated to $\underline{a}$.

Proof. We start by noting that, for $\underline{a} \in \mathcal{A}$ and $\mathfrak{g} = g_1^{f_1} \dots g_n^{f_n} \in G$, we have

$$\begin{aligned}&\flat_{\mathfrak{g}}\omega_{\underline{a}}(g_1^{j_1} \dots g_n^{j_n}, g_1^{k_1} \dots g_n^{k_n}) \\ &\quad = \prod_{l=1}^{n} \xi_{m_l}^{c_l f_l \lfloor \frac{j_l+k_l}{m_l} \rfloor} \prod_{1\le s<t\le n} \xi_{m_t}^{c_{st} f_t \lfloor \frac{j_s+k_s}{m_s} \rfloor} \prod_{1\le r<s<t\le n} \xi_{(m_r,m_s,m_t)}^{-c_{rst}(k_r j_s f_t + f_r k_s j_t - k_r f_s j_t)}.\end{aligned}$$

With $\bar{a}$ as in the statement of the proposition, we show that $\flat_{\mathfrak{g}}\omega_{\bar{a}}$ is a coboundary 2-cocycle. Indeed, by specializing the above formula for $\bar{a}$, we have that

$$\begin{aligned}\flat_{\mathfrak{g}}\omega_{\bar{a}}(g_1^{j_1} \dots g_n^{j_n}, g_1^{k_1} \dots g_n^{k_n}) &= \prod_{l=1}^{n} \xi_{m_l}^{c_l f_l \lfloor \frac{j_l+k_l}{m_l} \rfloor} \prod_{1\le s<t\le n} \xi_{m_t}^{c_{st} f_t \lfloor \frac{j_s+k_s}{m_s} \rfloor} \\ &= \prod_{l=1}^{n} \xi_{m_l}^{c_l f_l \frac{j_l+k_l-(j_l+k_l)'_{m_l}}{m_l}} \prod_{1\le s<t\le n} \xi_{m_t}^{c_{st} f_t \frac{j_s+k_s-(j_s+k_s)'_{m_s}}{m_s}} \\ &= \prod_{l=1}^{n} \frac{\xi_{m_l^2}^{c_l f_l j_l} \xi_{m_l^2}^{c_l f_l k_l}}{\xi_{m_l^2}^{c_l f_l (j_l+k_l)'_{m_l}}} \prod_{1\le s<t\le n} \frac{\xi_{m_t m_s}^{c_{st} f_t j_s} \xi_{m_t m_s}^{c_{st} f_t k_s}}{\xi_{m_t m_s}^{c_{st} f_t (j_s+k_s)'_{m_s}}} \\ &= \frac{f_{\mathfrak{g}}(g_1^{j_1} \dots g_n^{j_n})\ f_{\mathfrak{g}}(g_1^{k_1} \dots g_n^{k_n})}{f_{\mathfrak{g}}(g_1^{(j_1+k_1)'_{m_1}} \dots g_n^{(j_1+k_1)'_{m_n}})},\end{aligned}$$

where $f_{\mathfrak{g}} : G \to k^*$ is defined by $f_{\mathfrak{g}}(g_1^{j_1} \dots g_n^{j_n}) := \prod_{l=1}^{n} \xi_{m_l^2}^{c_l f_l j_l} \prod_{1\le s<t\le n} \xi_{m_t m_s}^{c_{st} f_t j_s}$, $(j_l + k_l)'_{m_l}$ denotes, for $1 \le l \le n$, the remainder of the division of $j_l + k_l$ by m_l and $\xi_{m_l^2} \in k$ is such that $\xi_{m_l^2}^{m_l} = \xi_{m_l}$; similarly, $\xi_{m_t m_s} \in k$ such that $\xi_{m_t m_s}^{m_s} = \xi_{m_t}$.

Next, define $\flat_{\mathfrak{g}}\omega_{\bar{\bar{a}}} : G \times G \to k^*$ by

$$\flat_{\mathfrak{g}}\omega_{\bar{\bar{a}}}(g_1^{j_1} \dots g_n^{j_n}, g_1^{k_1} \dots g_n^{k_n}) = \prod_{1\le r<s<t\le n} \xi_{(m_r,m_s,m_t)}^{-c_{rst}(k_r j_s f_t + f_r k_s j_t - k_r f_s j_t)}.$$

We claim that $\flat_{\mathfrak{g}}\omega_{\bar{\bar{a}}}$ is a coboundary 2-cocycle if and only if it is trivial, meaning that

$$\prod_{1\le r<s<t\le n} \xi_{(m_r,m_s,m_t)}^{-c_{rst}(k_r j_s f_t + f_r k_s j_t - k_r f_s j_t)} = 1, \tag{2.2.6}$$

for all $0 \le j_l, k_l \le m_l - 1$ and $1 \le l \le n$.

Indeed, assume that there exists a map $\rho : G \to k^*$ such that

$$\rho(g_1^{j_1} \dots g_n^{j_n})\rho(g_1^{k_1} \dots g_n^{k_n}) = \flat_{\mathfrak{g}}\omega_{\bar{\bar{a}}}(g_1^{j_1} \dots g_n^{j_n}, g_1^{k_1} \dots g_n^{k_n})\rho(g_1^{(j_1+k_1)'_{m_1}} \dots g_n^{(j_n+k_n)'_{m_n}}),$$

for all $0 \le j_l, k_l \le m_l - 1$, $1 \le l \le n$. For $1 \le i \le n-1$, we then have

$$\flat_{\mathfrak{g}}\omega_{\bar{\bar{a}}}(g_i^{a_i}, g_{i+1}^{a_{i+1}} \dots g_n^{a_n}) = \xi_M^{-\sum\limits_{1\le r<s<t\le n} M_{rst} k_r j_s f_t - \sum\limits_{1\le r<s<t\le n} M_{rst} j_t (f_r k_s - k_r f_s)} = 1,$$

since both sums that appear in the exponent of ξ_M are equal to zero (for the first sum the only possible non-zero summand is obtained when $s = i$ but then $r < i$ implies $k_r = 0$, while for the second sum the only possible non-zero summand is obtained when $t = i$, in which case $r < s < i$ implies $k_r = k_s = 0$). Thus, for any $1 \le i \le n-1$, we have that

$$\rho(g_i^{a_i})\rho(g_{i+1}^{a_{i+1}} \dots g_n^{a_n}) = \flat_{\mathfrak{g}}\omega_{\bar{\bar{a}}}(g_i^{a_i}, g_{i+1}^{a_{i+1}} \dots g_n^{a_n})\rho(g_i^{a_i} \dots g_n^{a_n}) = \rho(g_i^{a_i} \dots g_n^{a_n}),$$

for all $0 \le a_l \le m_l - 1$, $i \le l \le n$. By mathematical induction, it follows that $\rho(g_1^{a_1} \dots g_n^{a_n}) = \rho(g_1^{a_1}) \dots \rho(g_n^{a_n})$, for all $g_l^{a_l} \in C_l$, $1 \le l \le n$.

Likewise, for $1 \le l \le n$, $j_s = \delta_{s,l} a$ and $k_t = \delta_{t,l} b$ for all $1 \le s, t \le n$, we have

$$\begin{aligned} \flat_{\mathfrak{g}} \omega_{\overline{\overline{a}}}(g_l^a, g_l^b) &= \prod_{1 \le r<s<t \le n} \xi_{(m_r,m_s,m_t)}^{-c_{rst} k_r j_s f_t} \prod_{1 \le r<s<t \le n} \xi_{(m_r,m_s,m_t)}^{-c_{rst} f_r k_s j_t} \prod_{1 \le r<s<t \le n} \xi_{(m_r,m_s,m_t)}^{c_{rst} k_r f_s j_t} \\ &= \prod_{1 \le r<l<t \le n} \xi_{(m_r,m_l,m_t)}^{-c_{rlt} a k_r f_t} \prod_{1 \le r<s<l \le n} \xi_{(m_r,m_s,m_l)}^{-c_{rsl} a f_r k_s} \prod_{1 \le r<s<l \le n} \xi_{(m_r,m_s,m_l)}^{c_{rsl} a k_r f_s} = 1, \end{aligned}$$

and therefore $\rho(g_l^a)\rho(g_l^b) = \rho(g_l^{(a+b)'_{m_l}})$. It is now clear that

$$\begin{aligned} \rho(g_1^{(j_1+k_1)'_{m_1}} \dots g_n^{(j_n+k_n)'_{m_n}}) &= \rho(g_1^{j_1})\rho(g_1^{k_1}) \dots \rho(g_n^{j_n})\rho(g_n^{k_n}) \\ &= \rho(g_1^{j_1} \dots g_n^{j_n})\rho(g_1^{k_1} \dots g_n^{k_n}), \end{aligned}$$

and we conclude that $\flat_{\mathfrak{g}} \omega_{\overline{\overline{a}}}(g_1^{j_1} \dots g_n^{j_n}, g_1^{k_1} \dots g_n^{k_n}) = 1$, for all $g_1^{j_1} \dots g_n^{j_n}, g_1^{k_1} \dots g_n^{k_n} \in G$, as stated.

It remains to show that (2.2.6) is equivalent to the divisibility in (2.2.5). We are free to assume that $n \ge 3$, since for $n = 2$ there is nothing to prove.

Clearly, for all $0 \le k_l, j_l \le m_l - 1$, $1 \le l \le n$, (2.2.6) is equivalent to

$$M | S := - \sum_{1 \le r<s<t \le n} M_{rst}^c (k_r j_s f_t + f_r k_s j_t - k_r f_s j_t). \tag{2.2.7}$$

We will rewrite S so that we can get the scalar $k_r j_s$ as common factor, where r and s, for any $1 \le r < s \le n$, are considered fixed. Towards this end, we split S into a sum of three parts: $S_1 := -\sum_{1 \le r<s<t \le n} M_{rst}^c k_r j_s f_t$, $S_2 := -\sum_{1 \le r<s<t \le n} M_{rst}^c f_r k_s j_t$ and $S_3 := \sum_{1 \le r<s<t \le n} M_{rst}^c k_r f_s j_t$. For a given pair (r, s) with $1 \le r < s \le n$, the scalar $k_r j_s$ appears as a summand of S_1 defined by $r < s < w \le n$, as a summand of S_2 determined by $1 \le u < r < s$ and as a summand of S_3 given by $r < v < s$. Hence, (2.2.7) is equivalent to the fact that, for all $0 \le k_l, j_l \le m_l - 1$, $1 \le l \le n$,

$$M | \sum_{1 \le r<s \le n} k_r j_s \left(-\sum_{u=1}^{r-1} M_{urs}^c f_u + \sum_{v=r+1}^{s-1} M_{rvs}^c f_v - \sum_{w=s+1}^{n} M_{rsw}^c f_w \right).$$

It is now clear that this divisibility is equivalent to (2.2.5). □

Our next objective is translating condition (2.2.5) to a form suitable for concrete computations. We start by analyzing the case $n \ge 3$, where G is the product of more than 2 cyclic groups, as the case $n = 2$ is more complicated than it appears, as we will see later in this section and in Appendix A. We will deal with the simplest case, $n = 1$, last.

For $n \ge 3$, the divisibility in (2.2.5) reduces to a system of $\binom{n}{2}$ linear congruences mod M, in n unknowns $f_1, \dots, f_n$. Equivalently, (2.2.5) reduces to a homogeneous linear system in $n + \binom{n}{2}$ unknowns $f_1, \dots, f_n, x_{rs}$, where $1 \le r < s \le n$, and $\binom{n}{2}$ equations; namely,

$$XA = 0, \tag{2.2.8}$$

with $X = (f_1, \ldots, f_n, x_{12}, \ldots, x_{1n}, x_{23}, \ldots, x_{2n}, \ldots, x_{n-1n})$ and

$$A = \begin{pmatrix} A_{n-1} & A_{n-2} & \cdots & A_2 & A_1 \\ MI_{n-1} & 0 & \cdots & 0 & 0 \\ 0 & MI_{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & MI_1 \end{pmatrix},$$

where

$$A_{n-j} = \begin{pmatrix} M^c_{1jj+1} & M^c_{1jj+2} & \cdots & M^c_{1jn} \\ \vdots & \vdots & \vdots & \vdots \\ M^c_{j-1jj+1} & M^c_{j-1jj+2} & \cdots & M^c_{j-1jn} \\ 0 & 0 & \cdots & 0 \\ 0 & -M^c_{jj+1j+2} & \cdots & -M^c_{jj+1n} \\ M^c_{jj+1j+2} & 0 & \cdots & -M^c_{jj+2n} \\ M^c_{jj+1j+3} & M^c_{jj+2j+3} & \cdots & -M^c jj+3n \\ \vdots & \vdots & \vdots & \vdots \\ M^c_{jj+1n} & M^c_{jj+2n} & \cdots & 0 \end{pmatrix}$$

is an n by $n-j$ matrix, for all $1 \leq j \leq n-1$. The system (2.2.8) can be solved as follows.

A is an $n + \mathfrak{n}$ by $\mathfrak{n}$ matrix with integer coefficients, where, for simplicity, we have denoted $\begin{pmatrix} n \\ 2 \end{pmatrix}$ by $\mathfrak{n}$. As the ring of integers $\mathbb{Z}$ is a PID, there exist unimodular matrices U, V such that $UAV = \operatorname{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$; here U is an $n + \mathfrak{n}$-square matrix, V is an $\mathfrak{n}$-square matrix and $d_1, \ldots, d_r$ are non-zero integers such that $d_1 | d_2 | \ldots | d_r$, $1 \leq r \leq \mathfrak{n}$. Thus (2.2.8) is equivalent to $(XU^{-1})D = 0$, where $D = \operatorname{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$. This says that $XU^{-1} = (0, \ldots, 0, y_{r+1}, \ldots, y_{\mathfrak{n}})$ for some integers $y_{r+1}, \ldots, y_{\mathfrak{n}}$. Then $X = U(0, \ldots, 0, y_{r+1}, \ldots, y_{\mathfrak{n}})$ is the general solution of (2.2.8), solution determined by an arbitrary family of integers $y_{r+1}, \ldots, y_{\mathfrak{n}}$. Furthermore, we show that $r = \mathfrak{n}$ and that U can be somehow obtained from the Schmidt normal form $D' = U'A'V'$ of $A' := (A_{n-1}, \ldots, A_1)$, an n by $\mathfrak{n}$ matrix.

Indeed, if, in general, $O_{m,p}$ stands for the zero m by p matrix and $O_m = O_{m,m}$, we see that

$$\begin{pmatrix} U' & O_{n,\mathfrak{n}} \\ O_{\mathfrak{n},n} & V'^{-1} \end{pmatrix} \begin{pmatrix} A' \\ MI_{\mathfrak{n}} \end{pmatrix} V' = \begin{pmatrix} U'A' \\ MV'^{-1} \end{pmatrix} V' = \begin{pmatrix} D' \\ MI_{\mathfrak{n}} \end{pmatrix} =: B.$$

Assume now that $D' = \operatorname{diag}(d'_1, \ldots, d_{r'}, 0, \ldots, 0)$, for non-zero integers $d'_1, \ldots, d'_{r'}$ which satisfy $d'_1 | \ldots | d'_{r'}$; $r' \leq n$ is the rank of A'. For each $1 \leq j \leq r'$, consider integers α_j, β_j such that $\alpha_j d'_j + \beta_j M = (d'_j, M)$, and define U'' as being the matrix

$$\left(\begin{array}{cccccccccccc}
\alpha_1 & 0 & \cdots & 0 & 0 & & \beta_1 & 0 & \cdots & 0 & 0 & \\
0 & \alpha_2 & \cdots & 0 & 0 & & 0 & \beta_2 & \cdots & 0 & 0 & \\
\vdots & \vdots & \cdots & \cdots & \vdots & O_{r',n-r'} & \vdots & \vdots & \cdots & \vdots & \vdots & O_{r',\mathfrak{n}-r'} \\
0 & 0 & \cdots & \alpha_{r'-1} & 0 & & 0 & 0 & \cdots & \beta_{r'-1} & 0 & \\
0 & 0 & \cdots & 0 & \alpha_{r'} & & 0 & 0 & \cdots & 0 & \beta_{r'} & \\
 & & O_{\mathfrak{n}-r',r'} & & & O_{\mathfrak{n}-r',n-r'} & & & O_{\mathfrak{n}-r',r'} & & & I_{\mathfrak{n}-r'} \\
-\frac{M}{(d'_1,M)} & 0 & \cdots & 0 & 0 & & \frac{d'_1}{(d'_1,M)} & 0 & \cdots & 0 & 0 & \\
0 & -\frac{M}{(d'_2,M)} & \cdots & 0 & 0 & & 0 & \frac{d'_2}{(d'_2,M)} & \cdots & 0 & 0 & \\
\vdots & \vdots & \cdots & \vdots & \vdots & O_{r',n-r'} & \vdots & \vdots & \cdots & \vdots & \vdots & O_{r',\mathfrak{n}-r'} \\
0 & 0 & \cdots & -\frac{M}{(d'_{r'-1},M)} & 0 & & 0 & 0 & \cdots & \frac{d'_{r'-1}}{(d'_{r'-1},M)} & 0 & \\
0 & 0 & \cdots & 0 & -\frac{M}{(d'_{r'},M)} & & 0 & 0 & \cdots & 0 & \frac{d'_{r'}}{(d'_{r'},M)} & \\
 & & O_{n-r',r'} & & & I_{n-r'} & & & O_{n-r',r'} & & & O_{n-r',\mathfrak{n}-r'}
\end{array}\right).$$

The Laplace's cofactor expansion along the first two rows of U'', together with a mathematical induction on r', ensures that the determinant of U'' is equal to $(-1)^{r'}$. Also, by direct computation, we reach the equality

$$U''B = \mathrm{diag}((d'_1, M), \ldots, (d'_{r'}, M), M, \ldots, M) = D,$$

which says that the invariant factors of B, and therefore of A, are

$$(d'_1, M)|\ldots|(d'_{r'}, M)|M \ldots |M,$$

and thus in number of $\mathfrak{n}$. Finally, we can take $U = U''\begin{pmatrix} U' & O_{n,\mathfrak{n}} \\ O_{\mathfrak{n},n} & V'^{-1} \end{pmatrix}$ and $V = V'$ as unimodular matrices obeying $UAV = D$, from where we deduce that (2.2.8) has the solution $X = (0, \ldots, 0, y_1, \ldots, y_n)U$, after renumbering the y's. It follows now that

$$\begin{aligned}(f_1, \ldots, f_n) &= X \begin{pmatrix} I_n & O_{n,\mathfrak{n}} \\ O_{\mathfrak{n},n} & O_{\mathfrak{n},\mathfrak{n}} \end{pmatrix} \\ &= \left(-\frac{y_1 M}{(d'_1, M)}, \ldots, -\frac{y_{r'} M}{(d'_{r'}, M)}, y_{r'+1}, \ldots, y_n\right) U'\end{aligned} \tag{2.2.9}$$

is the solution of (2.2.5); here $y_1, \ldots, y_n$ are arbitrary integers.

The method presented above allows us to concretely determine the solution of (2.2.8), in particular, in the case when $n = 3$. In general, for an m by p matrix A with integer entries and $1 \le j \le \min(m, p)$, $\Delta_j(A)$ is our notation for the greatest common divisor of all j by j minors of A; we make the convention that if all j by j minors of A are equal to zero, then $\Delta_j(A) = 0$. It is well known that, modulo a sign, the invariants factors $d_1, \ldots, d_r$ of A are given by $d_j = \frac{\Delta_j(A)}{\Delta_{j-1}(A)}$, for all $1 \le j \le r$, where $\Delta_0(A) := 1$.

Example 2.2.5 Using the same notation as in Proposition 2.2.4, specializing it to the case $n = 3$, $\flat_{\mathfrak{g}}\omega_{\underline{a}}$ is coboundary if and only if

$$f_1 = \mu M', \quad f_2 = \nu M', \quad f_3 = \lambda M',$$

for some natural numbers λ, μ, ν, where $M' = \frac{M}{(M, c_{123})}$.

Proof. The matrix A takes the form $\begin{pmatrix} A' \\ MI_3 \end{pmatrix}$; here $A' = \begin{pmatrix} 0 & 0 & \upsilon \\ 0 & -\upsilon & 0 \\ \upsilon & 0 & 0 \end{pmatrix}$, where $\upsilon := M^c_{123} = \frac{c_{123}M}{(m_1,m_2,m_3)} = c_{123}$. If $\mathfrak{d} := (\upsilon, M)$, one can see easily that $\Delta_1(A) = \mathfrak{d}$, $\Delta_2(A) = (\upsilon^2, M^2, \upsilon M) = \mathfrak{d}^2\left(\left(\frac{\upsilon}{\mathfrak{d}}\right)^2, \left(\frac{M}{\mathfrak{d}}\right)^2, \frac{\upsilon}{\mathfrak{d}}\frac{M}{\mathfrak{d}}\right) = \mathfrak{d}^2$ and, similarly, $\Delta_3(A) = (\upsilon^3, M^3, \upsilon^2 M, \upsilon M^2) = \mathfrak{d}^3$. Thus, $d_1 = d_2 = d_3 = \mathfrak{d}$. Also, one can see directly that for $u, w \in \mathbb{Z}$ such that $u\upsilon + wM = \mathfrak{d}$, the matrix

$$U = \begin{pmatrix} 0 & 0 & u & w & 0 & 0 \\ 0 & -u & 0 & 0 & w & 0 \\ u & 0 & 0 & 0 & 0 & w \\ 0 & 0 & -M' & \upsilon' & 0 & 0 \\ 0 & M' & 0 & 0 & \upsilon' & 0 \\ -M' & 0 & 0 & 0 & 0 & \upsilon' \end{pmatrix} \text{ satisfies } UA = \begin{pmatrix} \mathfrak{d} & 0 & 0 \\ 0 & \mathfrak{d} & 0 \\ 0 & 0 & \mathfrak{d} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and that U is unimodular, provided that $M' = \frac{M}{\mathfrak{d}}$ and $\upsilon' = \frac{\upsilon}{\mathfrak{d}}$. Hence, for $n = 3$, the system (2.2.8) has the space of solutions

$$\begin{aligned} &\{(0,0,0,-\lambda,\nu-\mu)U|\mu,\nu,\lambda \in \mathbb{Z}\} \\ &\qquad = \{(\mu M', \nu M', \lambda M', -\lambda\upsilon', \nu\upsilon', -\mu\upsilon')|\mu,\nu,\lambda \in \mathbb{Z}\}, \end{aligned}$$

concluding the proof. □

Remark 2.2.6 By the proof of Proposition 2.2.4, $\flat_{\mathfrak{g}}\omega_{\underline{a}}$ is coboundary if and only if (2.2.6) holds, in which case $\flat_{\mathfrak{g}}\omega_{\underline{a}} = \flat_{\mathfrak{g}}\omega_{\bar{a}}$. It is easy to see that (2.2.6) holds in the case when $(m_r, m_s, m_t)|c_{rst}(f_r, f_s, f_t)$, for all $1 \le r < s < t \le n$. For $n = 3$, Example 2.2.5 tells us that $M = (m_1, m_2, m_3) \mid (M, c_{123})(f_1, f_2, f_3)$, which is equivalent to $(m_1, m_2, m_3)|c_{123}(f_1, f_2, f_3)$, is a necessary and sufficient condition for $\flat_{\mathfrak{g}}\omega_{\underline{a}}$ to be coboundary.

As a concrete example, take $m_1 = m_2 = m_3 = 4$. Then $\mathfrak{g} \in G$ satisfying (2.2.6) is $\mathfrak{g} = 1_G$ if $c_{123} \in \{1, 3\}$, and respectively $\mathfrak{g} \in \{g_1^i g_2^j g_3^k \mid i, j, k \in \{0, 2\}\}$ if $c_{123} = 2$.

Remark 2.2.7 The rank of A' is not, in general, equal to n. For instance, we are about to see that, for $n = 4$, the rank of A' is always less or equal to 3. To simplify the notation, if $a_1, \ldots, a_4 \in \mathbb{Z}$, we denote by $a_{i_1 \ldots i_k}$ the greatest common divisor of $a_{i_1}, \ldots, a_{i_k}$, for all $1 \le k \le 4$ and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, 4\}$.

For $n = 4$,

$$A = \begin{pmatrix} A' \\ MI_6 \end{pmatrix}, \quad \text{where } A' = \begin{pmatrix} 0 & 0 & 0 & -a_1 & -a_2 & -a_3 \\ 0 & a_1 & a_2 & 0 & 0 & -a_4 \\ -a_1 & 0 & a_3 & 0 & a_4 & 0 \\ -a_2 & -a_3 & 0 & -a_4 & 0 & 0 \end{pmatrix},$$

provided that $a_1 = -M^c_{123}$, $a_2 = -M^c_{124}$, $a_3 = -M^c_{134}$ and $a_4 = -M^c_{234}$. We have that $\Delta_1(A') = a_{1234}$,

$$\begin{aligned} \Delta_2(A') &= (a_i^2, a_i a_j | 1 \le i < j \le 4) \\ &= (a_1 a_{12}, a_1 a_{34}, a_2 a_{23}, a_2 a_{14}, a_3 a_{34}, a_3 a_{12}, a_4 a_{34}, a_4 a_{12}) \\ &= (a_1 a_{1234}, a_2 a_{1234}, a_3 a_{1234}, a_4 a_{1234}) = a_{1234}^2, \\ \Delta_3(A') &= (a_i a_j a_l | 1 \le i, j, l \le 4) \\ &= a_{1234}(a_t^2, a_i a_j | 1 \le t \le 4, 1 \le i < j \le 4) = a_{1234}^3, \end{aligned}$$

and $\Delta_4(A') = 0$. Thus, $r' \leq 3$ and $r' = 3$ if and only if at least one of the a_j is non-zero. Now, if U', V' are unimodular matrices satisfying

$$U'A'V' = \begin{pmatrix} a_{1234}I_3 & O_3 \\ O_{1,3} & O_{1,3} \end{pmatrix},$$

it follows that the solution of (2.2.8) is parametrized by $y_1, y_2, y_3, y_4 \in \mathbb{Z}$ such that

$$(f_1, f_2, f_3, f_4) = \Big(-\frac{y_1 M}{(a_{1234}, M)}, -\frac{y_2 M}{(a_{1234,M})}, -\frac{y_3 M}{(a_{1234}, M)}, y_4\Big)\, U'.$$

We now come back to the general case $n \geq 3$. For $G = C_{m_1} \times \ldots \times C_{m_n}$ a finite abelian group and $\omega_{\underline{a}}$ a 3-cocycle on G determined by $\underline{a} \in \mathcal{A}$ as in (2.2.4), denote by $Y_{\underline{y}}$ the 1 by n matrix defined as follows by $\underline{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n$:

$$Y_{\underline{y}} = \Big(\frac{-y_1 M}{(d'_1, M)}, \ldots, \frac{-y_{r'} M}{(d'_{r'}, M)}, y_{r'+1}, \ldots, y_n\Big).$$

As before, $M = LCM\{(m_r, m_s, m_t) | 1 \leq r < s < t \leq n\}$ and $d'_1 | \ldots | d'_{r'}$ are the invariant factors of the matrix A' that is part of the definition of the matrix A associated to the system (2.2.8). So, if U', V' are unimodular and such that $U'A'V' = \mathrm{diag}(d'1, \ldots, d'_{r'}, 0, \ldots, 0)$, then $Y_{\underline{y}}U'$ is a solution for (2.2.8) and, moreover, any solution of (2.2.8) is of this type.

Furthermore, denote by C the symmetric matrix which has the main diagonal given by the elements $\frac{c_1}{m_1^2}, \ldots, \frac{c_n}{m_n^2}$ and on the position (s, t) the element $\frac{c_{st}}{2m_s m_t}$, for all $1 \leq s < t \leq n$. Also, denote by C' the matrix $U'C^tU'$, where ${}^tZ$ stands for the transpose matrix of any matrix Z. It follows that C' is a symmetric matrix with rational entries.

Proposition 2.2.8 *Let $G = C_1 \times \ldots \times C_n$ be a finite abelian group as in the above, and $\omega_{\underline{a}}$ a 3-cocycle on G defined by $\underline{a} \in \mathcal{A}$. Let*

$$N = \mathrm{LCM}\{m_j^2, m_s m_t | 1 \leq j \leq n, 1 \leq s < t \leq n\}.$$

1. *If there exists a non-trivial 2-dimensional braided Hopf algebra in ${}^{k^G_{\omega_{\underline{a}}}}_{k^G_{\omega_{\underline{a}}}}\mathcal{YD}$, then N is even. Consequently, at least one of $m_1, \ldots, m_n$ is even.*

2. *Assuming N is even, there is a one-to-one correspondence between non-trivial 2-dimensional braided Hopf algebras in ${}^{k^G_{\omega_{\underline{a}}}}_{k^G_{\omega_{\underline{a}}}}\mathcal{YD}$ and $\underline{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ for which there exists $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that, for $\Lambda' := \left(\frac{\lambda_1}{m_1}, \ldots, \frac{\lambda_n}{m_n}\right) U'$,*

$$\Lambda'\ {}^tY_{\underline{y}} + Y_{\underline{y}}\ C'\ {}^tY_{\underline{y}} - \frac{1}{2} \in \mathbb{Z}. \tag{2.2.10}$$

Proof. Owing to Proposition 2.2.3, the non-trivial 2-dimensional Hopf algebras within ${}^{k^G_{\omega_{\underline{a}}}}_{k^G_{\omega_{\underline{a}}}}\mathcal{YD}$ are characterized by pairs $(\mathfrak{g}, \rho)$ consisting of an element $\mathfrak{g} = g_1^{f_1} \ldots g_n^{f_n} \in G$ and a map $\rho : G \to k^*$ fulfilling $\rho(e) = 1$, $\rho(\mathfrak{g}) = -1$ and $\flat_{\mathfrak{g}}\omega_{\underline{a}} = \partial\rho$. By keeping the notation established in the proof of Proposition 2.2.4, such a pair $(\mathfrak{g}, \rho)$ is determined by a solution $(f_1, \ldots, f_n)$ of (2.2.8) and a group morphism $\vartheta : G \to k^*$ obeying

$\vartheta(\mathfrak{g})f_{\mathfrak{g}}(\mathfrak{g}) = -1$; in general, if σ is a coboundary 2-cocycle and $\sigma = \partial\rho_1 = \partial\rho_2$, for some maps $\rho_1, \rho_2 : G \to k^*$, then $\vartheta : G \to k^*$ given by $\vartheta(g) = \rho_1(g)\rho_2(g)^{-1}$, for all $g \in G$, is a group morphism. But ϑ is given by an n-tuple $(\lambda_1, \ldots, \lambda_n)$ with $0 \leq \lambda_j \leq m_j - 1$, for all $1 \leq j \leq n$, in the sense that $\vartheta(g_1^{a_1} \ldots g_n^{a_n}) = \xi_{m_1}^{\lambda_1 a_1} \ldots \xi_{m_n}^{\lambda_n a_n}$, for all $0 \leq a_j \leq m_j - 1$, with $1 \leq j \leq n$.

Take ξ_N to be a primitive root of unity of order N in k and assume, without loss of generality, that

$$\xi_{m_j} = \xi_N^{\frac{N}{m_j}}, \quad \xi_{m_j^2} = \xi_N^{\frac{N}{m_j^2}}, \quad \text{and} \quad \xi_{m_s m_t} = \xi_N^{\frac{N}{m_s m_t}},$$

for all $1 \leq j \leq n$ and $1 \leq s < t \leq n$. Then,

$$\begin{aligned} \vartheta(g_1^{a_1} \ldots g_n^{a_n}) &= \xi_N^{\sum\limits_{j=1}^{n} \frac{\lambda_j a_j}{m_j} N} \quad \text{and} \\ f_{\mathfrak{g}}(g_1^{a_1} \ldots g_n^{a_n}) &= \xi_N^{\sum\limits_{j=1}^{n} \frac{c_j a_j f_j}{m_j^2} N + \sum\limits_{1 \leq s < t \leq n} \frac{c_{st} f_t a_s}{m_t m_s} N}, \end{aligned} \tag{2.2.11}$$

for all $0 \leq a_j \leq m_j - 1$, $1 \leq j \leq n$. Therefore, giving a non-trivial 2-dimensional Hopf algebra within ${}^{k^G_{\omega_{\underline{a}}}}_{k^G_{\omega_{\underline{a}}}}\mathcal{YD}$ is equivalent to giving a solution $(f_1, \ldots, f_n)$ of (2.2.8) such that, for some integers $\lambda_1, \ldots, \lambda_n$,

$$\zeta_N^{\sum\limits_{j=1}^{n} \frac{\lambda_j f_j}{m_j} N + \sum\limits_{j=1}^{n} \frac{c_j f_j^2}{m_j^2} N + \sum\limits_{1 \leq s < t \leq n} \frac{c_{st} f_s f_t}{m_s m_t} N} = -1. \tag{2.2.12}$$

Moreover, it is clear that (2.2.12) holds if and only if N is even and

$$\sum_{j=1}^{n} \frac{\lambda_j f_j}{m_j} + \sum_{j=1}^{n} \frac{c_j f_j^2}{m_j^2} + \sum_{1 \leq s < t \leq n} \frac{c_{st} f_s f_t}{m_s m_t} - \frac{1}{2} \in \mathbb{Z}.$$

As any solution $(f_1, \ldots, f_n)$ of (2.2.8) is of the form $Y_{\underline{y}}U'$, for some $\underline{y} \in \mathbb{Z}^n$, our proof follows. □

We use the isomorphism provided by (2.2.2) in order to return to our initial context: G is a finite abelian group and $k_{\Phi_{\underline{a}}}[G]$ is the Hopf group algebra of k and G considered as a quasi-Hopf algebra with reassociator

$$\Phi_{\underline{a}} = \sum_{\mathbf{i},\mathbf{j},\mathbf{l} \in \mathbf{N}} \omega_{\underline{a}}(g_1^{i_1} \ldots g_n^{i_n}, g_1^{j_1} \ldots g_n^{j_n}, g_1^{l_1} \ldots g_n^{l_n}) 1_{i_1,\ldots,i_n} \otimes 1_{j_1,\ldots,j_n} \otimes 1_{l_1,\ldots,l_n}.$$

Here, assuming $C = C_{m_1} \times \ldots \times C_{m_n}$ as in the above, we have denoted by $\mathbf{i}$ an n-tuple $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ with the property that $0 \leq i_l \leq m_l - 1$, for all $1 \leq l \leq n$, and by $\mathbf{N}$ the set of all these n-tuples $\mathbf{i}$.

