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Heat Kernels for a Class of Hybrid Evolution Equations

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Abstract
The aim of this paper is to construct (explicit) heat kernels for some *hybrid* evolution equations which arise in physics, conformal geometry and subelliptic PDEs. Hybrid means that the relevant partial differential operator appears in the form $\mathscr{L}_1 + \mathscr{L}_2 - \partial_t$, but the variables cannot be decoupled. As a consequence, the relative heat kernel cannot be obtained as the product of the heat kernels of the operators $\mathscr{L}_1 - \partial_t$ and $\mathscr{L}_2 - \partial_t$. Our approach is new and ultimately rests on the generalised Ornstein-Uhlenbeck operators in the opening of Hörmander's 1967 groundbreaking paper on hypoellipticity.

Keywords Heat kernel · CR extension problem · Cauchy problem

Mathematics Subject Classification (2010) 35K08 · 35R03 · 53C17

1 Introduction

Consider a second order partial differential operator $\mathscr{L}$ and the heat equation $\mathscr{L}u - \partial_t u = 0$ associated with it. Following a well-established tradition by heat kernel we mean a function $p(x, \xi, t)$ such that for any ξ the function $p(\cdot, \xi, \cdot)$ is a solution of the heat equation, and $p(x, \cdot, t) \longrightarrow \delta_x$ in the distributional sense as $t \to 0^+$. The aim of this paper is to construct explicit heat kernels for some *hybrid* evolution equations which arise in diverse frameworks such as e.g. sub-Riemannian geometry and problems from the applied sciences that are modelled by some classes of subelliptic equations. By hybrid we mean that the relevant partial differential operator appears in the form $\mathscr{L}_1 + \mathscr{L}_2 - \partial_t$, but the variables cannot be decoupled. Consequently, the relative heat kernel cannot be simply obtained as the product of the heat kernels of the operators $\mathscr{L}_1 - \partial_t$ and $\mathscr{L}_2 - \partial_t$. Our approach is completely

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self-contained, elementary, and it is purely based on PDE methods whose final objective is to emphasise the so far unexplored connection of the relevant class of hybrid equations with the generalised operators of Ornstein-Uhlenbeck type in the opening of Hörmander's groundbreaking 1967 work [30]. It is worth mentioning here that as a by-product we obtain a simple proof of the well-known (non-hybrid) cases of the heat operator in a stratified nilpotent Lie group of step two and of the Baouendi-Grushin operator (see respectively Sections 4 and 3 below).

To motivate our results we next discuss some prototypical examples which fall within the scope of our approach. We begin with an example from conformal CR geometry. In recent years the study of the so-called *extension operators* has received increasing attention from workers in analysis and geometry especially in connection with certain conformally invariant nonlocal operators. A typical situation of interest is represented by the Heisenberg group $\mathbb{H}^n \cong \mathbb{C}^n \times \mathbb{R}$ with real coordinates (z, σ) [1] and horizontal Laplacian

$$\mathscr{L} = \Delta_z + \frac{|z|^2}{4}\partial_{\sigma\sigma} + \sum_{i=1}^{n} \partial_\sigma \left(z_i \partial_{z_{n+i}} - z_{n+i}\partial_{z_i}\right). \tag{1.1}$$

In their seminal paper [19] Frank et al. have introduced the following extension problem: given a function $u \in C_0^\infty(\mathbb{H}^n)$, find a function $U \in C^\infty(\mathbb{H}^n \times \mathbb{R}^+_y)$ that solves the Dirichlet problem

$$\begin{cases} \partial_{yy}U + \frac{1-2s}{y}\partial_y U + \frac{y^2}{4}\partial_{\sigma\sigma}U + \mathscr{L}U = 0, & \text{in } \mathbb{H}^n \times \mathbb{R}^+_y, \\ U(g,0) = u(g), \end{cases} \tag{1.2}$$

where the fractional parameter $s \in (0,1)$. The term $\frac{y^2}{4}\partial_{\sigma\sigma}U$ has a geometric meaning whose explanation comes from the equivalence between (1.2) and the *scattering eigenvalue problem* in complex hyperbolic space. A fundamental aspect of the problem (1.2) is the following weighted Dirichlet-to-Neumann relation, proved in [19],

$$-\,2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}\lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y}((z,\sigma),y) = \mathscr{L}_s u(z,\sigma). \tag{1.3}$$

The pseudo-differential operator $\mathscr{L}_s$ in the right-hand side of Eq. 1.3 represents the fractional power of the conformal horizontal Laplacian on $\mathbb{H}^n$ defined via the spectral formula

$$\mathscr{L}_s = 2^s|\partial_\sigma|^s \frac{\Gamma(-\frac{1}{2}\mathscr{L}|\partial_\sigma|^{-1} + \frac{1+s}{2})}{\Gamma(-\frac{1}{2}\mathscr{L}|\partial_\sigma|^{-1} + \frac{1-s}{2})}.$$

This operator of order $2s$, which was first introduced by Branson, Fontana and Morpurgo in [9], is drastically different from the standard fractional powers defined by the well-known formula

$$(-\mathscr{L})^s = -\frac{s}{\Gamma(1-s)}\int_0^\infty \frac{1}{t^{1+s}}(Q_t u(g) - u(g))dt,$$

where $Q_t u(g) = e^{-t\mathscr{L}}u(g) = \int_{\mathbb{G}} q(g,g',t)u(g')dg'$ is the heat semigroup with kernel (4.6). The second order time-independent PDE in Eq. 1.2 is a notable example of the type of *hybrid* equations that are the object of interest of the present paper. To clarify this aspect we observe that if we formally think of w as a generic point in the space with fractal dimension $\mathbb{R}^{2(1-s)}$, and we let $y = |w|$ denote its "distance" to the origin, then the PDE in

[1] we explicitly mention here that traditionally the letter t is reserved for the vertical variable in $\mathbb{H}^n$. However, since we want to indicate the time variable with t, we have opted for the notation σ. The letter z instead indicates the horizontal variables in $\mathbb{R}^{2n}$.

Eq. 1.2 can be interpreted as the action of the differential operator $\Delta_w + \frac{|w|^2}{4}\partial_{\sigma\sigma} + \mathscr{L}$ on functions having spherical symmetry in w. If we consider the heat equation associated with such operator,

$$\Delta_w U + \frac{|w|^2}{4}\partial_{\sigma\sigma} + \mathscr{L}U - \partial_t U = 0, \tag{1.4}$$

it is immediate to recognise that in such equation the variables $(w, \sigma) \in \mathbb{R}^{2(1-s)} \times \mathbb{R}$ and $g = (z, \sigma) \in \mathbb{H}^n$ cannot be decoupled since the variable σ appears in both the limiting operators $\Delta_w + \frac{|w|^2}{4}\partial_{\sigma\sigma} - \partial_t$ and $\mathscr{L} - \partial_t$ (see Eq. 1.1). In Section 4 we will show that the heat kernel with pole at the origin associated with Eq. 1.4 is given by

$$\mathfrak{q}_{(s)}((z,\sigma),t,y) = \frac{2}{(4\pi t)^{\frac{m}{2}+2-s}} \int_{\mathbb{R}} e^{-\frac{i}{t}\sigma\lambda} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2+y^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda. \tag{1.5}$$

We emphasise that Eq. 1.2 is dramatically different from the extension problem à la Caffarelli-Silvestre

$$\begin{cases} \partial_{yy}U + \frac{1-2s}{y}\partial_y U + \mathscr{L}U = 0, & \text{in } \mathbb{H}^n \times \mathbb{R}^+_y, \\ U(g,0) = u(g), \end{cases} \tag{1.6}$$

in which the geometric term $\frac{y^2}{4}\partial_{\sigma\sigma}U$ is missing. The evolution PDE associated with Eq. 1.6 is

$$\Delta_w U + \mathscr{L}U - \partial_t U = 0,$$

and it should be clear to the reader that this is not of hybrid type since its fundamental solution with singularity at the origin

$$q^{(s)}((z,\sigma),t,y) = (4\pi t)^{-(1-s)} e^{-\frac{y^2}{4t}} p((z,\sigma),t)$$

is indeed the product of the fundamental solutions of the two heat operators $\Delta_w - \partial_t$ and $\mathscr{L} - \partial_t$ (the reader should note that we have used a superscript (s) to distinguish such heat kernel from that in Eq. 1.5, for which we have used a subscript (s)). Formula Eq. 1.5 (see also the more general case treated in Theorem 4.1 below) plays a critical role in the analysis of conformal properties of the pseudodifferential operator $\mathscr{L}_s$, and we refer the interested reader to the works [25, 26, 39, 43, 44] for more insights into this aspect.