Corollary 2.2.9 *With the same notation as in Proposition 2.2.8, if there exists a non-trivial 2-dimensional Hopf algebra within ${}^{k_{\Phi_{\underline{a}}}[G]}_{k_{\Phi_{\underline{a}}}[G]}\mathcal{YD}$, then N is even. Furthermore, assuming N even, a non-trivial Hopf algebra of dimension 2 in ${}^{k_{\Phi_{\underline{a}}}[G]}_{k_{\Phi_{\underline{a}}}[G]}\mathcal{YD}$ is of the form $B_{\sigma,v}$, where $\sigma : k[G] \to k$ is an algebra morphism and v is an element* *in $k[G]$, and both are determined by a solution $\mathfrak{f} = (f_1, \ldots, f_n) \in \mathbb{Z}^n$ of (2.2.8) and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ obeying (2.2.12) as follows:*

$$\begin{aligned}
\sigma(g_1^{j_1} \ldots g_n^{j_n}) &= \prod_{l=1}^{n} \xi_{m_l}^{f_l j_l}, \\
v &= \sum_{\mathfrak{l} \in \mathbf{N}} \zeta_N^{\left(\sum_{j=1}^{n} \frac{\lambda_j l_j}{m_j} + \sum_{j=1}^{n} \frac{c_j f_j l_j}{m_j^2} + \sum_{1 \le s < t \le n} \frac{c_{st} l_s f_t}{m_s m_t}\right) N} 1_{l_1, \ldots, l_n},
\end{aligned}$$

for all $g_1^{j_1} \ldots g_n^{j_n} \in G$.

Proof. It follows by Proposition 2.2.8 and the quasi-Hopf algebra identification $k^G \cong k[G]$ produced by the correspondence $P_{g_1^{j_1} \ldots g_n^{j_n}} \mapsto 1_{j_1, \ldots, j_n}$. Keeping in mind the proof of Lemma 2.2.2, for $\mathfrak{f}$ and $(\lambda_1, \ldots, \lambda_n)$ as in the statement, we deduce that, for ϑ and $f_{\mathfrak{g}}$ as in (2.2.11),

$$\sigma(g_1^{j_1} \ldots g_n^{j_n}) = \prod_{l=1}^{n} \left(\sum_{t=0}^{m_l - 1} \xi_{m_l}^{t j_l} \widetilde{P}_{g_l^t}(g_1^{f_1} \ldots g_n^{f_n}) \right) = \prod_{l=1}^{n} \xi_{m_l}^{f_l j_l},$$

and

$$\begin{aligned}
v &= \sum_{\mathfrak{l} \in \mathbf{N}} (\vartheta f_{\mathfrak{g}})(g_1^{l_1} \ldots g_n^{l_n}) 1_{l_1, \ldots, l_n} \\
&= \sum_{\mathfrak{l} \in \mathbf{N}} \zeta_N^{\left(\sum_{j=1}^{n} \frac{\lambda_j l_j}{m_j} + \sum_{j=1}^{n} \frac{c_j f_j l_j}{m_j^2} + \sum_{1 \le s < t \le n} \frac{c_{st} l_s f_t}{m_s m_t}\right) N} 1_{l_1, \ldots, l_n},
\end{aligned}$$

as stated. Note that $\sigma(v) = -1$ is equivalent to (2.2.12), since $\sigma(1_{j_1, \ldots, j_n}) = \delta_{\mathfrak{j}, \mathfrak{f}}$. □

Examples 2.2.10 Let $n = 3$, $G = C_{m_1} \times C_{m_2} \times C_{m_3}$ and $\omega_{\underline{a}}$ be a 3-cocycle on G. Recall that $M = (m_1, m_2, m_3)$, $M' = \frac{M}{(M, c_{123})}$ and that any solution of (2.2.8) is of the form $(f_1, f_2, f_3) = (\mu_1 M', \mu_2 M', \mu_3 M')$, with μ_1, μ_2, μ_3 integers. It is immediate that $M' | M | m_j$, for all $1 \le j \le 3$. Set now $m'_j := \frac{m_j}{M'}$, for all $1 \le j \le 3$.

In this particular case, (2.2.12) becomes

$$\sum_{j=1}^{3} \frac{\lambda_j \mu_j}{m'_j} + \sum_{j=1}^{3} \frac{c_j \mu_j^2}{m'^2_j} + \sum_{1 \le r < s \le 3} \frac{c_{rs} \mu_r \mu_s}{m'_r m'_s} - \frac{1}{2} \in \mathbb{Z}.$$

For this to happen, at least one of the integers m'_1, m'_2, m'_3 has to be even. Therefore, we assume that the set $J_e := \{1 \le j \le 3 | m'_j \in 2\mathbb{Z}\}$ is non-empty. Furthermore, we set $\mu_j = 0$, for all $j \in \{1, 2, 3\} \backslash J_e$, and $\mu_j = \frac{m'_j}{2}$, for all $j \in J_e$. Then, if we set

$$\mathfrak{c} := \sum_{j \in J_e} c_j + \sum_{r, s \in J_e; r < s} c_{rs},$$

(2.2.12) becomes

$$\sum_{j \in J_e} \frac{\lambda_j}{2} + \frac{\mathfrak{c}}{4} - \frac{1}{2} \in \mathbb{Z}.$$

The latest condition is fulfilled if and only if $\mathfrak{c}$ is even and $\sum_{j\in J_e}\lambda_j+\frac{\mathfrak{c}}{2}$ is an odd number. Thus, if J_e is non-empty and $\mathfrak{c}$ is even, there exists at least one non-trivial 2-dimensional Hopf algebra within ${}^{k_{\Phi_{\underline{a}}}[G]}_{k_{\Phi_{\underline{a}}}[G]}\mathcal{YD}$.

The condition "$\mathfrak{c}$ is even" is only sufficient. For instance, we can take $m_1=12$, $m_2=18$, $m_3=30$ and $c_{123}=4$, so that $M=6$, $M'=3$, $m'_1=4$, $m'_2=6$, $m'_3=10$ and $J_e=\{1,2,3\}$. Then, for $c_1=3$, $c_2=7$, $c_3=14$, $c_{12}=1$, $c_{13}=3$ and $c_{23}=5$, we compute $\mathfrak{c}=33$, an odd number. Furthermore, for $(f_1,f_2,f_3)=(3\mu_1,3\mu_2,3\mu_3)$ with $0\le\mu_1\le 3$, $0\le\mu_2\le 5$ and $0\le\mu_3\le 9$, a solution of (2.2.8) in this particular case, the condition (2.2.12) becomes

$$\frac{3}{4}\lambda_1\mu_1+\frac{1}{2}\lambda_2\mu_2+\frac{3}{10}\lambda_3\mu_3+\frac{27}{16}\mu_1^2+\frac{7}{4}\mu_2^2+ \\ +\frac{63}{50}\mu_3^2+\frac{3}{8}\mu_1\mu_2+\frac{27}{40}\mu_1\mu_3+\frac{3}{4}\mu_2\mu_3-\frac{1}{2}\in\mathbb{Z}.$$

For $\mu_2=2$ and $\mu_3=5$, this reduces to

$$\frac{3}{4}\lambda_1\mu_1+\frac{3}{2}\lambda_3+\frac{27}{16}\mu_1^2+\frac{33}{8}\mu_1-\frac{1}{2}\in\mathbb{Z},$$

and is satisfied in each of the following cases: (1) $\mu_1=0$, $0\le\lambda_1\le 11$, $0\le\lambda_2\le 17$ and $\lambda_3\in\{1,3,\dots,29\}$; (2) $\mu_1=2$, $0\le\lambda_2\le 17$, and $0\le\lambda_1\le 11$ and $0\le\lambda_3\le 29$ such that $\lambda_1+\lambda_3$ is odd. We conclude that this case provides plenty of non-trivial 2-dimensional braided Hopf algebras, although $\mathfrak{c}$ is odd.

We conclude the list of examples by pointing out that it might happen to have no non-trivial 2-dimensional braided Hopf algebras; for example, when $n=3$ and m'_j is odd, for all $1\le j\le 3$.

Next, we deal with the case $n=2$. We expected things to be simpler here but, as we will see, this is not the case. To start, assume $G=C_{m_1}\times C_{m_2}$ with C_{m_j} generated by g_j, an element of order $m_j\ge 2$, with $1\le j\le 2$. As before, set $M=(m_1,m_2)$, the greatest common divisor of m_1, m_2, and let $\omega_{\underline{a}}$ be the 3-cocycle on G defined by $\underline{a}=(c_1,c_2,c_{12})$, where $0\le c_j\le m_j-1$ and $0\le c_{12}\le M$, as follows:

$$\omega_{\underline{a}}(g_1^{i_1}g_2^{i_2},g_1^{j_1}g_2^{j_2},g_1^{k_1}g_2^{k_2})=\xi_{m_2}^{c_{12}i_2\lfloor\frac{j_1+k_1}{m_1}\rfloor}\prod_{l=1}^{2}\xi_{m_l}^{c_li_l\lfloor\frac{j_l+k_l}{m_l}\rfloor}.$$

As we have seen in the proof of Proposition 2.2.4, $\flat_{\mathfrak{g}}\omega_{\underline{a}}=\partial f_{\mathfrak{g}}$ is a coboundary 2-cocycle, for all $\mathfrak{g}=g_1^{f_1}g_2^{f_2}\in G$; here $f_{\mathfrak{g}}:G\to k^*$ is given, for all $g_1^{j_1}g_2^{j_2}\in G$, by

$$f_{\mathfrak{g}}(g_1^{j_1}g_2^{j_2}):=\xi_{m_1m_2}^{c_{12}f_2j_1}\prod_{l=1}^{2}\xi_{m_l^2}^{c_lf_lj_l}.$$

If $k_{\Phi_{\underline{a}}}[G]$ is the quasi-Hopf algebra with reassociator

$$\Phi_{\underline{a}}=\sum_{\mathrm{i},\mathrm{j},\mathrm{l}\in\mathbf{N}}\omega_{\underline{a}}(g_1^{i_1}g_2^{i_2},g_1^{j_1}g_2^{j_2},g_1^{k_1}g_2^{k_2})\,1_{i_1,i_2}\otimes 1_{j_1,j_2}\otimes 1_{l_1,l_2},$$

built on the Hopf group algebra $k[G]$, by using the same arguments as in the proof of Proposition 2.2.8, we get the following result.

Corollary 2.2.11 *Let $N = \mathrm{LCM}(m_1^2, m_2^2, m_1 m_2)$. If there exists a non-trivial 2-dimensional Hopf algebra within ${}^{k\Phi_{\underline{a}}[G]}_{k\Phi_{\underline{a}}[G]}\mathcal{YD}$ then N is even.*

Assuming N even, a non-trivial Hopf algebra of dimension 2 in ${}^{k\Phi_{\underline{a}}[G]}_{k\Phi_{\underline{a}}[G]}\mathcal{YD}$ is of the form $B_{\sigma,v}$, with $\sigma : k[G] \to k$ an algebra morphism and $v \in k[G]$, determined by $\mathfrak{f} = (f_1, f_2), \lambda = (\lambda_1, \lambda_2) \in \mathbf{N}$ satisfying

$$\frac{\lambda_1 f_1}{m_1} + \frac{\lambda_2 f_2}{m_2} + c_1 \frac{f_1^2}{m_1^2} + c_2 \frac{f_2^2}{m_2^2} + c_{12} \frac{f_1 f_2}{m_1 m_2} - \frac{1}{2} \in \mathbb{Z} \qquad (2.2.13)$$

as follows, for all $g_1^{j_1} g_2^{j_2} \in G$:

$$\begin{aligned} \sigma(g_1^{j_1} g_2^{j_2}) &= \xi_N^{\left(\frac{f_1 j_1}{m_1} + \frac{f_2 j_2}{m_2}\right)N}, \\ v &= \sum_{\mathfrak{l} \in \mathbf{N}} \zeta_N^{\left(\sum\limits_{j=1}^{2} \frac{\lambda_j l_j}{m_j} + \sum\limits_{j=1}^{n} \frac{c_j f_j l_j}{m_j^2} + \frac{c_{12} l_1 f_2}{m_1 m_2}\right)N} 1_{l_1, l_2}. \end{aligned}$$

The classification provided by Corollary 2.2.11 entails to condition (2.2.13). We will see in Appendix A that (2.2.13) reduces to finding the rational points of a conic section, an arithmetic problem with a rather laborious solution. However, there are particular situations when (2.2.13) can be solved on the spot, through the use of elementary methods. As an example of such a situation, we will analyze the case when $m_1 = m_2 = 2$; in other words, when G is the Klein group $C_2 \times C_2$. As before, denote by g_1 and g_2 the generators of G and by $\sigma_{f_1 f_2} : G \to k$ the algebra morphism determined by $\sigma_{f_1 f_2}(g_1^{j_1} g_2^{j_2}) = (-1)^{f_1 j_1 + f_2 j_2}$, for all $g_1^{j_1} g_2^{j_2} \in G$; here (f_1, f_2) is a fixed pair of integers f_1, f_2 in the set $\{0, 1\}$. We also denote by i a fixed primitive root of unity of order 4 in k.

Example 2.2.12 For G the Klein four group, the 2-dimensional non-trivial Hopf algebras in ${}^{k\Phi_{\underline{a}}[G]}_{k\Phi_{\underline{a}}[G]}\mathcal{YD}$ are of the form $B_{\sigma,v}$, with $\underline{a} \in \mathcal{A}$, $\sigma : k[G] \to k$ and $v \in k[G]$ given by the table below, using the notation established in Corollary 2.2.11.

$\underline{a} = (c_1, c_2, c_3)$	$\sigma_{f_1 f_2}$	(λ_1, λ_2)	v	$\sigma_{f_1 f_2}$	(λ_1, λ_2)	v
$(0,0,0)$	σ_{01}	$(0,1)$	g_2	σ_{10}	$(1,0)$	g_1
		$(1,1)$	$g_1 g_2$		$(1,1)$	$g_1 g_2$
	σ_{11}	$(0,1)$	g_2	σ_{11}	$(1,0)$	g_1
$(0,0,1)$	σ_{01}	$(0,1)$	$\mathfrak{g}^+ := \frac{1+\mathrm{i}}{2} g_2 + \frac{1-\mathrm{i}}{2} g_1 g_2$	σ_{10}	$(1,0)$	g_1
		$(1,1)$	$\mathfrak{g}^- := \frac{1-\mathrm{i}}{2} g_2 + \frac{1+\mathrm{i}}{2} g_1 g_2$		$(1,1)$	$g_1 g_2$
$(0,1,1)$	σ_{11}	$(0,0)$	$\mathfrak{h}^+ := \frac{1}{2}(g_1 + g_2) + \frac{1}{2}(1 - g_1 g_2)$	σ_{10}	$(1,0)$	g_1
		$(1,1)$	$\mathfrak{h}^- := \frac{1}{2}(g_1 + g_2) - \frac{1}{2}(1 - g_1 g_2)$		$(1,1)$	$g_1 g_2$
$(1,0,0)$	σ_{01}	$(0,1)$	g_2	σ_{01}	$(1,1)$	$g_1 g_2$
$(1,0,1)$	σ_{01}	$(0,1)$	$\mathfrak{g}^+$	σ_{11}	$(0,0)$	g_1
		$(1,1)$	$\mathfrak{g}^-$		$(1,1)$	g_2
$(0,1,0)$	σ_{10}	$(1,0)$	g_1	σ_{10}	$(1,1)$	$g_1 g_2$
$(1,1,0)$	σ_{11}	$(0,0)$	$\mathfrak{h}^+$	σ_{11}	$(1,1)$	$\mathfrak{h}^-$

Proof. The 2-dimensional non-trivial Hopf algebras in ${}^{k\Phi_{\underline{a}}[G]}_{k\Phi_{\underline{a}}[G]}\mathcal{YD}$ are parametrized by $f_1, f_2, \lambda_1, \lambda_2 \in \{0, 1\}$ such that $4 | 2(\lambda_1 f_1 + \lambda_2 f_2) + c_1 f_1^2 + c_2 f_2^2 + c_{12} f_1 f_2 - 2$. Equivalently, $2 | c_1 f_1^2 + c_2 f_2^2 + c_{12} f_1 f_2$ and $\lambda_1 f_1 + \lambda_2 f_2 + \frac{c_1 f_1^2 + c_2 f_2^2 + c_{12} f_1 f_2}{2}$ is an odd integer. But $c_1f_1^2, c_2f_2^2, c_{12}f_1f_2 \in \{0,1\}$, so, either all are equal to zero, or two of them are equal to one and the last one is equal to zero; in the former case, $\lambda_1f_1+\lambda_2f_2$ is odd, meaning that $\lambda_1f_1 = 1$ and $\lambda_2f_2 = 0$ or vice versa, while in the latter case $\lambda_1f_1+\lambda_2f_2$ is even, and so is $\lambda_1f_1 = \lambda_2f_2 \in \{0,1\}$.

When $c_1f_1^2 = c_2f_2^2 = c_{12}f_1f_2 = 0$ and $\lambda_1f_1+\lambda_2f_2$ is odd, we cannot have f_1, f_2, respectively λ_1, λ_2, simultaneously equal to zero. By considering the cases (f_1, f_2) equal to $(0,1)$, $(1,0)$ and respectively $(1,1)$, we obtain the solutions shown in the table for $\underline{a} \in \{(0,0,0),(0,0,1),(1,0,0),(0,1,0)\}$, as well as those mentioned for $\underline{a} = (1,0,1)$ and $\underline{a} = (0,1,1)$ in the case when $\sigma_{f_1f_2}$ is σ_{01} and σ_{10}, respectively. In a similar manner, we treat the cases: (i) $c_1f_1^2 = 0$, $c_2f_2^2 = c_{12}f_1f_2 = 1$ and $\lambda_1f_1+\lambda_2f_2$ is even; (ii) $c_1f_1^2 = c_2f_2^2 = 1$, $c_{12}f_1f_2 = 0$ and $\lambda_1f_1+\lambda_2f_2$ is even; (iii) $c_1f_1^2 = c_{12}f_1f_2 = 1$, $c_2f_2^2 = 0$ and $\lambda_1f_1+\lambda_2f_2$ is even. Following simple computations, they produce the remaining six parameterizations in the table. Take note that v can be computed in each case by using the formulas

$$1_{0,0} = \frac{1}{4}(1+g_1)(1+g_2); \quad 1_{1,0} = \frac{1}{4}(1-g_1)(1+g_2);$$
$$1_{0,1} = \frac{1}{4}(1+g_1)(1-g_2); \quad 1_{1,1} = \frac{1}{4}(1-g_1)(1-g_2).$$

□

It remains to study the case when $n = 1$, that is, when G is a cyclic group of order $m \geq 2$; we denote by g the generator of G. Towards this end, we fix a primitive root of unity $\mathfrak{q}$ of order m^2 in k and take $q = \mathfrak{q}^m$, a primitive root of unity of order m in k. Then, for $0 \leq c \leq m-1$, $k_{\Phi_c}[G]$ is the Hopf group algebra $k[G]$, considered as a quasi-Hopf algebra via the reassociator

$$\Phi_c = \sum_{i,j,l=0}^{m-1} q^{ci\lfloor \frac{j+l}{m} \rfloor} 1_i \otimes 1_j \otimes 1_l. \tag{2.2.14}$$

As before, $1_j = \frac{1}{m}\sum_{s=1}^{m-1} q^{-js}g^s$ denotes an idempotent of the group algebra, for all $0 \leq j \leq m-1$.

Using again the same arguments as in the proof of Proposition 2.2.8, we get the following result.

Proposition 2.2.13 *If there exists a non-trivial* 2*-dimensional Hopf algebra within* ${}^{k_{\Phi_c}[G]}_{k_{\Phi_c}[G]}\mathcal{YD}$*, the greatest common divisor of* c *and* m *is even.*

Assuming (c,m) *even, a non-trivial Hopf algebra of dimension* 2 *in* ${}^{k_{\Phi_c}[G]}_{k_{\Phi_c}[G]}\mathcal{YD}$ *is of the form* $B_{\sigma,v}$*, with* $\sigma : k[G] \to k$ *an algebra morphism and* $v \in k[G]$*, determined by* $f, \lambda \in \{0,\dots,m-1\}$ *satisfying*

$$\frac{\lambda f}{m} + \frac{cf^2}{m^2} - \frac{1}{2} \in \mathbb{Z}, \tag{2.2.15}$$

as follows, for all g^j *in* G*:*

$$\sigma(g^j) = q^{fj} \quad \textit{and} \quad v = \sum_{j=1}^{m} \mathfrak{q}^{m\lambda j + cfj} 1_j.$$

Proof. If there is a non-trivial Hopf algebra of dimension 2 in ${}^{k_{\Phi_c}[G]}_{k_{\Phi_c}[G]}\mathcal{YD}$, then there exist $f, \lambda \in \{0, \ldots, m-1\}$ and $k \in \mathbb{Z}$ such that $2cf^2 + 2m\lambda f - (2k+1)m^2 = 0$. It follows that m is even, so we only need to prove that c is also even. For $c = 0$, there is nothing to prove, while for $c \neq 0$ the equality $(2cf + m\lambda)^2 = 2c(2k+1)m^2 + \lambda^2 m^2$ says that $2c(2k+1) + \lambda^2$ is a square, let us say it is equal to ω^2, for some scalar ω. Then λ and ω have the same parity, which tells us that c must be even. □

Corollary 2.2.14 *Keeping the above notation, if (c, m) is even there is always a non-trivial Hopf algebra of dimension 2 in ${}^{k_{\Phi_c}[G]}_{k_{\Phi_c}[G]}\mathcal{YD}$.*

Proof. Let $d = (c, m)$, a non-zero even number, and let c' and m' be coprime integers such that $c = dc'$ and $m = dm'$. We saw that (2.2.15) is equivalent to the existence of $z \in \mathbb{Z}$ such that $(2cf + m\lambda)^2 = 2c(2z+1)m^2 + \lambda^2 m^2$. It follows that $m | 2cf + m\lambda$, so $\frac{m}{2} | cf$; if we write $2cf = mt$, for some $t \in \mathbb{Z}$, then $(t+\lambda)^2 = 2c(2z+1) + \lambda^2$, hence $t^2 + 2\lambda t = 2c(2z+1)$. Therefore, t is even and we can write $cf = mv$, for some integer v. Furthermore, as $c'f = m'v$, we find $s \in \mathbb{Z}$ such that $f = sm'$ and $v = sc'$. To conclude the proof, we show that we can always find s, λ, z such that $2s(c's + \lambda) = (2z+1)d$. Indeed, one can take $s = \frac{d}{2}$ and λ, z such that $\lambda = 2z + 1 - \frac{c}{2}$; note that, there is at least one $z \in \mathbb{Z}$ such that $\frac{c}{2} \leq 2z + 1 < m + \frac{c}{2}$, because $m \geq 2$. □

Remark 2.2.15 In the Hopf algebra case, meaning when $c = 0$, non-trivial Hopf algebras of dimension 2 in ${}^{k[G]}_{k[G]}\mathcal{YD}$ are determined by pairs (f, λ) with $\lambda f = (2z+1)\frac{m}{2}$, for some $z \in \mathbb{Z}$.

2.2.2 A non-abelian example

Next, we want to apply Lemma 2.2.2 to a finite non-abelian group G. There are two main obstacles we must avoid while choosing the group G; namely, we want the center of G to be not-trivial and, on the other hand, we need to be able to find explicitly the representatives for the 3-cocycles of G. A first candidate that satisfies both of these conditions is the dihedral group D_n with $2n$ elements, n an even natural number, since for n odd, D_n is centerless; but, although we have a full description for $H^3(D_n, k^*)$ (see [29, 30]) we don't have the concrete formulas for the representatives of the 3-cocycles of D_n. Nevertheless, we can overcome this issue by working with the so called double dihedral group $\bar{D}_n$, instead of D_n. We take $n \geq 2$ since, otherwise, $\bar{D}_1 \simeq C_4$ is an abelian group.

Definition 2.2.16 For any $n \geq 1$, the double dihedral group $\bar{D}_n$ is a group of order $4n$, generated by R and X with relations:

$$R^{2n} = e, \quad X^2 = R^n, \quad \text{and} \quad XR = R^{-1}X,$$

where e denotes the neutral element.

In the following, we sometimes identify the elements of $\bar{D}_n$ with pairs (A, a), in the sense that $(A, a) \equiv X^A R^a$; here $A \in \{0, 1\}$ and $a \in \{-n+1, \ldots, n\}$. Thus, a'_{2n}, the reminder of the division of a by $2n$, is replaced by $\langle a \rangle_{2n}$, a new reminder modulo $2n$ which lies in the range $-n+1, \ldots, n$. Note that $\langle a \rangle_{2n}$ is uniquely determined by the property that $R^a = R^{\langle a \rangle_{2n}}$ and $\langle a \rangle_{2n} \in \{-n+1, \ldots, n\}$, for all $a \in \mathbb{Z}$.

To find a relation between these two reminders, observe that $R^a = R^{a-2\lfloor \frac{a}{2n} \rfloor n}$ with $a - 2\lfloor \frac{a}{2n} \rfloor n \in \{0, \dots, 2n-1\}$, provided that $a \in \{-n+1, \dots, n\}$. Likewise, provided that $a \in \{0, \dots, 2n-1\}$, $R^a = R^{a+2\lfloor \frac{n-a}{2n} \rfloor n}$ with $a + 2\lfloor \frac{n-a}{2n} \rfloor n \in \{-n+1, \dots, n\}$. Therefore, for all $a \in \mathbb{Z}$, we can alternatively define the second remainder by

$$\langle a \rangle_{2n} := a'_{2n} + 2\lfloor \frac{n - a'_{2n}}{2n} \rfloor n = a + 2\lfloor \frac{n-a}{2n} \rfloor n. \tag{2.2.16}$$

Indeed, $R^{2n} = e$ implies $R^a = R^{\langle a \rangle_{2n}}$, and since $\langle a \rangle_{2n}$ can also be written in the form $\langle a \rangle_{2n} = n - (n-a)'_{2n}$, it follows that $\langle a \rangle_{2n} \in \{-n+1, \dots, n\}$.

Lemma 2.2.17 *For any $n \in \mathbb{N}$, the center $Z(\bar{D}_n)$ of the double dihedral group $\bar{D}_n$ equals $\{e, R^n\}$.*

Proof. For $a \in \{0, \dots, 2n-1\}$ we have that $RXR^a = XR^{a-1}$ and $XR^aR = XR^{a+1}$, so XR^a does not belong to $Z(\bar{D}_n)$, unless $n = 1$. On the other hand, $XR^a = R^{-a}X$, so R^a is in the center of $\bar{D}_n$ if and only if $R^{2a} = e$, which is equivalent to $a \in \{0, n\}$. □

Owing to [42], $H^3(\bar{D}_n, k^*) \simeq \mathbb{Z}_{4n}$. However, only half of the representatives for the cohomology classes of $H^3(\bar{D}_n, k^*)$ can be defined in an explicit way. To use this presentation, we need to assume that k has characteristic different from 2 and contains a $2n^2$ primitive root of unity, denoted by ξ_{2n^2}. We also fix the primitive roots $\xi_n = \xi^{2n}$, $\xi_{2n} = \xi^n$ and $\xi_4 = \xi^{\frac{n^2}{2}}$, and for $p \in \{0, \dots, 2n-1\}$, we define $\omega_p : \bar{D}_n \times \bar{D}_n \times \bar{D}_n \to k^*$ by

$$\begin{aligned}
&\omega_p(X^A R^a, X^B R^b, X^C R^c) \\
&\quad = \xi_{2n^2}^{p\left((-1)^{B+C} a((-1)^C b + c - \langle (-1)^C b + c + nBC \rangle_{2n}) - \frac{n^2}{2} ABC\right)} \\
&\quad \overset{(2.2.16)}{=} \xi_{2n^2}^{p\left((-1)^{B+C+1} a(2\lfloor \frac{n-(-1)^C b - c - nBC}{2n} \rfloor n + nBC) - \frac{n^2}{2} ABC\right)} \\
&\quad = \xi_{2n}^{p(-1)^{B+C+1} aBC} \xi_n^{p(-1)^{B+C+1} a \lfloor \frac{n-(-1)^C b - c - nBC}{2n} \rfloor} \xi_4^{-pABC},
\end{aligned} \tag{2.2.17}$$

for all couples $(A, a), (B, b), (C, c)$ with $A, B, C \in \{0, 1\}$ and $a, b, c \in \{-n+1, \dots, n\}$. Then $(\omega_p)_{0 \le p \le 2n-1}$ are non-cohomologous normalized 3-cocycles for $\bar{D}_n$.

Lemma 2.2.18 *The normalized 2-cocycle $\flat_{R^n}\omega_p$ on $\bar{D}_n$ is determined by*

$$\flat_{R^n}\omega_p(X^A R^a, X^B R^b) = (-1)^{pAB} \xi_n^{\frac{pa}{2}((-1)^B - 1)},$$

for all $A, B \in \{0, 1\}$ and $a, b \in \{-n+1, \dots, n\}$. Furthermore, if $f : \bar{D}_n \to k^*$ is given by $f(X^A R^a) = (-1)^A \xi_{2n}^{-pa}$, for all $A \in \{0, 1\}$ and $a \in \{-n+1, \dots, n\}$, then $\flat_{R^n}\omega_p = \partial f$; therefore, $\flat_{R^n}\omega_p$ is a coboundary 2-cocycle.

Proof. We compute, for all (A, a) and (B, b) as in the statement of the lemma, that

$$\begin{aligned}
\flat_{R^n}\omega_p(X^A R^a, X^B R^b) &= \frac{\omega_p(R^n, X^A R^a, X^B R^b)\omega_p(X^A R^a, X^B R^b, R^n)}{\omega_p(X^A R^a, R^n, X^B R^b)} \\
&= \frac{\xi_{2n}^{p(-1)^{A+B+1} nAB} \xi_n^{p(-1)^{A+B+1} n \lfloor \frac{n-(-1)^B a - b - nAB}{2n} \rfloor} \xi_n^{p(-1)^{B+1} a \lfloor -\frac{b}{2n} \rfloor}}{\xi_n^{p(-1)^{B+1} a \lfloor \frac{n-(-1)^B n - b}{2n} \rfloor}} \\
&= (-1)^{pAB} \xi_n^{p(-1)^{B+1} a \left(\lfloor -\frac{b}{2n} \rfloor - \lfloor \frac{n-(-1)^B n - b}{2n} \rfloor\right)}.
\end{aligned}$$

Our assertion follows from the fact that

$$\lfloor -\frac{b}{2n} \rfloor - \lfloor \frac{n-(-1)^B n - b}{2n} \rfloor = \begin{cases} 0, & \text{if } B=0 \\ -1, & \text{if } B=1 \end{cases} = \frac{1}{2}((-1)^B - 1).$$

Finally, for $f : \bar{D}_n \to k^*$ as in the statement of the lemma, we have that

$$\begin{aligned} \partial f(X^A R^a, X^B R^b) &= \frac{f(X^A R^a) f(X^B R^b)}{f(X^A R^a X^B R^b)} = \frac{(-1)^{A+B} \xi_{2n}^{-p(a+b)}}{f(X^{(A+B)'_2} R^{\langle (-1)^B a + b + nAB \rangle_{2n}})} \\ &\overset{(2.2.16)}{=} (-1)^{2\lfloor \frac{A+B}{2} \rfloor} \xi_{2n}^{-p(a-(-1)^B a - nAB)} = (-1)^{pAB} \xi_n^{\frac{pa}{2}((-1)^B - 1)} \\ &= \flat_{R^n} \omega_p (X^A R^a, X^B R^b), \end{aligned}$$

for all $A, B \in \{0,1\}$ and $a, b \in \{-n+1, \ldots, n\}$, as stated. So our proof is complete. □

Proposition 2.2.19 *For the quasi-Hopf algebra $(k^{\bar{D}_n}, \Phi_{\omega_p})$, there are non-trivial 2-dimensional braided Hopf algebras in ${}^{(k^{\bar{D}_n}, \Phi_{\omega_p})}_{(k^{\bar{D}_n}, \Phi_{\omega_p})}\mathcal{YD}$ if and only if either p is odd or p and n have different parities.*

Proof. Owing to Lemma 2.2.17 and Proposition 2.2.3, giving a non-trivial braided Hopf algebra B of dimension 2 in ${}^{(k^{\bar{D}_n}, \Phi_{\omega_p})}_{(k^{\bar{D}_n}, \Phi_{\omega_p})}\mathcal{YD}$ is equivalent to defining $\rho : \bar{D}_n \to k^*$ such that $\rho(R^n) = -1$ and $\flat_{R^n} \omega_p = \partial \rho$. Moreover, as we have observed in the proof of Proposition 2.2.8, any two maps $\rho_1, \rho_2 : \bar{D}_n \to k^*$ satisfying $\partial \rho_1 = \partial \rho_2$ are related through a group homomorphism $\vartheta : \bar{D}_n \to k^*$, in the sense that $\rho_1(g) = \rho_2(g)\vartheta(g)$, for all $g \in G$. Therefore, by Lemma 2.2.18, B is determined by a map $\rho : \bar{D}_n \to k^*$ given by $\rho(X^A R^a) = (-1)^A \xi_{2n}^{-pa} \vartheta(X^A R^a)$, for all $A \in \{0,1\}$ and $a \in \{-n+1, \ldots, n\}$, where $\vartheta : \bar{D}_n \to k^*$ is a suitable group homomorphism.

It is immediate that a group homomorphism $\vartheta : \bar{D}_n \to k^*$ is parametrized by two scalars υ and ξ_{2n}^s in k, with $0 \le s \le 2n-1$, obeying $\upsilon^2 = (-1)^s$ and $\upsilon \xi_{2n}^s = \upsilon \xi_{2n}^{-s}$. If follows that $s \in \{0, n\}$ and $\upsilon^2 = (-1)^s$, and therefore there are four group homomorphisms $\vartheta : \bar{D}_n \to k^*$; namely, they are determined by the couples

$$(\vartheta(X), \vartheta(R)) \in \{(1,1), (-1,1), (\xi_4^n, -1), (-\xi_4^n, -1)\}.$$

Summing up, there are exactly four maps $\rho : \bar{D}_n \to k^*$ for which $\flat_{R^n} \omega_p = \partial \rho$. They are given by

$$\begin{aligned} &\rho_1 : X^A R^a \mapsto (-1)^A \xi_{2n}^{-pa}, \\ &\rho_2 : X^A R^a \mapsto \xi_{2n}^{-pa}, \\ &\rho_3 : X^A R^a \mapsto (-\xi_4^n)^A (-1)^a \xi_{2n}^{-pa}, \\ &\rho_4 : X^A R^a \mapsto \xi_4^{nA} (-1)^a \xi_{2n}^{-pa}, \end{aligned}$$

for all $A \in \{0,1\}$ and $a \in \{-n+1, \ldots, n\}$. Our assertion follows now from the fact that $\rho_1(R^n) = \rho_2(R^n) = (-1)^p$ and $\rho_3(R^n) = \rho_4(R^n) = (-1)^{n+p}$. □

To conclude, we present a concrete example, using the most famous double dihedral group.

Example 2.2.20 (Quaternion group) When $n = 2$, $\bar{D}_2$ is the quaternion group Q, the group generated by X and R with relations $R^4 = e$, $X^2 = R^2$ and $XR = R^3X$; we can obtain the original presentation of Q by taking $-1 = R^2$, $\bar{i} = X$, $\bar{j} = R$ and $\bar{k} = XR$; indeed, we can easily verify that $\bar{i}^2 = \bar{j}^2 = \bar{k}^2 = \bar{i}\bar{j}\bar{k} = -1$.

Owing to Proposition 2.2.19, there are non-trivial 2-dimensional braided Hopf algebras within ${}^{(k^{\bar{D}_2},\Phi_{\omega_p})}_{(k^{\bar{D}_2},\Phi_{\omega_p})}\mathcal{YD}$ if and only if p is either 1 or 3. If this is the case, we get 8 non-trivial 2-dimensional braided Hopf algebras which give rise to 8 quasi-Hopf algebras of dimension 16, all of the form $H(\theta)_{v,\sigma}$; recall that $v = \sum_{A,a}\rho(X^AR^a)P_{X^AR^a}$, and that $\sigma : \bar{k}^{D_2} \to k$ is given by $\sigma(P_{X^AR^a}) = \delta_{A,0}\delta_{a,n}$, for all $A \in \{0, 1\}$ and $a \in \{-n+1, \dots, n\}$.

2.3 Some non-commutative, non-cocommutative examples

The classification of quasi-Hopf algebras with radical of codimension 2 was obtained in [19] by P. Etingof and S. Gelaki; namely, they proved that there are precisely four types of such quasi-Hopf algebras, denoted by $H(2)$, $H_+(8)$, $H_-(8)$ and $H(32)$. Note that, as we are about to see, $H_+(8)$, $H_-(8)$ and $H(32)$ contain $H(2)$ as a quasi-Hopf subalgebra. Recall that $H(2) := (k[C_2], \Phi)$, where $\Phi = 1\otimes 1\otimes 1 - 2p_-\otimes p_-\otimes p_-$.

Throughout this section, k is a field which contains a primitive 4-th root of unit $\mathfrak{q}$. Our next aim is to construct new classes of quasi-Hopf algebras from an $H \in \{H(2), H_\pm(8), H(32)\}$ and a pair (σ, v) of H satisfying conditions (2.1.22) to (2.1.25), for all $h \in H$; see Proposition 2.1.9 for all the detail on the construction.

By using the classification of the 4-dimensional quasi-Hopf algebras, see Section 3.2, if follows that, for $H(2)$, a pair (σ, v) like the one above cannot exist. To obtain a direct argument for this assertion, we can first look at the elements $v = \mu 1 + \nu g \in H(2)$ for which $\varepsilon(v) = 1$ and $\sigma(v) = -1$. As we cannot have $\sigma(g) = 1$, otherwise this would imply $\sigma = \varepsilon$, and so the contradiction $1 = -1$, we get $\sigma(g) = -1$; thus $\mu + \nu = 1$ and $\mu - \nu = -1$, and hence $v = g$. Then $\sigma(p_-) = 1$ and (2.1.24) read as $\Delta(g) = \sigma(x^3)x^1g \otimes x^2g$, in turn equivalent to $g\otimes g = g\otimes g - 2p_-\otimes p_-$. This latter equality is false.

So, for $H = H(2)$, Proposition 2.1.9 only produces quasi-Hopf algebras of the form $k[C_2]\otimes H(2)$, which are quasi-Hopf algebras built on the Hopf group algebra structure of $k[C_2\times C_2]$, with the help of the reassociator Φ of $H(2)$. We will see below that a similar situation also occurs when $H \in \{H_\pm(8), H(32)\}$.

Recall from [19] that $H(32)$ is the k-algebra generated by g, x and y with relations

$$g^2 = 1, \quad gx = -xg, \quad gy = -yg, \quad x^4 = y^4 = 0 \quad \text{and} \quad yx = \mathfrak{q}xy.$$

The quasi-Hopf algebra structure of $H(32)$ is defined in such a way that $k[\langle g\rangle]$ is a quasi-Hopf subalgebra of $H(32)$, and by

$$\begin{aligned}
\Delta(x) &= x\otimes(p_+ + \mathfrak{q}p_-) + 1\otimes p_+x + g\otimes p_-x,\\
\Delta(y) &= y\otimes(p_+ - \mathfrak{q}p_-) + 1\otimes p_+y + g\otimes p_-y,\\
S(x) &= -x(p_+ + \mathfrak{q}p_-),\ S(y) = -y(p_+ - \mathfrak{q}p_-),\\
\epsilon(x) &= \epsilon(y) = 0.
\end{aligned}$$

$H_{\mathfrak{q}}(8)$ is the unital k-algebra generated by g, x subject to the relations

$$g^2 = 1, \quad x^4 = 0 \quad \text{and} \quad gx = -xg;$$

the quasi-Hopf algebra structure of $H_{\mathfrak{q}}(8)$ is defined in a similar way to that of $H(32)$, with structure maps determined by:

$$\Delta(x) = x \otimes (p_+ + \mathfrak{q}p_-) + 1 \otimes p_+x + g \otimes p_-x,$$
$$S(x) = -x(p_+ + \mathfrak{q}p_-), \quad \epsilon(x) = 0.$$

Following the notation of [19], $H_+(8)$ and $H_-(8)$ correspond to the case when $\mathfrak{q} = i$ and $\mathfrak{q} = -i$, respectively.

Lemma 2.3.1 *For $H \in \{H_{\mathfrak{q}}(8), H(32)\}$, a 2-dimensional braided Hopf algebra in ${}_H^H\mathcal{YD}$ is isomorphic to the Hopf group algebra $k[C_2]$, regarded as an object in ${}_H^H\mathcal{YD}$ through the trivial action and coaction.*

Proof. An algebra morphism $\sigma : H \to k$ is determined by $\sigma(g) \in \{\pm 1\}$ and $\sigma(x) = 0$, provided that $H = H_{\mathfrak{q}}(8)$, which follows by the relation $x^4 = 0$; in the case when $H = H(32)$, in addition we must have $\sigma(y) = 0$.

Let $v \in H$ be such that $\varepsilon(v) = 1$ and $\sigma(v) = -1$. We cannot have $\sigma(g) = 1$; otherwise $\sigma = \varepsilon$, and so $1 = -1$, a contradiction. We conclude that $\sigma(g) = -1$. Set $v = v_0 + v_1$ with $v_0 \in k[\langle g\rangle]$ and $v_1 = \sum_{j=0,l=1}^{1,3} \lambda_{jl} g^j x^l$, for some λ_{jl} in k provided that $H = H_{\mathfrak{q}}(8)$, otherwise, when $H = H(32)$, set $v_1 = \sum_{j,l,t=0;l+t\neq 0}^{1,3,3} \lambda_{jlt} g^j x^l y^t$ for some λ_{jlt} in k. We have $\varepsilon(v_0) = 1$ and $\sigma(v_0) = -1$, relations which force $v_0 = g$; thus $v = g + v_1$.

We now ask for v to satisfy the additional relation $\sigma(h_2)h_1v = \sigma(h_1)vh_2$, for all $h \in H$. By taking $h = g$, we get that $vg = gv$, hence $gv_1 = v_1g$, a fact which gives us, in the case when $H = H_{\mathfrak{q}}(8)$, $v_1 = (\sum_{j=0}^{1} \lambda_j g^j)x^2$ for some $\lambda_j \in k$, and respectively $v_1 = \sum_{j,l,t=0;l+t\in\{2,4\}}^{1,3,3} \lambda_{jlt} g^j x^l y^t$ in the case when $H = H(32)$. Likewise, by taking $h = x$, as $\sigma(p_+) = 0$ and $\sigma(p_-) = 1$ we get that $\mathfrak{q}xv = vp_+x - vp_-x$, relation equivalent to $\mathfrak{q}xv = vgx$; for $H = H_{\mathfrak{q}}(8)$, this is equivalent to

$$-\mathfrak{q}gx + \mathfrak{q}(\sum_{j=0}^{1}(-1)^j\lambda_j g^j)x^3 = x + (\sum_{j=0}^{1}\lambda_j g^{j+1})x^3,$$

and, for $H = H(32)$, it yields

$$-\mathfrak{q}gx + \mathfrak{q}\sum_{\substack{j,l,t=0\\ l+t\in\{2,4\}}}^{1,3,3} \lambda_{jlt}(-1)^j g^j x^{l+1} y^t = x + \sum_{\substack{j,l,t=0\\ l+t\in\{2,4\}}}^{1,3,3} \lambda_{jlt}\mathfrak{q}^t g^{j+1} x^{l+1} y^t.$$

In both cases we are lead to a contradiction, thus concluding our proof. $\square$

Hence, for $H = H_{\mathfrak{q}}(8)$, our techniques give rise to a single quasi-Hopf algebra of dimension 16, which will be denoted by $H_{\mathfrak{q}}(16)$ in what follows, through the trivial biproduct construction, meaning that $H_{\mathfrak{q}}(16)$ equals $H_{\mathfrak{q}}(8)_{g_2} = k[C_2] \otimes H_{\mathfrak{q}}(8)$; see Proposition 2.1.9 for all the details. In particular, we recall that $H_{\mathfrak{q}}(16)$ contains $H_{\mathfrak{q}}(8)$ as a quasi-Hopf subalgebra, whose generators we now denote by g_1 and x, and a second group-like element g_2, which commutes with all the elements of $H_{\mathfrak{q}}(8)$ and such that g_1, g_2, x generate $H_{\mathfrak{q}}(16)$ as an algebra.

In the next proposition, we show how, starting from a quasi-Hopf algebra H which admits only trivial 2-dimensional braided Hopf algebras, we can obtain non-trivial structures by taking the trivial rank 2 biproduct of H as the base quasi-Hopf algebra. Namely, we consider $H = H_{\mathfrak{q}}(8)$ and are able to recover one non-trivial 2-dimensional braided Hopf algebra.

Proposition 2.3.2 *For $H_{\mathfrak{q}}(16)$, the only pair (σ, v) satisfying (2.1.22) to (2.1.25) is given by the k-algebra morphism $\sigma : H_{\mathfrak{q}}(16) \to k$, determined by $\sigma(g_1) = 1$, $\sigma(g_2) = -1$ and $\sigma(x) = 0$, and the element $v = g_2$. Furthermore, $H_{\mathfrak{q}}(8)(\theta)_{(\sigma,v)}$ is a 32-dimensional quasi-Hopf algebra isomorphic to the tensor product quasi-Hopf algebra of H_4 and $H_{\mathfrak{q}}(8)$.*

Proof. We write v as

$$v = \sum_{j,l=0}^{1,1} \mu_{jl} g_1^j g_2^l + \sum_{\substack{j,l=0\\t=1}}^{1,1;3} \lambda_{jlt} g_1^j g_2^l x^t,$$

for some $\mu_{jl}, \lambda_{jlt} \in k$. We also denote $\sigma(g_j) = \nu_j \in \{\pm 1\}$, $0 \leq j \leq 1$; as we have already remarked, we cannot have both ν_1, ν_2 equal to 1, otherwise we would obtain $\sigma = \varepsilon$, and thus $1 = -1$. Since $\varepsilon(v) = 1$, we have $\sum_{j,l=0}^{1,1} \mu_{ij} = 1$; similarly, since $\sigma(v) = -1$ we have $\sum_{j,l=0}^{1,1} \nu_1^j \nu_2^l \mu_{jl} = -1$.

For $h = g_1$, the equality $\sigma(h_2)h_1 v = \sigma(h_1) v h_2$ reads as $v g_1 = g_1 v$, which is in turn equivalent to

$$\sum_{j,l=0}^{1,1} \mu_{jl} g_1^{j+1} g_2^l + \sum_{\substack{j,l=0\\t=1}}^{1,1;3} (-1)^t \lambda_{jlt} g_1^{j+1} g_2^l x^t = \sum_{j,l=0}^{1,1} \mu_{jl} g_1^j g_2^j + \sum_{\substack{j,l=0\\t=1}}^{1,1;3} \lambda_{jlt} g_1^{j+1} g_2^l x^t.$$

It follows that $\lambda_{jlt} = 0$ for any odd t, and so

$$v = \sum_{j,l=0}^{1,1} \mu_{jl} g_1^j g_2^l + \left(\sum_{j,l=0} \lambda_{jl} g_1^j g_2^l \right) x^2,$$

for some $\mu_{jl}, \lambda_{jl} \in k$. With v written this way, one can easily check that the relation (2.1.25) is satisfied by $h = g_2$ too. When we check (2.1.25) for $h = x$, we land at $\omega x v = v(p_+ + \nu_1 p_-)x$, where $\omega := \left(\frac{1}{2}(1+\lambda_1) + \frac{\mathfrak{q}}{2}(1-\nu_1)\right)$. This last equality reduces to

$$\begin{aligned}
\omega\Big(\sum_{j,l=0}^{1,1} (-1)^j \mu_{jl} g_1^j g_2^l\Big) x + \omega\Big(\sum_{j,l=0}^{1,1} (-1)^j \lambda_{jl} g_1^j g_2^l\Big) x^3 \\
= \Big(\sum_{j,l=0}^{1,1} \mu_{jl}(p_+ g_2^l + (-1)^j \nu_1 p_- g_2^l)\Big) x + \Big(\sum_{j,l=0}^{1,1} \lambda_{jl}(p_+ g_l^2 + (-1)^j \nu_1 p_- g_2^l)\Big) x^3,
\end{aligned}$$

where we adopted the notation $p_\pm = \frac{1}{2}(1 \pm g_1)$. For $0 \leq j \leq 1$, the above relation leads to the system

$$\begin{cases} \omega(\mu_{0j} 1 - \mu_{1j} g_1) = \mu_{0j}(p_+ + \nu_1 p_-) + \mu_{1j}(p_+ - \nu_1 p_-) \\ \omega(\lambda_{0j} 1 - \lambda_{1j} g_1) = \lambda_{0j}(p_+ + \nu_1 p_-) + \lambda_{1j}(p_+ - \nu_1 p_-) \end{cases},$$

which is equivalent to

$$\begin{cases} \omega(\mu_{0j}-\mu_{1j})=\mu_{0j}+\mu_{1j}\\ \omega(\mu_{0j}+\mu_{1j})=\nu_1(\mu_{0j}-\mu_{1j})\\ \omega(\lambda_{0j}-\lambda_{1j})=\lambda_{0j}+\lambda_{1j}\\ \omega(\lambda_{0j}+\lambda_{1j})=\nu_1(\lambda_{0j}-\lambda_{1j}) \end{cases};$$

we reduce to two cases, $\nu_1\in\{-1,1\}$, that will be analyzed separately in what follows.

Case 1: $\nu_1=1$. We get $\omega=1$, $\nu_2=-1$ and $\mu_{1j}=\lambda_{1j}=0$, for all $0\leq j\leq 1$. As in the above, we conclude that $v=g_2+(\lambda 1+\mu g_2)x^2$, for some $\lambda,\mu\in k$. On the other hand, the condition (2.1.24) on v holds if and only if $\Delta(v)=v\otimes v$, because $\sigma(p_-)=0$; recall that $p_-=\frac{1}{2}(1-g_1)$ and $\sigma(g_1)=\nu_1=1$. A simple computation yields

$$\Delta(x^2)=x^2\otimes g_1+(1+\mathfrak{q})x\otimes p_+x+(1-\mathfrak{q})g_1x\otimes p_-x+g_1\otimes x^2,$$

and with this we obtain by direct computation that $\Delta(v)=v\otimes v$ is equivalent to $\lambda=\mu=0$.

Summing up, $v=g_2$ and σ is determined by $\sigma(g_1)=1$, $\sigma(g_2)=-1$ and $\sigma(x)=0$. Furthermore, $H_{\mathfrak{q}}(8)(\theta)_{(\sigma,v)}$ is the quasi-Hopf algebra generated as a k-algebra by g_1,g_2,x,θ, subject to relations

$$g_1^2=g_2^2=1,\quad x^2=\theta^2=0,\quad g_1g_2=g_2g_1,\quad xg_1=-g_1x,$$
$$g_2x=xg_2,\quad g_1\theta=\theta g_1,\quad g_2\theta=-\theta g_2,\quad x\theta=\theta p_+x+\theta p_-x=\theta x.$$

The quasi-coalgebra structure is given in such a way that g_1 and g_2 are grouplike elements, $\Delta(x)=x\otimes(p_++\mathfrak{q}p_-)+1\otimes p_+x+g_1\otimes p_-x$, $\Delta(\theta)=g_2\otimes\theta+\theta\otimes 1$, $\varepsilon(x)=\varepsilon(\theta)=0$ and the reassociator is the same as $H(2)$, $\Phi=1-2p_-\otimes p_-\otimes p_-$; the antipode is defined in a similar manner to the one of $H_{\mathfrak{q}}(8)$, namely by $S(x)=-x(p_++\mathfrak{q}p_-)$. A simple inspection shows us that the k-algebra generated by g_2 and θ is a Hopf subalgebra of $H_{\mathfrak{q}}(16)(\theta)_{(\sigma,v)}$ which coincides with H_4, and that $H_{\mathfrak{q}}(16)(\theta)_{(\sigma,v)}=H_4\otimes H_{\mathfrak{q}}(8)$ as quasi-Hopf algebras.

Case 2: $\nu_1=-1$. Then, we are provided with $\omega=\mathfrak{q}$, $\mu_{1j}=\mathfrak{q}\mu_{0j}$ and $\lambda_{1j}=\mathfrak{q}\lambda_{0j}$, for all $0\leq j\leq 1$, where $\mathfrak{q}$ denotes here any primitive 4-th root of unity (we can assume it to be $\mathfrak{q}$ without loss of generality). Thus

$$v=\sum_{j=0}^{1}\mu_{0j}(1+\mathfrak{q}g_1)g_2^j+\Big(\sum_{j=0}^{2}\lambda_{0j}(1+\mathfrak{q}g_1)g_2^j\Big)x^2.$$

Now, $\varepsilon(v)=1$ implies $\mu_{00}+\mu_{01}=\frac{1}{2}(1-\mathfrak{q})$, while $\sigma(v)=-1$ implies $\mu_{00}+\nu_2\mu_{01}=-\frac{1}{2}(1+\mathfrak{q})$. Hence, $(1-\nu_2)\mu_{01}=1$, and therefore $\nu_2=-1$, $\mu_{00}=-\frac{\mathfrak{q}}{2}$ and $\mu_{01}=\frac{1}{2}$. We conclude that

$$v=-\frac{\mathfrak{q}}{2}(1+\mathfrak{q}g_1)+\frac{1}{2}(1+\mathfrak{q}g_1)g_2+\Big(\sum_{j=0}^{1}\tau_j(1+\mathfrak{q}g_1)g_2^j\Big)x^2,$$

for some $\tau_1,\tau_2\in k$. Finally, we have $\sigma(p_-)=1$ and a simple computation shows that $p_-v=vp_-$. Hence, (2.1.24) reduces to

$$\Delta(v)=v\otimes v-2p_-v\otimes p_-v=p_+v\otimes v+p_-v\otimes g_1v.$$

With v as presented above, $1 \otimes 1$ appears in the formula of $\Delta(v)$ with coefficient $-\frac{\mathfrak{q}}{2}$. On the other hand, since

$$p_+v = \frac{1}{2}(1-\mathfrak{q})p_+ + \frac{1}{2}(1+\mathfrak{q})p_+g_2 + (1+\mathfrak{q})p_+\big(\sum_{j=0}^{1}\tau_j g_2^j\big)x^2,$$

$$p_-v = -\frac{1}{2}(1+\mathfrak{q})p_- + \frac{1}{2}(1-\mathfrak{q})p_-g_2 + (1-\mathfrak{q})p_-\big(\sum_{j=0}^{1}\tau_j g_2^j\big)x^2,$$

$$g_1v = \frac{1}{2}(1-\mathfrak{q}g_1) + \frac{\mathfrak{q}}{2}(1-\mathfrak{q}g_1)g_2 + \mathfrak{q}(1-\mathfrak{q}g_1)\big(\sum_{j=0}^{1}\tau_j g_2^j\big)x^2,$$

it follows that $1 \otimes 1$ appears in the formula of $p_+v \otimes v + p_- \otimes g_1v$ with coefficient $-\frac{\mathfrak{q}}{8}(1-\mathfrak{q}) - \frac{1}{8}(1+\mathfrak{q}) = -\frac{1}{4}(1+\mathfrak{q})$. We conclude that $-\frac{1}{4}(1+\mathfrak{q}) = -\frac{\mathfrak{q}}{2}$, which implies $\mathfrak{q} = 1$, so this case cannot occur, since $\mathfrak{q}$ is a primitive root. $\square$

We end this section by pointing out that, provided H has radical of codimension 2, we can find non-trivial 2-dimensional Hopf algebras within ${}_H^H\mathcal{Y}D$ only when H is a twist deformation of a Hopf algebra. To this end, recall that, for n a non-zero natural number, the Nichols Hopf algebra H_{2^n} is the k-algebra generated by $g, x_1, \ldots, x_n$ with relations $g^2 = 1$, $x_i^2 = 0$, $gx_i = -x_ig$, for all $1 \le i \le n-1$, and $x_ix_j = -x_jx_i$, for all $1 \le i \ne j \le n-1$. The Hopf algebra structure of H_{2^n} is designed in such a way that g is a grouplike element and x_i is $(g,1)$-skew primitive, for all $1 \le i \le n$.

Proposition 2.3.3 *Let H be a finite dimensional quasi-Hopf algebra with radical of codimension 2. There are non-trivial 2-dimensional braided Hopf algebras in ${}_H^H\mathcal{Y}D$ if and only if H is twist equivalent to a Nichols Hopf algebra H_{2^n}, for some $n \ge 1$.*

Proof. We start with the direct implication. By Lemma 2.3.1 and the comments made at the beginning of this section, H is not twist equivalent to one of the quasi-Hopf algebras $H(2)$, $H_{\mathfrak{q}}(8)$ or $H(32)$. In other words, owing to [19, Theorem 3.4], the reassociator of H in trivial on 1-dimensional H-modules; by [19, Theorem 3.1], it follows now that H is twist equivalent to H_{2^n}, for some $n \ge 1$.

Conversely, by [11] it suffices to show that there are non-trivial 2-dimensional Hopf algebras within ${}_{H_{2^n}}^{H_{2^n}}\mathcal{Y}D$. Indeed, Theorem 2.1.8 says that 2-dimensional braided Hopf algebras over H_{2^n} are parametrized by couples (σ, v) consisting of an algebra morphism $\sigma : H_{2^n} \to k$ and a grouplike element $v \in H_{2^n}$ such that $\sigma(v) = -1$ and $\sigma(h_2)h_1v = \sigma(h_1)vh_2$, for all $h \in H_{2^n}$. Thus, $v = g$ and σ must obey $\sigma(g) = -1$; the second condition reduces to a tautology, as it is equivalent to $x_ig = -gx_i$, for all $1 \le i \le n$. Concluding, (σ, g) provides a non-trivial braided Hopf algebra of dimension 2 in ${}_{H_{2^n}}^{H_{2^n}}\mathcal{Y}D$, as needed. Furthermore, the resulting Hopf algebra $H_{2^n}(\theta)_{\sigma,g}$ identifies with $H_{2^{n+1}}$, so that the Nichols algebras H_{2^n} can be viewed as a tower of biproduct Hopf algebras. $\square$

Chapter 3

On the Sweedler Hopf algebra

The aim of this section is to illustrate how the characterization obtained in Theorem 2.1.8 can be applied to the classification of quasi-Hopf algebras in low dimension. This is done by providing an alternative path to prove that non-semisimple 4-dimensional quasi-Hopf algebras are twist equivalent to the Sweedler Hopf algebra H_4, a result reached in [19] by Etingof and Gelaki, through cohomological means. A similar purely algebraic approach will be used later, in Chapter 4, to complete the classification of 6 dimensional quasi-Hopf algebras. Additionally, we construct a infinite class of non-isomorphic quasi-Hopf algebras of dimension 4.

3.1 Twist-equivalence for the cyclic group algebra

In this short section, we prove a Lemma, originally from [10], which will play a role in Section 3.2 and in Chapter 4. The base field k will be here an algebraically closed field of characteristic 0.

Let G be a finite group and $\omega \in Z^3(G, k^*)$ a normalized 3-cocycle of G; the quasi-Hopf algebra structure of k^G_ω produces a monoidal structure on the category $\mathrm{Rep}(k^G_\omega)$ of representations of k^G_ω; by [24, Remark 2.6.2], there is a correspondence between equivalence classes of $\mathrm{Rep}(k^G_\omega)$ and twist equivalence classes of k^G_ω. Furthermore, owing to [24, Remark 2.6.2], the equivalence classes of these monoidal categories $\mathrm{Rep}(k^G_\omega)$, for any $\omega \in Z^3(G, k^*)$, are in a one-to-one correspondence with the orbits of the group action

$$\mathrm{Out}(G) \times H^3(G, k^*) \to H^3(G, k^*) : (\widehat{f}, \overline{\omega}) \mapsto \overline{f^*\omega}. \tag{3.1.1}$$

Here $\mathrm{Out}(G)$ is the group of outer automorphisms of G and $f^*\omega \in Z^3(G, k^*)$ is defined by $f^*\omega(g_1, g_2, g_3) = \omega(f(g_1), f(g_2), f(g_3))$, for all $g_1, g_2, g_3 \in G$. Therefore, for a finite group G, the number of quasi-Hopf algebras of the form k^G_ω, with $\omega \in Z^3(G, k^*)$, which are not pairwise twist equivalent is equal to the number of orbits $\mathcal{O}_{\overline{\omega}}$ of the group action defined by (3.1.1).

We are interested in applying the previously mentioned result in [24, Remark 2.6.2] to an arbitrary finite cyclic group C_n. For this, let ξ be a n-th root of unity and σ a generator for C_n.

It is well known that 3-cocycle representatives for the group C_n are given by $\{\phi_{\xi^a} \mid 0 \leq a \leq n-1\}$, where, for all $0 \leq i, j, l \leq n-1$,

$$\phi_{\xi^a}(\sigma^i, \sigma^j, \sigma^l) = \xi^{ai\lfloor \frac{j+l}{n} \rfloor}. \tag{3.1.2}$$

Since C_n is abelian, $\mathrm{Out}(C_n) = \mathrm{Aut}(C_n) \simeq U(\mathbb{Z}_n)$; $\mathrm{Aut}(C_n)$ is the group automorphisms of C_n and $U(\mathbb{Z}_n)$ is the group of invertible residue classes modulo n: $U(\mathbb{Z}_n) = \{s \mid 1 \leq s \leq n-1\ ,\ (s,n) = 1\}$, where (s,n) is the greatest common divisor of s and n. Finally, denote $n_b := \frac{n}{(n,b)}$, provided that $1 \leq b \leq n$, and let $\varphi(n)$ be the order of the group $U(\mathbb{Z}_n)$ (so φ is the Euler's function).

Lemma 3.1.1 *For $G = C_n$, the group action in (3.1.1) can be written, for $0 \leq a \leq n-1$, as follows:*

$$(f, \overline{\phi_{\xi^a}}) \mapsto \overline{\phi_{\xi^{s^2 a}}}, \tag{3.1.3}$$

provided that $f \in \mathrm{Aut}(C_n) \simeq U(\mathbb{Z}_n)$ is defined by $1 \leq s \leq n-1$ such that $(s,n) = 1$ (that is, f is determined by $f(\sigma) = \sigma^s$). Consequently,

$$|\mathcal{O}_{\widehat{\xi^a}}| = \begin{cases} \frac{\varphi(n_a)}{2^t} & \text{if } k \in \{0,1\} \\ \frac{\varphi(n_a)}{2^{t+1}} & \text{if } k = 2 \\ \frac{\varphi(n_a)}{2^{t+2}} & \text{if } k \geq 3 \end{cases},$$

where $n_a = 2^k p_1^{l_1} \dots p_t^{l_t}$ is the factorization of n_a as a product of distinct primes.

Proof. If f is defined by s as in the statement of the lemma, we compute that

$$f^*\phi_{\xi^a}(\sigma^i, \sigma^j, \sigma^l) = \phi_{\xi^a}(\sigma^{(si)'}, \sigma^{(sj)'}, \sigma^{(sl)'}) = \xi^{a(si)'\lfloor \frac{(sj)'+(sl)'}{n} \rfloor} = \xi^{asi\lfloor \frac{(sj)'+(sl)'}{n} \rfloor},$$

where, for a natural number m, m' stands for the remainder of the division of m by n. We have $m = \lfloor \frac{m}{n} \rfloor n + m'$, and by applying this formula for suitable m's we get that

$$\begin{aligned} \lfloor \frac{(sj)'+(sl)'}{n} \rfloor &= \frac{(sj)'+(sl)'-((sj)'+(sl)')'}{n} = \frac{(sj)'+(sl)'-(s(j+l)')'}{n} \\ &= \frac{sj+sl-s(j+l)'}{n} - \lfloor \frac{sj}{n} \rfloor - \lfloor \frac{sl}{n} \rfloor + \lfloor \frac{s(j+l)'}{n} \rfloor \\ &= s\lfloor \frac{j+l}{n} \rfloor - \lfloor \frac{sj}{n} \rfloor - \lfloor \frac{sl}{n} \rfloor + \lfloor \frac{s(j+l)'}{n} \rfloor, \end{aligned}$$

for all $0 \leq j, l \leq n-1$, and therefore

$$f^*\phi_{\xi^a}(\sigma^i, \sigma^j, \sigma^l) = \phi_{\xi^{s^2 a}}(\sigma^i, \sigma^j, \sigma^l)\ \xi^{-sai\left(\lfloor \frac{sj}{n} \rfloor + \lfloor \frac{sl}{n} \rfloor - \lfloor \frac{s(j+l)'}{n} \rfloor\right)}. \tag{3.1.4}$$

Define $\omega : C_n \times C_n \times C_n \to k^*$ by

$$\omega(\sigma^i, \sigma^j, \sigma^l) = \xi^{-sai\left(\lfloor \frac{sj}{n} \rfloor + \lfloor \frac{sl}{n} \rfloor - \lfloor \frac{s(j+l)'}{n} \rfloor\right)},$$

for all $0 \leq i, j, l \leq n-1$. We claim that there exists a map $h : C_n \times C_n \to k^*$ such that, for all $0 \leq i, j, l \leq n-1$,

$$\omega(\sigma^i, \sigma^j, \sigma^l) = h(\sigma^j, \sigma^l) h(\sigma^{(i+j)'}, \sigma^l)^{-1} h(\sigma^i, \sigma^{(j+l)'}) h(\sigma^i, \sigma^j)^{-1}.$$

This would imply that ω is a coboundary 3-cocycle, and so $f^*\phi_{\xi^a}$ and $\phi_{\xi^{s^2 a}}$ are cohomologous.

Indeed, take $h(\sigma^i,\sigma^j) := \xi^{sai\lfloor\frac{sj}{n}\rfloor}$, for all $0 \leq i,j \leq n-1$. We have, for all $0 \leq i,j,l \leq n-1$, that

$$\begin{aligned} &h(\sigma^j,\sigma^l)h(\sigma^{(i+j)'},\sigma^l)^{-1}h(\sigma^i,\sigma^{(j+l)'})h(\sigma^i,\sigma^j)^{-1} \\ &= \xi^{saj\lfloor\frac{sl}{n}\rfloor}\xi^{-sa(i+j)'\lfloor\frac{sl}{n}\rfloor}\xi^{sai\lfloor\frac{s(j+l)'}{n}\rfloor}\xi^{-sai\lfloor\frac{sj}{n}\rfloor} \\ &= \xi^{-sai\lfloor\frac{sl}{n}\rfloor+sai\lfloor\frac{s(j+l)'}{n}\rfloor-sai\lfloor\frac{sj}{n}\rfloor} = \omega(\sigma^i,\sigma^j,\sigma^l), \end{aligned}$$

as stated. This ends the first part of the proof.

It follows now that the orbit $\mathcal{O}_{\overline{\xi^a}}$ equals $\{\overline{\xi^{s^2a}} \mid 1 \leq s \leq n, (s,n)=1\}$. In addition, for $s,s' \in U(\mathbb{Z}_n)$ we have $s^2a \equiv s'^2a\ (mod\ n)$ if and only if $s^2 \equiv s'^2\ (mod\ n_a)$. We conclude that $|\mathcal{O}_{\overline{\xi^a}}| = |U(\mathbb{Z}_{n_a})^{(2)}|$, where, for a finite multiplicative group G, we denote $G^{(2)} := \{g^2 \mid g \in G\}$; if G is additive, we denote $(2)G := \{2g \mid g \in G\}$. Summarizing, the size of $\mathcal{O}_{\overline{\xi^a}}$ is equal to $\psi(n_a)$; for an integer $u \geq 2$, $\psi(u)$ is the number of quadratic residue *mod* u. It is well known that $\psi(uv) = \psi(u)\psi(v)$ for any $(u,v)=1$, so if $n_a = 2^kp_1^{l_1}\dots p_t^{l_t}$ is the factorization of n_a as a product of distinct prime numbers then $\psi(n_a) = \psi(2^k)\psi(p_1^{l_1})\dots\psi(p_t^{l_t})$. On the other hand, by [46, Theorem 7.3], we know that $U(\mathbb{Z}_2)=1$, $U(\mathbb{Z}_4)\simeq\mathbb{Z}_2$, $U(\mathbb{Z}_{2^l})\simeq\mathbb{Z}_2\times\mathbb{Z}_{2^{l-2}}$ for $l \geq 3$, and that $U(\mathbb{Z}_{p^i})\simeq\mathbb{Z}_{\varphi(p^i)}$ for any odd prime p (all are group isomorphisms). Hence, at the level of sets, $U(\mathbb{Z}_2)^{(2)}\simeq\{1\}$, $U(\mathbb{Z}_4)^{(2)}\simeq(2)\mathbb{Z}_2\simeq\{1\}$, and $U(\mathbb{Z}_{2^l})^{(2)}\simeq(2)\left(\mathbb{Z}_2\times\mathbb{Z}_{2^{l-2}}\right)\simeq(2)\mathbb{Z}_{2^{l-2}}\simeq\mathbb{Z}_{2^{l-3}}$, for all for $l\geq 3$ (note that $\mathbb{Z}_1 := 1$, the trivial group). Likewise, if p is an odd prime number and m a natural number, $U(\mathbb{Z}_{p^m})^{(2)}\simeq(2)\mathbb{Z}_{\varphi(p^m)}\simeq\mathbb{Z}_{\frac{\varphi(p^m)}{2}}$ at the level of sets. We conclude that

$$\psi(n_a) = \psi(2^k)\frac{\varphi(p_1^{l_1}\dots p_t^{l_t})}{2^t} = \begin{cases} \frac{\varphi(n_a)}{2^t} & \text{if } k \in \{0,1\} \\ \frac{\varphi(n_a)}{2^{t+1}} & \text{if } k = 2 \\ \frac{\varphi(n_a)}{2^{t+2}} & \text{if } k \geq 3 \end{cases},$$

as stated. □

We are now free to proceed with the first application to the classification of quasi-Hopf ablgebras of the characterization result obtained in the previous chapter.

3.2 Quasi-Hopf algebras of dimension 4

Throughout this section k is an algebraic closed field of characteristic 0. We need this assumption on k in order to make use of the classification of Hopf algebras H in dimension 4: if H is not semisimple then H is isomorphic to H_4; if H is semisimple, H is isomorphic to a Hopf group algebra. The original aim that lead us to prove Proposition 2.1.9 was to provide a similar result for quasi-Hopf algebras, and especially to produce a quasi-Hopf analogue for H_4; as we will see, a result of Etingof and Gelaki in [19] says that the latter is not possible. Nevertheless, we will use the same result to show that, in dimension 4, the class of quasi-Hopf algebras twist equivalent to H_4 provides infinitely many non-isomorphic quasi-Hopf algebras.