Another significant model of the class of equations encompassed by the present paper is the following:

$$\Delta_w f + \frac{|w|^2}{4}\Delta_\sigma f + \langle w, \nabla_y f\rangle - \partial_t f = 0, \tag{1.7}$$

where $w, y \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^k$ and $t > 0$. The Eq. 1.7 is a hybrid between the time-dependent Baouendi-Grushin operator in $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^+_t$, $\Delta_w + \frac{|w|^2}{4}\Delta_\sigma - \partial_t$, and the famous degenerate Kolmogorov operator in $\mathbb{R}^{2n} \times \mathbb{R}^+_t$, $\Delta_w + \langle w, \nabla_y\rangle - \partial_t$. Our interest in Eq. 1.7, and the corresponding heat kernel (1.10) below, stems from its connection with a notable class of PDEs arising in the physics of human vision and polymers. Consider in fact the Mumford operator [40]

$$\mathscr{M} = \Delta_{\mathbb{S}^1} + \cos\theta\,\partial_\xi + \sin\theta\,\partial_\eta - \partial_t = \partial_\theta^2 + \cos\theta\,\partial_\xi + \sin\theta\,\partial_\eta - \partial_t = V_1^2 + V_2, \tag{1.8}$$

where $V_1 = \partial_\theta$ and $V_2 = \cos\theta\,\partial_\xi + \sin\theta\,\partial_\eta - \partial_t$. The operator (1.8) plays a critical role in the physics of semiflexible polymers and it is of interest to understand the relevant heat kernel. There is a natural Lie algebra associated with the vector fields V_1, V_2, the *roto-translation* *algebra*, but it is not nilpotent. However, one can check that, in view of Hörmander's theorem in [30], the Mumford operator $\mathscr{M}$ is hypoelliptic. Its nilpotent approximation is the equation

$$f_{ww} + \frac{w^2}{2} f_\sigma + w f_y - f_t = 0. \tag{1.9}$$

The PDE (1.9) differs from Eq. 1.7 in the fact that it contains the term $\frac{w^2}{2} f_\sigma$, instead of $\frac{w^2}{2} f_{\sigma\sigma}$, but one can link one to the other by means of transmutation formulas. Our approach produces the following explicit heat kernel for Eq. 1.7 (with pole at a generic point $(w_0, \sigma_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n$)

$$\begin{aligned} & h((w,\sigma,y),(w_0,\sigma_0,y_0),t) \\ &= (4\pi)^{-n} \int_{\mathbb{R}^k} e^{2\pi i \langle \sigma - \sigma_0, \lambda\rangle} e^{-\frac{\pi|\lambda|}{2}(|w|^2 - |w_0|^2 + 2nt)} (\det t K_\lambda(t))^{-1/2} \times \\ & \times \exp\left\{ -\frac{1}{4} \langle (t K_\lambda(t))^{-1} \begin{pmatrix} w_0 - e^{-2\pi t|\lambda|} w \\ y_0 - y - \frac{1-e^{-2\pi t|\lambda|}}{2\pi|\lambda|} w \end{pmatrix}, \begin{pmatrix} w_0 - e^{-2\pi t|\lambda|} w \\ y_0 - y - \frac{1-e^{-2\pi t|\lambda|}}{2\pi|\lambda|} w \end{pmatrix} \rangle \right\} d\lambda \end{aligned} \tag{1.10}$$

where

$$t K_\lambda(t) = e^{-2\pi t|\lambda|} \begin{pmatrix} \frac{\sinh(2\pi t|\lambda|)}{2\pi|\lambda|} I_n & \frac{\cosh(2\pi t|\lambda|)-1}{4\pi^2|\lambda|^2} I_n \\ \frac{\cosh(2\pi t|\lambda|)-1}{4\pi^2|\lambda|^2} I_n & \frac{e^{2\pi t|\lambda|}(2\pi t|\lambda| - \sinh(2\pi t|\lambda|)) + (\cosh(2\pi t|\lambda|)-1)(e^{2\pi t|\lambda|}-1)}{8\pi^3|\lambda|^3} I_n \end{pmatrix}.$$

Formula Eq. 1.10 is a special case of the more general Theorem 3.6 below, to which we refer the reader.

We now briefly discuss the organisation of the paper. In Section 2 we recall the class (2.1) of generalised Ornstein-Uhlenbeck operators in the opening of Hörmander's cited paper [30], and for completeness provide a short proof of Proposition 2.1 since this result constitutes the backbone of the present work. Section 3 introduces the hybrid class Eq. 3.1, of which the Eq. 1.7 discussed above is a prototypical representative. Besides its own interest, such section is instrumental to the rest of the paper. In the Section 3.1 we solve the Cauchy problem Eq. 3.4 for a generalised harmonic oscillator. The main result is Proposition 3.2 that establishes a generalisation of the classical formula of Mehler. In Section 3.2 we use this result to derive the heat kernel for the Baouendi-Grushin operator, see Eq. 3.19 in Theorem 3.4. In Theorem 3.6 we finally construct the heat kernel for the class of hybrid equations in Eq. 3.1. Section 4 represents the more geometric part of the paper. There we construct the heat kernel for the conformal extension problem Eq. 4.1. The latter represents a time-dependent generalisation to arbitrary groups of Heisenberg type of the above discussed conformal extension problem Eq. 1.2 from [19]. The main result of the section is Theorem 4.1. To prove it we follow a pattern similar to that in Section 3. We first construct the heat kernel for a generalised harmonic oscillator with a complex drift. This step serves as a building block in the proof of the main Theorem 4.1. In the process, and as a by-poduct of our approach, we also provide a new elementary proof of the famous formula of Hulanicki-Gaveau-Cygan for the heat kernel on a Carnot group of step two, see Theorem 4.6.

2 The Generalised Ornstein-Uhlenbeck Operators of Hörmander

In this section we recall a well-known explicit heat kernel that constitutes the essential ingredient of the present work. Consider the class of differential equations in $\mathbb{R}^N \times (0, \infty)$,

$$\mathscr{K} u = \mathscr{A} u - \partial_t u \overset{def}{=} \operatorname{tr}(Q\nabla^2 u) + \langle Bz, \nabla u\rangle - \partial_t u = 0. \tag{2.1}$$

Here, the $N \times N$ matrices Q and B have real, constant coefficients, and moreover $Q = Q^\star \geq 0$. A basic feature of the operator $\mathscr{K}$ is the invariance with respect to the following non-Abelian group law in $\mathbb{R}^{N+1}$

$$(z, s) \circ (\zeta, t) = (\zeta + e^{-tB} z, s + t),$$

see [36]. We emphasise that the evolution equation $\mathscr{K} u = \mathscr{A} u - \partial_t u = 0$ encompasses operators that are very different in nature. Besides of course the classical heat equation ($Q = I_N$ and $B = O_N$), it contains the Ornstein-Uhlenbeck equation $\Delta_z u - \langle z, \nabla_z u\rangle - \partial_t u = 0$ in [41] ($Q = I_N$ and $B = -I_N$), but also the very degenerate equation of Kolmogorov from the kinetic theory of gases $\Delta_v u + \langle v, \nabla_x u\rangle - \partial_t u = 0$ in $\mathbb{R}^{2n} \times (0, \infty)$, see [35] ($Q = \begin{pmatrix} I_n & O_n \\ O_n & O_n \end{pmatrix}$ and $B = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$), as well as the degenerate Ornstein-Uhlenbeck equation $\Delta_v u - \langle v, \nabla_v u\rangle + \langle v, \nabla_x u\rangle - \partial_t u = 0$ in $\mathbb{R}^{2n} \times (0, \infty)$ which arises in the Smoluchowski-Kramers approximation of Brownian motion with friction, see [10] ($Q = \begin{pmatrix} I_n & O_n \\ O_n & O_n \end{pmatrix}$ and $B = \begin{pmatrix} -I_n & O_n \\ I_n & O_n \end{pmatrix}$).

In [30] Hörmander proved that Eq. 2.1 is hypoelliptic if and only if its *covariance matrix* satisfies the following Kalman condition for one (and therefore every) $t > 0$

$$K(t) = \frac{1}{t}\int_0^t e^{sB} Q e^{sB^\star} ds \ > \ 0. \tag{2.2}$$

The hypothesis (2.2) will henceforth be assumed in this section. Under such assumption we note that $t \to tK(t)$ is strictly increasing in the sense of quadratic forms: one has in fact for any $t > t_0 > 0$

$$\begin{aligned} tK(t) - t_0 K(t_0) &= \int_{t_0}^t e^{sB} Q e^{sB^\star} ds = e^{-t_0 B}\left(\int_0^{t-t_0} e^{\sigma B} Q e^{\sigma B^\star} d\sigma\right) e^{-t_0 B^\star} \\ &= e^{-t_0 B}(t - t_0) K(t - t_0) e^{-t_0 B^\star} > 0. \end{aligned}$$

It follows that

$$t \mapsto (tK(t))^{-1} \quad \text{is strictly decreasing.} \tag{2.3}$$

Therefore, the matrix

$$K_\infty^{-1} = \lim_{t\to\infty} (tK(t))^{-1}$$

is well-defined, and of course it is symmetric and nonnegative definite. Formally, K_∞^{-1} is the inverse of the matrix $\int_0^\infty e^{sB} Q e^{sB^\star} ds$, but it is well-known that the latter is well-defined if and only if all the eigenvalues of B have strictly negative real part (see, e.g., [11, Proposition 2.3]). On the other hand K_∞^{-1} is well-defined for any choice of Q, B satisfying Eq. 2.2, even if it possibly has a non-trivial kernel.