3.2.1 The nonsemisimple case

It was shown in [19, Corollary 3.6] that a 4-dimensional nonsemisimple quasi-Hopf algebra A is twist equivalent to H_4. The proof in [19] was done in two steps. Namely, the authors show that:

1. $\mathrm{gr}(A)$ is a Hopf algebra isomorphic to H_4, and therefore a selfdual Hopf algebra;
2. the third Hochschild cohomology of H_4 with coefficients in k, $H^3(H_4, k)$, is trivial.

Then, the classification result follows from [21, Proposition 2.3] which states that a finite dimensional quasi-Hopf algebra A with $\mathrm{gr}(A)$ a Hopf algebra is twist equivalent to a Hopf algebra, provided that $H^3(\mathrm{gr}(A)^*, k) = 0$ ($\mathrm{gr}(A)$ is the graded algebra associated to A, more details will be given below).

We present in this subsection an alternative proof for 1. Towards this end, for a 4-dimensional non-semisimple quasi-Hopf algebra A we show that $\mathrm{gr}(A) \simeq A \simeq H_4$ as an algebra, and then we use the representative 3-cocycles of $k[C_2]$ to see that $\mathrm{gr}(A) \simeq H_4$ as a Hopf algebra.

Consider now the graded algebra $\mathrm{gr}(A)$ associated with A; in the following, we will use the notation established in Section 1.4.1. Other than what previously discussed, another notable thing about $\mathrm{gr}(A)$ is the fact that each A_m can be seen as a two-sided two-cosided quasi-Hopf module over A_0, with natural A_0-actions and coactions; we refer to [5] for the detailed definition of the category ${}^{A_0}_{A_0}\mathcal{M}^{A_0}_{A_0}$. By the structure theorems proved in [47], it follows that A_m is a free left and right A_0-module, for all $m \geq 0$.

When $\dim_k A = 4$, $\frac{A}{J}$ is a semisimple algebra of dimension at most 3, thus A is basic and the results mentioned above apply. In particular, they allow us to show that $\dim_k J = 2$. To this end, we need another classical result, proved in [1, Theorem 1.4.9]. Namely, if $\mathbb{A}$ is a k-algebra with Jacabson radical J and $\frac{\mathbb{A}}{J}$ is a separable k-algebra then there exists a semisimple k-subalgebra $\mathbb{S}$ of $\mathbb{A}$ such that $\mathbb{A}$ can be written as the direct sum of k-spaces $\mathbb{A} = J \oplus \mathbb{S}$; obviously, $\frac{\mathbb{A}}{J} \simeq \mathbb{S}$ as k-spaces.

Lemma 3.2.1 *Let A be a nonsemisimple quasi-Hopf algebra of dimension 4 with Jacobson radical J. Then $\dim_k J = 2$.*

Proof. Since $\dim_k(J)$ is at most 3, we show that it cannot be equal to 1 or 3.

If $\dim_k J = 1$, as J^2 is properly included in J, it follows that $J^2 = 0$. Hence $\mathrm{gr}(A) = \frac{A}{J} \oplus J \simeq A$ as k-vector spaces. We get form here that $3 = \dim_k(A_0)$ is a divisor of $4 = \dim_k \mathrm{gr}(A)$, a contradiction.

If $\dim_k J = 3$, A_0 is a 1-dimensional quasi-Hopf algebra. This means that $A_0 = k$, the trivial Hopf algebra, and therefore $\mathrm{gr}(A) \simeq B \times A_0 \simeq B$ is an ordinary Hopf algebra. If $J^2 = 0$, as above we get that $\mathrm{gr}(A) = \frac{A}{J} \oplus J \simeq A$ as k-spaces, hence $\mathrm{gr}(A)$ is a 4-dimensional Hopf algebra whose Jacobson radical is of dimension 3. We are lead to a contradiction, since the classification of 4-dimensional Hopf algebras says that the Jacobson radical must be zero or of dimension 2. Thus $\dim_k J^2 \in \{1, 2\}$.

Assume that $\dim_k J^2 = 1$; this forces $J^3 = 0$, and therefore $\mathrm{gr}(A) = A_0 \oplus \frac{J}{J^2} \oplus J^2$ is a 4-dimensional Hopf algebra whose Jacobson radical $\frac{J}{J^2} \oplus J^2$ has dimension 3; we saw that this is false. So it remains that $\dim_k J^2 = 2$, in which case either $\mathrm{gr}(A) = A_0 \oplus A_1 \oplus J^2$ (provided that $J^3 = 0$) or $\mathrm{gr}(A) = A_0 \oplus A_1 \oplus A_2 \oplus J^3$ (provided that $J^3 \neq 0$, and so $\dim_k J^3 = 1$); in both cases we get that $\mathrm{gr}(A)$ is a 4-dimensional Hopf algebra whose Jacobson radical is of dimension 3, a contradiction we met for several times during this proof. □

We now have all the necessary ingredients to show that $\mathrm{gr}(A)$ is a Hopf algebra.

Proposition 3.2.2 *Let A be a nonsemisimple quasi-Hopf algebra of dimension 4. Then A and $\mathrm{gr}(A)$ are isomorphic as algebras and $\mathrm{gr}(A)$ is a Hopf algebra isomorphic to H_4.*

Proof. As before, J stands for the Jacobson radical of A. Then $\dim_k J = 2$ and $J^2 = 0$; we cannot have $\dim_k J^2 = 1$, otherwise $A_1 = \frac{J}{J^2}$ is a one dimensional vector space and at the same time a free module over A_0, a 2-dimensional vector space.

Thus $\mathrm{gr}(A) = \frac{A}{J} \oplus J \simeq A$ is a 4-dimensional quasi-Hopf algebra, and we will next see that it is isomorphic to A as an algebra. Towards this end, for all $a \in A$, consider $a = \mathbb{S}(a) + J(a)$ the decomposition of a in $A = \mathbb{S} \oplus J$, where $\mathbb{S}$ is the semisimple subalgebra of A given by [1, Theorem 1.4.9]. Since $J^2 = 0$, for all $a, a' \in A$, $aa' = \mathbb{S}(a)\mathbb{S}(a') + \mathbb{S}(a)J(a') + J(a)\mathbb{S}(a')$, so $\mathbb{S}(aa') = \mathbb{S}(a)\mathbb{S}(a')$ and $J(aa') = \mathbb{S}(a)J(a') + J(a)\mathbb{S}(a')$, for all $a, a' \in A$. It follows that $\mathbb{S}$ induces an algebra morphism from A to $\mathbb{S}$, which we denote by $\pi_{\mathbb{S}}$, such that $\pi_{\mathbb{S}}\iota = \mathrm{Id}_{\mathbb{S}}$ ($\iota : \mathbb{S} \to A$ is the inclusion, an algebra morphism, too). In addition, the canonical k-isomorphism $p : \mathbb{S} \to A_0 : s \mapsto p(s)$ turns into a k-algebra isomorphism, its inverse being determined by $A_0 \to \mathbb{S} : p(a) \mapsto \mathbb{S}(a)$. We deduce from here that $\zeta : A = \mathbb{S} \oplus J \to A_0 \oplus J = \mathrm{gr}(A)$ given by $\zeta(a) = p(\mathbb{S}(a)) + J(a) = p(a) + J(a)$, for all $a \in A$, is a k-isomorphism. It is a k-algebra morphism as well, since the graded algebra structure of $\mathrm{gr}(A)$ is defined by $p_m(x)p_n(y) = p_{m+n}(xy)$, for all positive integers m and n, $x \in J^m$ and $y \in J^n$, and therefore in our case $p(a)x = p_1(ax) = ax$ and $xp(a) = p_1(xa) = xa$, for all $a \in A$, $x \in J$; then we compute that, for all $a, a' \in A$,

$$
\begin{aligned}
\zeta(a)\zeta(a') &= p(aa') + p(a)J(a') + J(a)p(a') \\
&= p(aa') + \mathbb{S}(a)J(a') + J(a)\mathbb{S}(a') = p(aa') + J(aa') = \zeta(aa'),
\end{aligned}
$$

as needed. Clearly, $\zeta(1_A) = p(1_A)$ is equal to the unit of $\mathrm{gr}(A)$, hence $A \simeq \mathrm{gr}(A)$ as k-algebras.

Consequently, $A \simeq B \times A_0$ as an algebra, for a certain braided Hopf algebra $B \in {}^{A_0}_{A_0}\mathcal{YD}$. Since A is not semisimple and A_0 is, Corollary 2.1.10 implies that $A \simeq A_0(\theta)_{\sigma,v}$, for an algebra morphism $\sigma : A_0 \to k$ and $v \in A_0$ such that $\varepsilon(v) = 1$, $\sigma(v) = -1$. But, as an algebra, $A_0 = k[C_2] = k[\langle g \rangle]$, so σ is determined by $\sigma(1) = 1_k$, $\sigma(g) = -1$ (if $\sigma(g) = 1$, $\sigma = \varepsilon$, fact that implies $1 = \varepsilon(v) = \sigma(v) = -1$, a contradiction); this forces $v = g$. It is clear now that, up to isomorphism, $A = H_4$ as an algebra.

On the other hand, there exists a quasi-Hopf algebra morphism $p : A \to A_0$ and an algebra morphism $\upsilon : A_0 \simeq \mathbb{S} \hookrightarrow A$ given by $\upsilon(p(a)) = \mathbb{S}(a)$, for all $a \in A$, such that $p\upsilon = \mathrm{Id}_{A_0}$. We proved that $A = H_4$ as an algebra, so $\mathbb{S} = k[\langle g \rangle]$ and J is the subspace of A generated by x and gx, where g, x are the algebra generators of H_4. By identifying A_0 to $k_\phi[\langle g \rangle]$ as a quasi-Hopf algebra (for a certain 3-cocycle ϕ of $\langle g \rangle$), we deduce that there exists a surjective quasi-Hopf algebra morphism $p : A \to k_\phi[\langle g \rangle]$ that acts as identity on the subalgebra $\mathbb{S} = k[\langle g \rangle]$ of A. In particular, if the quasi-coalgebra structure of A is defined by (Δ, ε) then $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$.

Denote by $(\widetilde{\Delta}, \widetilde{\varepsilon})$ the quasi-coalgebra structure of $\mathrm{gr}(A)$ induced by the quasi-coalgebra structure (Δ, ε) of A. Since $x \in J$ and $J \subseteq A$ is a quasi-Hopf ideal, $\Delta(x) = a \otimes x + b \otimes gx + x \otimes c + gx \otimes d$, for some elements a, b, c, d of A such that $\varepsilon(a) = \varepsilon(c) = 1$ and $\varepsilon(b) = \varepsilon(d) = 0$. As $\widetilde{\Delta}(x)^2 = 0$ and

$$
\begin{aligned}
\widetilde{\Delta}(x) &= (p \otimes \mathrm{Id} + \mathrm{Id} \otimes p)\Delta(x) \\
&= p(a) \otimes x + p(b) \otimes gx + x \otimes p(c) + gx \otimes p(d),
\end{aligned}
$$

a direct computation yields $p(a) = p_+ + \lambda p_-$, $p(b) = \mu p_-$, $p(c) = p_+ - \lambda p_-$ and $p(d) = -\mu p_-$, for some $\lambda, \mu \in k$ such that $\lambda^2 - \mu^2 = 1$.

It is well known that, for k a field of characteristic different from 2, the third cohomology group $H^3(\langle g\rangle, k^*)$ is isomorphic to C_2. The representative 3-cocycles of $\langle g\rangle$ are given by the trivial one and $\phi = 1 - 2p_- \otimes p_- \otimes p_-$. Since ϕ is also the reassociator of $\mathrm{gr}(A)$, in the former case we get that the "$p_- \otimes p_- \otimes -$" component of $\phi(\widetilde{\Delta} \otimes \mathrm{Id})(\widetilde{\Delta}(x))$ has $-$ equal to $x - 2p_- x$ while the "$p_- \otimes p_- \otimes -$" component of $(\mathrm{Id} \otimes \widetilde{\Delta})(\widetilde{\Delta}(x))\phi$ has $-$ equal to $\lambda^2 x + \lambda\mu gx - \lambda\mu gx - \mu^2 x - 2p_+ x = x - 2p_+ x$. The quasi-coassociativity of $\widetilde{\Delta}$ says that $\phi(\widetilde{\Delta} \otimes \mathrm{Id})(\widetilde{\Delta}(x)) = (\mathrm{Id} \otimes \widetilde{\Delta})(\widetilde{\Delta}(x))\phi$, and this implies $p_+ x = p_- x$, a contradiction.

We reduce to the case in which ϕ is trivial, hence $\phi = (1 \otimes F)(\mathrm{Id} \otimes \Delta)(F)(\Delta \otimes \mathrm{Id})(F^{-1})(F^{-1} \otimes 1)$, for some twist F of $k[C_2]$. It can be easily seen that $F = 1 - \kappa p_- \otimes p_-$, for some $\kappa \in k$ different from 1_k, with inverse $F^{-1} = 1 - \frac{\kappa}{\kappa - 1} p_- \otimes p_-$. A straightforward computation leads us to $\phi = 1 \otimes 1 \otimes 1$. Therefore, as $\mathrm{gr}(A) \simeq B \times k[C_2]$ is a 4-dimensional nonsemisimple Hopf algebra (once more, $\mathrm{gr}(A) \simeq A$ as an algebra), from the classification of 4-dimensional Hopf algebras we conclude that $\mathrm{gr}(A)$ and H_4 are isomorphic Hopf algebras. □

As we have already mentioned, owing to Etingof and Gelaki, a nonsemisimple 4-dimensional quasi-Hopf algebra is twist equivalent to H_4. In what follows we want to determine when two twist-deformations of H_4 are isomorphic as quasi-Hopf algebras.

For a general quasi-Hopf algebra H, a situation when this fact occurs is when the twists $\mathbb{F}$, $\mathbb{G}$ of H are related through a special invertible element σ, as described by the following lemma, which is the quasi-Hopf analogue of Propositions 2.3.3 and 2.3.5 in [36].

Lemma 3.2.3 *Let H be a quasi-Hopf algebra; if there exists an invertible $\sigma \in H$ such that*

$$\varepsilon(\sigma) = 1, \quad \sigma^{-1} = S(\sigma) \quad \textit{and} \quad \mathbb{G} = (\sigma^{-1} \otimes \sigma^{-1})\mathbb{F}\Delta(\sigma), \tag{3.2.1}$$

then $\varphi : H_{\mathbb{F}} \to H_{\mathbb{G}}$ defined by $\varphi(h) = \sigma^{-1} h \sigma$, for all $h \in H$, is a quasi-Hopf algebra isomorphism.

Proof. Denote by ϕ the reassociator of H, by $\phi_{\mathbb{F}}$ the reassociator of the twisted quasi-Hopf algebra $H_{\mathbb{F}}$ and $\mathbb{F} = \mathcal{F}^1 \otimes \mathcal{F}^2$, $\mathbb{G} = \mathcal{G}^1 \otimes \mathcal{G}^2$, $\mathbb{F}^{-1} = \mathcal{P}^1 \otimes \mathcal{P}^2$, $\mathbb{G}^{-1} = \mathcal{Q}^1 \otimes \mathcal{Q}^2$. By the third equality in (3.2.1), we obtain the relations, in Sweedler notation, $\mathbb{F} = \sigma \mathcal{G}^1 \sigma_1^{-1} \otimes \sigma \mathcal{G}^2 \sigma_2^{-1}$ and $\mathbb{F}^{-1} = \sigma_1 \mathcal{Q}^1 \sigma^{-1} \otimes \sigma_2 \mathcal{Q}^2 \sigma^{-1}$. Since the arguments in the proof of [36, Propositions 2.3.5] hold for quasi-Hopf algebras too, we conclude by checking

$$\begin{aligned}
\varphi^{\otimes 3}(\phi_{\mathbb{F}}) &= \varphi^{\otimes 3}((1 \otimes \mathbb{F})(\mathrm{Id} \otimes \Delta)(\overline{\mathbb{F}})\phi(\Delta \otimes \mathrm{Id})(\mathbb{F})^{-1}((\overline{\mathbb{F}})^{-1} \otimes 1)) \\
&= \sigma^{-1}\mathcal{F}^1 X^1 \mathcal{P}_1^1 \overline{\mathcal{P}}^1 \sigma \otimes \sigma^{-1}\mathcal{F}^1 \overline{\mathcal{F}}_1^2 X^2 \mathcal{P}_2^1 \overline{\mathcal{P}}^2 \sigma \otimes \sigma^{-1}\mathcal{F}^2 \overline{\mathcal{F}}_2^2 X^3 \mathcal{P}^2 \sigma \\
&\overset{(3.2.1)}{=} \mathcal{G}^1 \sigma_1^{-1} X^1 \sigma_{1_1} \mathcal{Q}_1^1 \overline{\mathcal{Q}}_1 \otimes \mathcal{G}^1 \overline{\mathcal{G}}_1^2 \sigma_{2_1}^{-1} X^2 \sigma_{1_2} \mathcal{Q}_2^1 \overline{\mathcal{Q}}^2 \otimes \mathcal{G}^2 \overline{\mathcal{G}}_2 \sigma_{2_2}^{-1} X^3 \sigma_2 \mathcal{Q}^2 \\
&\overset{(1.2.16)}{=} \mathcal{G}^1 X^1 \mathcal{Q}_1^1 \overline{\mathcal{Q}}_1 \otimes \mathcal{G}^1 \overline{\mathcal{G}}_1^2 X^2 \mathcal{Q}_2^1 \overline{\mathcal{Q}}^2 \otimes \mathcal{G}^2 \overline{\mathcal{G}}_2 X^3 \mathcal{Q}^2 \\
&= (1 \otimes \mathbb{G})(\mathrm{Id} \otimes \Delta)(\overline{\mathbb{G}}))\phi(\Delta \otimes \mathrm{Id})(\mathbb{G})^{-1}((\overline{\mathbb{G}})^{-1} \otimes 1) = \phi_{\mathbb{G}},
\end{aligned}$$

where we put a line over a twist or its components to distinguish multiple copies. □

For $H = H_4$, an invertible element $\sigma \in H_4$ satisfying $\varepsilon(\sigma) = 1$ has the form $\sigma = p_+ + \lambda p_- + \rho p_+ x + \varrho p_- x$, with $0 \neq \lambda, \rho, \varrho \in k$; then $\sigma^{-1} = p_+ + \overline{\lambda} p_- + \overline{\rho} p_+ x + \overline{\varrho} p_- x$, where $\overline{\lambda} = \lambda^{-1}$, $\overline{\rho} = -\rho\lambda^{-1}$ and $\overline{\varrho} = -\rho\lambda^{-1}$. If follows that $\sigma^{-1} = S(\sigma)$ if and only if $\sigma = p_+ + \lambda p_-$ and $\lambda^2 = 1$, thus $\sigma \in \{1, g\}$.

The existence of σ as in the above is only a sufficient condition for $H_{\mathbb{F}}$ and $H_{\mathbb{G}}$ to be isomorphic quasi-Hopf algebras. An example in this sense is given by H_4; for this, we need the lemma below.

Lemma 3.2.4 *Let $\mathcal{F}$ be a twist on H_4 such that H_4 and $(H_4)_{\mathcal{F}}$ are isomorphic as quasi-Hopf algebras, where H_4 is seen as a quasi-Hopf algebra with trivial reassociator. Then $\mathcal{F} = 1 - \tau g x \otimes x$, for some $\tau \in k$, and the isomorphism φ between H_4 and $(H_4)_{\mathcal{F}}$ is determined by a non-zero scalar ω, in the sense that $\varphi(g) = g$ and $\varphi(x) = \omega x$.*

Proof. Recall that $p_\pm = \frac{1}{2}(1 \pm g)$; they are orthogonal idempotent elements of H_4 such that $1 = p_+ + p_-$ and $p_+ - p_- = g$. Then $\{p_\pm, p_\pm x\}$ is a basis for H_4 and, with respect to this basis, any element $\mathcal{F} \in H_4 \otimes H_4$ obeying $\varepsilon(\mathcal{F}^1)\mathcal{F}^2 = 1 = \varepsilon(\mathcal{F}^2)\mathcal{F}^1$ is of the form

$$\begin{aligned} \mathcal{F} = p_+ \otimes 1 + p_- \otimes p_+ + a p_- \otimes p_+ + b p_- \otimes p_- x \\ + c p_- \otimes p_+ x + \mu p_- \otimes p_- + \nu p_- x \otimes p_- x, \\ + \tau p_- x \otimes p_+ x + u p_+ x \otimes p_- + v p_+ x \otimes p_- x + w p_+ x \otimes p_+ x, \end{aligned} \tag{3.2.2}$$

for some scalars $a, b, c, \mu, \nu, \tau, u, v, w \in k$; in what follows, we will refer to them as being the scalars defining $\mathcal{F}$. It is easy to see that $\mathcal{F}$ is a twist (that is, in this case, $\mathcal{F}$ is invertible in $H_4 \otimes H_4$) if and only if $a \neq 0$, in which case the scalars defining $\mathcal{F}^{-1}$, denoted by $\overline{a}, \ldots, \overline{w}$, are given by

$$\begin{aligned} &\overline{a} = \frac{1}{a}, \quad \overline{b} = -\frac{b}{a}, \quad \overline{c} = -\frac{c}{a}, \quad \overline{\mu} = -\frac{\mu}{a}, \quad \overline{\nu} = -\frac{\nu}{a}, \\ &\overline{\tau} = \frac{c\mu}{a} - \tau, \quad \overline{u} = -\frac{u}{a}, \quad \overline{v} = \frac{ub}{a} - v, \quad \overline{w} = -\frac{w}{a}. \end{aligned}$$

Now, let $\varphi : H_4 \to (H_4)_{\mathcal{F}}$ be a quasi-Hopf algebra isomorphism. It follows from the definition of a quasi-Hopf isomorphism that $\alpha_{\mathcal{F}} = \beta_{\mathcal{F}} = 1$ and $\Phi_{\mathcal{F}} = 1 \otimes 1 \otimes 1$. In other words, $\mathcal{F}^1 S(\mathcal{F}^2) = 1 = S(\mathcal{G}^1)\mathcal{G}^2$ and $(1 \otimes \mathcal{F})(\mathrm{Id} \otimes \Delta)(\mathcal{F}) = (\mathcal{F} \otimes 1)(\Delta \otimes \mathrm{Id})(\mathcal{F})$, where $\mathcal{F}^{-1} := \mathcal{G}^1 \otimes \mathcal{G}^2$.

We have $S(p_\pm) = p_\pm$, $S(p_- x) = -p_+ x$ and $S(p_+ x) = p_- x$, as $p_\pm x = x p_\mp$. Therefore, $\mathcal{F}^1 S(\mathcal{F}^2) = 1$ holds if and only if $a = 1$ and $u = c = 0$. Likewise, $S(\mathcal{G}^1)\mathcal{G}^2 = 1$, which is equivalent to $S(\mathcal{F}^1)\mathcal{F}^2 = 1$, holds if and only if $a = 1$ and $b = \mu = 0$. Thus

$$\mathcal{F} = 1 \otimes 1 + \nu p_- x \otimes p_- x + \tau p_- x \otimes p_+ x + v p_+ x \otimes p_- x + w p_+ x \otimes p_+ x.$$

With this simplified formula of $\mathcal{F}$ in mind, since

$$\begin{aligned} \Delta(p_-) &= p_+ \otimes p_- + p_- \otimes p_+, \\ \Delta(p_- x) &= p_+ \otimes p_- x + p_+ x \otimes p_- - p_- \otimes p_+ x + p_- x \otimes p_+, \\ \Delta(p_+) &= p_+ \otimes p_+ + p_- \otimes p_-, \\ \Delta(p_+ x) &= p_+ \otimes p_+ x + p_+ x \otimes p_+ - p_- \otimes p_- x + p_- x \otimes p_-, \end{aligned}$$

we see that $(1\otimes\mathcal{F})(\mathrm{Id}\otimes\Delta)(\mathcal{F})$ equals to

$$\begin{aligned}1\otimes 1\otimes 1&+(\nu p_-x\otimes v p_+x)\otimes(p_+\otimes p_-x+p_+x\otimes p_--p_-\otimes p_+x+p_-x\otimes p_+)\\&+(\tau p_-x+wp_+x)\otimes(p_+\otimes p_+x+p_+x\otimes p_+-p_-\otimes p_-x+p_-x\otimes p_-)\\&+1\otimes(\nu p_-x\otimes p_-x+\tau p_-x\otimes p_+x+vp_+x\otimes p_-x+wp_+x\otimes p_+x),\end{aligned}$$

and that $(\mathcal{F}\otimes 1)(\Delta\otimes\mathrm{Id})(\mathcal{F})$ equals to

$$\begin{aligned}1\otimes 1\otimes 1&+(p_+\otimes p_-x+p_+x\otimes p_--p_-\otimes p_+x+p_-x\otimes p_+)\otimes(\nu p_-x\otimes\tau p_+x)\\&+(p_+\otimes p_+x+p_+x\otimes p_+-p_-\otimes p_-x+p_-x\otimes p_-)\otimes(vp_-x\otimes wp_+x)\\&+(\nu p_-x\otimes p_-x+\tau p_-x\otimes p_+x+vp_+x\otimes p_-x+wp_+x\otimes p_+x)\otimes 1.\end{aligned}$$

Hence, we have $(1\otimes\mathcal{F})(\mathrm{Id}\otimes\Delta)(\mathcal{F})=(\mathcal{F}\otimes 1)(\Delta\otimes\mathrm{Id})(\mathcal{F})$ if and only if

$$\begin{aligned}&\nu(p_+\otimes p_-x+p_+x\otimes p_--p_-\otimes p_+x+p_-x\otimes p_+)\\&\quad+\tau(p_+\otimes p_+x+p_+x\otimes p_+-p_-\otimes p_-x+p_-x\otimes p_-)\\&\quad=\nu p_+\otimes p_-x+\tau p_+\otimes p_+x+vp_-\otimes p_-x+wp_-\otimes p_+x+\nu p_-x\otimes\tau p_+x\otimes 1;\end{aligned}$$

$$\begin{aligned}&v(p_+\otimes p_-x+p_+x\otimes p_--p_-\otimes p_+x+p_-x\otimes p_+)\\&\quad+w(p_+\otimes p_+x+p_+x\otimes p_+-p_-\otimes p_-x+p_-x\otimes p_-)\\&\quad=\nu p_-\otimes p_-x+\tau p_-\otimes p_+x+vp_+\otimes p_-x+wp_+\otimes p_+x+vp_-x\otimes 1+wp_+x\otimes 1;\end{aligned}$$

$$\begin{aligned}&1\otimes(\nu p_-x\otimes p_-x+\tau p_-x\otimes p_+x+vp_+x\otimes p_-x+wp_+x\otimes p_+x)\\&\quad=p_+\otimes\big(p_-x\otimes(\nu p_-x+\tau p_+x)+p_+x\otimes(vp_-x+wp_+x)\big)\\&\quad-p_-\otimes\big(p_+x\otimes(\nu p_-x+\tau p_+x)+p_-x\otimes(vp_-x+wp_+x)\big).\end{aligned}$$

A simple inspection ensures us that the three equalities above if and only if $w=v=-\tau=-\nu$, in which case

$$\mathcal{F}=1+\tau p_-x\otimes(p_-x+p_+x)-\tau p_+x\otimes(p_-x+p_+x)=1-\tau gx\otimes x,$$

as stated. Now, for $\mathcal{F}=1-\tau gx\otimes x$, it can be easily checked that $\Delta_{\mathcal{F}}=\Delta$; thus $(H_4)_{\mathcal{F}}$ is a Hopf algebra equal to H_4. Consequently, defining a quasi-Hopf algebra isomorphism between H_4 and $(H_4)_{\mathcal{F}}$ is equivalent to giving a Hopf algebra automorphism of H_4. Perhaps it is folklore that any Hopf algebra automorphism φ of H_4 is defined by a non-zero scalar as in the statement of the lemma. If this is not the case, as an algebra automorphism of H_4, φ is determined by $\varphi(g)=g+\lambda x+\rho gx$ and $\varphi(x)=\omega x+\delta gx$, for some $\lambda,\rho,\omega,\delta\in$ k with $(\omega,\delta)\neq 0$, since $g^2=1$, $x^2=0$, $\varepsilon(g)=1$ and $\varepsilon(x)=0$. From $(\varphi\otimes\varphi)(g)=\varphi(g)\otimes\varphi(g)$ we get $\lambda=\rho=0$, and from $\Delta(\varphi(x))=(\varphi\otimes\varphi)(g\otimes x+x\otimes 1)$ we deduce that $\delta=0$. Summing up, $\varphi(g)=g$ and $\varphi(x)=\omega x$ with $0\neq\omega\in k$. □

Proposition 3.2.5 *Let $\mathbb{F}$, $\mathbb{G}$ be twists on H_4 determined by $a,\ldots,w\in k$ and by $a',\ldots,w'\in k$ respectively. Then $(H_4)_{\mathbb{F}}$ and $(H_4)_{\mathbb{G}}$ are isomorphic as quasi-Hopf algebras if and only if $a=a'$ and there exist $0\neq\omega,\kappa\in k$ such that $(b',c',\mu',u')=\omega(b,c,\mu,u)$, $\nu'=\omega^2\nu+\kappa a$, $\tau'=\omega^2\tau+\kappa$, $v'=\omega^2v-\kappa$ and $w'=\omega^2w-\kappa$.*

Proof. Take note that $\varphi:(H_4)_{\mathbb{F}}\to(H_4)_{\mathbb{G}}$ is a quasi-Hopf algebra isomorphism if and only if $\varphi:H_4\to(H_4)_{\mathbb{F}_\varphi^{-1}\mathbb{G}}$ is also a isomorphism of quasi-Hopf algebras, where $\mathbb{F}_\varphi:=(\varphi\otimes\varphi)(\mathbb{F})$ with inverse $\mathbb{F}_\varphi^{-1}$. Denote $\mathcal{F}=\mathbb{F}_\varphi^{-1}\mathbb{G}$; by Lemma 3.2.4, there exist

$0 \neq \omega, \kappa \in k$ such that $\mathbb{G} = \mathbb{F}_\varphi(1 \otimes 1 - \kappa gx \otimes x)$, where φ is defined by $\varphi(g) = g$, $\varphi(x) = \omega x$. Equivalently, $\mathbb{G} = \mathbb{F}_\varphi - \kappa \mathbb{F}_\varphi(gx \otimes x)$, hence $\mathbb{G}$ equals to

$$\begin{aligned} &p_+ \otimes 1 + p_- \otimes p_+ + ap_- \otimes p_- + \omega bp_- \otimes p_- x + \omega cp_- \otimes p_+ x \\ &\quad + \omega\mu p_- x \otimes p_- + (\omega^2\nu + \kappa a)p_- x \otimes p_- x + (\omega^2\tau + \kappa)p_- x \otimes p_+ x \\ &\quad + \omega u p_+ x \otimes p_- + (\omega^2 v - \kappa)p_+ x \otimes p_- + (\omega^2 w - \kappa)p_+ x \otimes p_+ x, \end{aligned}$$

as previously stated. □

Remarks 3.2.6 1. As a continuation to the comments made before Lemma 3.2.4 and of Lemma 3.2.3, note that the case when $\mathbb{F}$ and $\mathbb{G}$ are related through a special element $\sigma \in H_4$ (which, in H_4, can be either 1 or g) occurs if and only if $\kappa = 0$ and $\omega^2 = 1$.

2. For $a \neq a'$, the quasi-Hopf algebras $(H_4)_\mathbb{F}$ and $(H_4)_\mathbb{G}$ are not isomorphic. We conclude that there are infinitely many non-isomorphic nonsemismple quasi-Hopf algebras of dimension 4, all twist equivalent to H_4.

3.2.2 The semisimple case

For the sake of completeness, we conclude this section by pointing out the classes of non twist equivalent semisimple quasi-Hopf algebras H of dimension 4, over an algebraically closed field k of characteristic 0.

Proposition 3.2.7 *Let H be a semisimple quasi Hopf algebra of dimension 4. Then, H is twist equivalent to one of the following non-twist equivalent quasi-Hopf algebras:*

- $k_{\phi_{\xi^a}}[C_4]$ *for any $a \in \{0, 1, 2, 3\}$, the usual Hopf algebra $k[C_4]$, seen as a quasi-Hopf algebra with reassociator defined by*

$$\Phi_a = \sum_{i,j,l=0}^{3} \xi^{ai\lfloor \frac{j+l}{n} \rfloor}\, 1_i \otimes 1_j \otimes 1_l.$$

- $k_{\phi_{a,b,c}}[C_2 \times C_2]$ *for any $a, b, c \in \{0, 1\}$, the Hopf algebra $k[C_2 \times C_2]$ seen as a quasi-Hopf algebra with reassociator*

$$\Phi_{a,b,c} = \sum_{\substack{i,j,s\\ t,l,r \in \{0,1\}}} (-1)^{ai\lfloor \frac{l+s}{2} \rfloor + bj\lfloor \frac{l+s}{2} \rfloor + cj\lfloor \frac{r+t}{2} \rfloor}\, \mathbf{1}_{ij} \otimes \mathbf{1}_{st} \otimes \mathbf{1}_{lr}, \tag{3.2.3}$$

where $\{\mathbf{1}_{ij} := \frac{1}{4}\sum_{t,s\in\{0,1\}} (-1)^{it+js} g_1^t g_2^s\}_{i,j\in\{0,1\}}$ and g_1, g_2 denote two fixed generators of $C_2 \times C_2$.

In particular, there are 12 semisimple quasi-Hopf algebras in dimension 4 which are not pairwise twist equivalent.

Proof. Since any irreducible representation of H is one dimensional, it follows that, as an algebra, H is the group algebra of k and a group of order 4. Consequently, H is a commutative algebra, and therefore a usual Hopf algebra. Equivalently, H is a the group Hopf algebra of k and a group G with 4 elements, seen as a quasi-Hopf algebra via a normalized 3-cocycle on G. G is either the cyclic group C_4 or the Klein group $C_2 \times C_2$.

• For $G = C_n$, it is well known that 3-cocycle representatives for C_n are given by $\{\phi_{\xi^a} \mid 0 \leq a \leq n-1\}$ (se e.g. [15, Example 1.16]), where ξ is a primitive root of unity of degree n in k and, for all $0 \leq i, j, l \leq n-1$,

$$\phi_{\xi^a}(\sigma^i, \sigma^j, \sigma^l) = \xi^{ai\lfloor \frac{j+l}{n} \rfloor}. \tag{3.2.4}$$

Then, by Lemma 3.1.1, the action in (3.1.1) can be restated, for $0 \leq a \leq n-1$, as

$$(f, \overline{\phi_{\xi^a}}) \mapsto \overline{\phi_{\xi^{s^2 a}}}, \tag{3.2.5}$$

provided that $f \in \mathrm{Aut}(C_n) \simeq U(\mathbb{Z}_n)$ is defined by $1 \leq s \leq n-1$ such that $(s, n) = 1$, meaning that f is determined by $f(\sigma) = \sigma^s$. If we take now $n = 4$, as $s^2 \equiv 1$ $(mod\ 4)$ for all $s \in \{1, 3\}$, and so the action in (3.2.5) is trivial, then it follows by [24, Remark 2.6.2] that we have 4 quasi-Hopf algebra structures on $k[C_4]$ which are not pairwise twist equivalent.