To introduce the main result of this section we next recall the time-dependent intertwined pseudo-distance

$$m_t(z, \zeta) = \sqrt{\langle K(t)^{-1}(\zeta - e^{tB} z), \zeta - e^{tB} z\rangle}, \qquad t > 0.$$

The behaviour for large t of $tK(t)$ and $m_t(\cdot,\cdot)$ has been instrumental in our previous works [22, 23] in establishing several functional inequalities related to the differential operator $\mathscr{A}$.

Proposition 2.1 *The heat kernel of* Eq. 2.1 *is given by*

$$p(z,\zeta,t)=\frac{(4\pi)^{-N/2}}{(\det(tK(t)))^{1/2}}\exp\left(-\frac{m_t(z,\zeta)^2}{4t}\right). \tag{2.4}$$

More precisely, for any $f\in C(\mathbb{R}^N)$ *such that*

$$e^{-\frac{1}{4}\langle K_\infty^{-1}z,z\rangle}f(z)\in L^\infty\left(\mathbb{R}^N\right), \tag{2.5}$$

the function

$$u(z,t)=P_tf(z)=\int_{\mathbb{R}^N}p(z,\zeta,t)f(\zeta)d\zeta \tag{2.6}$$

solves the Cauchy problem $\mathscr{K}u=\mathscr{A}u-\partial_t u=0$ *in* $\mathbb{R}^N\times(0,\infty)$, $u(z,0)=f(z)$.

Proof The proof of Eq. 2.4 is known and fairly elementary. Denoting by ξ the dual variable of z, and letting $\hat{u}(\xi,t)=\int_{\mathbb{R}^N}e^{-2\pi i\langle\xi,z\rangle}u(z,t)dz$, then on the Fourier transform side the Cauchy problem reduces to solving

$$\begin{cases}\partial_t\hat{u}+\langle B^\star\xi,\nabla_\xi\hat{u}\rangle+\left(4\pi^2\langle Q\xi,\xi\rangle+\operatorname{tr}B\right)\hat{u}=0 & \text{in } \mathbb{R}^N\times(0,\infty),\\ \hat{u}(\xi,0)=\hat{f}(\xi).\end{cases} \tag{2.7}$$

Now Eq. 2.7 can be easily solved via the method of characteristics. Fixing $(\xi,t)\in\mathbb{R}^N\times(0,\infty)$, one considers $g(s)=\hat{u}(e^{sB^\star}\xi,s+t)$. This function, in turn, solves

$$g'(s)+\left(4\pi^2\langle e^{sB}Qe^{sB^\star}\xi,\xi\rangle+\operatorname{tr}B\right)g(s)=0,\qquad g(0)=\hat{u}(\xi,t).$$

Recalling (2.2), we see that $g(s)$ is given by

$$g(s)=\hat{u}(\xi,t)e^{-4\pi^2s\langle K(s)\xi,\xi\rangle}e^{-s\operatorname{tr}B}.$$

Since $g(-t)=\hat{f}(e^{-tB^\star}\xi)$ and $K(-t)=e^{-tB}K(t)e^{-tB^*}$, this implies the remarkable formula

$$\hat{u}(\xi,t)=e^{-t\operatorname{tr}B}e^{-4t\pi^2\langle K(t)e^{-tB^*}\xi,e^{-tB^*}\xi\rangle}\hat{f}(e^{-tB^\star}\xi). \tag{2.8}$$

The representation formula Eq. 2.6 follows from Eq. 2.8 by taking the inverse Fourier transform and straightforward manipulations.

It is proved in [17, Theorem 1.4] that, if $|f(z)|\le Ce^{c|z|^2}$ for some positive constants c,C, then the function $u=P_tf$ is solution to $\mathscr{K}u=0$ in a suitable strip $\mathbb{R}^N\times(0,T)$ and it attains the initial datum f. We need to prove that, given f satisfying Eq. 2.5, the function $u=P_tf$ is well defined for every $t>0$ and it defines in fact a solution of the Cauchy problem in the whole $\mathbb{R}^N\times(0,\infty)$. To see this, for any $(z,t)\in\mathbb{R}^N\times(0,\infty)$ we write

$$\begin{aligned}u(z,t)&=\int_{\mathbb{R}^N}\frac{(4\pi)^{-N/2}}{(\det(tK(t)))^{1/2}}\exp\left(-\frac{m_t(z,\zeta)^2}{4t}\right)f(\zeta)d\zeta\\&=\exp^{-\frac{1}{4}\left\langle(tK(t))^{-1}e^{tB}z,e^{tB}z\right\rangle}\int_{\mathbb{R}^N}\frac{(4\pi)^{-N/2}}{(\det(tK(t)))^{1/2}}\exp^{\frac{1}{2}\left\langle(tK(t))^{-1}\zeta,e^{tB}z\right\rangle}\exp^{-\frac{1}{4}\left\langle(tK(t))^{-1}\zeta,\zeta\right\rangle}f(\zeta)d\zeta.\end{aligned}$$

Exploiting (2.5) we thus infer

$$|u(z,t)| \le \sup_{\zeta} \left| e^{-\frac{1}{4}\langle K_\infty^{-1}\zeta,\zeta\rangle} f(\zeta) \right| \exp^{-\frac{1}{4}\left\langle (tK(t))^{-1}e^{tB}z, e^{tB}z \right\rangle} \times \tag{2.9}$$
$$\times \int_{\mathbb{R}^N} \frac{(4\pi)^{-N/2}}{(\det(tK(t)))^{1/2}} \exp^{\frac{1}{2}\left\langle (tK(t))^{-1}\zeta, e^{tB}z \right\rangle} \exp^{-\frac{1}{4}\left\langle (tK(t))^{-1}\zeta,\zeta \right\rangle} e^{\frac{1}{4}\langle K_\infty^{-1}\zeta,\zeta\rangle} d\zeta.$$

Property (2.3) ensures, for every fixed $t > 0$, the existence of $\mu_t > 0$ such that

$$\left\langle \left((tK(t))^{-1} - K_\infty^{-1} \right) \eta, \eta \right\rangle \ge \mu_t |\eta|^2 \quad \forall \eta \in \mathbb{R}^N.$$

Inserting this information in Eq. 2.9 we deduce that

$$\begin{aligned} |u(z,t)| &\le \sup_{\zeta} \left| e^{-\frac{1}{4}\langle K_\infty^{-1}\zeta,\zeta\rangle} f(\zeta) \right| \exp^{-\frac{1}{4}\left\langle (tK(t))^{-1}e^{tB}z, e^{tB}z \right\rangle} \times \\ &\quad \times \int_{\mathbb{R}^N} \frac{(4\pi)^{-N/2}}{(\det(tK(t)))^{1/2}} \exp^{\frac{1}{2}\left\langle (tK(t))^{-1}\zeta, e^{tB}z \right\rangle} \exp^{-\frac{1}{4}\mu_t|\zeta|^2} d\zeta \\ &= \sup_{\zeta} \left| e^{-\frac{1}{4}\langle K_\infty^{-1}\zeta,\zeta\rangle} f(\zeta) \right| \exp^{\frac{1}{4}\left\langle \left(\frac{1}{\mu_t}(tK(t))^{-1} - I_N\right)(tK(t))^{-1}e^{tB}z, e^{tB}z \right\rangle} \times \\ &\quad \times \int_{\mathbb{R}^N} \frac{(4\pi)^{-N/2}}{(\det(tK(t)))^{1/2}} \exp^{-\frac{1}{4}\left| \sqrt{\mu_t}\zeta - \frac{1}{\sqrt{\mu_t}}(tK(t))^{-1}e^{tB}z \right|^2} d\zeta < \infty. \end{aligned}$$

By arguing in a similar way one can compute the derivatives of u at any point $(z,t) \in \mathbb{R}^N \times (0,\infty)$ by exchanging the order of derivation and integration: this shows that u is solution and completes the proof of the theorem. □

It is easy to see that for the heat equation ($Q = I_N$ and $B = O_N$) the matrix $tK(t) = tI_N$ and therefore $K_\infty^{-1} = O_N$. The matrix K_∞^{-1} is the null matrix also for the Kolmogorov equation ($Q = \begin{pmatrix} I_n & O_n \\ O_n & O_n \end{pmatrix}$ and $B = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$) since $tK(t) = \begin{pmatrix} tI_n & \frac{t^2}{2}I_n \\ \frac{t^2}{2}I_n & \frac{t^3}{3}I_n \end{pmatrix}$ and thus $(tK(t))^{-1} = \begin{pmatrix} \frac{4}{t}I_n & -\frac{6}{t^2}I_n \\ -\frac{6}{t^2}I_n & \frac{12}{t^3}I_n \end{pmatrix}$. Instead, for the Ornstein-Uhlenbeck equation ($Q = I_N$ and $B = -I_N$) the matrix $tK(t) = \frac{1}{2}(1 - e^{-2t})I_N$ and therefore $K_\infty^{-1} = 2I_N$. In the following two examples we discuss two situations that will be useful in the remainder of the present work. Henceforth in this paper we indicate by $j : \mathbb{R} \to \mathbb{R}$ the real analytic function defined by

$$j(\tau) = \frac{\tau}{\sinh(\tau)}. \tag{2.10}$$

Given a $N \times N$ matrix C with real coefficients, the notation $j(C)$ will denote the matrix identified by the power series of the function j. It is worth noting that $j(C)$ is invertible with inverse matrix given by $j(C)^{-1} = \sum_{k=0}^{\infty} \frac{C^{2k}}{(2k+1)!}$. Similar interpretation for the matrix $\cosh C$.

Example 2.2 (Ornstein-Uhlenbeck with a possibly degenerate drift) Let D be a real $N \times N$ matrix, that is symmetric and nonnegative definite, and consider the operator obtained by Eq. 2.1 with the choice

$$Q = I_N \qquad \text{and} \qquad B = -2D.$$

Then

$$K_\infty^{-1} = 4D. \tag{2.11}$$

Proof With Q and B given above we have $tK(t) = \int_0^t e^{-4sD} ds$. Keeping in mind that $e^\tau = \sum_{k=0}^\infty \frac{\tau^k}{k!}$ and $\sinh(\tau) = \sum_{k=0}^\infty \frac{\tau^{2k+1}}{(2k+1)!}$, we now make the following observation

$$e^{tD}\,(tK(t))\,e^{tD} = \int_0^t e^{(2t-4s)D} ds = \frac{1}{4}\int_{-2t}^{2t} e^{\tau D} d\tau = \frac{1}{4}\sum_{k=0}^\infty \frac{D^{2k}}{(2k)!}\int_{-2t}^{2t} \tau^{2k} d\tau$$
$$= t\sum_{k=0}^\infty \frac{(2tD)^{2k}}{(2k+1)!} = t\,(j(2tD))^{-1},$$

where j is as in Eq. 2.10. Since the previous identity can be rewritten as follows

$$tK(t) = te^{-tD}\,(j(2tD))^{-1}\,e^{-tD},$$

we then obtain

$$(tK(t))^{-1} = \frac{1}{t} e^{tD} j(2tD) e^{tD}. \tag{2.12}$$

We notice that $(tK(t))^{-1}$ and D diagonalize simultaneously, and the function $\frac{e^{2\mu t}}{t} j(2\mu t)$ converges to 4μ as $t \to \infty$ for any $\mu \geq 0$. Then, from Eq. 2.12 we obtain

$$K_\infty^{-1} = \lim_{t\to\infty} (tK(t))^{-1} = 4D,$$

which proves Eq. 2.11. □

The Smoluchowski-Kramers equation ($Q = \begin{pmatrix} I_n & O_n \\ O_n & O_n \end{pmatrix}$ and $B = \begin{pmatrix} -I_n & O_n \\ I_n & O_n \end{pmatrix}$) mentioned in the opening of the section falls within the class considered in the following example.

Example 2.3 (Degenerate Ornstein-Uhlenbeck) Let $\mathbb{N} = n + n_1$ with $n, n_1 \in \mathbb{N}$ and $n_1 \leq n$. Consider a symmetric and positive definite $n \times n$ matrix D_1 and a $n_1 \times n$ matrix B_0 with rank n_1. For the operator $\mathscr{K}$ in Eq. 2.1 corresponding to the choice

$$Q = \begin{pmatrix} I_n & O_{n\times n_1} \\ O_{n_1\times n} & O_{n_1\times n_1} \end{pmatrix}, \qquad B = \begin{pmatrix} -2D_1 & O_{n\times n_1} \\ B_0 & O_{n_1\times n_1} \end{pmatrix},$$

one has

$$K_\infty^{-1} = \begin{pmatrix} 4D_1 & O_{n\times n_1} \\ O_{n_1\times n} & O_{n_1\times n_1} \end{pmatrix}. \tag{2.13}$$

Proof As a first step we observe that with Q and B given above, where the lower indices indicate the dimensions of the various zero matrices, the Kalman condition Eq. 2.2 is satisfied. Since the kernel of Q is n_1-dimensional, the operator $\operatorname{tr}(Q\nabla^2) + \langle Bz, \nabla\rangle$ is degenerate-elliptic. Denoting $V_i = \partial_{z_i}$ for $i \in \{1, \ldots, n\}$ and $V_0 = \langle Bz, \nabla\rangle$ one has

$$\operatorname{tr}(Q\nabla^2) + \langle Bz, \nabla\rangle = \sum_{i=1}^n V_i^2 + V_0.$$

Moreover, the commutator between V_i and V_0 is given by

$$[V_i, V_0] = \left(B^*\nabla\right)_i = -2\sum_{j=1}^n (D_1)_{ij}\,\partial_{z_j} + \sum_{j=1}^{n_1} (B_0)_{ji}\,\partial_{z_{n+j}}.$$

From this relation and the fact that $\mathrm{Im}(B_0) = \mathbb{R}^{n_1}$ we deduce that the vector fields $V_0, V_1, ...V_n$ satisfy Hörmander's finite rank condition on the Lie algebra in [30] and therefore the operator $\mathscr{K}$ is hypoelliptic. As we have recalled, this is equivalent to saying that Eq. 2.2 hold.

Next, we compute

$$e^{tB} = \begin{pmatrix} e^{-2tD_1} & O_{n\times n_1} \\ \frac{1}{2}B_0 D_1^{-1}\left(I_n - e^{-2tD_1}\right) & I_{n_1} \end{pmatrix},$$

and

$$tK(t) = \begin{pmatrix} K_{11}(t) & K_{12}(t) \\ K_{12}^*(t) & K_{22}(t) \end{pmatrix},$$

where

$$\begin{aligned} K_{11}(t) &= \frac{1}{4}D_1^{-1}\left(I_n - e^{-4tD_1}\right) \\ K_{12}(t) &= \frac{1}{8}D_1^{-2}\left(I_n - e^{-2tD_1}\right)^2 B_0^* \\ K_{22}(t) &= \frac{1}{4}B_0 D_1^{-2}\left(tI_n + \frac{1}{4}D_1^{-1}\left(-3I_n + 4e^{-2tD_1} - e^{-4tD_1}\right)\right)B_0^*. \end{aligned}$$

By means of known formulas for the inverse of a partitioned matrix (see, e.g., [31, Section 0.7.3]), we obtain

$$(tK(t))^{-1} = \begin{pmatrix} \left(K_{11}(t) - K_{12}(t)K_{22}^{-1}(t)K_{12}^*(t)\right)^{-1} & O_{n\times n_1} \\ O_{n_1\times n} & \left(K_{22}(t) - K_{12}^*(t)K_{11}^{-1}(t)K_{12}(t)\right)^{-1} \end{pmatrix} \times$$
$$\times \begin{pmatrix} I_n & -K_{12}(t)K_{22}^{-1}(t) \\ -K_{12}^*(t)K_{11}^{-1}(t) & I_{n_1} \end{pmatrix}.$$

We now notice that, as $t \to \infty$, we have the limiting relations

$$K_{11}^{-1}(t) \to 4D_1, \quad K_{22}(t) = \frac{t}{4}B_0 D_1^{-2} B_0^* + O(1), \quad K_{12}(t) = O(1).$$

Since $B_0 D_1^{-2} B_0^*$ is invertible, we conclude that

$$K_\infty^{-1} = \lim_{t\to\infty}(tK(t))^{-1} = \begin{pmatrix} 4D_1 & O_{n\times n_1} \\ O_{n_1\times n} & O_{n_1\times n_1} \end{pmatrix},$$

which establishes Eq. 2.13. □

3 Baouendi met Kolmogorov

In this section we discuss a first interesting class of hybrid evolution equations which, remarkably, is directly amenable to the setting of Proposition 2.1 by means of partial Fourier transform and a suitable exponential transformation, see Eq. 3.30 below. The reader should bear in mind that the work in this section is also instrumental to the remainder of the paper. Consider a symmetric and positive definite $n \times n$ matrix Q_1. Let $n_1 \leq n$, and fix also a $n_1 \times n$ matrix B_0 having maximum rank n_1. We denote the relevant variables