• For $G = C_2 \times C_2$, the 3-cocycle representatives where described explicitly in [8, Theorem 3.5] and [31, Proposition 3.9]. Following the presentation of the second paper, these are determined by the parameters $a, b, c \in \{0, 1\}$ as follows:

$$\phi_{a,b,c}(g_1^i g_2^j, g_1^s g_2^t, g_1^l g_2^r) = (-1)^{ai\lfloor \frac{l+s}{2} \rfloor + bj\lfloor \frac{l+s}{2} \rfloor + cj\lfloor \frac{r+t}{2} \rfloor},$$

for all $i, j, s, t, l, r \in \{0, 1\}$; since $\{\mathbf{1}_{ij}\}_{i,j \in \{0,1\}}$ form a complete system of orthogonal idempotents on $k[C_2 \times C_2]$, we obtain (3.2.3) by identifying $k^{C_2 \times C_2}$ with $k[C_2 \times C_2]$ through the isomorphism $P_{g_1^i g_2^j} \mapsto \mathbf{1}_{ij}$ Furthermore, by the comments made after the proof of [8, Theorem 3.5], since $Out(C_2 \times C_2) = \mathrm{Aut}(C_2 \times C_2) \simeq S_3$, we conclude that for $G = C_2 \times C_2$ the group action in (3.1.1) is trivial, and therefore we have 8 quasi-Hopf algebra structures on $k[C_2 \times C_2]$ which are not pairwise twist equivalent.

To conclude, we notice that $k[C_4]$ and $k[C_2 \times C_2]$ are isomorphic as algebras, both are isomorphic to k^4, but not as Hopf algebras. Indeed, if there exists a Hopf algebra isomorphism $\varphi : k[C_2 \times C_2] \to k[C_4]$ then φ behaves well with respect to the antipodes $S_{k[C_4]}$ and $S_{k[C_2 \times C_2]}$ of $k[C_4]$ and $k[C_2 \times C_2]$, respectively. Since $S_{k[C_2 \times C_2]}$ is the identity morphism, we get that $S_{k[C_4]} \circ \phi = \phi$ or, equivalently, that $S_{k[C_4]} = \mathrm{Id}_{k[C_4]}$; this is false. $\square$

Chapter 4

Quasi-Hopf algebras of dimension 6

In this chapter, we complete the classification of 6-dimensional quasi-Hopf algebras, over an algebraically closed field of characteristic 0, by proving that all of them are semisimple. We do this mainly by characterizing YD-bialgebras in the category of Yetter-Drinfeld modules over $H(2)$.

Finally, by translating the main result of [22] from the language of fusion categories to that of algebras, we are able to produce a complete list of quasi-Hopf algebras in dimension 6, up to twist equivalence.

4.1 Free biproduct (quasi-)Hopf algebras of rank 3 over C_2

We are working over a field k which is algebraically closed and of characteristic 0; we need this condition to make use of results in the classification of Fusion categories and Hopf algebras. For n a non-zero natural number, unless otherwise stated, we keep denoting by C_n the cyclic group of order n and by g one of its generators.

Let $H(2)$ be the group Hopf algebra $k[C_2]$, seen as a quasi-Hopf algebra with reassociator $\Phi_2 = 1 \otimes 1 \otimes 1 - 2p_- \otimes p_- \otimes p_-$, where $p_\pm = \frac{1}{2}(1 \pm g)$. The antipode is the identity map and the distinguished elements are $\alpha = g$ and $\beta = 1$; in what follows, we denote this quasi-Hopf structure on $k[C_2]$ by $H(2)$.

As a preliminary step in proving our main result, we describe all the 6-dimensional quasi-Hopf algebras with a projection onto $H(2)$, by classifying braided Hopf algebras of dimension 3 in ${}_{H(2)}^{H(2)}\mathcal{YD}$. The other case, of 6-dimensional quasi-Hopf algebras A with a projection that covers the embedding of a 3-dimensional quasi-Hopf subalgebra of A, was considered in Chapter 2, see in particular Proposition 2.2.13, since all 3 dimensional quasi-Hopf algebras are group quasi-Hopf algebras over C_3.

It is well known that, up to isomorphism, there are three 6-dimensional Hopf algebras: $k[C_6]$, the group Hopf algebra of the cyclic group of order 6, $k[S_3]$, the group Hopf algebra of the symmetric group with 3 letters and, finally, the dual Hopf algebra to $k[S_3]$, denoted by k^{S_3}; for all details on the classification of low dimensional Hopf algebras we refer to [50]. As we will see, this classification yields for free the classification of the 3-dimensional Hopf algebras in ${}_{k[C_2]}^{k[C_2]}\mathcal{YD}$.

Proposition 4.1.1 *Let H denote the Hopf algebra $k[C_2]$. If B is a 3-dimensional braided Hopf algebra in ${}_H^H\mathcal{YD}$, then B is isomorphic to one of the following braided Hopf algebras:*

- *B_{C_6}, the group Hopf algebra $k[C_3]$ viewed as an object of ${}^H_H\mathcal{Y}D$ via trivial H-action and H-coaction.*

- *B_{S_3}, the group Hopf algebra $k[C_3]$ considered in ${}^H_H\mathcal{Y}D$ via the trivial H-coaction and H-action defined by $g \cdot x = x^2$, where x stands for a generator of C_3.*

- *B_*, the group Hopf algebra $k[C_3]$ regarded as an object of ${}^H_H\mathcal{Y}D$ via the trivial H-action and H-coaction determined by*

$$\lambda(x^i) = p_+ \otimes x^i + p_- \otimes x^{(2i)'}, \tag{4.1.1}$$

where x is the generator of C_3 and $(2i)'$ is the remainder of the division of $2i$ by 3, $i \in \{0, 1, 2\}$.

Moreover, via the biproduct construction, we have a one-to-one correspondence between B_{C_6}, B_{S_3}, B_* and, respectively, $k[C_6]$, $k[S_3]$, k^{S_3}.

Proof. The group isomorphism $C_6 \simeq C_3 \times C_2$ produces a Hopf algebra isomorphism $k[C_6] \simeq k[C_3 \times C_2] \simeq k[C_3] \otimes k[C_2]$, where $k[C_3] \otimes k[C_2]$ has the tensor product Hopf algebra structure. It follows form here that $k[C_6]$ can be realized as the biproduct Hopf algebra between B_{C_6} and $k[C_2]$, since the trivial H-action and H-coaction translate, through the biproduct construction, to the tensor product Hopf algebra structure.

The symmetric group S_3 identifies to the semidirect product of groups $C_3 \rtimes C_2$. Moreover, if $\sigma \in S_3$ is a cycle of length 3 and $\tau \in S_3$ is a transposition, $C_3 = \langle \sigma \rangle$, $C_2 = \langle \tau \rangle$ and $\tau\sigma = \sigma^{-1}\tau$. In Hopf algebra language, the inclusion $i : k[C_2] \to k[S_3]$ is a Hopf algebra morphism and the projection $\pi : k[S_3] \to k[C_2]$ given by $\pi(\sigma^i\tau^j) = \tau^j$, $0 \le i \le 2$ and $0 \le j \le 1$, is a Hopf algebra morphism that covers i. Thus, $k[S_3]$ is a biproduct Hopf algebra between a braided Hopf algebra B within ${}^H_H\mathcal{Y}D$ and $k[C_2]$. As $B = \mathrm{Im}(E)$, with $E : k[S_3] \to k[S_3]$ given by $E(\sigma^i\tau^j) = \sigma^i\tau^j S(\pi(\sigma^i\tau^j)) = \sigma^i$, for all $0 \le i \le 2$ and $0 \le j \le 1$, we obtain that $B = k[C_3]$ as a k-vector space, and an object of ${}^H_H\mathcal{Y}D$ with structure determined by $\tau \triangleright \sigma^i = \tau\sigma^i\tau = \sigma^{-i} = \sigma^{(2i)'}$ and $\sigma^i \mapsto \pi(\sigma^i) \otimes \sigma^i = 1 \otimes \sigma^i$, for all $0 \le i \le 2$; $\triangleright$ here denotes the conjugation, meaning that, for $g_1, g_2 \in G$ a group, $g_1 \triangleright g_2 := g_1 g_2 g_1^{-1}$. The structure of B as a Hopf algebra in ${}^H_H\mathcal{Y}D$ equals the structure of the group Hopf algebra $k[C_3]$, and therefore $B = B_{S_3}$.

With the same notation as above, we have a Hopf algebra morphism $i^* : k^{S_3} \to k[C_2]^* \equiv k[C_2]$ that covers the Hopf algebra inclusion $\pi^* : k[C_2] \equiv k[C_2]^* \to k^{S_3}$; the Hopf algebra identification $k[C_2]^* \equiv k[C_2]$ is given by $1 \mapsto P_1 + P_\tau$ and $\tau \mapsto P_1 - P_\tau$, where $\{P_1, P_\tau\}$ is the basis of $k[C_2]^*$ dual to the basis $\{1, \tau\}$ of $k[C_2]$. Note that, the inverse of this correspondence is determined by $P_1 \mapsto p_+$ and $P_\tau \mapsto p_-$. So, as in the previous case, k^{S_3} can be characterized as a biproduct Hopf algebra between a braided Hopf algebra B' in ${}^H_H\mathcal{Y}D$ and $k[C_2]$. Explicitly, if $\{P_{ij}\}$ is the basis of k^{S_3} dual to the basis $\{\sigma^i\tau^j\}$ of $k[S_3]$, we have that $\pi^* : k[C_2] \to k^{S_3}$ is given by $\pi^*(1) = \varepsilon$ and $\pi^*(\tau) = \sum_{i,j=0}^{2,1}(-1)^j P_{ij}$, where ε is the counit of $k[S_3]$ or, alternatively, the unit of k^{S_3}. Likewise, $i^* : k^{S_3} \to k[C_2]$ is determined by $i^*(P_{i0}) = \delta_{i,0}p_+$ and $i^*(P_{i1}) = \delta_{i,0}p_-$, $0 \le i \le 2$, where $\delta_{i,j}$ denotes the Kronecker's delta of i and j. By using the definition of the projection $E : k^{S_3} \to B'$, we find that $E(P_{i0}) = P_{i0} + P_{i1}$ and $E(P_{i1}) = 0$, for all $0 \le i \le 2$, and this fact implies that B' is spanned by $\{P_{i0} + P_{i1} \mid 0 \le i \le 2\}$. Moreover, if we denote $e_i := P_{i0} + P_{i1}$, $0 \le i \le 2$, then $\{e_0, e_1, e_2\}$ is a basis of

B' defined by idempotents such that $e_0 + e_1 + e_2 = 1$. In addition, if we consider $C_3 = \langle\sigma\rangle$, $e_i(\sigma^l) = \delta_{i,l}$, for all $0 \le i, l \le 2$, then $\{e_0, e_1, e_2\}$ can be seen as the dual basis of the basis $\{1, \sigma, \sigma^2\}$ of $k[C_3]$. More explicitly, the isomorphism between B' and $k[C_3]^* \cong k[C_3]$ can be uncovered as follows.

Let ω be a primitive root of unity of order 3 and set $x := e_0 + \omega e_1 + \omega^2 e_2$. We compute that $x^2 := e_0 + \omega^2 e_1 + \omega e_2$ and $x^3 = e_0 + e_1 + e_2 = 1$. Since $\{1, x, x^2\}$ is another basis of B', we conclude that $B' \simeq k[C_3] = k[\langle x\rangle]$ as k-algebras. Furthermore, the braided coalgebra structure of B' in ${}^H_H\mathcal{Y}D$, with coproduct map denoted by $\underline{\Delta}$, is determined by

$$\underline{\Delta}(e_i) = \sum_{(u+v)'=i} e_u \otimes e_v, \ \forall\, i \in \{0, 1, 2\},$$

where the sum is over $u, v \in \{0, 1, 2\}$ such that the remainder $(u+v)'$ of the division of $u + v$ by 3 equals i. By a brief direct computation, we obtain that $\underline{\Delta}(x) = x \otimes x$, and therefore $B' \simeq k[C_3]$ as Hopf algebras. Finally, the H-action on B' is trivial becasue B' is a commutative algebra and τ has order 2, therefore $\tau \triangleright e_i = \pi^*(\tau)e_i\pi^*(\tau) = \pi^*(\tau^2)e_i = e_i$. Considering the left H-coaction on B', it is given by $e_i \mapsto (i^* \otimes \mathrm{Id}_{B'})(\Delta(e_i)) = p_+ \otimes e_i + p_- \otimes e_{(2i)'}$, for all $0 \le i \le 2$, where Δ is the comultiplication of k^{S_3}. In terms of x, we have $x^i \mapsto p_+ \otimes x^i + p_- \otimes x^{(2i)'}$, for all $0 \le i \le 2$.

Hence, any 6-dimensional Hopf algebra is a biproduct Hopf algebra, and as such it is determined by a braided Hopf algebra in ${}^H_H\mathcal{Y}D$. Then, by the classification result mentioned above, we have that B_{C_6}, B_{S_3} and B_* are the only types of 3-dimensional braided Hopf algebras in ${}^H_H\mathcal{Y}D$. $\square$

It is well-known that, for a finite dimensional Hopf algebra H, the category of left-right Yetter-Drinfeld modules over H, ${}_H\mathcal{Y}D^H$, is isomorphic, even as a braided category, to the category of left modules over the quantum double $D(H)$ of H; for more details, we refer the reader to [34]. As far as we are concerned, if $M \in {}_H\mathcal{Y}D^H$ then M is a left $D(H)$-module via the action $(h^* \bowtie h) \cdot m = h^*((h \cdot m)_{(1)})(h \cdot m)_{(0)}$, for all $h^* \in H^*$ and $h \in H$, where $\cdot$ is the left H-action on M and $M \ni m \mapsto m_{(0)} \otimes m_{(1)} \in M \otimes H$ denotes the right action of H on M. Conversely, any left $D(H)$-module M can be seen in ${}_H\mathcal{Y}D^H$ via the following structure, for all $h \in H$ and $m \in M$,

$$h \cdot m = (\varepsilon \bowtie h)m \quad \text{and} \quad m \mapsto \sum_{i=1}^{n} (e^i \bowtie 1)m \otimes e_i,$$

where $\{e^i, e_i\}$ denote the elements of dual bases in H^* and H respectively, and the juxtaposition denotes the action of $D(H)$ on M.

On the other hand, by considering [15, Theorem 8.14] for our Hopf algebra H, the categories ${}^H_H\mathcal{Y}D$ and ${}_H\mathcal{Y}D^H$ are monoidally isomorphic; the inverse functors that provide the isomorphism act as identity on modules and morphisms, and transport the left (respectively right) H-coaction into a right (respectively left) one with the help of the inverse of the antipode S of H (respectively, with the help of S). Consequently, ${}^H_H\mathcal{Y}D$ is monoidally isomorphic to the category of left $D(H)$-modules.

If we now take H to be equal to the Hopf algebra $k[C_2]$, one can see that

$$D(H) = k[C_2]^* \otimes k[C_2] \cong k[C_2] \otimes k[C_2] \cong k[C_2 \times C_2]$$

as a Hopf algebras. It is well known that $k[C_2 \times C_2] \cong k[C_4]$ as algebras, so, for $H = k[C_2]$, we have four one dimensional non-isomorphic simple $D(H)$-modules by the Wedderburn–Artin theorem, and therefore four non-isomorphic simple left Yetter-Drinfeld modules over H, which will be denoted in what follows by M_j^i, where $i, j \in \{0, 1\}$. As the M_i^j's are all one dimensional, we consider $M_j^i = km_j^i$ for some non-zero vector m_i^j. By using the correspondences mentioned above between $D(H)$-modules and left Yetter-Drinfeld modules over H, we have the following left Yetter-Drinfeld module structure for M_i^j:

$$g \cdot m_j^i = (-1)^j m_j^i \quad \text{and} \quad \lambda(m_j^i) = g^i \otimes m_j^i. \tag{4.1.2}$$

Using the same notation as in Proposition 4.1.1, we have that

$$\begin{aligned} B_{S_3} &= k1 \oplus k(x + x^2) \oplus k(x - x^2) \\ &= M_0^0 \oplus M_0^0 \oplus M_1^0, \end{aligned} \tag{4.1.3}$$

$$\begin{aligned} B_* &= k(1 + x + x^2) \oplus k(x + x^2) \oplus k(x - x^2) \\ &= M_0^0 \oplus M_0^0 \oplus M_0^1. \end{aligned} \tag{4.1.4}$$

We now move to the quasi-Hopf setting. In what follows, $\mathfrak{q}$ is a primitive fourth root of unity in k or, equivalently, $\mathfrak{q}^2 = -1$.

Proposition 4.1.2 *The left Yetter-Drinfeld module category ${}_{H(2)}^{H(2)}\mathcal{YD}$ is a semisimple monoidal category with 4 simple objects, denoted by M_i, for any $0 \leq i \leq 3$. Each M_i is one dimensional and if m_i is a generator of M_i, M_i is in ${}_H^H\mathcal{YD}$ with structure given by*

$$g \cdot m_i = (-1)^i m_i \quad \text{and} \quad \lambda(m_i) = (p_+ + \mathfrak{q}^i p_-) \otimes m_i. \tag{4.1.5}$$

Proof. By [7, Proposition 2.10], $D(H(2))$ is generated as an algebra by $X = \varepsilon \bowtie g$ and $Y = (P_1 - P_g) \bowtie 1$; here $\{P_1, P_g\}$ is the basis of $k[C_2]^*$ dual to the basis $\{1, g\}$ of $k[C_2]$. The generators X and Y verify the relations $X^2 = 1$ and $Y^2 = X$, thus we have $D(H(2)) = k[\langle Y \rangle] \cong k[C_4]$ as an algebra. Then, the quasi-Hopf algebra structure of $D(H(2))$ is defined by

$$\begin{aligned} &\Delta(Y) = -\frac{1}{2}(Y \otimes Y + Y^3 \otimes Y + Y \otimes Y^3 - Y^2 \otimes Y^3), \\ &\varepsilon(Y) = -1, \\ &\Phi = 1 - 2p_-^X \otimes p_-^X \otimes p_-^X, \\ &S_D(Y) = Y, \quad \alpha_D = X, \quad \beta_D = 1, \end{aligned}$$

where, in reference to the notation used before for $k[C_2]$, we set $p_-^X = \frac{1}{2}(1 - X)$; note that $X = Y^2$ is a grouplike element of $D(H(2))$ of order two, so p_-^X is an idempotent element of $D(H(2))$.

As in the Hopf case, we have an isomorphism of categories

$${}_{H(2)}^{H(2)}\mathcal{YD} \simeq {}_{H(2)}\mathcal{YD}^{H(2)} \simeq {}_{D(H(2))}\mathcal{M} \simeq {}_{k[C_4]}\mathcal{M};$$

see [15, Theorems 8.14 and 8.29]. Since ${}_{k[C_4]}\mathcal{M}$ is a semisimple category with four one dimensional non-isomorphic simple objects, we deduce that ${}_{H(2)}^{H(2)}\mathcal{YD}$ is also a semisimple category with four non-isomorphic simple objects, all of dimension one.

Denote by M_j, for $j \in \{0,1,2,3\}$, the simple modules over $k[C_4]$, whose structure is determined, for any $m \in M_j$, by $Y \cdot m = \mathfrak{q}^j m$. In what follows, we describe the four simple objects of ${}^{H(2)}_{H(2)}\mathcal{YD}$.

The correspondence in [15, Lemma 8.28] allows us to compute the simple objects of ${}_{H(2)}\mathcal{YD}^{H(2)}$, still denoted by $\{M_j\}_j$. Since $H(2) = k[\langle X\rangle] \cong k[C_2]$ as a subalgebra of $D(H(2))$, the left $H(2)$-action on M_j is given, for all $m \in M_j$, by

$$X \cdot m = Y^2 \cdot m = \mathfrak{q}^{2j} m = (-1)^j m.$$

Also, by the quoted result, the right $H(2)$-coaction on M_j is defined by

$$\rho(m) = (P_1 \bowtie p_2^1)m \otimes S^{-1}(p^2)p_1^1 + (P_g \bowtie p_2^1)m \otimes S^{-1}(p^2)gp_1^1,$$

where, calling back to (1.2.25), $p_R = p^1 \otimes p^2 = x^1 \otimes x^2\beta S(x^3) = 1 - 2p_- \otimes p_-$. It is immediate that $\Delta(p^1) \otimes p^2 = g \otimes g \otimes p_- + 1 \otimes 1 \otimes p_+$, hence

$$\begin{aligned}
\rho(m) &= -(P_1 \bowtie g)m \otimes p_- + (P_g \bowtie g)m \otimes p_- \\
&\quad +(P_1 \bowtie 1)m \otimes p_+ + (P_g \bowtie 1)m \otimes p_+ \\
&= -((P_1 - P_g) \bowtie g)m \otimes p_- + ((P_1 + P_g) \bowtie 1)m \otimes p_+ \\
&= -(Y(\varepsilon \bowtie g))m \otimes p_- + (\varepsilon \bowtie 1)m \otimes p_+ \\
&= -Y^3 m \otimes p_- + m \otimes p_+ \\
&= m \otimes p_+ - \mathfrak{q}^{3j} m \otimes p_- = m \otimes p_+ - \mathfrak{q}^{-j} m \otimes p_-.
\end{aligned}$$

To achieve our aim, we use the functor described in [15, Theorem 8.14] in order to get the left Yetter-Drinfeld structure over $H(2)$ of M_j. To this end, we need the elements defined in (1.2.26) and (1.2.27), namely

$$\begin{aligned}
q_R &= q^1 \otimes q^2 := X^1 \otimes S^{-1}(\alpha X^3)X^2 \\
&= 1 \otimes g + 2p_- \otimes p_- \\
&= 1 \otimes p_+ - g \otimes p_-, \\
p_L &= \tilde{p}^1 \otimes \tilde{p}^2 := X^2 S^{-1}(X^1\beta) \otimes X^3 \\
&= 1 \otimes p_+ + g \otimes p_-.
\end{aligned}$$

A simple computation ensures us that

$$(\Delta(q^1) \otimes q^2)\Phi^{-1} = 1 - 2p_+ \otimes p_+ \otimes p_-$$

and since $p_\pm \cdot m = \frac{1}{2}(1 \pm \mathfrak{q}^{2j})m$, for all $m \in M_j$, we get that, for all $m \in M_j$,

$$\begin{aligned}
&(\tilde{p}^2 \cdot m)_{(1)}\tilde{p}^1 \otimes (\tilde{p}^2 \cdot m)_{(0)} \\
&\quad = (p_+ \cdot m)_{(1)} \otimes (p_+ \cdot m)_{(0)} + (p_- \cdot m)_{(1)} g \otimes (p_- \cdot m)_{(0)} \\
&\quad = \frac{1}{2}(1 + \mathfrak{q}^{2j})(p_+ - \mathfrak{q}^{-j}p_-) \otimes m + \frac{1}{2}(1 - \mathfrak{q}^{2j})(p_+ - \mathfrak{q}^{-j}p_-)g \otimes m \\
&\quad = (p_+ - \mathfrak{q}^j p_-) \otimes m.
\end{aligned}$$

Now, by [15, Lemma 8.28], the left $H(2)$-coaction on M_j is given, for $m \in M_j$, by

$$\begin{aligned}
\lambda(m) &= q_1^1 x^1 S(q^2 x^3 (\tilde{p}^2 \cdot m)_1 \tilde{p}^1) \otimes q_2^1 x^2 \cdot (\tilde{p}^2 \cdot m)_0 \\
&= S((\tilde{p}^2 \cdot m)_{(1)}\tilde{p}^1) \otimes (\tilde{p}^2 \cdot m)_{(0)} \\
&\quad -2p_+ S(p_-(\tilde{p}^2 \cdot m)_{(1)}\tilde{p}^1) \otimes p_+ \cdot (\tilde{p}^2 \cdot m)_{(0)} \\
&= (p_+ - \mathfrak{q}^{-j}p_-) \otimes m.
\end{aligned}$$

The left $H(2)$-module structure on M_j, considered as a left Yetter-Drinfeld module, remains the same, namely $g \cdot m = (-1)^j m$, for all $m \in M_j$. By switching 1 and 3 and, respectively, 0 and 2, we obtain the structure of the simple objects $\{M_j\}_{0\leq j\leq 3}$ of ${}^{H(2)}_{H(2)}\mathcal{YD}$ as in the statement of the proposition. □

Corollary 4.1.3 *Let B be a 3-dimensional bialgebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$. B is isomorphic as a Yetter-Drinfeld module to one of the following:*

$$M_0 \oplus M_{2i} \oplus M_{2j}; \tag{4.1.6}$$
$$M_0 \oplus M_{2i} \oplus M_{2j+1}; \tag{4.1.7}$$
$$M_0 \oplus M_{2i+1} \oplus M_{2j+1}, \tag{4.1.8}$$

where $i, j \in \{0, 1\}$ and $\{M_l\}_l$ are as in Proposition 4.1.2.

Proof. The Corollary follows directly from Proposition 4.1.2, by considering the last two relations in (1.3.41), which imply that one of the Yetter-Drinfeld submodules of B, containing the unit, must be isomorphic to M_0. □

In the next lemma we show that, for a braided bialgebra, the case (4.1.8) does not occur.

Lemma 4.1.4 *Let B be a 3-dimensional bialgebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$. In the notation of Proposition 4.1.2, B is not isomorphic as a Yetter-Drinfeld module to $M_0 \oplus M_{2i+1} \oplus M_{2j+1}$.*

Proof. Suppose the contrary; namely, that $B = k1 \oplus kx \oplus ky$ is a bialgebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$ with unit 1 and $x, y \in B$ such that $kx \simeq M_{2i+1}$ and $ky \simeq M_{2j+1}$ as left Yetter-Drinfeld $H(2)$-modules, for some $i, j \in \{0, 1\}$. We move to characterize the braided bialgebra structure of B in terms of its generators x, y. For instance, by applying (1.3.41) to x, y, we deduce that

$$g \cdot x^2 = x^2, \quad g \cdot y^2 = y^2, \quad g \cdot xy = xy, \quad g \cdot yx = yx, \tag{4.1.9}$$

implying that $x^2, y^2, xy, yx \in k1$; equivalently,

$$x^2 = \omega 1, \quad y^2 = \omega' 1, \quad xy = l1, \quad yx = r1, \tag{4.1.10}$$

for some scalars $\omega, \omega', l, r \in k$. Secondly, $p_+ \cdot x = p_+ \cdot y = 0$ and $p_- \cdot x = x$, $p_- \cdot y = y$. These relations together with (1.3.42) applied to the generators x, y lead to the following equalities:

$$\omega x = (xx)x = -x(xx) = -\omega x, \quad \omega' y = (yy)y = -y(yy) = -\omega' y; \tag{4.1.11}$$
$$\omega y = (xx)y = -x(xy) = -lx, \quad rx = (yx)x = -y(xx) = -\omega y; \tag{4.1.12}$$
$$lx = (xy)x = -x(yx) = -rx, \quad ly = (xy)y = -x(yy) = -\omega' x; \tag{4.1.13}$$
$$ry = (yx)y = -y(xy) = -ly, \quad \omega' x = (yy)x = -y(yx) = -ry. \tag{4.1.14}$$

Thus $l = r = \omega = \omega' = 0$, and therefore B is generated as an algebra by x, y with relations $x^2 = y^2 = xy = yx = 0$.

From $\underline{\varepsilon}_B(g \cdot b) = \varepsilon(g)\,\underline{\varepsilon}_B(b) = \underline{\varepsilon}_B(b)$, we get $\underline{\varepsilon}_B(x) = \underline{\varepsilon}_B(y) = 0$. Thus, if we consider $\underline{\Delta}_B(x) = \sum_{a,b\in\{1,x,y\}} \sigma_{ab}\, a \otimes b$, the counit properties say that

$$\underline{\Delta}_B(x) = 1 \otimes x + x \otimes 1 + \sigma_{xx} x \otimes x + \sigma_{yy} y \otimes y + \sigma_{xy} x \otimes y + \sigma_{yx} y \otimes x. \tag{4.1.15}$$

The first relation in (1.3.44) applied to x reduces to $-\underline{\Delta}_B(x) = (g \otimes g) \cdot \underline{\Delta}_B(x)$, and is equivalent to

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -\sigma_{xx} & -\sigma_{xy} \\ 0 & -\sigma_{yx} & -\sigma_{yy} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & \sigma_{xx} & \sigma_{xy} \\ 0 & \sigma_{yx} & \sigma_{yy} \end{pmatrix}. \tag{4.1.16}$$

We conclude that $\underline{\Delta}(x) = x \otimes 1 + 1 \otimes x$ and $\underline{\varepsilon}(x) = 0$, that is, x is a primitive element. Consequently, by considering (1.3.48) for $\mathfrak{b} = x$ and $\mathfrak{b}' = x$, since $x^2 = 0$, $p_- \cdot x = x$ and $p_+ \cdot x = 0$, we get that $0 = \underline{\Delta}_B(x^2)$ equals

$$\begin{aligned}
&(y^1X^1 \cdot 1)(y^2Y^1(x^1X^2 \cdot x)_{[-1]}x^2X_1^3 \cdot 1) \otimes (y_1^3Y^2 \cdot (x^1X^2 \cdot x)_{[0]})(y_2^3Y^3x^3X_2^3 \cdot x) \\
&+ (y^1X^1 \cdot x)(y^2Y^1(x^1X^2 \cdot 1)_{[-1]}x^2X_1^3 \cdot 1) \otimes (y_1^3Y^2 \cdot (x^1X^2 \cdot 1)_{[0]})(y_2^3Y^3x^3X_2^3 \cdot x) \\
&+ (y^1X^1 \cdot 1)(y^2Y^1(x^1X^2 \cdot x)_{[-1]}x^2X_1^3 \cdot x) \otimes (y_1^3Y^2 \cdot (x^1X^2 \cdot x)_{[0]})(y_2^3Y^3x^3X_2^3 \cdot 1) \\
&+ (y^1X^1 \cdot x)(y^2Y^1(x^1X^2 \cdot 1)_{[-1]}x^2X_1^3 \cdot x) \otimes (y_1^3Y^2 \cdot (x^1X^2 \cdot 1)_{[0]})(y_2^3Y^3x^3X_2^3 \cdot 1) \\
=&1 \otimes x^2 + x \otimes x + x_{[-1]} \cdot x \otimes x_{[0]} + x^2 \otimes 1 = (1 + \mathfrak{q}^{2i+1})x \otimes x.
\end{aligned}$$

Hence, $1 + \mathfrak{q}^{2i+1} = 0$ for a certain $i \in \{0, 1\}$, a contradiction. $\square$

We study now the case (4.1.7). For simplicity, we first look at the algebra case.

Lemma 4.1.5 *Let B be a 3-dimensional algebra in ${}_{H(2)}^{H(2)}\mathcal{YD}$, isomorphic as a Yetter-Drinfeld module to $M_0 \oplus M_{2i} \oplus M_{2j+1}$. Then B is isomorphic to one of the following algebras in ${}_{H(2)}^{H(2)}\mathcal{YD}$:*

- $B_{2o}^{00} := k1 \oplus k\bar{x} \oplus k\bar{y} \simeq M_0 \oplus M_2 \oplus M_{2j+1}$ *in ${}_{H(2)}^{H(2)}\mathcal{YD}$, with relations*
$$\bar{x}^2 = \bar{y}^2 = \bar{x}\bar{y} = \bar{y}\bar{x} = 0;$$
- $B_{0o}^{10} := k1 \oplus k\bar{x} \oplus k\bar{y} \simeq M_0 \oplus M_0 \oplus M_{2j+1}$ *in ${}_{H(2)}^{H(2)}\mathcal{YD}$, with relations*
$$\bar{x}^2 = 1, \quad \bar{y}^2 = 0, \quad \bar{x}\bar{y} = \bar{y}, \quad \bar{y}\bar{x} = -\bar{y};$$
- $B_{0o}^{0} := k1 \oplus k\bar{x} \oplus k\bar{y} \simeq M_0 \oplus M_0 \oplus M_{2j+1}$ *in ${}_{H(2)}^{H(2)}\mathcal{YD}$, with relations*
$$\bar{x}^2 = 0, \quad \bar{y}^2 = \bar{x}, \quad \bar{x}\bar{y} = \bar{y}\bar{x} = 0;$$
- $B_{0o}^{00} = k1 \oplus k\bar{x} \oplus k\bar{y} \simeq M_0 \oplus M_0 \oplus M_{2j+1}$ *in ${}_{H(2)}^{H(2)}\mathcal{YD}$, with relations*
$$\bar{x}^2 = 0, \quad \bar{y}^2 = 0, \quad \bar{x}\bar{y} = \bar{y}\bar{x} = \bar{y};$$
- $B_{0o}^{10c} := k1 \oplus k\bar{x} \oplus k\bar{y} \simeq M_0 \oplus M_0 \oplus M_{2j+1}$ *in ${}_{H(2)}^{H(2)}\mathcal{YD}$, with relations*
$$\bar{x}^2 = 1, \quad \bar{y}^2 = 0, \quad \bar{x}\bar{y} = \bar{y}\bar{x} = \bar{y}.$$

Proof. We use arguments similar to the ones in the proof of Lemma 4.1.4. Assume that $B = k1 \oplus kx \oplus ky \simeq M_0 \oplus M_{2i} \oplus M_{2j+1}$ in ${}_{H(2)}^{H(2)}\mathcal{YD}$, for some $i, j \in \{0, 1\}$.

We have $g \cdot x = x$ and $g \cdot y = -y$, so the first relation in (1.3.41) yields $g \cdot x^2 = x^2$ and $g \cdot y^2 = y^2$. Thus, $x^2 = \alpha 1 + \beta x$ and $y^2 = \overline{\alpha} 1 + \overline{\beta} x$, for some scalars $\alpha, \overline{\alpha}, \beta, \overline{\beta} \in k$.

Likewise, $g\cdot(xy)=-xy$ and $g\cdot(yx)=-yx$, hence $xy=\gamma y$ and $yx=\overline{\gamma}y$, for some scalars $\gamma,\overline{\gamma}\in k$.

Clearly, $p_-\cdot x=0$ and $p_-\cdot y=y$. Therefore, the associativity of the multiplication of B, expressed by the relation in (1.3.42), is equivalent to the following equalities:

$$\begin{array}{llll}(xx)x=x(xx), & (yy)y=-y(yy), & (yx)x=y(xx), & (xx)y=x(xy),\\ (xy)x=x(yx), & (xy)y=x(yy), & (yx)y=y(xy), & (yy)x=y(yx).\end{array}$$

These are satisfied if and only if

$$\begin{gathered}\overline{\beta}(\gamma+\overline{\gamma})=-2\overline{\alpha},\quad \overline{\gamma}^2=\alpha+\beta\overline{\gamma},\quad \gamma^2=\alpha+\beta\gamma,\\ \alpha\overline{\beta}=\overline{\alpha}\gamma=\overline{\alpha}\overline{\gamma},\quad \overline{\alpha}+\beta\overline{\beta}=\overline{\beta}\gamma=\overline{\beta}\overline{\gamma}.\end{gathered}\tag{4.1.17}$$

The last condition remaining to be studied is the colinearity of the multiplication of B. By (1.3.43), applied to our structure, we find that $\lambda_B(x^2)$ equals

$$\begin{aligned}&X^1(x^1Y^1\cdot x)_{[-1]}x^2(Y^2\cdot x)_{[-1]}Y^3\otimes(X^2\cdot(x^1Y^1\cdot x)_{[0]})(X^3x^3\cdot(Y^2\cdot x)_{[0]})\\ &\quad= X^1(p_++(-1)^ip_-)^2\otimes(X^2\cdot x)(X^3\cdot x)=1\otimes x^2,\end{aligned}$$

and so $\beta(p_++(-1)^ip_-)\otimes x=\beta1\otimes x$; we conclude that $i=0$ or $\beta=0$.