$w \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^k$, $y \in \mathbb{R}^{n_1}$ and $t \in (0, \infty)$. Our objective is to solve the Cauchy problem in $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{n_1} \times (0, \infty)$,

$$\begin{cases} \Delta_w f + \frac{1}{4} \langle Q_1 w, w \rangle \Delta_\sigma f + \langle B_0 w, \nabla_y f \rangle - \partial_t f = 0, \\ f(w, \sigma, y, 0) = f_0(w, \sigma, y), \end{cases} \tag{3.1}$$

where f_0 is a suitably assigned function in $\mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{n_1}$. We stress that, in our terminology, the partial differential operator $\mathscr{L} - \partial_t$ in Eq. 3.1 is hybrid since $\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_2$, where $\mathscr{L}_1 f = \frac{1}{2}\Delta_w f + \frac{1}{4} \langle Q_1 w, w \rangle \Delta_\sigma f$ and $\mathscr{L}_2 f = \frac{1}{2}\Delta_w f + \langle B_0 w, \nabla_y f \rangle$, and the term $\Delta_w f$ appears in both $\mathscr{L}_1$ and $\mathscr{L}_2$. Before proceeding we emphasise that the hybrid PDE in Eq. 3.1 encompasses equations as diverse as the parabolic Baouendi-Grushin equation in $\mathbb{R}^n \times \mathbb{R}^k \times (0, \infty)$

$$\Delta_w f + \frac{|w|^2}{4} \Delta_\sigma f - \partial_t f = 0, \tag{3.2}$$

see [1, 28, 29], and the already mentioned degenerate Kolmogorov equation in $\mathbb{R}^{2n} \times (0, \infty)$,

$$\Delta_w f + \langle w, \nabla_y f \rangle - \partial_t f = 0, \tag{3.3}$$

see [35]. The former of these two limiting cases is obtained by taking $n_1 = 0$ and $Q_1 = I_n$ in Eq. 3.1, whereas the latter corresponds to taking $k = 0$, $n = n_1$, and $B_0 = I_n$ in Eq. 3.1. To ease the reader's understanding we first discuss in detail our approach to constructing the fundamental solution of Eq. 3.2 since this allows to present some of the ideas in a significant, yet simpler model. This will be accomplished in the next two Sections 3.1 and 3.2, the former of which contains a self-contained construction of the Mehler fundamental solution for the generalised harmonic oscillator in Eq. 3.4 below by reducing such operator to a special case of Proposition 2.1. We mention that when the matrix D is a multiple of the identity such fundamental solution is well-known and we could have simply lifted its expression from the literature, see Remark 3.3 below. In line with the declared self-contained spirit of the present paper, our objective is to show that Proposition 3.2 below can be derived from Proposition 2.1 by elementary considerations.

3.1 The Harmonic Oscillator aka Ornstein-Uhlenbeck

In what follows given a number $n \in \mathbb{N}$ we denote by $D \in M_{n\times n}(\mathbb{R})$ a matrix such that $D = D^\star$, $D \geq 0$. We consider the Cauchy problem for the generalised harmonic oscillator

$$\begin{cases} \Delta_z v - |Dz|^2 v - \partial_t v = 0, \\ v(z, 0) = v_0(z) \qquad z \in \mathbb{R}^n, \ t > 0, \end{cases} \tag{3.4}$$

where v_0 is suitably chosen. We have the following key lemma.

Lemma 3.1 *Suppose that the functions v and w are connected by the transformation*

$$v(z, t) = e^{-(\frac{1}{2}\langle Dz, z\rangle + t \operatorname{tr} D)} w(z, t). \tag{3.5}$$

Then, v is a solution to the PDE in Eq. 3.4 *if and only if w is a solution to the following equation of Ornstein-Uhlenbeck type*

$$\Delta w - 2\langle Dz, \nabla w \rangle - w_t = 0.$$

Proof We compute

$$\nabla(e^{-(\frac{1}{2}\langle Dz,z\rangle+t\,\mathrm{tr}\,D)}) = -e^{-(\frac{1}{2}\langle Dz,z\rangle+t\,\mathrm{tr}\,D)})Dz,$$
$$\Delta(e^{-(\frac{1}{2}\langle Dz,z\rangle+t\,\mathrm{tr}\,D)}) = (|Dz|^2 - \mathrm{tr}\,D)e^{-(\frac{1}{2}\langle Dz,z\rangle+t\,\mathrm{tr}\,D)},$$
$$\partial_t(e^{-(\frac{1}{2}\langle Dz,z\rangle+t\,\mathrm{tr}\,D)}) = -e^{-(\frac{1}{2}\langle Dz,z\rangle+t\,\mathrm{tr}\,D)}\,\mathrm{tr}\,D.$$

This gives

$$\Delta v - |Dz|^2 v - \partial_t v = e^{-(\frac{1}{2}\langle Dz,z\rangle+t\,\mathrm{tr}\,D)}(\Delta w - 2\langle Dz, \nabla w\rangle - w_t). \tag{3.6}$$

The Eq. 3.6 proves the lemma. □

The next proposition is the main result of this subsection.

Proposition 3.2 (generalised Mehler formula) *Let $\mathscr{M}$ be given by the following formula*

$$\mathscr{M}(z,\zeta,t) = (4\pi t)^{-\frac{n}{2}}\sqrt{\det j(2tD)} \tag{3.7}$$
$$\times \exp\left\{-\frac{1}{4t}\Big(\langle j(2tD)\cosh 2tD\, z, z\rangle + \langle j(2tD)\cosh 2tD\,\zeta,\zeta\rangle - 2\langle j(2tD)\, z,\zeta\rangle\Big)\right\},$$

with j as in Eq. 2.10. *Then, for any $v_0 \in C(\mathbb{R}^n)$ such that*

$$e^{-\frac{1}{2}\langle Dz,z\rangle}v_0(z) \in L^\infty\left(\mathbb{R}^n\right), \tag{3.8}$$

the function

$$v(z,t) = \int_{\mathbb{R}^n}\mathscr{M}(z,\zeta,t)v_0(\zeta)d\zeta$$

is solution of Eq. 3.4.

Proof In view of Lemma 3.1 we see that if v solves the Cauchy problem Eq. 3.4, then $w = e^{\frac{1}{2}\langle Dz,z\rangle+\mathrm{tr}\,D\,t}v$ solves the Cauchy problem

$$\begin{cases}\Delta w - 2\langle Dz,\nabla w\rangle - w_t = 0,\\ w(z,0) = e^{\frac{1}{2}\langle Dz,z\rangle}v_0(z).\end{cases} \tag{3.9}$$

To solve Eq. 3.9 we intend to apply Proposition 2.1 with $N = n$, $Q = I_n$ and $B = -2D$. Keeping Example 2.2 in mind, we know from Eqs. 2.12 and 2.11 that with this choice we have

$$K(t)^{-1} = e^{tD}j(2tD)e^{tD} \tag{3.10}$$

and

$$K_\infty^{-1} = 4D.$$

Therefore, the initial datum in Eq. 3.9 is equal to

$$w(z,0) = e^{\frac{1}{2}\langle Dz,z\rangle}v_0(z) = e^{\frac{1}{4}\langle K_\infty^{-1}z,z\rangle}\left(e^{-\frac{1}{2}\langle Dz,z\rangle}v_0(z)\right)$$

and, since v_0 is continuous and satisfies Eq. 3.8, it nicely fits the assumption of Proposition 2.1. According to Eqs. 2.4-2.6 the solution of Eq. 3.9 is thus given by

$$w(z,t) = \int_{\mathbb{R}^m} p(z,\zeta,t)e^{\frac{1}{2}\langle D\zeta,\zeta\rangle}v_0(\zeta)d\zeta,$$

where we have let

$$p(z,\zeta,t) = (4\pi t)^{-\frac{n}{2}}(\det K(t))^{-1/2}e^{-\frac{1}{4t}\langle (K(t))^{-1}(\zeta - e^{-2tD}z),(\zeta - e^{-2tD}z)\rangle}. \tag{3.11}$$

By Eq. 3.10 we have in particular that

$$(\det K(t))^{-1/2} = e^{t\,\mathrm{tr}\,D}\sqrt{\det j(2tD)}. \tag{3.12}$$

Using Eqs. 3.10 and 3.12 in Eq. 3.11, we find

$$p(z,\zeta,t) \,=\, (4\pi t)^{-\frac{n}{2}} e^{t\,\mathrm{tr}\,D}\sqrt{\det j(2tD)}e^{-\frac{1}{4t}\langle j(2tD)(e^{tD}\zeta-e^{-tD}z),e^{tD}\zeta-e^{-tD}z\rangle}. \tag{3.13}$$

From Eqs. 3.13 and 3.5 we now see that the solution of Eq. 3.4 is given by

$$v(z,t) = \int_{\mathbb{R}^n} \mathscr{M}(z,\zeta,t)v_0(\zeta)d\zeta,$$

where

$$\begin{aligned}\mathscr{M}(z,\zeta,t) \,&=\, (4\pi t)^{-\frac{n}{2}}\sqrt{\det j(2tD)}e^{-\frac{1}{4t}\langle 2tDz,z\rangle}e^{\frac{1}{4t}\langle 2tD\zeta,\zeta\rangle} \\ &\quad\times e^{-\frac{1}{4t}\langle j(2tD)(e^{tD}\zeta-e^{-tD}z),e^{tD}\zeta-e^{-tD}z\rangle}.\end{aligned} \tag{3.14}$$

Using the tautological identity

$$2tD = j(2tD)\sinh 2tD = j(2tD)\frac{e^{2tD}-e^{-2tD}}{2},$$

and the fact that

$$e^{\pm tD}j(2tD) = j(2tD)e^{\pm tD},$$

we can now write the argument in the exponentials in Eq. 3.14 as

$$\begin{aligned}&\langle 2tDz,z\rangle - \langle 2tD\zeta,\zeta\rangle + \langle j(2tD)(e^{tD}\zeta - e^{-tD}z), e^{tD}\zeta - e^{-tD}z\rangle \\ =\;& \langle j(2tD)\sinh 2tDz,z\rangle - \langle j(2tD)\sinh 2tD\zeta,\zeta\rangle + \langle j(2tD)e^{-tD}z, e^{-tD}z\rangle \\ &+\langle j(2tD)e^{tD}\zeta, e^{tD}\zeta\rangle - \langle j(2tD)e^{-tD}z, e^{tD}\zeta\rangle - \langle j(2tD)e^{tD}\zeta, e^{-tD}z\rangle \\ =\;& \langle j(2tD)\sinh 2tDz,z\rangle - \langle j(2tD)\sinh 2tD\zeta,\zeta\rangle + \langle j(2tD)e^{-2tD}z,z\rangle \\ &+\langle j(2tD)e^{2tD}\zeta,\zeta\rangle - \langle j(2tD)z,\zeta\rangle - \langle j(2tD)\zeta,z\rangle \\ =\;& \langle j(2tD)\cosh 2tD\; z,z\rangle + \langle j(2tD)\cosh 2tD\;\zeta,\zeta\rangle - 2\langle j(2tD)\; z,\zeta\rangle.\end{aligned}$$

This shows Eq. 3.7. We have finally proved Proposition 3.2. □

In what follows we will use the following alternative expression of Eq. 3.7

$$\begin{aligned}\mathscr{M}(z,\zeta,t) \,&=\, (4\pi t)^{-\frac{n}{2}}\sqrt{\det j(2tD)}\,\times \\ &\quad\times\exp\left\{-\frac{1}{4t}\left(\left|\sqrt{j(2tD)\cosh 2tD}\;\zeta - \sqrt{j(2tD)\cosh^{-1}2tD}\; z\right|^2\right.\right. \\ &\quad\left.\left.+\langle j(2tD)\left(\cosh 2tD - \cosh^{-1}2tD\right)\; z,z\rangle\right)\right\}.\end{aligned} \tag{3.15}$$

In particular, by performing the change of variable

$$\zeta \mapsto \eta = \sqrt{j(2tD)\cosh 2tD}\;\frac{\zeta}{\sqrt{4t}} - \sqrt{j(2tD)\cosh^{-1}2tD}\;\frac{z}{\sqrt{4t}},$$

from Eq. 3.15 it is immediate to recognise that

$$\int_{\mathbb{R}^n}\mathscr{M}(z,\zeta,t)d\zeta \,=\, \frac{1}{\sqrt{\det\cosh 2tD}}\exp\left\{-\frac{1}{4t}\langle j(2tD)\left(\cosh 2tD - \cosh^{-1}2tD\right)\; z,z\rangle\right\}. \tag{3.16}$$

Formula Eq. 3.16 will be useful in the proof of Theorem 3.4 below.

Remark 3.3 The reader may find it interesting to compare Eq. 3.7 with the classical 1866 Mehler formula for the harmonic oscillator $\Delta u - \omega|x|^2 u - u_t = 0$, see e.g. [7, Section 4.2],

$$\mathscr{M}(x, y, t) = (4\pi t)^{-n/2} \left(\frac{2\sqrt{\omega}t}{\sinh 2\sqrt{\omega}t}\right)^{n/2} e^{-\frac{\sqrt{\omega}}{2}(\operatorname{cotanh} 2\sqrt{\omega}t\,(|x|^2+|y|^2)-2\operatorname{csch} 2\sqrt{\omega}t\,\langle x,y\rangle)}.$$

This formula follows immediately from Eq. 3.7 by taking $D = \sqrt{\omega} I_n$ in its expression.

3.2 The Heat Kernel of the Baouendi-Grushin Operator

In his 1967 Ph.D. Dissertation [1] under the supervision of B. Malgrange, S. Baouendi first studied the Dirichlet problem in L^2 for a class of degenerate elliptic operators that includes the following model

$$\Delta_w + \frac{|w|^2}{4}\Delta_\sigma, \tag{3.17}$$

where $(w, \sigma) \in \mathbb{R}^n \times \mathbb{R}^k$. At that time M. Vishik was visiting Malgrange, who discussed with him the thesis project of Baouendi. Vishik subsequently asked Malgrange permission to suggest to his own Ph.D. student, Grushin, to work on some questions related to the hypoellipticity of operators modelled on Eq. 3.17, see [28, 29]. This is how the operator Eq. 3.17 became known as the *Baouendi-Grushin operator*. This operator is also important since it is connected to harmonic functions with special symmetries in a group of Heisenberg type $\mathbb{G}$. We notice, in this respect, that there is no global group law underlying Eq. 3.17, but the operator is invariant with respect to standard translations $(w, \sigma) \to (w, \sigma + \sigma')$ along the manifold of degeneracy $M = \{0\} \times \mathbb{R}^k$. We consider the Cauchy problem

$$\begin{cases} \Delta_w u + \frac{|w|^2}{4}\Delta_\sigma u - \partial_t u = 0 & \text{in } \mathbb{R}^{n+k} \times (0, \infty), \\ u((w, \sigma), 0) = f(w, \sigma). \end{cases} \tag{3.18}$$

The next result provides an explicit heat kernel for Eq. 3.17.

Theorem 3.4 *Let*

$$\begin{aligned} \mathscr{B}((w, \sigma), (w', \sigma'), t) = \frac{2^k}{(4\pi t)^{\frac{n}{2}+k}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle \lambda, \sigma'-\sigma\rangle} \left(\frac{|\lambda|}{\sinh |\lambda|}\right)^{\frac{n}{2}} \\ \times e^{-\frac{|\lambda|}{4t \tanh |\lambda|}((|w|^2+|w'|^2)-2\langle w, w'\rangle \operatorname{sech} |\lambda|)} d\lambda. \end{aligned} \tag{3.19}$$

Then for every $f \in \mathscr{S}(\mathbb{R}^{n+k})$ *the function*

$$u((w, \sigma), t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \mathscr{B}((w, \sigma), (w', \sigma'), t) f(w', \sigma') dw' d\sigma'$$

is a solution of Eq. 3.18.

Proof We indicate with $\hat{u}(w, \lambda, t) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \lambda, \sigma\rangle} u(w, \sigma, t) d\sigma$ the partial Fourier transform of u with respect to the variable $\sigma \in \mathbb{R}^k$, with dual variable $\lambda \in \mathbb{R}^k$. Applying such Fourier transform to Eq. 3.18, for any fixed $\lambda \in \mathbb{R}^k$ we obtain

$$\begin{cases} \Delta_w \hat{u} - \pi^2|\lambda|^2|w|^2\hat{u} - \partial_t \hat{u} = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \hat{u}((w, \lambda), 0) = \hat{f}(w, \lambda). \end{cases}$$

This is a Cauchy problem for a harmonic oscillator such as Eq. 3.4 above, with matrix $D = D(\lambda) = \pi|\lambda| I_n$. From Proposition 3.2 we know that the solution of such problem is given by the formula

$$\begin{aligned}\hat{u}((w,\lambda),t) &= \int_{\mathbb{R}^n} \mathscr{M}_\lambda(w,w',t)\hat{f}(w',\lambda)dw' \\ &= \int_{\mathbb{R}^k} e^{-2\pi i\langle\lambda,\sigma'\rangle}\int_{\mathbb{R}^n} \mathscr{M}_\lambda(w,w',t)f(w',\sigma')dw'd\sigma',\end{aligned}$$

where $\mathscr{M}_\lambda(w,w',t)$ is Mehler's fundamental solution given by

$$\mathscr{M}_\lambda(w,w',t) = (4\pi)^{-n/2}\left(\frac{2\pi|\lambda|}{\sinh 2\pi t|\lambda|}\right)^{n/2} e^{-\frac{\pi|\lambda|}{2}\left((|w|^2+|w'|^2)\operatorname{cotanh}(2\pi t|\lambda|)-2\langle w,w'\rangle \operatorname{csch}(2\pi t|\lambda|)\right)}, \tag{3.20}$$

see Remark 3.3. We know that $\hat{u}((w,\lambda),t) \underset{t\to 0^+}{\longrightarrow} \hat{f}(w,\lambda)$ in the pointwise sense. We will now show that, for every fixed $w \in \mathbb{R}^n$, such convergence also holds in $L^1(\mathbb{R}^k, d\lambda)$. To see this we write

$$\begin{aligned}&\int_{\mathbb{R}^k} |\hat{u}((w,\lambda),t) - \hat{f}(w,\lambda)|d\lambda \\ &= \int_{\mathbb{R}^k}\left|\hat{u}((w,\lambda),t) - \hat{f}(w,\lambda)\left(\int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)dw' + 1 - \int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)dw'\right)\right|d\lambda \\ &\le \int_{\mathbb{R}^k}\int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)\left|\hat{f}(w',\lambda)-\hat{f}(w,\lambda)\right|dw'd\lambda + \int_{\mathbb{R}^k}\left|1-\int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)dw'\right|\left|\hat{f}(w,\lambda)\right|d\lambda.\end{aligned}$$