In a similar way, we compute that

$$\begin{aligned}\lambda_B(y^2)&= X^1(x^1\cdot y)_{[-1]}x^2y_{[-1]}g\otimes(X^2\cdot(x^1\cdot y)_{[0]})(X^3x^3\cdot y_{[0]})\\ &= (p_+-p_-)\otimes y^2+2p_-\otimes y^2=1\otimes y^2,\end{aligned}$$

hence $\overline{\beta}(p_++(-1)^ip_-)tx=\overline{\beta}1\otimes x$; so, either $i=0$ or $\overline{\beta}=0$.

By using (1.3.43) for x and y, we get

$$\gamma(p_++\mathfrak{q}^{2j+1}p_-)\otimes y=\lambda_B(xy)=\gamma(p_++\mathfrak{q}^{2(i+j)+1}p_-)\otimes y,$$

and therefore $i=0$ or $\gamma=0$. Similarly, for y and x,

$$\overline{\gamma}(p_++\mathfrak{q}^{2j+1}p_-)\otimes y=\lambda_B(yx)=\overline{\gamma}(p_++\mathfrak{q}^{2(j+i)+1}p_-)\otimes y,$$

which reduces to $i=0$ or $\overline{\gamma}=0$.

<u>Case 1</u>: i=1. We get $\beta=\overline{\beta}=\gamma=\overline{\gamma}=0$, implying $\alpha=\overline{\alpha}=0$, too. Hence $B=B^{00}_{2o}$.

<u>Case 2</u>: i=0. We get the relations stated in (4.1.17); we distinguished the following possibilities:

<u>2.1</u>: $\gamma\neq\overline{\gamma}$. Then $\overline{\beta}=0$, so $\overline{\alpha}=0$, too. In addition, $\overline{\gamma}=\beta-\gamma$, $\gamma^2=\alpha+\beta\gamma$, $4\alpha\neq-\beta^2$ and x,y verify $x^2=\alpha1+\beta x$, $y^2=0$, $xy=\gamma y$ and $yx=(\beta-\gamma)y$. If we set $x':=\frac{\beta}{2}1-x$ and $\delta:=\frac{\beta}{2}-\gamma$ then $\delta\neq0$ and $x'^2=\delta^21$, $x'y=\delta y$ and $yx'=-\delta y$. Take now $\overline{x}:=\delta^{-1}x'$ to get $\overline{x}^2=1$, $y^2=0$, $\overline{x}y=y$ and $y\overline{x}=-y$. It is clear at this point that the correspondence defined by $\overline{x}\mapsto\delta^{-1}x'=\delta^{-1}(\frac{\beta}{2}1-x)$ and $\overline{y}\mapsto y$ defines an algebra isomorphism between B^{10}_{0o} and B in ${}^{H(2)}_{H(2)}\mathcal{YD}$. Note that $B^{10}_{0o}\simeq B=k1\oplus k(\delta^{-1}(\frac{\beta}{2}1-x))\oplus ky\simeq M_0\oplus M_0\oplus M_{2j+1}$ in ${}^{H(2)}_{H(2)}\mathcal{YD}$.

<u>2.2</u>: $\gamma=\overline{\gamma}$. In this case, we are reducing to $\overline{\beta}\gamma=-\overline{\alpha}$, $\gamma^2=\alpha+\beta\gamma$, $\alpha\overline{\beta}=\overline{\alpha}\gamma$ and $\overline{\alpha}+\beta\overline{\beta}=\overline{\beta}\gamma$. We must stress the fact that $\gamma=\overline{\gamma}$ means that γ and $\overline{\gamma}$ are both equal to one of the solutions of the equation $t^2-\beta t-\alpha=0$; in other words, the

assumption $\gamma = \overline{\gamma}$ does not impose any extra condition on α and β like, for instance, $\beta^2 = -4\alpha$.

Assume $\overline{\beta} \neq 0$. We have $\gamma = \overline{\gamma} = -\frac{\overline{\alpha}}{\overline{\beta}}$, $\alpha = -\left(\frac{\overline{\alpha}}{\overline{\beta}}\right)^2$, $\beta = -\frac{2\overline{\alpha}}{\overline{\beta}}$ and $\gamma^2 = \alpha + \beta\gamma$ is automatically satisfied. Then B is generated by x, y with relations $\left(x + \frac{\overline{\alpha}}{\overline{\beta}}1\right)^2 = 0$, $y^2 = \overline{\alpha}1 + \overline{\beta}x$ and $xy = yx = -\frac{\overline{\alpha}}{\overline{\beta}}$. Hence, if we take $x' := x + \frac{\overline{\alpha}}{\overline{\beta}}1$ then $x'^2 = 0$, $y^2 = \overline{\beta}x'$ and $x'y = yx' = 0$, with $\overline{\beta} \neq 0$. We deduce from here that the correspondence $\bar{x} \mapsto \overline{\beta}x' = \overline{\alpha}1 + \overline{\beta}x = y^2$, $\bar{y} \mapsto y$, produces an isomorphism of algebras between B^0_{0o} and B in ${}^{H(2)}_{H(2)}\mathcal{YD}$. Note that $B^0_{0o} \simeq B = k1 \oplus ky^2 \oplus ky \simeq M_0 \oplus M_0 \oplus M_{2j+1}$ in ${}^{H(2)}_{H(2)}\mathcal{YD}$.

In the case when $\overline{\beta} = 0$, we get $\overline{\alpha} = 0$, so B is generated by x, y with relations $x^2 = \alpha 1 + \beta x$, $y^2 = 0$ and $xy = yx = \gamma y$, where α, β, γ are scalars obeying $\gamma^2 = \alpha + \beta\gamma$. For $x' := \frac{\beta}{2}1 - x$ and $\delta := \frac{\beta}{2} - \gamma$, we have that $x'^2 = \delta^2 1$ and $x'y = yx' = \delta y$. If we have $\beta^2 = -4\alpha$, this leads to $\delta^2 = 0$, and thus the correspondence $\bar{x} \mapsto x' = \frac{\beta}{2}1 - x$ and $\bar{y} \mapsto y$ defines an algebra isomorphism between B^{00}_{0o} and B in ${}^{H(2)}_{H(2)}\mathcal{YD}$. Note that, in this case $B^{00}_{0o} \simeq B = k1 \oplus k(\frac{\beta}{2}1 - x) \oplus ky \simeq M_0 \oplus M_0 \oplus M_{2j+1}$ in ${}^{H(2)}_{H(2)}\mathcal{YD}$. Finally, if we have that $\beta^2 \neq -4\alpha$, and therefore $\delta \neq 0$, we take $\overline{x} := \delta^{-1}x'$ which satisfies $\overline{x}^2 = 1$, $\overline{x}y = y\overline{x} = y$ and $y^2 = 0$. We conclude that the correspondence $\bar{x} \mapsto \delta^{-1}x' = \delta^{-1}(\frac{\beta}{2}1 - x)$ and $\bar{y} \mapsto y$ defines an algebra isomorphism between B^{10c}_{0o} and B in ${}^{H(2)}_{H(2)}\mathcal{YD}$. Note that $B^{10c}_{0o} \simeq B = k1 \oplus k(\frac{\beta}{2}1 - x) \oplus ky \simeq M_0 \oplus M_0 \oplus M_{2j+1}$ in ${}^{H(2)}_{H(2)}\mathcal{YD}$. □

Proposition 4.1.6 *Let B be a 3-dimensional bialgebra in ${}^{H(2)}_{H(2)}\mathcal{YD}$. If B is isomorphic as a Yetter-Drinfeld module to $M_0 \oplus M_{2i} \oplus M_{2j+1}$, then B is isomorphic to one of the following braided bialgebras:*

1. *B^{10}_{0o}, with comultiplication and counit given by*

$$\begin{aligned} &\underline{\Delta}(x) = -\tfrac{1}{2}(1 \otimes 1 - 1 \otimes x - x \otimes 1 - x \otimes x),\ \underline{\varepsilon}(x) = 1;\\ &\underline{\Delta}(y) = \tfrac{1}{2}(1 \otimes y + y \otimes 1 + x \otimes y + y \otimes x),\ \underline{\varepsilon}(y) = 0. \end{aligned} \tag{4.1.18}$$

2. *B^{10}_{0o}, with comultiplication and counit given by*

$$\begin{aligned} &\underline{\Delta}(x) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x),\ \underline{\varepsilon}(x) = -1;\\ &\underline{\Delta}(y) = \tfrac{1}{2}(1 \otimes y + y \otimes 1 - x \otimes y - y \otimes x),\ \underline{\varepsilon}(y) = 0. \end{aligned} \tag{4.1.19}$$

3. *B^{10c}_{0o}, with comultiplication and counit given by*

$$\begin{aligned} &\underline{\Delta}(x) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x),\ \underline{\varepsilon}(x) = -1;\\ &\underline{\Delta}(y) = \tfrac{1}{2}(1 \otimes y + y \otimes 1 - x \otimes y - y \otimes x),\ \underline{\varepsilon}(y) = 0. \end{aligned} \tag{4.1.20}$$

Proof. We use the results obtained in Lemma 4.1.5; for simplicity, we consider, notation wise, the generators of the braided bialgebra B as being x, y, rather that $\bar{x}, \bar{y}$. Also, we set $\underline{\Delta}_B(x) = \sum_{p,q \in \{1,x,y\}} \sigma_{p,q}\, p \otimes q$ and $\underline{\Delta}_B(y) = \sum_{p,q \in \{1,x,y\}} \tau_{p,q}\, p \otimes q$.

Since $g \cdot y = -y$, by the second relation in (**??**) we obtain $\underline{\varepsilon}_B(y) = 0$. The equality $-\underline{\Delta}(y) = \underline{\Delta}(g \cdot y) = (g \otimes g)\underline{\Delta}(y)$ implies $\tau_{1,1} = \tau_{x,x} = \tau_{y,y} = 0$ and $(1 + (-1)^i)\tau_{1,x} = (1 + (-1)^i)\tau_{x,1} = 0$, $(1 + (-1)^{i+1})\tau_{x,y} = (1 + (-1)^{i+1})\tau_{y,x} = 0$. We use now the counit properties to get $\tau_{1,x} = \tau_{x,1} = 0$ and $\tau_{y,1} + \underline{\varepsilon}(x)\tau_{y,x} = 1 = \tau_{1,y} + \underline{\varepsilon}(x)\tau_{x,y}$. Summing up, we have

$$\underline{\Delta}(y) = \tau_{1,y}1 \otimes y + \tau_{x,y}x \otimes y + \tau_{y,1}y \otimes 1 + \tau_{y,x}y \otimes x,$$

with $(1+(-1)^{i+1})\tau_{x,y} = (1+(-1)^{i+1})\tau_{y,x} = 0$ and $\tau_{y,1}+\underline{\varepsilon}(x)\tau_{y,x} = 1 = \tau_{1,y}+\underline{\varepsilon}(x)\tau_{x,y}$.

From now on, for a bialgebra B in ${}_H^H\mathcal{YD}$ with comultiplication $\underline{\Delta}_B$, if b, b' are elements in B, the product $\underline{\Delta}_B(b)\underline{\Delta}_B(b')$ in $B \otimes B$ is made with respect to the tensor product algebra structure of $B \otimes B$ within ${}_H^H\mathcal{YD}$; see the right hand side of the equality (1.3.48). There are situations when the braided tensor algebra $B \otimes B$ equals the usual tensor product k-algebra between B and itself, for instance when the left H-action on B is trivial, but, as we are about to see, this is generally not the case.

Denote $\underline{\Delta}(y) = y_{\underline{1}} \otimes y_{\underline{2}} = y'_{\underline{1}} \otimes y'_{\underline{2}}$. With the help of (1.3.48), which, we recall, expresses the multiplicativity of the morphism $\underline{\Delta}$, we compute that

$$\begin{aligned}
\underline{\Delta}(y^2) &= (y^1X^1 \cdot y_{\underline{1}})(y^2Y^1(x^1X^2 \cdot y_{\underline{2}})_{[-1]}x^2X^3_1 \cdot y'_{\underline{1}}) \\
&\quad \otimes (y^3_1Y^2 \cdot (x^1X^2 \cdot y_{\underline{2}})_{[0]})(y^3_2Y^3x^3X^3_2 \cdot y'_{\underline{2}}) \\
&= \tau_{1,y}Y^1(x^1 \cdot y)_{[-1]}x^2 \cdot y'_{\underline{1}} \otimes (Y^2 \cdot (x^1 \cdot y)_{[0]})(Y^3x^3 \cdot y'_{\underline{2}}) \\
&\quad +\tau_{x,y}x(Y^1(x^1 \cdot y)_{[-1]}x^2 \cdot y'_{\underline{1}}) \otimes (Y^2 \cdot (x^1 \cdot y)_{[0]})(Y^3x^3 \cdot y'_{\underline{2}}) \\
&\quad +\tau_{y,1}(y^1 \cdot y)(y^2 \cdot y'_{\underline{1}}) \otimes y^3 \cdot y'_{\underline{2}} \\
&\quad +\tau_{y,x}(y^1 \cdot y)(y^2Y^1x_{[-1]} \cdot y'_{\underline{1}}) \otimes (y^3_1Y^2 \cdot x_{[0]})(y^3_2Y^3 \cdot y'_{\underline{2}}) \\
&= \tau^2_{1,y}1 \otimes y^2 + \tau_{1,y}\tau_{x,y}y_{[-1]} \cdot x \otimes y_{[0]}y \\
&\quad +\tau_{1,y}\tau_{y,1}y_{[-1]} \cdot y \otimes y_{[0]} + \tau_{1,y}\tau_{y,x}y_{[-1]} \cdot y \otimes y_{[0]}x \\
&\quad +\tau_{x,y}\tau_{1,y}x \otimes y^2 + \tau^2_{x,y}x(y_{[-1]} \cdot x) \otimes y_{[0]}y \\
&\quad +\tau_{x,y}\tau_{y,1}x(y_{[-1]} \cdot y) \otimes y_{[0]} + \tau_{x,y}\tau_{y,x}x(y_{[-1]} \cdot y) \otimes y_{[0]}x \\
&\quad +\tau_{y,1}\tau_{1,y}y \otimes y + \tau_{y,1}\tau_{x,y}yx \otimes y + \tau^2_{y,1}y^2 \otimes 1 \\
&\quad +\tau_{y,1}\tau_{y,x}y^2 \otimes x + \tau_{y,x}\tau_{1,y}y \otimes xy + \tau_{y,x}\tau_{x,y}y(x_{[-1]} \cdot x) \otimes x_{[0]}y \\
&\quad +\tau_{y,x}\tau_{y,1}(y^1 \cdot y)(y^2x_{[-1]} \cdot y) \otimes y^3 \cdot x_{[0]} \\
&\quad +\tau^2_{y,x}(y^1 \cdot y)(y^2x_{[-1]} \cdot y) \otimes y^3 \cdot (x_{[0]}x) \\
&= \tau^2_{1,y}1 \otimes y^2 + \tau_{1,y}\tau_{x,y}\Big(1 + \frac{1+(-1)^i}{2} + \mathfrak{q}^{2j+1}\frac{1-(-1)^i}{2}\Big)x \otimes y^2 \\
&\quad +(1+\mathfrak{q}^{2j+1})\tau_{1,y}\tau_{y,1}y \otimes y + 2\tau_{y,1}\tau_{y,x}y^2 \otimes x \\
&\quad +\tau^2_{x,y}\Big(\frac{1+(-1)^i}{2} + \mathfrak{q}^{2j+1}\frac{1-(-1)^i}{2}\Big)x^2 \otimes y^2 \\
&\quad +(-1)^i\tau^2_{y,x}y^2 \otimes x^2 + \tau^2_{y,1}y^2 \otimes 1 + \tau_{y,1}\tau_{x,y}(\gamma\mathfrak{q}^{2j+1} + \overline{\gamma})y \otimes y \\
&\quad +\tau_{1,y}\tau_{y,x}(\overline{\gamma}\mathfrak{q}^{2j+1} + \gamma)y \otimes y + \tau_{x,y}\tau_{y,x}\gamma\overline{\gamma}(1+\mathfrak{q}^{2j+1})y \otimes y.
\end{aligned}$$

If $i = 1$, $\tau_{x,y} = \tau_{y,x} = 0$ and $\tau_{1,y} = \tau_{y,1} = 1$. So $\underline{\Delta}(y^2) = 1 \otimes y^2 + (1+\mathfrak{q}^{2j+1})y \otimes y + y^2 \otimes 1$, and this forces $y^2 \neq 0$, a contradiction. We conclude that $i = 0$, and therefore

$$\begin{aligned}
\underline{\Delta}(y^2) = \tau^2_{1,y}1 \otimes y^2 &+ 2\tau_{1,y}\tau_{x,y}x \otimes y^2 + (1+\mathfrak{q}^{2j+1})\tau_{1,y}\tau_{y,1}y \otimes y + 2\tau_{y,1}\tau_{y,x}y^2 \otimes x \\
&+\tau^2_{x,y}x^2 \otimes y^2 + \tau^2_{y,x}y^2 \otimes x^2 + \tau^2_{y,1}y^2 \otimes 1 + \tau_{y,1}\tau_{x,y}(\gamma\mathfrak{q}^{2j+1} + \overline{\gamma})y \otimes y \\
&+\tau_{1,y}\tau_{y,x}(\overline{\gamma}\mathfrak{q}^{2j+1} + \gamma)y \otimes y + \tau_{x,y}\tau_{y,x}\gamma\overline{\gamma}(1+\mathfrak{q}^{2j+1})y \otimes y.
\end{aligned}$$

Similarly, by using the fact that $g \cdot x = x$ and $g \cdot y = -y$, we see that

$$\underline{\Delta}(x) = \sigma_{1,1} 1 \otimes 1 + \sigma_{1,x} 1 \otimes x + \sigma_{x,1} x \otimes 1 + \sigma_{x,x} x \otimes x + \sigma_{y,y} y \otimes y$$

with $\sigma_{1,1} = -\underline{\varepsilon}(x)\sigma_{1,x}$ and $\sigma_{1,x} = \sigma_{x,1} = 1 - \underline{\varepsilon}(x)\sigma_{x,x}$. Owing to the above relations, the coassociativity of $\underline{\Delta}$ on x, see (1.3.45), reduces to

$$\sigma_{y,y}\sigma_{x,1} = \sigma_{y,y}\tau_{y,1} = \sigma_{y,y}\tau_{1,y}, \quad \sigma_{x,x}\sigma_{y,y} = \sigma_{y,y}\tau_{x,y} = \sigma_{y,y}\tau_{y,x}.$$

Likewise, the coassociativity of $\underline{\Delta}$ on y is equivalent to

$$\tau_{y,1}\tau_{y,x} = \tau_{y,x}\sigma_{x,1}, \quad \sigma_{x,1}\tau_{x,y} = \tau_{1,y}\tau_{x,y}, \quad \tau_{x,y}^2 = \sigma_{x,x}\tau_{x,y},$$
$$\tau_{y,x}^2 = \tau_{y,x}\sigma_{x,x}, \quad \sigma_{y,y}\tau_{y,x} = -\sigma_{y,y}\tau_{x,y}.$$

Case 1: $\sigma_{y,y} \neq 0$. Then $\tau_{x,y} = \tau_{y,x} = \sigma_{x,x} = 0$, $\tau_{1,y} = \tau_{y,1} = \sigma_{x,1} = \sigma_{1,x} = 1$ and $\sigma_{1,1} = -\underline{\varepsilon}(x)$. Thus, $\underline{\Delta}(y) = 1 \otimes y + y \otimes 1$ and $\underline{\Delta}(y^2) = 1 \otimes y^2 + y^2 \otimes 1 + (1 + \mathfrak{q}^{2j+1}) y \otimes y$. We cannot have $y^2 = 0$, so $y^2 = x$. Consequently,

$$\underline{\Delta}(x) = 1 \otimes x + x \otimes 1 + (1 + \mathfrak{q}^{2j+1}) y \otimes y;$$

together with $\underline{\Delta}(x) = \sigma_{1,1} 1 \otimes 1 + 1 \otimes x + x \otimes 1 + \sigma_{y,y} y \otimes y$, this implies $\sigma_{1,1} = \underline{\varepsilon}(x) = 0$ and $\sigma_{y,y} = 1 + \mathfrak{q}^{2j+1} \neq 0$.

Summarizing, $B = B_{0o}^0$ as an algebra in ${}_{H(2)}^{H(2)}\mathcal{Y}D$, and is a coalgebra in ${}_{H(2)}\mathcal{M}$ with structure determined by

$$\begin{aligned} \underline{\Delta}(x) &= x \otimes 1 + 1 \otimes x + (1 + \mathfrak{q}^{2j+1}) y \otimes y, \\ \underline{\Delta}(y) &= 1 \otimes y + y \otimes 1, \\ \underline{\varepsilon}(x) &= 0, \ \underline{\varepsilon}(y) = 0. \end{aligned}$$

Moreover, $\underline{\Delta}(y^2) = \underline{\Delta}(y)\underline{\Delta}(y)$. Similarly to the computation of $\underline{\Delta}(y^2)$ carried out above, one can find that

$$\begin{aligned} \underline{\Delta}(x)\underline{\Delta}(x) &= xx_{\underline{1}} \otimes x_{\underline{2}} + x_{\underline{1}} \otimes xx_{\underline{2}} \\ &\quad + \sigma_{y,y}(y^1 \cdot y)(y^2 Y^1 (x^1 \cdot y)_{[-1]} x^2 \cdot x_{\underline{1}}) \\ &\quad \otimes (y_1^3 Y^2 \cdot (x^1 \cdot y)_{[0]})(y_2^3 Y^3 x^3 \cdot x_{\underline{2}}) \\ &= 2x \otimes x + \sigma_{y,y} y (y_{[-1]} \cdot x) \otimes y_{[0]} + \sigma_{y,y} y \otimes yx \\ &\quad + \sigma_{y,y}^2 (y^1 \cdot y)(y^2 Y^1 (x^1 \cdot y)_{[-1]} x^2 \cdot y) \\ &\quad \otimes (y_1^3 Y^2 \cdot (x^1 \cdot y)_{[0]})(y_2^3 Y^3 x^3 \cdot y) \\ &= 2x \otimes x - 2\mathfrak{q}^{2j+1}(y^1 \cdot y)(y^2 Y^1 y_{[-1]} \cdot y) \otimes (y_1^3 Y^2 \cdot y_{[0]})(y_2^3 Y^3 \cdot y) \\ &= 2x \otimes x + 2(y^1 \cdot y)(y^2 Y^1 \cdot y) \otimes y^3 \cdot ((Y^2 \cdot y)(Y^3 \cdot y)) \\ &= 2x \otimes x - 2(y^1 \cdot y)(y^2 \cdot y) \otimes y^3 \cdot x = 2x \otimes x - 2y^2 \otimes x = 0, \end{aligned}$$

and therefore $\underline{\Delta}(x^2) = \underline{\Delta}(x)\underline{\Delta}(x)$. Likewise, $\underline{\Delta}(x)\underline{\Delta}(y) = \mathfrak{q}^{2j+1}(x \otimes y - y \otimes x)$ and $\underline{\Delta}(y)\underline{\Delta}(x) = \mathfrak{q}^{2j+1}(y \otimes x - x \otimes y)$, so $\underline{\Delta}$ is not multiplicative.

Case 2: $\sigma_{y,y} = 0$. We start by noting that (1.3.46) is already satisfied by x, y, so we will pay attention only to the other remaining relations. We distinguish two possibilities.

21: $\tau_{x,y} = 0$. We reduce to $\tau_{1,y} = 1$, $\tau_{yx}^2 = \tau_{yx}\sigma_{xx}$ and $\tau_{y,1}\tau_{y,x} = \tau_{y,x}\sigma_{x,1}$. Furthermore, we cannot have $\tau_{y,x} = 0$. Indeed, for $\tau_{y,x} = 0$ we have $\tau_{y,1} = 1$ and

$\underline{\Delta}(y) = 1\otimes y + y\otimes 1$; thus $\underline{\Delta}(y)\underline{\Delta}(y) = 1\otimes y^2 + y^2\otimes 1 + (1+\mathfrak{q}^{2j+1})y\otimes y$, which implies $y^2 \neq 0$, and so $y^2 = x$. We obtain that $\underline{\Delta}(x) = \underline{\Delta}(y^2) = 1\otimes x + x\otimes 1 + (1+\mathfrak{q}^{2j+1})y\otimes y$, hence $\sigma_{1,1} = 0$, $\sigma_{1,x} = \sigma_{x,1} = 1$ and $\sigma_{x,x} = 1 + \mathfrak{q}^{2j+1} \neq 0$. We get $\underline{\varepsilon}(x) = 0$, and this contradicts the fact that $x^2 = 1$.

Thus, in this case we have that $\tau_{y,1} = \sigma_{x,1} = \sigma_{1,x} := a$, $\tau_{y,x} = \sigma_{x,x} := b$ and $\sigma_{1,1} = -\underline{\varepsilon}(x)a$, with $a + \underline{\varepsilon}(x)b = 1$, $b \neq 0$. If $y^2 = x$, $\gamma = \overline{\gamma} = 0$ and $x^2 = 0$, from the general formula of $\underline{\Delta}(y^2)$ computed above we deduce that

$$\underline{\Delta}(x) = \underline{\Delta}(y^2) = 1\otimes x + a(1+\mathfrak{q}^{2j+1})y\otimes y + 2abx\otimes x + a^2x\otimes 1.$$

Comparing this with $\underline{\Delta}(x) = \sigma_{1,1}1\otimes 1 + a1\otimes x + ax\otimes 1 + bx\otimes x$, we arrive at the contradiction $a = 0$ and $a = 1$.

We conclude that $y^2 = 0$, and, by using again the formula for $\underline{\Delta}(y^2)$ found above, we land at $a(1+\mathfrak{q}^{2j+1}) + b(\overline{\gamma}\mathfrak{q}^{2j+1} + \gamma) = 0$.

Suppose we are dealing with B^{00}_{0o}. Then $a = -b$ and $\sigma_{1,1} = \underline{\varepsilon}(x)b$; we compute

$$\begin{aligned}\underline{\Delta}(x)\underline{\Delta}(x) &= \sigma_{1,1}x_{\underline{1}}\otimes x_{\underline{2}} - bx_{\underline{1}}\otimes x_{\underline{2}} - bx_{\underline{1}}\otimes xx_{\underline{2}} + bxx_{\underline{1}}\otimes xx_{\underline{2}}\\ &= \sigma_{1,1}^2 1\otimes 1 - 2b\sigma_{1,1}1\otimes x - 2b\sigma_{1,1}x\otimes 1 + (2b^2 + b\sigma_{1,1})x\otimes x,\end{aligned}$$

which forces $\sigma_{1,1} = 0$ and $b = 0$, a contradiction.

The only possibility left is $B = B^{10}_{0o}$, when we have $a = \mathfrak{q}^{2j+1}b$ and

$$\underline{\Delta}(x)\underline{\Delta}(x) = (\sigma_{1,1}^2 - b^2)1\otimes 1 + 2\mathfrak{q}^{2j+1}(b+\sigma_{1,1})(1\otimes x + x\otimes 1) - 2b(b-\sigma_{1,1})x\otimes x.$$

As $x^2 = 1$ and $b \neq 0$, $\sigma_{1,1}^2 = b^2 + 1$ and $\sigma_{1,1} = b = 0$; a contradiction. So this case is not possible.

<u>22</u>: Suppose that $\tau_{x,y} \neq 0$. Then it follows $\sigma_{x,x} = \tau_{x,y} \neq 0$, $\tau_{1,y} = \sigma_{x,1} = \sigma_{1,x}$, $\tau_{y,x}^2 = \tau_{y,x}\sigma_{x,x}$ and $\tau_{y,1}\tau_{y,x} = \tau_{y,x}\sigma_{x,1}$.

<u>22</u>$_1$: If $\tau_{y,x} = 0$, we have $\tau_{y,1} = 1$, $\tau_{1,y} = \sigma_{x,1} = \sigma_{1,x} := a$ and $\tau_{x,y} = \sigma_{x,x} = b \neq 0$. Therefore, the coproduct of B reduces to

$$\begin{aligned}\underline{\Delta}(x) &= \sigma_{1,1}1\otimes 1 + a1\otimes x + ax\otimes 1 + bx\otimes x,\\ \underline{\Delta}(y) &= a1\otimes y + y\otimes 1 + bx\otimes y.\end{aligned}$$

In a manner similar to the one above, we compute that

$$\begin{aligned}\underline{\Delta}(x)\underline{\Delta}(x) &= \sigma_{1,1}1\otimes 1 + ax_{\underline{1}}\otimes xx_{\underline{2}} + axx_{\underline{1}}\otimes x_{\underline{2}} + bxx_{\underline{1}}\otimes xx_{\underline{2}}\\ &= \sigma_{1,1}^2 1\otimes 1 + 2a\sigma_{1,1}1\otimes x + 2a\sigma_{1,1}x\otimes 1\\ &\quad +2(a^2 + b\sigma_{1,1})x\otimes x + a^2 1\otimes x^2 + 2abx\otimes x^2\\ &\quad +a^2x^2\otimes 1 + 2abx^2\otimes x + b^2x^2\otimes x^2.\end{aligned}$$

- If $x^2 = 0$, we get $\sigma_{1,1} = a = 0$, and therefore $\underline{\Delta}(y) = y\otimes 1 + bx\otimes y$ and $\underline{\Delta}(y)\underline{\Delta}(y) = y^2\otimes 1 + b(\gamma\mathfrak{q}^{2j+1} + \overline{\gamma})y\otimes y$. We cannot have $y^2 = 0$, otherwise $\gamma\mathfrak{q}^{2j+1} + \overline{\gamma} = 0$, which is false for any scalars $\overline{\gamma}$, γ. Thus $y^2 = x$, and since we have $\underline{\Delta}(x) = bx\otimes x$, we get again a contradiction.
- If $x^2 = 1$, $a(b+\sigma_{1,1}) = 0$, $\sigma_{1,1}^2 + b^2 + 2a^2 = 1$ and $a^2 + b\sigma_{1,1} = 0$. Since $y^2 = 0$, $a(1+\mathfrak{q}^{2j+1}) + b(\mathfrak{q}^{2j+1} - 1) = 0$ or, equivalently, $a = -\mathfrak{q}^{2j+1}b$. Then, as $xy = y$,

$$\begin{aligned}\underline{\Delta}(y) &= \underline{\Delta}(x)\underline{\Delta}(y) = \sigma_{1,1}y_{\underline{1}}\otimes y_{\underline{2}} + ay_{\underline{1}} + xy_{\underline{2}} + axy_{\underline{1}}\otimes y_{\underline{2}} + bxy_{\underline{1}}\otimes xy_{\underline{2}}\\ &= (a^2 + b^2 + ab + b\sigma_{1,1})1\otimes y + (a+\sigma_{1,1})y\otimes 1\\ &\quad +(a^2 + 2ab + b\sigma_{1,1})x\otimes y + (a+b)y\otimes x.\end{aligned}$$

Consequently, $b = -a = b\mathfrak{q}^{2j+1}$, so $b = 0$, which is not the case.

$\underline{22}_2$: Suppose now $\tau_{y,x} \neq 0$; here, we have $\tau_{y,x} = \sigma_{x,x} = \tau_{x,y} := b \neq 0$ and $\tau_{1,y} = 1 - \underline{\varepsilon}(x)\tau_{x,y} = 1 - \underline{\varepsilon}(x)\tau_{y,x} = \tau_{y,1} = \sigma_{x,1} = \sigma_{1,x} := a$. Therefore,

$$\underline{\Delta}(x) = \sigma_{1,1}1 \otimes 1 + a1 \otimes x + ax \otimes 1 + bx \otimes x,$$
$$\underline{\Delta}(y) = a1 \otimes y + ay \otimes 1 + bx \otimes y + by \otimes x.$$

The formula for $\underline{\Delta}(x)$ is the same as in the subcase $\underline{22}_1$. Thus, assuming $x^2 = 0$, we get $\sigma_{1,1} = a = 0$ and $\underline{\Delta}(y)\underline{\Delta}(y) = b^2\gamma\overline{\gamma}(1 + \mathfrak{q}^{2j+1})y \otimes y$. We cannot have $y^2 = 0$, otherwise, $\gamma = \overline{\gamma} = 1$ and $b = 0$. It follows that $y^2 = x$, implying $\gamma = \overline{\gamma} = 0$ and $\underline{\Delta}(x) = \underline{\Delta}(y)\underline{\Delta}(y) = 0$. So, $b = 0$ which is false.

Hence we have, by necessity, $x^2 = 1$, and this fact forces

$$a(b + \sigma_{1,1}) = 0, \quad 2a^2 + b^2 + \sigma_{1,1}^2 = 1 \quad \text{and} \quad a^2 + b\sigma_{1,1} = 0.$$

In particular, we deal with B^{10}_{0o} or B^{10c}_{0o}, hence $y^2 = 0$, $xy = y$ and $yx = \overline{\gamma}y$ with $\overline{\gamma}$ either 1 or -1. The fact that $0 = \underline{\Delta}(y^2) = \underline{\Delta}(y)\underline{\Delta}(y)$ is equivalent to $a^2 = b^2$ if $\overline{\gamma} = -1$ and to $a = -b$ if $\overline{\gamma} = 1$. Then, by $\underline{\Delta}(y) = \underline{\Delta}(xy)$ and

$$\begin{aligned}
\underline{\Delta}(x)\underline{\Delta}(y) &= \sigma_{1,1}y_{\underline{1}} \otimes y_{\underline{2}} + ay_{\underline{1}} \otimes xy_{\underline{2}} + axy_{\underline{1}} \otimes y_{\underline{2}} + bxy_{\underline{1}} \otimes xy_{\underline{2}} \\
&= (a\sigma_{1,1} + a^2 + ab + b^2)(1 \otimes y + y \otimes 1) \\
&\quad +(a^2 + 2ab + b\sigma_{1,1})(x \otimes y + y \otimes x),
\end{aligned}$$

we obtain that $a\sigma_{1,1} + a^2 + ab + b^2 = a$ and $a^2 + 2ab + b\sigma_{1,1} = b$. Similarly, $\overline{\gamma}\underline{\Delta}(y) = \underline{\Delta}(yx)$ and

$$\begin{aligned}
\underline{\Delta}(y)\underline{\Delta}(x) &= \sigma_{1,1}y_{\underline{1}} \otimes y_{\underline{2}} + ay_{\underline{1}} \otimes y_{\underline{2}}x + ay_{\underline{1}}((y_{\underline{2}})_{[-1]} \cdot x) \otimes (y_{\underline{2}})_{[0]} \\
&\quad +by_{\underline{1}}((y_{\underline{2}})_{[-1]} \cdot x) \otimes (y_{\underline{2}})_{[0]}x \\
&= (a\sigma_{1,1} + \overline{\gamma}a^2 + ab + \overline{\gamma}b^2)(1 \otimes y + y \otimes 1) \\
&\quad +(b\sigma_{1,1} + \overline{\gamma}2ab + a^2)(x \otimes y + y \otimes x)
\end{aligned}$$

reduces to $a\sigma_{1,1} + \overline{\gamma}a^2 + ab + \overline{\gamma}b^2 = \overline{\gamma}a$ and $b\sigma_{1,1} + \overline{\gamma}2ab + a^2 = \overline{\gamma}b$. The eight relations that guarantee a braided bialgebra structure on B^{10}_{0o} or B^{10c}_{0o} take place simultaneously in the following cases:

- For B^{10}_{0o}, if $a = b \neq 0$ then they reduce to $\sigma_{1,1} = -a$, $4a^2 = 1$ and $2a^2 = a$. Thus, $\sigma_{1,1} = -\frac{1}{2}$ and $a = b = \frac{1}{2}$, leading to the braided bialgebra structure on B^{10}_{0o} stated in (4.1.18).
- For B^{10}_{0o}, if $a = -b \neq 0$, the above mentioned eight relations reduce to $\sigma_{1,1} = a$, $4a^2 = 1$ and $2a^2 = a$. Therefore, $\sigma_{1,1} = a = \frac{1}{2}$ and $b = -\frac{1}{2}$, leading to the second braided bialgebra structure on B^{10}_{0o} stated in (4.1.19).
- For B^{10c}_{0o}, these relations lead to $\sigma_{1,1} = a = \frac{1}{2}$ and $b = -\frac{1}{2}$. We then get the last braided bialgebra in statement; namely, the one described in (4.1.20).