Applying Eq. 3.16 with $D = \pi|\lambda| I_n$ we easily obtain

$$\int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)dw' = \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{n}{2}} e^{-\frac{|w|^2}{4t}\frac{2\pi t|\lambda|}{\tanh 2\pi t|\lambda|}\left(1-\operatorname{sech}^2 2\pi t|\lambda|\right)}.$$

From this identity we immediately see that

(i) $0 \le \int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)dw' \le 1$;
(ii) $\int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)dw' \to 1$ as $t \to 0^+$.

From (i) and (ii) we infer by dominated convergence theorem that

$$\int_{\mathbb{R}^k}\left|1-\int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)dw'\right|\left|\hat{f}(w,\lambda)\right|d\lambda \underset{t\to 0^+}{\longrightarrow} 0.$$

On the other hand, by applying Eq. 3.15 with $D = \pi|\lambda| I_n$ and performing the change of variables $w' \mapsto \eta$ with $w' - w\operatorname{sech} 2\pi t|\lambda| = \eta\sqrt{4t\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}$, we deduce

$$\begin{aligned}&\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}\mathscr{M}_\lambda(w,w',t)\left|\hat{f}(w',\lambda)-\hat{f}(w,\lambda)\right|dw'd\lambda \\ &= \pi^{-\frac{n}{2}}\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}\left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{n}{2}} e^{-\frac{|w|^2}{4t}\frac{2\pi t|\lambda|}{\tanh 2\pi t|\lambda|}\left(1-\operatorname{sech}^2 2\pi t|\lambda|\right)} e^{-|\eta|^2} \times \\ &\times\left|\hat{f}\left(w\operatorname{sech} 2\pi t|\lambda| + \eta\sqrt{4t\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}},\lambda\right) - \hat{f}(w,\lambda)\right|d\eta d\lambda \\ &\le \pi^{-\frac{n}{2}}\int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{-|\eta|^2}\left|\hat{f}\left(w\operatorname{sech} 2\pi t|\lambda|+\eta\sqrt{4t\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}},\lambda\right)-\hat{f}(w,\lambda)\right|d\eta d\lambda \underset{t\to 0^+}{\longrightarrow} 0,\end{aligned}$$

where the last limiting relation can be again justified via dominated convergence theorem since $\hat{f} \in \mathscr{S}$ and therefore $\hat{f}(w', \lambda)$ is continuous at (w, λ) and it can be bounded by an integrable function uniformly in w'. This proves that

$$\lim_{t \to 0^+} \int_{\mathbb{R}^k} |\hat{u}((w, \lambda), t) - \hat{f}(w, \lambda)| d\lambda = 0 \qquad \forall\, w \in \mathbb{R}^n. \tag{3.21}$$

If we now take the inverse Fourier transform of $\hat{u}((w, \lambda), t)$, we find the following representation for the solution of problem Eq. 3.18

$$u((w, \sigma), t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^k} e^{-2\pi i \langle \lambda, \sigma' - \sigma \rangle} \mathscr{M}_\lambda(w, w', t) d\lambda \right) f(w', \sigma') dw' d\sigma'. \tag{3.22}$$

We stress that Eq. 3.21 ensures uniform in $\sigma \in \mathbb{R}^k$, and therefore pointwise convergence of $u((w, \sigma), t)$ to $f(w, \sigma)$ as $t \to 0^+$. From Eq. 3.22 it is clear that the heat kernel of the parabolic Baouendi-Grushin equation in Eq. 3.18 is thus given by

$$\mathscr{B}((w, \sigma), (w', \sigma'), t) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \lambda, \sigma' - \sigma \rangle} \mathscr{M}_\lambda(w, w', t) d\lambda, \tag{3.23}$$

where $\mathscr{M}_\lambda(w, w', t)$ is as in Eq. 3.20. Changing variable $\lambda \to 2\pi t \lambda$ in the integral over $\mathbb{R}^k$, we finally obtain Eqs. 3.19 from 3.23. □

We mention the works [2, 12, 13] for various derivations and integral representations of the heat kernel for Eq. 3.17 when $k = 1$. For related discussions about the fundamental solutions of more general (time-independent) Baouendi-Grushin operators we refer the reader to [3, 5, 20].

3.3 Back to the Hybrid Equation

In this subsection we finally solve the Cauchy problem Eq. 3.1. In preparation for our main result, Theorem 3.6 below, we introduce for every $\lambda \in \mathbb{R}^k$ the $(n + n_1) \times (n + n_1)$ matrices

$$Q = \begin{pmatrix} I_n & O_{n \times n_1} \\ O_{n_1 \times n} & O_{n_1 \times n_1} \end{pmatrix}, \qquad B_\lambda = \begin{pmatrix} -2\pi |\lambda| \sqrt{Q_1} & O_{n \times n_1} \\ B_0 & O_{n_1 \times n_1} \end{pmatrix}. \tag{3.24}$$

For every $t > 0$ we next consider the covariance matrix associated with Q and B_λ

$$t K_\lambda(t) \overset{def}{=} \int_0^t e^{s B_\lambda} Q e^{s B_\lambda^*} ds. \tag{3.25}$$

If we keep in mind Example 2.3 with the choice $D_1 = \pi |\lambda| \sqrt{Q_1}$, we know that the matrix $K_\lambda(t)$ is positive definite for every $t > 0$, i.e. it satisfies the Kalman condition Eq. 2.2 above. Moreover, by Eq. 2.13 we have

$$K_{\lambda,\infty}^{-1} = \begin{pmatrix} 4\pi |\lambda| \sqrt{Q_1} & O_{n \times n_1} \\ O_{n_1 \times n} & O_{n_1 \times n_1} \end{pmatrix}. \tag{3.26}$$

We next establish a lemma that will play a crucial role in the proof of Theorem 3.6.

Lemma 3.5 *For any $t > 0$ and $\lambda \in \mathbb{R}^k$ we have*

$$(t K_\lambda(t))^{-1} + \frac{1}{2} e^{-t B_\lambda^*} K_{\lambda,\infty}^{-1} e^{-t B_\lambda} \geq \left(t K_\lambda(t) - \frac{1}{2} t K_\lambda(t) K_{\lambda,\infty}^{-1} t K_\lambda(t) \right)^{-1}.$$

Proof The case $\lambda = 0$ is trivial since $tK_\lambda(t)$ is still a positive definite matrix and $K^{-1}_{\lambda,\infty}$ is the null matrix. We can thus assume $\lambda \neq 0$. Keeping in mind the explicit form of e^{-tB_λ} (see Example 2.3 with $D_1 = \pi|\lambda|\sqrt{Q_1}$), we notice that

$$\frac{1}{2}e^{-tB^*_\lambda}K^{-1}_{\lambda,\infty}e^{-tB_\lambda} = \begin{pmatrix} 2\pi|\lambda|\sqrt{Q_1}e^{4\pi t|\lambda|\sqrt{Q_1}} & O_{n\times n_1} \\ O_{n_1\times n} & O_{n_1\times n_1} \end{pmatrix}$$

is a $(n+n_1)\times(n+n_1)$ matrix of rank n. We can then exploit the formula for the inverse of a small-rank adjustment (see, e.g., [31, Section 0.7.4]), which is sometimes referred to as the Sherman-Morrison-Woodbury formula, to infer that

$$\begin{aligned}&\left((tK_\lambda(t))^{-1} + \frac{1}{2}e^{-tB^*_\lambda}K^{-1}_{\lambda,\infty}e^{-tB_\lambda}\right)^{-1}\\ &= tK_\lambda(t) - tK_\lambda(t)\begin{pmatrix} \left(\frac{1}{2\pi|\lambda|}Q_1^{-\frac{1}{2}}e^{-4\pi t|\lambda|\sqrt{Q_1}} + \frac{1}{4\pi|\lambda|}Q_1^{-\frac{1}{2}}\left(I_n - e^{-4\pi t|\lambda|\sqrt{Q_1}}\right)\right)^{-1} & O_{n\times n_1} \\ O_{n_1\times n} & O_{n_1\times n_1}\end{pmatrix} tK_\lambda(t),\end{aligned}$$

where we have used the fact that the first block $K_{11}(t)$ in $tK_\lambda(t)$ is equal to

$$\frac{1}{4\pi|\lambda|}Q_1^{-\frac{1}{2}}\left(I_n - e^{-4\pi t|\lambda|\sqrt{Q_1}}\right)$$