It remains to show that the three braided bialgebra structures that we just obtained are not isomorphic. Assume the opposite, and take ψ a braided bialgebra isomorphism with target bialgebra in 2. and source bialgebra in 1. or 3.; then $\psi(x)^2 = 1$ and $\psi(y)^2 = 0$. For $z = a1 + bx + cy \in B^{10}_{0o}$ we have

$$z^2 = (a^2 + b^2)1 + 2abx + bc(xy + yx) + 2ac = (a^2 + b^2)1 + 2abx + 2ac.$$

Thus, $z^2 = 1$ if and only if $z \in \{\pm 1, \pm x + cy\}$, and $z^2 = 0$ if and only if $z = c'y$. If the source of ψ is, for instance 1., owing to the definition of the counits, we get that $\psi(x) = -x + cy$ and $\psi(y) = c'y$, $c' \neq 0$. Then $\psi(y) = \psi(xy) = \psi(x)\psi(y)$ is equivalent to $c'y = -c'y$, so $c' = 0$, and this is false. Likewise, if the source of ψ is 3. then $\psi(x) = x + cy$ and $\psi(y) = c'y$, $c' \neq 0$. Thus $c'y = \psi(y) = \psi(yx) = \psi(y)\psi(x) = c'y(x + cy)$ in B^{10}_{00}, so $c'y = -c'y$. We get $c' = 0$, a contradiction. □

If $(B, \underline{m}_B, \underline{\eta}_B, \underline{\Delta}_B, \underline{\varepsilon}_B)$ is a bialgebra in a braided category $(\mathcal{C}, c)$, so is the bialgebra $B^{op+,cop-}$, defined as the same object B equipped with the multiplication $\underline{m}_B c_{B,B}$, unit $\underline{\eta}_B$, comultiplication $c^{-1}_{B,B}\underline{\Delta}_B$ and counit $\underline{\varepsilon}_B$.

Remarks 4.1.7 1). For the bialgebra B^{10}_{0o} in 1. above, set $\overline{x} := \frac{1}{2}(1+x)$. One can see easily that

$$\begin{aligned} &\overline{x}^2 = \overline{x},\ y^2 = 0,\ \overline{x}y = y,\ y\overline{x} = 0,\\ &\underline{\Delta}(\overline{x}) = \overline{x} \otimes \overline{x},\ \underline{\varepsilon}(\overline{x}) = 1,\\ &\underline{\Delta}(y) = \overline{x} \otimes y + y \otimes \overline{x},\ \underline{\varepsilon}(y) = 0. \end{aligned} \tag{4.1.21}$$

We denote by $\underline{B}$ the braided bialgebra copy of B^{10}_{0o} in 1., defined by the relations in (4.1.21).

In the same spirit, for the bialgebra B^{10}_{0o} in 2. above, set $\overline{x} := \frac{1}{2}(1-x)$. Then

$$\begin{aligned} &\overline{x}^2 = \overline{x},\ y^2 = 0,\ \overline{x}y = 0,\ y\overline{x} = y,\\ &\underline{\Delta}(\overline{x}) = \overline{x} \otimes \overline{x},\ \underline{\varepsilon}(\overline{x}) = 1,\\ &\underline{\Delta}(y) = \overline{x} \otimes y + y \otimes \overline{x},\ \underline{\varepsilon}(y) = 0. \end{aligned} \tag{4.1.22}$$

We denote by $\underline{\underline{B}}$ the braided bialgebra copy of B^{10}_{0o} in 2., defined by the relations in (4.1.22). Observe that $\underline{\underline{B}} = \underline{B}^{op+,cop-}$ as braided bialgebras.

Finally, for the bialgebra B^{10c}_{0o} in 3. above, set again $\overline{x} := \frac{1}{2}(1-x)$. Then

$$\begin{aligned} &\overline{x}^2 = \overline{x},\ y^2 = 0,\ \overline{x}y = 0,\ y\overline{x} = 0,\\ &\underline{\Delta}(\overline{x}) = \overline{x} \otimes \overline{x},\ \underline{\varepsilon}(\overline{x}) = 1,\\ &\underline{\Delta}(y) = \overline{x} \otimes y + y \otimes \overline{x},\ \underline{\varepsilon}(y) = 0. \end{aligned} \tag{4.1.23}$$

We denote by $\underline{\underline{B}}^c$ the braided bialgebra copy of B^{10c}_{0o} in 3., defined by the relations in (4.1.23). Note that $\underline{\underline{B}}^c = (\underline{\underline{B}}^c)^{op+,cop-}$ as braided bialgebras.

2). Neither $\underline{B}$, $\underline{\underline{B}}$ nor $\underline{\underline{B}}^c$ is a braided Hopf algebra; indeed, if $\underline{S}$ would be an antipode for $\underline{B}$, $\underline{\underline{B}}$ or $\underline{\underline{B}}^c$, $\underline{S}(\overline{x})$ would be a bilateral inverse of $\overline{x}$, which contradicts the fact that $\overline{x}$ is a zero divisor in any of the three mentioned bialgebras. Consequently, there are no braided Hopf algebras within ${}^{H(2)}_{H(2)}\mathcal{YD}$, which as objects in ${}^{H(2)}_{H(2)}\mathcal{YD}$ are isomorphic to $M_0 \oplus M_{2i} \oplus M_{2j+1}$.

Proposition 4.1.8 *We describe the structure of three quasi-Hopf algebras with a projection on $H(2)$:*

- *The biproduct quasi-bialgebra $\underline{H} := \underline{B} \times H(2)$ can be described as follows. As an algebra, it is unital and generated by $G = 1 \times g$, $X = \overline{x} \times 1$ and $Y = y \times 1$ with relations $G^2 = 1$, $X^2 = X$, $Y^2 = 0$, $GX = XG$, $GY = -YG$, $XY = Y$ and $YX = 0$. The quasi-coalgebra structure of $\underline{H}$ is determined by the reassociator $\Phi = 1 - 2P_- \otimes P_- \otimes P_-$, where $P_\pm := \frac{1}{2}(1 \pm G)$, the comultiplication*

$$\begin{gathered} \Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes X,\\ \Delta(Y) = (P_+ + \mathfrak{q}^{2j+1}P_-)X \otimes Y + Y \otimes P_+X - GY \otimes P_-X, \end{gathered} \tag{4.1.24}$$

and counit $\varepsilon(G) = \varepsilon(X) = 1$ and $\varepsilon(Y) = 0$.

- *Similarly, the biproduct quasi-bialgebra* $\underline{\underline{H}} := \underline{\underline{B}} \times H(2)$ *is the unital algebra generated by* $G := 1 \times g$, $X = 1 \times x$ *and* $Y = 1 \times y$ *with relations* $G^2 = 1$, $X^2 = X$, $Y^2 = 0$, $GX = XG$, $GY = -YG$, $XY = 0$ *and* $YX = Y$. *Its quasi-bialgebra structure is given by the reassociator* $\Phi = 1 - 2P_- \otimes P_- \otimes P_-$, *comultiplication*

$$\Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes X,$$
$$\Delta(Y) = (P_+ + \mathfrak{q}^{2j+1} P_-)X \otimes Y + Y \otimes P_+X - GY \otimes P_-X$$

and counit $\varepsilon(G) = \varepsilon(X) = 1$ *and* $\varepsilon(Y) = 0$, *where* $P_\pm := \frac{1}{2}(1 \pm G)$. $\underline{\underline{H}}$ *is the opposite version of* $\underline{H}$.

- *Last but not least,* $\underline{\underline{H}}^c := \underline{\underline{B}}^c \times H(2)$ *is the unital algebra generated by* $G := 1 \times g$, $X = 1 \times x$ *and* $Y = 1 \times y$ *with relations* $G^2 = 1$, $X^2 = X$, $Y^2 = 0$, $GX = XG$, $GY = -YG$ *and* $XY = YX = 0$. *Its quasi-bialgebra structure is given by the reassociator* $\Phi = 1 - 2P_- \otimes P_- \otimes P_-$, *comultiplication*

$$\Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes X,$$
$$\Delta(Y) = (P_+ + \mathfrak{q}^{2j+1} P_-)X \otimes Y + Y \otimes P_+X - GY \otimes P_-X$$

and counit $\varepsilon(G) = \varepsilon(X) = 1$ *and* $\varepsilon(Y) = 0$. *As before,* $P_\pm := \frac{1}{2}(1 \pm G)$.

Proof. Follows through a direct computation by the biproduct construction, see the previously cited results in [11]. $\square$

Proposition 4.1.9 *A* 3*-dimensional Hopf algebra within the category of left Yetter-Drinfeld modules over* $H(2)$ *is isomorphic, in the notation of Proposition 4.1.1, to either* B_{C_6} *or* B_*.

Proof. It follows from the results proved so far that all 3-dimensional Hopf algebras in ${}^{H(2)}_{H(2)}\mathcal{YD}$ are isomorphic, as Yetter-Drinfeld modules, to $M_0 \otimes M_{2i} \otimes M_{2j}$. In particular, $H(2)$ acts trivially on all 3-dimensional Hopf algebras in ${}^{H(2)}_{H(2)}\mathcal{YD}$; therefore, the reassociator disappears in all the relations from (1.3.41) to (1.3.48). Consequently, we reduced to the relations that determine the Hopf algebras within ${}^{k[C_2]}_{k[C_2]}\mathcal{YD}$. This implies that the 3-dimensional Hopf algebras in ${}^{H(2)}_{H(2)}\mathcal{YD}$ coincide with those in ${}^{k[C_2]}_{k[C_2]}\mathcal{YD}$ on which $H(2)$ acts trivially. $\square$

Remark 4.1.10 Proposition 4.1.9 implies that all 6-dimensional quasi-Hopf algebras with a projection on $H(2)$ are semisimple. Indeed, we know from Proposition 4.1.1 that the biproduct quasi-Hopf algebras $B_{C_6} \times H(2)$ and $B_* \times H(2)$ are isomorphic as algebras to $k[C_6]$ and k^{S_3}, respectively.

Corollary 4.1.11 *All biproduct quasi-Hopf algebras of dimension* 6 *are semisimple.*

Proof. Let $B \times H$ be a 6-dimensional biproduct quasi-Hopf algebra that is free of rank 3 over the 2-dimensional quasi-Hopf algebra H. Up to twisting, H is either $k[C_2]$ or $H(2)$, see for instance [15, Example 1.16]. By [11, Theorem 5.1], $B \times H$ is isomorphic to a biproduct quasi-Hopf algebra, a free module of rank 3 over either $k[C_2]$ or $H(2)$. Then, our assertion follows from Remark 4.1.10 and the fact that all 6-dimensional Hopf algebras are semisimple. $\square$

4.2 Quasi-Hopf algebras of dimension 6

Let $p < q$ be distinct prime numbers. In [22, Theorem 6.3], Etingof and Gelaki classify the fusion categories of Frobenius-Perron dimension pq with all the simple objects having integer dimension. As a byproduct, they also obtain the classification of the pq-dimensional semisimple quasi-Hopf algebras in terms of their category of representations. For $p = 2$, a special situation occurs in the result quoted above, and more types of object appear in the classification. We are concerned with the case $pq = 6$; in [22], this special case of the theorem is attributed to T. Chmutova.

In this section we prove that any 6-dimensional quasi-Hopf algebra is semisimple, and this fact completes the classification of quasi-Hopf algebras in dimension 6, over a field k of characteristic zero and algebraically closed. Equivalently, in the language of thensor categories, we also get the classification of finite tensor categories with Frobenius-Perron dimension 6 with all the objects having integer Frobenius-Perron dimension.

To prove the classification results mentioned above, we need the lemma below. Recall that a k-algebra A is basic if all the irreducible representations of A are 1-dimensional or, equivalently, if $\frac{A}{J_A} \simeq k \times \ldots \times k$ as an algebra, where J_A is the Jacobson radical of A.

Lemma 4.2.1 *Any 6-dimensional non-semisimple quasi-Hopf algebra is basic.*

Proof. By way of contradiction, assume that there is a 6-dimensional non-semisimple quasi-Hopf algebra that is not basic. Then, if we denote by J the Jacobson radical of H, J is a non-zero nilpotent ideal of H and $H_0 := \frac{H}{J}$ is a semisimple algebra of dimension 4 or 5, owing to the Wedderburn-Artin theorem. The case $\dim_k H_0 = 4$ cannot occur since ε, the counit of H, provides a 1-dimensional representation of H, forcing $H_0 \simeq k^4$; thus H is basic, which is false.

So, we reduced to the case $\dim_k H_0 = 5$, when $H_0 \simeq k \times M_2(k)$ as an algebra; $M_2(k)$ is the algebra of 2-square matrices. Also, $\dim_k J = 1$, and we assume that J is generated by the non-zero element x. Last but not least, denote by $p : H \to H_0$ the canonical projection, a k-algebra morphism.

Owing to [1, Theorem 1.4.9], there exists a semisimple subalgebra $\mathbb{S}$ of H such that $H = \mathbb{S} \oplus J$ as k-vector spaces; we can see immediately that, $\mathbb{S} \simeq H_0$ as k-vector spaces. Since $J^2 = 0$, it is easy to check that $\mathbb{S} \to H_0 : s \mapsto p(s)$ is a k-algebra isomorphism, hence $\mathbb{S} \simeq k \times M_2(k)$ as algebras. Clearly, J nilpotent implies $\varepsilon(x) = 0$. Therefore, it follows that ε, the counit of H, restricts to an algebra morphism $\overline{\varepsilon} : \mathbb{S} \to k$: $\varepsilon(s \oplus \lambda x) = \overline{\varepsilon}(s)$, for all $h = s \oplus \lambda x \in H = \mathbb{S} \oplus J$. As $\mathbb{S} \simeq k \times M_2(k)$ has a unique 1-dimensional representation, it follows that $\overline{\varepsilon}$ is the unique algebra morphism from $\mathbb{S}$ to k, and that $\mathrm{Ker}(\overline{\varepsilon}) \simeq M_2(k)$. This last fact allows us to see that the multiplication of H can be expressed in terms of the multiplication of $\mathbb{S}$ and x as

$$(s \oplus \lambda x)(s' \oplus \lambda' x) = ss' \oplus (\lambda' \overline{\varepsilon}(s) + \lambda \overline{\varepsilon}(s'))x,$$

for all $s, s' \in \mathbb{S}$ and $\lambda x, \lambda' x \in J$; here λ and λ' are arbitrary scalars. We wrote an element in $H = \mathbb{S} \oplus J$ as $s \oplus \lambda x$ in order to distinguish the $\mathbb{S}$ and J components of the elements in H. Thus, if we take $\{1, E_{ij} \mid 1 \leq i, j \leq 2\}$ as the canonical basis of $\mathbb{S}$ obtained from the canonical basis of $k \times M_2(k)$ through the algebra isomorphism $\mathbb{S} \simeq k \times M_2(k)$, then $\{1, E_{ij}, x \mid 1 \leq i, j \leq 2\}$ is a basis of H and the algebra structure of H is determined by the following rules: 1 is the unit of H and, since $\varepsilon(x)=0$ and $\varepsilon(E_{ij})=0$ for all $1\leq i,j\leq 2$, for any $1\leq i,j,s,t\leq 2$ we have

$$x^2=0,\quad xE_{ij}=E_{ij}x=0\quad\text{and}\quad E_{ij}E_{st}=\delta_{j,s}E_{it}.$$

We look now at $\Delta(x)$, an element in $H\otimes H$ which we will write as

$$\begin{aligned}\Delta(x)=&1\otimes(\lambda 1+\sum_{i,j}\lambda_{ij}E_{ij}+\mu x)\\&+\sum_{i,j}E_{ij}\otimes(\lambda'_{ij}1+\sum_{s,t}\lambda^{ij}_{st}E_{st}+\mu_{ij}x)+x\otimes(\gamma 1+\sum_{i,j}\gamma_{ij}E_{ij}+\theta x),\end{aligned}$$

for some scalars $\lambda,\mu,\lambda_{ij},\lambda'_{ij},\lambda^{ij}_{st},\mu_{ij},\gamma,\gamma_{ij}$ and θ. By using that ε is counit for Δ, we get $\lambda=0$, $\mu=\gamma=1$ and $\lambda_{ij}=\lambda'_{ij}=0$, for all $1\leq i,j\leq 2$. Thus

$$\Delta(x)=1\otimes x+\sum_{i,j}E_{ij}\otimes(\sum_{s,t}\lambda^{ij}_{st}E_{st}+\mu_{ij}x)+x\otimes(1+\sum_{i,j}\gamma_{ij}E_{ij}+\theta x).$$

But Δ is multiplicative, and therefore $\Delta(x)\Delta(x)=\Delta(x^2)=0$. On the other hand,

$$\Delta(x)\Delta(x)\quad=\quad 2x\otimes x+\sum_{i,j}E_{ij}\otimes\sum_{s,t,u,v}\lambda^{iu}_{st}\lambda^{uj}_{tv}E_{sv},$$

and thus $\Delta(x^2)$ cannot be zero. Thus, we landed to a last contradiction, and so our proof is finished. □

Let H be a basic quasi-Hopf algebra which is not semisimple, and take J the Jacobson radical of H. We recall that J is also a quasi-Hopf ideal of H, thus $H_0=\frac{H}{J}$ has a unique quasi-Hopf algebra structure with respect to which the canonical projection $p:H\to H_0$, which becomes a quasi-Hopf algebra morphism.

Moreover, we denote here by A the graded algebra $\mathrm{gr}(H)=\bigoplus_{m\geq 1}H_m$ associated to the filtration of H by powers of J, introduced in Section 1.4.1, where $H_m=\frac{J^m}{J^{m+1}}$, for all $m\geq 0$, and we use again the convention $J^0=H$. Then A has a canonical quasi-Hopf algebra structure, with reassociator Φ defined by a 3-cocycle of the group G obtained through algebra isomorphisms $H_0\simeq k\times\ldots\times k\simeq k^G\simeq k[G]$. Consequently, when H is not semisimple and basic we have quasi-Hopf algebra morphisms $H_0\underset{\pi}{\overset{i}{\rightleftarrows}}\mathrm{gr}(H)$ such that $\pi i=\mathrm{Id}_{H_0}$, hence $\mathrm{gr}(H)$ is a biproduct quasi-Hopf algebra between a braided Hopf algebra B in ${}^{k_\Phi[G]}_{k_\Phi[G]}\mathcal{YD}$ and $k_\Phi[G]$. Furthermore, each homogenous component H_m of H can be seen as an object in the category of two-sided two-cosided Hopf module over H_0. Since any object of this category is free as a left and right H_0-module, it follows that so is every H_m; we refer to [47] for all the details.

We are now ready to present the main result of this section.

Theorem 4.2.2 *Any* 6*-dimensional quasi-Hopf algebra is semisimple.*

Proof. Assume that there exists a non-semisimple quasi-Hopf algebra of dimension 6. Since Lemma 4.2.1 guarantees us that H is basic, we can than apply all the results mentioned above. We distinguish the following cases.

Case 1: $\dim_k J = 1$. We have $\dim_k H_0 = 5$ and $J^2 = 0$, thus H_1 cannot be free over H_0.

Case 2: $\dim_k J = 2$. Similarly, $\dim_k H_0 = 4$ and since $\dim_k H_1$ is at most 2 and so it cannot be a non-zero power of 4, H_1 is not free over H_0.

Case 3: $\dim_k J = 3$. As $\dim_k H_0 = 3$, we must have $J^2 = 0$. Therefore, $\mathrm{gr}(H) = H_0 \oplus J$ has dimension 6 and is a biproduct quasi-Hopf algebra between a braided Hopf algebra B in ${}_{k_\Phi[C_3]}^{k_\Phi[C_3]}\mathcal{YD}$ and $k_\Phi[C_3]$, for a certain 3-cocycle Φ of $k[C_3]$. Then B has dimension 2 and by the results obtained in [9] it follows that $\mathrm{gr}(H)$ is semisimple. This is false because in this particular case the Jacobson radical of $\mathrm{gr}(H)$ is J and J is not the null ideal.

Case 4: $\dim_k J = 4$. We have $\dim_k H_0 = 2$, which imposes $\dim_k J^2 \in \{0, 2\}$. If $\dim_k J^2 = 0$, $\mathrm{gr}(H) = H_0 \oplus J$ has again dimension 6 and it is a biproduct between a braided Hopf algebra B in ${}_{k_\Phi[C_2]}^{k_\Phi[C_2]}\mathcal{YD}$ and $k_\Phi[C_2]$, for a certain 3-cocycle Φ of $k[C_2]$. Then B has dimension 3 and by the results obtained in the previous section it follows that $\mathrm{gr}(H)$ is semisimple. But the Jacobson radical of $\mathrm{gr}(H)$ is $J \neq 0$, a contradiction. We also reach the same contradiction when $\dim_k J^2 = 2$, because $J^3 = 0$ and so $\mathrm{gr}(H) = H_0 \oplus H_1 \oplus H_2$ is of dimension 6, too.

Case 5: $\dim_k J = 5$. Since $H_0 \simeq k$ is the trivial Hopf algebra, $\mathrm{gr}(H)$ is an ordinary Hopf algebra. By the definition of $\mathrm{gr}(H)$, $\dim_k(\mathrm{gr}(H)) = \dim_k(H) = 6$, and it follows from here that $\mathrm{gr}(H)$ is a semisimple Hopf algebra. On the other hand, the Jacobson radical of $\mathrm{gr}(H)$ has dimension 5, so it cannot be the null ideal.

So our assumption is false, therefore any quasi-Hopf algebra of dimension 6 is semisimple. □

Remark 4.2.3 Since the quasi-bialgebras in Proposition 4.1.8 are all non-semisimple algebras, it follows that they do not admit a quasi-Hopf algebra structure.

We translate Theorem 4.2.2 to the language of tensor categories.

Corollary 4.2.4 *Let $\mathcal{C}$ be a finite tensor category of Frobenius-Perron dimension 6 such that every object of $\mathcal{C}$ has integer Frobenius-Perron dimension. Then $\mathcal{C}$ is a fusion category.*

Proof. It is a direct consequence of Theorem 4.2.2 and [23, Proposition 2.6]. □

We end this section by giving an explicit presentation for all 6-dimensional quasi-Hopf algebras. Our presentation is in concordance with the classification of fusion categories produced in [22].

The cohomology groups of the dihedral group were characterized in [29] and [30]; consequently, we have a description for $H^3(S_3, k^*)$. Explicit formulas for 3-cocycle representatives for the group S_3 can be deduced from the formula (3.2.8) in [42]. Namely, they are given by

$$\omega_p = \sum_{I,J,L=0}^{1} \sum_{i,j,l=0}^{2} \mathfrak{q}^{p(-1)^{J+L} i \lfloor \frac{(-1)^L j + l}{3} \rfloor} (-1)^{pIJL}, \tag{4.2.1}$$

where $\mathfrak{q}$ is a primitive 3-rd root of unity and p is in the range $0, \ldots, 5$. This fact completes [22, Corollary 4.2], where it is shown that $H^3(S_3, \mathbb{C}^*) \simeq C_6$, but 3-cocycle representatives were not given.

In what follows, we consider the symmetric group S_3 generated by τ and σ, where τ is a transposition, while σ is a cycle of length 3.

Theorem 4.2.5 *Let $\mathfrak{q}$ be a primitive 3-rd root of unity and ξ a primitive 6-th root of unity. Then, any 6 dimensional quasi-Hopf algebra is twist equivalent to one of the following:*

- $(k[C_6], \Phi_a)$, *the group Hopf algebra $k[C_6]$, seen as a quasi-Hopf algebra with reassociator*
$$\Phi_a = \sum_{i,j,l=0}^{5} \xi^{ai\lfloor\frac{j+l}{3}\rfloor} 1_i \otimes 1_j \otimes 1_l, \tag{4.2.2}$$
where, if g generates C_6, $\{1_i := \frac{1}{6}\sum_{t=0}^{5} \xi^{-ti} g^t \mid 0 \leq i \leq 5\}$ is the set of primitive idempotents of C_6. There are 6 quasi-Hopf algebras of this type which are not twist equivalent, given by $0 \leq a \leq 5$.

- (k^{S_3}, Φ_p), *the dual of the group Hopf algebra $k[S_3]$, endowed with the reassociator*
$$\Phi_p = \sum_{I,J,L=0}^{1} \sum_{i,j,l=0}^{2} \mathfrak{q}^{p(-1)^{J+L} i \lfloor\frac{(-1)^L j+l}{3}\rfloor} (-1)^{pIJL}\ P_{\tau^I\sigma^i} \otimes P_{\tau^J\sigma^j} \otimes P_{\tau^L\sigma^l}. \tag{4.2.3}$$
Here $\{P_{\tau^I\sigma^i} \mid 0 \leq I \leq 1, 0 \leq i \leq 2\}$ is the basis of k^{S_3} dual to the basis $\{\tau^I\sigma^i \mid 0 \leq I \leq 1, 0 \leq i \leq 2\}$ of $k[S_3]$. There are 6 quasi-Hopf algebras of this type which are not twist equivalent, given by $0 \leq p \leq 5$.

- *The group Hopf algebra $k[S_3]$.*

- $(k[S_3], \Psi_a)$, *the group Hopf algebra $k[S_3]$ with reassociator Ψ_a obtained from a non-trivial 3-cocycle of $C_3 = \langle\sigma\rangle$ which commutes with $\tau \otimes \tau \otimes \tau$; namely,*
$$\Psi_a = \sum_{i,j,l=0}^{2} \mathfrak{q}^{ai\lfloor\frac{j+l}{3}\rfloor + ai(\lfloor\frac{(j+l)'}{2}\rfloor - \lfloor\frac{j}{2}\rfloor - \lfloor\frac{l}{2}\rfloor)}\ \mathbf{1}_i \otimes \mathbf{1}_j \otimes \mathbf{1}_l, \tag{4.2.4}$$
where $\{\mathbf{1}_i := \frac{1}{3}\sum_{t=0}^{2} \mathfrak{q}^{-ti}\sigma^t \mid 0 \leq i \leq 2\}$ is the set of primitive idempotents of $C_3 = \langle\sigma\rangle$ and $(m)'$ denotes the remainder of the division of the integer m by 3. There are 2 quasi-Hopf algebras of this type which are not twist equivalent, defined by $a \in \{1, 2\}$.

Proof. Follows by Theorem 4.2.2 and [22, Theorem 6.3]. To get the reassociator (4.2.2), we have identified the Hopf algebra of functions on C_6, k^{C_6}, with the group Hopf algebra $k[C_6]$ by using the usual isomorphism provided by the correspondence $P_{g^i} \mapsto 1_i$, for all $0 \leq i \leq 5$. Then, we made use of (3.2.4) to get for free all the non-twist equivalent structures on $k[C_6]$.

The fact that, for $0 \leq a \neq b \leq 5$, the quasi-Hopf algebras $(k[C_6], \Phi_a)$ and $(k[C_6], \Phi_b)$ are not twist equivalent can be proved as follows. By the proof of Lemma 3.1.1, $\mid \mathcal{O}_{\overline{\xi^a}} \mid = |\{\overline{\xi^{s^2a}} \mid 1 \leq s \leq 5, (s, 6) = 1\}| = 1$, for all $0 \leq a \leq 5$, since $s^2a \equiv a\ (mod\ 6)$ for all $s \in U(\mathbb{Z}_6)$. Hence, for $G = C_6$ the group action defined by (3.1.1) has 6 orbits; now our claim follows from [24, Remark 2.6.2].

Likewise, to get (4.2.3) we used the fact that $k[S_3]$ is a dual quasi-Hopf algebra (for the definition, see for instance [15, Section 3.7]; it is the dual structure to a quasi-Hopf algebra) with reassociator defined by (4.2.1). Then, passing to the dual of $k[S_3]$ we obtain k^{S_3} with th e reassociator mentioned in (4.2.3). As in the previous case, by using the fact that $\mathrm{Out}(S_3) = 1$, see [46, Theorem 7.5], and [24, Remark 2.6.2], we get that $(k^{S_3}, \Phi_p)_{0\leq p\leq 5}$ are 6 quasi-Hopf algebras which are not pairwise twist equivalent.

The only thing left to prove is the fact that ψ_a, $1 \leq a \leq 2$, defined by

$$\psi_a(\sigma^i, \sigma^j, \sigma^l) = \mathfrak{q}^{ai\lfloor \frac{j+l}{3} \rfloor + a(\lfloor \frac{(j+l)'}{2} \rfloor - \lfloor \frac{j}{2} \rfloor - \lfloor \frac{l}{2} \rfloor)},$$

is a 3-cocycle for the group C_3 and, moreover, it is invariant under the action of C_2; the latter means that $\psi_a(\sigma^i, \sigma^j, \sigma^l) = \psi_a(\sigma^{-i}, \sigma^{-j}, \sigma^{-l})$, for all $0 \leq i, j, l \leq 2$. Towards this end, we show that ψ_a is cohomologus to $\phi_{\mathfrak{q}^a}$. Indeed, if we take $g_a : C_3 \times C_3 \to k$ defined by $g_a(\sigma^i, \sigma^j) := \mathfrak{q}^{ai\lfloor \frac{j}{2} \rfloor}$, for all $0 \leq i, j \leq 2$, then

$$\begin{aligned}
&\phi_{\mathfrak{q}^a}(\sigma^i, \sigma^j, \sigma^l) g_a(\sigma^j, \sigma^l) g_a(\sigma^{(i+j)'}, \sigma^l)^{-1} g_a(\sigma^i, \sigma^{(j+l)'}) g_a(\sigma^i, \sigma^j)^{-1} \\
&\quad = \mathfrak{q}^{ai\lfloor \frac{j+l}{3} \rfloor} \mathfrak{q}^{ai\lfloor \frac{j}{2} \rfloor}\ \mathfrak{q}^{aj\lfloor \frac{l}{2} \rfloor}\ \mathfrak{q}^{-a(i+j)\lfloor \frac{l}{2} \rfloor}\ \mathfrak{q}^{ai\lfloor \frac{(j+l)'}{2} \rfloor}\ \mathfrak{q}^{-ai\lfloor \frac{j}{2} \rfloor} = \psi_a(\sigma^i, \sigma^j, \sigma^l).
\end{aligned}$$

Moreover, $\psi_a(\sigma, \sigma, \sigma) = \mathfrak{q}^a = \psi_a(\sigma^2, \sigma^2, \sigma^2)$, $\psi_a(\sigma, \sigma, \sigma^2) = 1 = \psi_a(\sigma^2, \sigma^2, \sigma)$, $\psi_a(\sigma, \sigma^2, \sigma) = 1 = \psi_a(\sigma^2, \sigma, \sigma^2)$, $\psi_a(\sigma^2, \sigma, \sigma) = \mathfrak{q}^{2a} = \psi_a(\sigma, \sigma^2, \sigma^2)$. The rest of the invariance conditions for ψ_a follow from ψ_a being normalized, as $g_a(\sigma^i, 1) = 1 = g_a(1, \sigma^j)$ for any i and j. □

Remark 4.2.6 Ψ_a produces a quasi-Hopf algebra structure on $k[\langle\sigma\rangle] \simeq k[C_3]$, twist equivalent to the one produced by $\Phi_{\mathfrak{q}^a}$, defined in (3.2.4), on $k[C_3]$. In addition, the natural embedding of $k[C_3]$ in $k[S_3]$ can be seen as a quasi-Hopf algebra embedding of $(k[C_3], \Psi_a)$ into $(k[S_3], \Psi_a)$. But there is no quasi-Hopf projection covering this embedding, because $(k[S_3], \Psi_a)$ cannot be described as a biproduct quasi-Hopf algebra.

Appendix A

Braided bialgebras over two cyclic groups

This section is focused on devising a strategy to find solutions to the relation (2.2.13), each of which yelds a biproduct quasi-Hopf algebra over the product of two finite cyclic groups. In what follows, assume we are in the context and notation of Corollary 2.2.11. In addition, we write $\frac{f_1}{m_1} = \frac{x_0}{L}$ and $\frac{f_2}{m_2} = \frac{y_0}{L}$, for some integers x_0 and y_0, where L denotes the least common multiple between m_1 and m_2.

For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$, define the map $\mathcal{C}_{\lambda,k} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, for all $(x, y) \in \mathbb{Z}^2$, by

$$\mathcal{C}_{\lambda,k}(x, y) = 2c_1x^2 + 2c_2y^2 + 2c_{12}xy + 2\lambda_1Lx + 2\lambda_2Ly - (2k + 1)L^2; \tag{A.0.1}$$

one can immediately see that (2.2.13) reduces to finding parameters λ, k for which the conic section $\mathcal{C}_{\lambda,k}(x, y) = 0$ has rational points in $\mathbb{Z}^2$; in other words, we study the existence of integer solutions for the parametric equation $\mathcal{C}_{\lambda,k}(x, y) = 0$. In the following, we will show a way to determine the solutions of this equation.

A.1 Rational points

We can determine the rational points $\mathcal{C}_{\lambda,k}(x, y) = 0$ by reducing $\mathcal{C}_{\lambda,k}$ to its standard form. More precisely, we must deal with one of the following cases.

Case I: $c_1 \neq 0$. We rewrite $\mathcal{C}_{\lambda,k}(x, y)$ as

$$\begin{aligned}\mathcal{C}_{\lambda,k}(x, y) = 2c_1\left(x + \frac{c_{12}}{2c_1}y + \frac{\lambda_1}{2c_1}L\right)^2 + \frac{4c_1c_2 - c_{12}^2}{2c_1}y^2 \\ + \frac{2\lambda_2c_1 - \lambda_1c_{12}}{c_1}Ly - \left(\frac{\lambda_1^2}{2c_1} + 2k + 1\right)L^2.\end{aligned}$$

We denote by $\delta := 4c_1c_2 - c_{12}^2$ the discriminant of $\mathcal{C}_{\lambda,k}$.

Case I_1: $\delta = 0$. We reduce to solving in $\mathbb{Z}$ the system

$$\begin{cases} 2c_1x + c_{12}y + \lambda_1L = \widetilde{x} \\ \widetilde{x}^2 + 2(2\lambda_2c_1 - \lambda_1c_{12})Ly - (\lambda_1^2 + 2(2k + 1)c_1)L^2 = 0. \end{cases} \tag{A.1.2}$$

The Diophantine equation $2c_1x+c_{12}y = \gamma$ has solutions in $\mathbb{Z}$ if and only if $(2c_1, c_{12})|\gamma$. If this is the case and (x_0, y_0) is a particular solution of

$$2c_1x + c_{12}y = \gamma \tag{A.1.3}$$

then the general solution of (A.1.3) is $(x,y) = (-c_{12}u + x_0, 2c_1u + y_0)$, for any $u \in \mathbb{Z}$. So, for $\widetilde{x} = \lambda_1 L + (2c_1, c_{12})v$, we have to look for those integers λ_1, λ_2, k for which the equation in $(u,v) \in \mathbb{Z}^2$,

$$(\lambda_1 L + (2c_1, c_{12})v)^2 + 2(2\lambda_2 c_1 - \lambda_1 c_{12})L(2c_1 u + y_0) - (\lambda_1^2 + 2(2k+1)c_1)L^2 = 0$$

has solutions. This equation is solved case by case, by finding first u such that $(\lambda_1^2 + 2(2k+1)c_1)L^2 - 2(2\lambda_2 c_1 - \lambda_1 c_{12})L(2c_1 u + y_0) = \omega^2$, for any integer ω, and then v obeying $\lambda_1 L + (2c_1, c_{12})v = \pm\omega$.

Case I_2: $\delta \neq 0$. We rewrite $\mathcal{C}_{\lambda,k}(x,y)$ as

$$\frac{1}{2c_1}(2c_1x + c_{12}y + \lambda_1 L)^2 + \frac{1}{2c_1\delta}(\delta y + (2\lambda_2 c_1 - \lambda_1 c_{12})L)^2 - \frac{1}{2c_1\delta}d,$$

where $d := (\lambda_1^2\delta + (2\lambda_2 c_1 - \lambda_1 c_{12})^2 + 2(2k+1)c_1\delta)L^2$. We have to solve in $\mathbb{Z}$ the equation

$$\widetilde{x}^2 + \delta\widetilde{y}^2 = d, \tag{A.1.4}$$

and then to find the integers λ_1, λ_2, k for which the linear system

$$\begin{cases} \delta y + (2\lambda_2 c_1 - \lambda_1 c_{12})L = \widetilde{x} \\ 2c_1 x + c_{12}y + \lambda_1 L = \widetilde{y} \end{cases}$$

has solutions (x,y) in $\mathbb{Z}^2$. To be more accurate, the values of x, y are limited by m_1, m_2, respectively, so the values of $\widetilde{x}$, $\widetilde{y}$ are also limited. We get in this way bounds for d, and therefore for k. Then, for any possible k, we solve first (A.1.4) and then the linear system. Note also that the Pell-type equation (A.1.4) has a finite number of solutions, provided that $\delta > 0$ or $\delta < 0$ and $-\delta$ is a square. When $\delta < 0$ and square free, the equation (A.1.4) has no solution or an infinite number of solutions; in the latter case, we refer to [39] for the algorithm to find those solutions.

Case II: $c_1 = 0$. We distinguish four possibilities.

Case II_1: $c_2 \neq 0$ and $c_{12} = 0$. We write

$$\mathcal{C}_{\lambda,k}(x,y) = \frac{1}{2c_2}(2c_2y + \lambda_2 L)^2 + 2\lambda_1 Lx - \frac{1}{2c_2}(\lambda_2^2 + 2(2k+1)c_2)L^2.$$

Our problem reduces to the finding of those integers λ_1, λ_2, k for which there exists $x, y, \omega \in \mathbb{Z}$ such that $(\lambda_2^2 + 2(2k+1)c_2)L^2 - 4\lambda_1 c_2 Lx = \omega^2$ and $2c_2y + \lambda_2 L = \pm\omega$.

Case II_2: $c_2 \neq 0$ and $c_{12} \neq 0$. We have now that

$$\begin{aligned} \mathcal{C}_{\lambda,k}(x,y) &= \frac{2}{c_{12}^2}(c_{12}y + \lambda_1 L)(c_{12}^2 x + c_2c_{12}y - (\lambda_1 c_2 - \lambda_2 c_{12})L) \\ &\quad + \left(\frac{2\lambda_1(\lambda_1 c_2 - \lambda_2 c_{12})}{c_{12}^2} - 2k - 1\right)L^2. \end{aligned}$$

So, we have to find integers λ_1, λ_2, k such that the elementary equation

$$\begin{aligned} &2(c_{12}y + \lambda_1 L)(c_{12}^2 x + c_2c_{12}y - (\lambda_1 c_2 - \lambda_2 c_{12})L) \\ &\qquad = ((2k+1)c_{12}^2 - 2\lambda_1(\lambda_1 c_2 - \lambda_2 c_{12}))L^2 \end{aligned}$$

has solutions in $\mathbb{Z}$.

Case II_3: $c_2 = 0$ and $c_{12} \neq 0$. We have that

$$\mathcal{C}_{\lambda,k}(x,y) = \frac{2}{c_{12}}(c_{12}x + \lambda_2 L)(c_{12}y + \lambda_1 L) - \frac{1}{c_{12}}\left((2k+1)c_{12} + 2\lambda_1\lambda_2\right)L^2,$$

and so we must find integers λ_1, λ_2, k for which the elementary equation

$$2(c_{12}x + \lambda_2 L)(c_{12}y + \lambda_1 L) = (2k+1)c_{12} + 2\lambda_1\lambda_2$$

has solutions in $\mathbb{Z}$. Observe that there is no solution in this case, provided that c_{12} is odd.

Case II_4: $c_2 = 0$ and $c_{12} = 0$. It is easy to see that in this case we reduce our problem to the solving of the Diophantine equation $2\lambda_1 x + 2\lambda_2 y = (2k+1)L$, which has solutions in $\mathbb{Z}$ if and only if we choose λ_1, λ_2, k such that $(\lambda_1, \lambda_2) | (2k+1)\frac{L}{2}$; where we notice that L is even.

A.2 An example

We exemplify the method of finding rational points for a conic section as in the above, for the case when $m_1 = 2$ and $m_2 = 6$. Thus, $M = 2$, $L = 6$, $N = 36$, $\lambda_1, c_1, c_{12} \in \{0, 1\}$, $\lambda_2, c_2 \in \{0, \cdots, 5\}$ and $\frac{f_1}{2} = \frac{x}{6}$, $f_2 = y$, so that $x \in \{0, 3\}$ and $y \in \{0, \cdots, 5\}$. The desired braided Hopf algebra structures are then parametrized by $(x, y) \in \mathbb{Z}^2$ for which $\mathcal{C}_{\lambda,k}(x,y) = 0$ for some λ and k, where

$$\mathcal{C}_{\lambda,k}(x,y) = 2c_1x^2 + 2c_2y^2 + 2c_{12}xy + 12\lambda_1 x + 12\lambda_2 y - 36(2k+1).$$

Note that $\delta = 4c_1c_2 - c_{12}^2$ is equal to zero if and only if $c_1c_2 = c_{12} = 0$. Parallel to the cases considered in Subsection A.1, we have the following possibilities. The notation below is the one we used in the subsection that we just mentioned. In addition, by $(x, y; \lambda_1, \lambda_2, k)$ we understand that (x, y) is a rational point for $\mathcal{C}_{\lambda,k}$, determined by $\lambda = (\lambda_1, \lambda_2)$ and k.

Case I_1: $c_1 = 1$ and $\delta = 0$, that is, $c_1 = 1$ and $c_2 = c_{12} = 0$. The Diophantine equation reduces to $2x + 6\lambda_1 = \widetilde{x}$, and forces $36(\lambda_1^2 + 4k + 2) - 24\lambda_2 y$ to be a square. Thus, $3 \mid \lambda_2 y$ and $\lambda_1^2 + 4k + 2 - 2\frac{\lambda_2 y}{3}$ is a square. Observe that we cannot have $\lambda_2 = 0$, for otherwise $4k+2$ or $4k+3$ is a square, a contradiction, by recalling that $\lambda_1 \in \{0, 1\}$.

• If $\lambda_2 \in \{1, 2, 4, 5\}$, $y = 3$ and we must have $\lambda_1^2 - 2\lambda_2 + 4k + 2$ a square. A square is congruent to 0 or 1 modulo 4. If we consider our square congruent to 0 modulo 4 then $(\lambda_1, \lambda_2, k) \in \{(0, 1, t^2), (0, 5, t^2 + 2)\}$, $t \in \mathbb{Z}$. When $(\lambda_1, \lambda_2, k) = (0, 1, t^2)$ we get $x = \pm 6t$, $y = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$ and $k = t^2$, and we are forced to take $t = 0$. Hence, $(0, 3; 0, 1, 0)$ is a rational point. Likewise, when $(\lambda_1, \lambda_2, k) = (0, 5, t^2 + 2)$ we deduce that $x = \pm 6t$, $y = 3$; we get that $(0, 3; 0, 5, 2)$ is a rational point, too. Analogously, if the above mentioned square is congruent to 1 modulo 4 we get that $(\lambda_1, \lambda_2, k) \in \{(1, 1, t^2 + t), (1, 5, t^2 + t + 1)\}$, $t \in \mathbb{Z}$; only the first triple yields a rational point, namely $(0, 3; 1, 1, 0)$.

• If $\lambda_2 = 3$, $\lambda_1^2 - 2y + 4k + 2$ is a square; we proceed as in the previous case. We deduce that $(\lambda_1, y, k) \in \{(1, 1, t^2+t), (1, 3, t^2+t+1), (1, 5, t^2+t+2), (0, 1, t^2), (0, 3, t^2+1), (0, 5, t^2+2) | t \in \mathbb{Z}\}$. Since $\widetilde{x}^2 = 36(\lambda_1^2 + 4k + 2) - 72y$ and $\widetilde{x} = 2x + 6\lambda_1 \in \{6, 12\}$ for $\lambda_1 = 1$ and $\widetilde{x} \in \{0, 6\}$ for $\lambda_1 = 0$, we conclude that the rational points are: $(0, 1; 1, 3, 0)$, $(0, 3; 1, 3, 1)$, $(0, 5; 1, 3, 2)$, $(0, 1; 0, 3, 0)$, $(0, 3; 0, 3, 1)$ and $(0, 5; 0, 3, 2)$.

Case I_2: $c_1 = 1$ and $\delta \neq 0$; equivalently, $c_1 = c_{12} = 1$ or $c_1 = 1$, $c_{12} = 0$ and $c_2 \in \{1, \cdots, 5\}$.

I_{2_1}: $c_1 = c_{12} = 1$. We have $\delta < 0$ if and only if $c_2 = 0$, and this is why we treat this case separately.

• For $c_2 = 0$, the Pell's type equation reads $\widetilde{x}^2 - \widetilde{y}^2 = d$, with $\widetilde{x} = -y + 6(2\lambda_2 - \lambda_1)$, $\widetilde{y} = 2x + y + 6\lambda_1$ and $d = 36(4\lambda_2^2 - 4\lambda_1\lambda_2 - 4k - 2)$. Having this particular shape, it can be solved directly by considering its equivalent form

$$(x + 6\lambda_2)(x + y - 6(\lambda_2 - \lambda_1)) = 18(-2\lambda_2^2 + 2\lambda_1\lambda_2 + 2k + 1).$$

As we observed, $x \in \{0, 3\}$. If $x = 0$, we reduce to $\lambda_2 y = 6k + 3$. So $(\lambda_2, y) \in \{(3, 1), (1, 3)\}$, and therefore $(0, 3; \lambda_1, 1, 0)$ and $(0, 1; \lambda_1, 3, 0)$, $\lambda_1 \in \{0, 1\}$, are rational points. Likewise, for $x = 3$ we reduce to $(2\lambda_2 + 1)y + 6\lambda_1 = 3(4k+1)$, which ensures us that $3|(1+2\lambda_2)y$. If $\lambda_2 = 0$, $y = 3$ and $\lambda_1 = 2k$. If $\lambda_2 = 1$ then $y + 2\lambda_1 = 4k+1$, thus $(y, \lambda_1, k) \in \{(1, 0, 0), (3, 1, 1), (5, 0, 1)\}$. If $\lambda_2 \in \{2, 3\}$, $y = 3$ and $\lambda_1 + \lambda_2 = 2k$. If $\lambda_2 = 4$ then $3y + 2\lambda_1 = 4k+1$, hence $(y, \lambda_1, k) = (1, 1, 1)$. Finally, if $\lambda_2 = 5$ then $y = 3$ and $\lambda_1 = 2k - 5$. We deduce that the rational points are $(3, 3; 0, 0, 0)$, $(3, 1; 0, 1, 0)$, $(3, 3; 1, 1, 1)$, $(3, 5; 0, 1, 1)$, $(3, 3; 0, 2, 1)$, $(3, 3; 1, 3, 2)$, $(3, 1; 1, 4, 1)$, $(3, 3; 1, 5, 3)$.

• Consider $c_2 \in \{1, \cdots, 5\}$, so that $\delta = 4c_2 - 1 \in \{3, 7, 11, 15, 19\}$. As $\lambda_1 \in \{0, 1\}$ and $x \in \{0, 3\}$, to treat this case easily, we consider the four cases we might have. More precisely, we first notice that, in terms of x and y, our equation $\widetilde{x}^2 + \delta\widetilde{y}^2 = d$ reads as

$$(\delta + 1)y^2 + 4(x + 6\lambda_2)y + 4x(x + 6\lambda_1) - 72(2k + 1) = 0.$$

- If $\lambda_1 = 0$, for $x = 0$ we get $c_2y^2 + 6\lambda_2 y - 18(2k + 1) = 0$ which provides solutions only when $c_2 \in \{2, 4\}$: $(0, 3; 0, 0, 0)$, $(0, 3; 0, 2, 1)$, $(0, 3; 0, 4, 2)$ and, respectively, $(0, 3; 0, 1, 1)$, $(0, 3; 0, 3, 2)$, $(0, 3, 0, 5, 3)$. For $x = 3$ the equation becomes $c_2y^2 + 3(2\lambda_2 + 1)y - 9(4k + 1) = 0$, and admits solutions only when $c_2 \in \{2, 4\}$; namely: $(3, 3; 0, 1, 1)$, $(3, 3; 0, 3, 2)$, $(3, 3; 0, 5, 3)$ and, respectively, $(3, 3; 0, 0, 1)$, $(3, 3; 0, 2, 2)$, $(3, 3; 0, 4, 3)$.

- If $\lambda_1 = 1$, for $x = 0$ we have $c_2y^2 + 6\lambda_2 y - 18(2k+1) = 0$, the same equation as in the case when $\lambda_1 = x = 0$; so, it gives the rational points $(0, 3; 1, 0, 0)$, $(0, 3; 1, 2, 1)$, $(0, 3; 1, 4, 2)$ when $c_2 = 2$, and $(0, 3; 1, 1, 1)$, $(0, 3; 1, 3, 2)$, $(0, 3; 1, 5, 3)$ when $c_2 = 4$. Likewise, $x = 3$ yields $c_2y^2 + 3(2\lambda_2 + 1)y - 9(4k - 1) = 0$, and we have solutions only when $c_2 \in \{2, 4\}$: $(3, 3; 1, 0, 1)$, $(3, 3; 1, 2, 2)$, $(3, 3; 1, 4, 3)$ and, respectively, $(3, 3; 1, 1, 2)$, $(3, 3; 1, 3, 3)$, $(3, 3; 1, 5, 4)$.

I_{2_2}: $c_1 = 1$, $c_{12} = 0$ and $c_2 \in \{1, \cdots, 5\}$. It is more convenient to turn $\widetilde{x}^2 + \delta\widetilde{y}^2 = d$ in terms of x, y; we have that $c_2y^2 + 6\lambda_2 y + x^2 + 6\lambda_1 x - 18(2k+1) = 0$. For $x = 0$, we obtain that $6|c_2y^2$ and $y \neq 0$; we have $y = 3$, otherwise y is even, and so $y^2 + 2\lambda_2 y$ is a multiple of 4 and equal to $6(2k + 1)$ which is false. Thus, for $x = 0$, we must have $c_2 \in \{2, 4\}$ and $y = 3$, and this leads to $c_2 + 2(\lambda_2 - 2k - 1) = 0$; so, for $c_2 = 2$ we get the rational points $(0, 3; \lambda_1, 2t, t)$, $\lambda_1 \in \{0, 1\}$ and $t \in \{0, 1, 2\}$, while for $c_2 = 4$ we obtain the rational points $(0, 3; \lambda_1, 2t + 1, t + 1)$, $\lambda_1 \in \{0, 1\}$ and $t \in \{0, 1, 2\}$.

Similarly, for $x = 3$, we have $c_2y^2 + 6\lambda_2 y + 18\lambda_1 - 9(4k + 1) = 0$, so $y \neq 0$ and $3|c_2y^2$. It follows that either $c_2 = 3$ or $y = 3$. One can see that in the first case, when $c_2 = 3$, we have the rational points $(3, 1; 0, 1, 0)$, $(3, 3; 0, 2t - 1, t)$ with $t \in \{1, 2, 3\}$ and $(3, 5; 0, 5, 6)$, provided that $\lambda_1 = 0$, and respectively $(3, 1; 1, 4, 1)$, $(3, 3; 1, 2(t - 1), t)$ with $t \in \{1, 2, 3\}$ and $(3, 5; 1, 2, 4)$, provided that $\lambda_1 = 1$. The second case, when $y = 3$, leads to the following rational points: $(3, 3; \lambda_1, \lambda_2, k)$ with $\lambda_1 + \lambda_2 = 2k + \frac{1-c_2}{2}$, since there are 6 solutions for each possible value of c_2, hence 18 in total; more precisely: for $c_2 = 1$ they are $(3,3;0,2t,t)$ and $(3,3;1,2t{+}1,t{+}1)$ with $t \in \{0,1,2\}$; for $c_2 = 3$ they are given by $(3,3;0,2t+1,t+1)$ and $(3,3;1,2t,t)$ with $t \in \{0,1,2\}$; for $c_2 = 5$ the rational points are $(3,3;0,2t,t{+}1)$ and $(3,3;1,2t{+}1,t{+}2)$ with $t \in \{0,1,2\}$.

Case II$_1$: $c_2 \in \{1,\cdots,5\}$, $c_1 = c_{12} = 0$. We check when $9(\lambda_2^2 + 2(2k+1)c_2) - 6\lambda_1 c_2 x$ is a square.

• When $\lambda_1 x = 0$, if $\lambda_2^2 - \omega^2 = -2(2k+1)c_2$, for some integer ω, then λ_2 and ω have the same parity; this forces c_2 even, so $c_2 \in \{2,4\}$. Note that $\pm 3\omega = c_2 y + 3\lambda_2$.

– For $c_2 = 2$, since any square is 0, 1 or 4 modulo 8, $\lambda_2^2 + 4(2k+1)$ a square implies that λ_2^2 is 0 or 4 modulo 8, so $\lambda_2 \in \{0,2,4\}$. When $\lambda_2 = 0$, $2k+1$ is a square, hence $k = 2t(t+1)$, $t \in \mathbb{Z}$; we get that $\omega = \pm 2(2t+1)$ and $y = \pm 3(2t+1)$, forcing $y = 3$ and $t = 0$. Concluding, we have $(x,3;\lambda_1,0,0)$ as rational points, provided that $\lambda_1 x = 0$; they are 3 in total. Likewise, when $\lambda_2 = 4$, we have $2k+2 = \left(\frac{\omega}{12}\right)^2$ and $2y+6 = \pm\frac{\omega}{2}$, for some even integer ω, so $k = 2t^2 - 1$ and $y = \pm 6t - 3$, for some $t \in \mathbb{Z}$, forcing $y = 3$ and $k = 1$; we get in this case the rational points $(x,3;\lambda_1,2,1)$ with $\lambda_1 x = 0$; again, they are 3. Finally, for $\lambda_2 = 4$, $2k+5 = \left(\frac{\omega}{12}\right)^2$ is a square and $y+6 = \pm\frac{\omega}{4}$, and therefore $k = 2(t^2+t-1)$ and $y+6 = \pm 3(2t+1)$, for some $t \in \mathbb{Z}$; these facts entail to $y = 3$ and $k = 2$, and so we get another 3 rational points: $(x,3;\lambda_1,4,2)$, whenever $\lambda_1 x = 0$.

– For $c_2 = 4$, $\lambda_2^2 + 8(2k+1)$ a square implies that $\lambda_2 \in \{1,3,5\}$, since a square equals 0, 1, 4 or 9 modulo 16. For $\lambda_2 = 1$, $16k+9 = \left(\frac{\omega}{6}\right)^2$ and $4y+3 = \pm\frac{\omega}{2}$, for some even integer ω, hence $k = 4t^2+3t$ or $k = 4t^2+5t+1$, for some $t \in \mathbb{Z}$; we get that $4y+3 = \pm 3(8t+3)$ or $4y+3 = \pm 3(8t+5)$, $t \in \mathbb{Z}$, thus $y = 3$ and $k = 1$, leading to the rational points $(x,3;\lambda_1,1,1)$, where $\lambda_1 x = 0$; there are 3 in total. In the case when $\lambda_2 = 3$, $16(k+1)+1 = \left(\frac{\omega}{6}\right)^2$ is a square and $4y+9 = \pm\frac{\omega}{2}$, and so $k = 4t^2+t-1$ or $k = 4t^2+7t+2$, $t \in \mathbb{Z}$; we obtain that $4y+9 = \pm 3(8t+1)$ or $4y+9 = \pm 3(8t+7)$, for some $t \in \mathbb{Z}$, so $y = 3$ and $k = 2$, and we have the rational points $(x,3;\lambda_1,3,2)$ whenever $\lambda_1 x = 0$. Similarly, when $\lambda_1 = 5$, $16(k+2)+1 = \left(\frac{\omega}{6}\right)^2$ is a square and $4y+15 = \pm\frac{\omega}{2}$, and we must have $k = 4t^2+t-2$ or $k = 4t^2+7t+1$, for some $t \in \mathbb{Z}$; we deduce that $4y+15 = \pm 3(8t+1)$ or $4y+15 = \pm 3(8t+7)$, hence $y = 3$ and $k = 3$, and we have the rational points $(x,3;\lambda_1,5,3)$, whenever $\lambda_1 x = 0$.

• Assume that $\lambda_1 x \neq 0$ or, equivalently, that $\lambda_1 = 1$ and $x = 3$. We have $\lambda_2^2 + 4kc_2 = \left(\frac{\omega}{6}\right)^2$ and $c_2 y + 3\lambda_2 = \pm\frac{\omega}{2}$, for some even integer ω. Clearly, $3|c_2 y$ and we distinguish two possibilities.

– If $c_2 = 3$, $\lambda_2^2 + 12k = (y+\lambda_2)^2$, and therefore $y^2 + 2\lambda_2 y - 12k = 0$. We get $y \in \{0,2,4\}$, and the rational points $(3,0;1,\lambda_2,0)$, $0 \le \lambda_2 \le 5$, and the points in $\{(3,2;1,3t-1,t),(3,4;1,3t-2,2t) | t \in \{1,2\}\}$.

– If $c_2 \neq 3$, we have $y = 3$ and $c_2 + 2\lambda_2 = 4k$. This leads us to the following rational points: $(3,3;1,2t-1,t)$ with $t \in \{1,2,3\}$, provided that $c_2 = 2$, and $(3,3;1,2t-2,t)$ with $t \in \{1,2,3\}$, provided that $c_2 = 4$.

Case II$_2$: $c_1 = 0$, $c_2 \neq 0$ and $c_{12} \neq 0$. We must solve in $\mathbb{Z}$ the equation

$$(y+6\lambda_1)(x+c_2 y - 6(\lambda_1 c_2 - \lambda_2)) = 18(2k+1-2\lambda_1(\lambda_1 c_2 - \lambda_2)).$$

• If $\lambda_1 = 0$, we reduce to $y(x+c_2 y+6\lambda_2) = 18(2k+1)$, which imposes the following discussion.

– $y = 1$: $x + c_2 + 6\lambda_2 = 18(2k+1)$, with the unique solution $x = c_2 = 3$, $\lambda_2 = 2$ and $k = 0$; the associated rational point is $(3,1;0,2,0)$.

– $y = 2$: $x+2c_2+6\lambda_2 = 9(2k+1)$, which imposes $x = 3$; thus $c_2+3\lambda_2 = 3(3k+1)$, leading to $c_2 = 3$ and $\lambda_2 = 3k$, and so to the rational points $(3, 2; 0, 3t, t)$, $t \in \{0, 1\}$.

– $y = 3$: $x + 3c_2 + 6\lambda_2 = 6(2k + 1)$, which for $x = 0$ gives $c_2 \in \{2, 4\}$ such that $c_2+2\lambda_2 = 2(2k+1)$, while for $x = 3$ gives $c_2 \in \{1, 3, 5\}$ such that $c_2+2\lambda_2 = 4k+1$; so we get the following rational points: $(0, 3; 0, 2k, k)$ with $k \in \{0, 1, 2\}$, provided that $c_2 = 2$; $(0, 3; 0, 2k-1, k)$ with $k \in \{1, 2, 3\}$, provided that $c_2 = 4$; $(3, 3; 0, 2k, k)$ with $k \in \{0, 1, 2\}$, provided that $c_2 = 1$; $(3, 3; 0, 2k - 1, k)$ with $k \in \{1, 2, 3\}$, provided that $c_2 = 3$; $(3, 3; 0, 2k - 2, k)$ with $k \in \{1, 2, 3\}$, provided that $c_2 = 5$.

• If $\lambda_1 = 1$, we proceed as in the previous case. The equation becomes $(y + 6)(x + c_2y - 6c_2 + 6\lambda_2) = 18(2k + 1 - 2c_2 + 2\lambda_2)$ and we cannot have $y = 2$.

– $y = 0$: $x = 3(2k + 1)$; thus $x = 3$, $0 \neq c_2, \lambda_2 \in \{0, \cdots, 5\}$ and $k = 0$, so we have 30 rational points of the form $(3, 0; 1, \lambda_2, 0)$, no matter how we choose $c_2 \in \{1, \cdots, 5\}$.

– $y = 1$: $c_2 + 7x + 6\lambda_2 = 18(2k + 1)$, hence $6|c_2 + x$; we get $c_2 = x = 3$ and $\lambda_2 = 6k - 1$, so the rational points are $(3, 1; 1, 5, 1)$, provided that $c_2 = 3$.

– $y = 3$: $x + c_2 + 2\lambda_2 = 2(2k + 1)$, which yields the rational points $(0, 3; 1, 2k, k)$ with $k \in \{0, 1, 2\}$, provided that $c_2 = 2$; $(0, 3; 1, 2k - 1, k)$ with $k \in \{1, 2, 3\}$, provided that $c_2 = 4$; $(3, 3; 1, 2k - 1, k)$ with $k \in \{1, 2, 3\}$, provided that $c_2 = 1$; $(3, 3; 1, 2k - 2, k)$ with $k \in \{1, 2, 3\}$, provided that $c_2 = 3$; $(3, 3; 1, 2k - 3, k)$ with $k \in \{2, 3, 4\}$, provided that $c_2 = 5$;

– $y = 4$: $5(x - 2c_2 + 6\lambda_2) = 9(2k + 1 - 2c_2 + 2\lambda_2)$, with general solution $x-2c_2+6\lambda_2 = 9t$ and $2k+1-2c_2+2\lambda_2 = 5t$, $t \in \mathbb{Z}$. It follows that $x-(2k+1)+4\lambda_2 = 4t$, so $x = 3$, and since $x - 3(2k + 1) + 4c_2 = -6t$ it follows that $c_2 = 3$, too. We reduce to $t = k - 2$ and $2\lambda_2 = 3k - 5$, and so we get the following rational points: $(3, 4; 1, 3s - 1, 2s + 1)$ with $s \in \{1, 2\}$, provided that $c_2 = 3$.

– $y = 5$: $11(x-c_2+6\lambda_2) = 18(2k+1-2c_2+2\lambda_2)$, and therefore $x-c_2+6\lambda_2 = 18t$ and $k + 1 - 2c_2 + 2\lambda_2 = 11t$, $t \in \mathbb{Z}$. As $6|x - c_2$, we have $x = c_2 = 3$, which leads to $\lambda_2 = 3t$ and $2k - 5 = 5t$. We get the rational point $(3, 5; 1, 3, 5)$, provided that $c_2 = 3$.

Case II$_3$: $c_1 = c_2 = 0$ and $c_{12} \neq 0$. There is no rational point in this case since $c_{12} = 1$ is odd.

Case II$_4$: $c_1 = c_2 = c_{12} = 0$. We must solve the Diophantine equation $\lambda_1x+\lambda_2y = 3(2k + 1)$.

– If $\lambda_1x = 0$, $\lambda_2y = 3(2k + 1)$ forces $k \in \{0, 1, 2, 3\}$. We obtain the following 15 rational points $(x, 1; \lambda_1, 3, 0)$, $(x, 3; \lambda_1, 1, 0)$, $(x, 3; \lambda_1, 3, 1)$, $(x, 3; \lambda_1, 5, 2)$ and $(x, 5; \lambda_1, 3, 2)$, whenever $\lambda_1x = 0$.

– If $\lambda_1x \neq 0$, we have $\lambda_1 = 1$, $x = 3$ and $\lambda_2y = 6k$ with $k \in \{0, 1, 2\}$. When $k = 0$, either $y = 0$ or $\lambda_2 = 0$, giving 12 rational points: $(3, 0; 1, \lambda_2, 0)$, $0 \leq \lambda_2 \leq 5$, and $(3, y; 1, 0, 0)$ with $0 \leq y \leq 5$. For $k = 1$, either $\lambda_2 = 3$ or $y = 3$, and we get the rational points $(3, 2; 1, 3, 1)$ and $(3, 3; 1, 2, 1)$. For $k = 2$, $(y, \lambda_2) \in \{(3, 4), (4, 3)\}$, and we get the rational points $(3, 3; 1, 4, 2)$ and $(3, 4; 1, 3, 2)$.

We summarize our classification result in the table below. The notation we use is the same as in Example 2.2.12.

$\underline{a} = (c_1, c_2, c_{12})$	$\sigma_{f_1 f_2}$	(λ_1, λ_2)	$\underline{a} = (c_1, c_2, c_{12})$	$\sigma_{f_1 f_2}$	(λ_1, λ_2)
	σ_{01}	$(0,3),(1,3)$		σ_{03}	$(0,1),(1,1),(0,3)$
	σ_{12}	$(1,0),(1,3)$			$(1,3),(0,5),(1,5)$
$(0,0,0)$	σ_{05}	$(0,3),(1,3)$	$(0,0,0)$	σ_{10}	$(1,0),(1,1),(1,2)$
	σ_{11}	$(0,3),(1,0)$			$(1,3),(1,4),(1,5)$
	σ_{15}	$(0,3),(1,0)$		σ_{13}	$(0,1),(0,3),(0,5)$
	σ_{14}	$(1,0),(1,3)$			$(1,0),(1,2),(1,4)$
$(1,0,0)$	σ_{01}	$(0,3),(1,3)$	$(1,0,0)$	σ_{03}	$(0,1),(0,3),(0,5)$
	σ_{05}	$(0,3),(1,3)$			$(1,1),(1,3)$
$(0,1,1)$	σ_{10}	$(1,0),(1,1),(1,2)$	$(0,1,1)$	σ_{13}	$(0,0),(0,2),(0,4)$
		$(1,3),(1,4),(1,5)$			$(1,1),(1,3),(1,5)$
	σ_{01}	$(0,3),(1,3)$		σ_{03}	$(0,1),(1,1)$
$(1,0,1)$	σ_{11}	$(0,1),(0,2),(1,4)$	$(1,0,1)$	σ_{13}	$(0,0),(0,2)$
	σ_{15}	$(0,1)$			$(1,1),(1,3),(1,5)$
$(1,1,0)$	σ_{13}	$(0,0),(0,2),(0,4)$	$(1,2,0)$	σ_{03}	$(0,0),(0,2),(0,4)$
		$(1,1),(1,3),(1,5)$			$(1,0),(1,2),(1,4)$
$(1,3,0)$	σ_{13}	$(0,1),(0,3),(0,5)$	$(1,3,0)$	σ_{11}	$(0,1),(1,4)$
		$(1,0),(1,2),(1,4)$		σ_{15}	$(0,5),(1,2)$
$(1,4,0)$	σ_{03}	$(0,1),(0,3),(0,5)$	$(1,5,0)$	σ_{13}	$(0,0),(0,2),(0,4)$
		$(1,1),(1,3),(1,5)$			$(1,1),(1,3),(1,5)$
	σ_{03}	$(0,0),(0,2),(0,4)$		σ_{03}	$(0,1),(0,3),(0,5)$
$(1,2,1)$		$(1,0),(1,2),(1,4)$	$(1,4,1)$		$(1,1),(1,3),(1,5)$
	σ_{13}	$(0,1),(0,3),(0,5)$		σ_{13}	$(0,0),(0,2),(0,4)$
		$(1,0),(1,2),(1,4)$			$(1,1),(1,3),(1,5)$
	σ_{03}	$(0,0),(0,2),(0,4)$		σ_{10}	$(1,0),(1,1),(1,2)$
$(0,2,0)$		$(1,0),(1,2),(1,4)$	$(0,3,0)$		$(1,3),(1,4),(1,5)$
	σ_{13}	$(0,0),(0,2),(0,4)$		σ_{12}	$(1,2),(1,5)$
		$(1,1),(1,3),(1,5)$		σ_{14}	$(1,1),(1,4)$
	σ_{03}	$(0,1),(0,3),(0,5)$		σ_{03}	$(0,0),(0,2),(0,4)$
$(0,4,0)$		$(1,1),(1,3),(1,5)$	$(0,2,1)$		$(1,0),(1,2),(1,4)$
	σ_{13}	$(0,1),(0,3),(0,5)$		σ_{10}	$(1,0),(1,1),(1,2)$
		$(1,0),(1,2),(1,4)$			$(1,3),(1,4),(1,5)$
	σ_{10}	$(1,0),(1,1),(1,2)$		σ_{03}	$(0,1),(0,3),(0,5)$
		$(1,3),(1,4),(1,5)$	$(0,4,1)$		$(1,1),(1,3),(1,5)$
	σ_{11}	$(0,2),(1,5)$		σ_{10}	$(1,0),(1,1),(1,2)$
	σ_{12}	$(0,0),(0,3)$			$(1,3),(1,4),(1,5)$
$(0,3,1)$	σ_{13}	$(0,1),(0,3),(0,5)$		σ_{13}	$(0,0),(0,2),(0,4)$
		$(1,0),(1,2),(1,4)$	$(0,5,1)$		$(1,1),(1,3),(1,5)$
	σ_{14}	$(1,2),(1,5)$		σ_{10}	$(1,0),(1,1),(1,2)$
	σ_{15}	$(1,3)$			$(1,3),(1,4),(1,5)$

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