(see again Example 2.3). By exploiting the simple inequality $\left(I_n + e^{-4\pi t|\lambda|\sqrt{Q_1}}\right)^{-1} \geq \frac{1}{2}I_n$, we deduce that

$$\begin{aligned}&\left((tK_\lambda(t))^{-1} + \frac{1}{2}e^{-tB^*_\lambda}K^{-1}_{\lambda,\infty}e^{-tB_\lambda}\right)^{-1}\\ &= tK_\lambda(t) - tK_\lambda(t)\begin{pmatrix} 4\pi|\lambda|\sqrt{Q_1}\left(I_n + e^{-4\pi t|\lambda|\sqrt{Q_1}}\right)^{-1} & O_{n\times n_1} \\ O_{n_1\times n} & O_{n_1\times n_1}\end{pmatrix} tK_\lambda(t)\\ &\leq tK_\lambda(t) - tK_\lambda(t)\begin{pmatrix} 2\pi|\lambda|\sqrt{Q_1} & O_{n\times n_1} \\ O_{n_1\times n} & O_{n_1\times n_1}\end{pmatrix} tK_\lambda(t) = tK_\lambda(t) - \frac{1}{2}tK_\lambda(t)K^{-1}_{\lambda,\infty}tK_\lambda(t)\end{aligned}$$

which implies the desired conclusion. □

Theorem 3.6 *For $X = (w, y)$ and $X_0 = (w_0, y_0)$ we let*

$$p_\lambda((w,y),(w_0,y_0),t) = (4\pi)^{-\frac{n+n_1}{2}}(\det tK_\lambda(t))^{-1/2}e^{-\frac{1}{4}\langle (tK_\lambda(t))^{-1}\left(X_0 - e^{tB_\lambda}X\right), X_0 - e^{tB_\lambda}X\rangle},$$

and define

$$h((w,\sigma,y),(w_0,\sigma_0,y_0),t) = \int_{\mathbb{R}^k} e^{2\pi i\langle \sigma-\sigma_0,\lambda\rangle}e^{-\pi|\lambda|\left(\frac{1}{2}\langle\sqrt{Q_1}(w+w_0),w-w_0\rangle + \operatorname{tr}\sqrt{Q_1}t\right)} \times p_\lambda((w,y),(w_0,y_0),t)\,d\lambda. \quad (3.27)$$

Given $f_0 \in \mathscr{S}(\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^{n_1})$, the function

$$f(w,\sigma,y,t) = \int_{\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^{n_1}} h((w,\sigma,y),(w_0,\sigma_0,y_0),t)\,f_0(w_0,\sigma_0,y_0)\,dw_0d\sigma_0dy_0, \quad (3.28)$$

solves the Cauchy problem Eq. 3.1.

Proof As before, our first step is to take the Fourier transform of Eq. 3.1 with respect to the variable σ, with dual variable $\lambda \in \mathbb{R}^k$. If we let

$$v_\lambda(w, y, t) = \hat{f}(w, \lambda, y, t) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \lambda, \sigma \rangle} f(w, \sigma, y, t) d\sigma,$$

then for every fixed $\lambda \in \mathbb{R}^k$ the problem Eq. 3.1 becomes in $\mathbb{R}^n \times \mathbb{R}^{n_1} \times (0, \infty)$

$$\begin{cases} \Delta_w v_\lambda - \pi^2 |\lambda|^2 \langle Q_1 w, w \rangle v_\lambda + \langle B_0 w, \nabla_y v_\lambda \rangle - \partial_t v_\lambda = 0, \\ v_\lambda(w, y, 0) = \hat{f}_0(w, \lambda, y), \qquad (w, y) \in \mathbb{R}^n \times \mathbb{R}^{n_1}. \end{cases} \tag{3.29}$$

Our second step is to make the following change of dependent variable $v_\lambda \to u_\lambda$, where the two functions are linked by the relation

$$v_\lambda(w, y, t) = e^{-\pi|\lambda|(\frac{1}{2}\langle \sqrt{Q_1} w, w \rangle + \operatorname{tr} \sqrt{Q_1} t)} u_\lambda(w, y, t). \tag{3.30}$$

This step represents a generalised version of Eq. 3.5 in Lemma 3.1. After some straightforward computations one recognises that in terms of the function u_λ the problem Eq. 3.29 becomes in $\mathbb{R}^n \times \mathbb{R}^{n_1} \times (0, \infty)$

$$\begin{cases} \Delta_w u_\lambda - 2\pi|\lambda| \langle \sqrt{Q_1} w, \nabla_w u_\lambda \rangle + \langle B_0 w, \nabla_y u_\lambda \rangle - \partial_t u_\lambda = 0, \\ u_\lambda(w, y, 0) = e^{\frac{\pi}{2}|\lambda| \langle \sqrt{Q_1} w, w \rangle} \hat{f}_0(w, \lambda, y), \qquad (w, y) \in \mathbb{R}^n \times \mathbb{R}^{n_1}. \end{cases} \tag{3.31}$$

Remarkably, the PDE in Eq. 3.31 can be cast in the form Eq. 2.1, where now $N = n + n_1$, and the matrices Q and $B = B_\lambda$ are given by Eq. 3.24. The covariance matrix is given by the positive definite matrix $K_\lambda(t)$ in Eq. 3.25. By Eq. 3.26 we can rewrite the initial datum in Eq. 3.31 as

$$u_\lambda(w, y, 0) = e^{\frac{\pi}{2}|\lambda| \langle \sqrt{Q_1} w, w \rangle} \hat{f}_0(w, \lambda, y) = e^{\frac{1}{8}\left\langle K_{\lambda,\infty}^{-1}(w,y),(w,y)\right\rangle} \hat{f}_0(w, \lambda, y).$$

Since $\hat{f}_0$ is bounded, we can apply Proposition 2.1: if we thus let $X = (w, y)$ and $X_0 = (w_0, y_0)$, and define $p_\lambda((w, y), (w_0, y_0), t)$ as in Eq. 2.4 above, we infer that the function

$$u_\lambda(w, y, t) = \int_{\mathbb{R}^n \times \mathbb{R}^{n_1}} p_\lambda((w, y), (w_0, y_0), t) e^{\frac{1}{2}\pi|\lambda| \langle \sqrt{Q_1} w_0, w_0 \rangle} \hat{f}_0(w_0, \lambda, y_0) dw_0 dy_0$$

solves the problem Eq. 3.31. In view of Eq. 3.30 this implies that

$$\begin{aligned} v_\lambda(w, y, t) &= e^{-\pi|\lambda|(\frac{1}{2}\langle \sqrt{Q_1} w, w \rangle + \operatorname{tr} \sqrt{Q_1} t)} \times \\ &\times \int_{\mathbb{R}^n \times \mathbb{R}^{n_1}} p_\lambda((w, y), (w_0, y_0), t) e^{\frac{1}{2}\pi|\lambda| \langle \sqrt{Q_1} w_0, w_0 \rangle} \hat{f}_0(w_0, \lambda, y_0) dw_0 dy_0. \end{aligned}$$

As we intend to take the inverse Fourier transform of $v_\lambda(w, y, t)$ with respect to λ, we want to understand the behaviour of $v_\lambda(w, y, t)$ with respect to this variable. Since f_0 belongs to the Schwartz class, we know that $\hat{f}_0(w_0, \lambda, y_0)$ decays faster than any polynomial in λ in a uniform way with respect to (w_0, y_0). Our objective is thus to analyse the function

$$I_X(\lambda, t) = e^{-\pi|\lambda|(\frac{1}{2}\langle \sqrt{Q_1} w, w \rangle + \operatorname{tr} \sqrt{Q_1} t)} \int_{\mathbb{R}^n \times \mathbb{R}^{n_1}} p_\lambda((w, y), (w_0, y_0), t) e^{\frac{1}{2}\pi|\lambda| \langle \sqrt{Q_1} w_0, w_0 \rangle} dw_0 dy_0.$$

Using Eq. 3.26 and the explicit expression of $p_\lambda(X, X_0, t)$ in Eq. 2.4, we write

$$I_X(\lambda, t) = \frac{e^{-\pi t|\lambda| \operatorname{tr} \sqrt{Q_1}}}{(4\pi)^{\frac{n+n_1}{2}}} \frac{e^{-\frac{1}{8}\left\langle K_{\lambda,\infty}^{-1} X, X \right\rangle}}{\sqrt{\det t K_\lambda(t)}}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}^{n+n_1}} e^{-\frac{1}{4}\langle (tK_\lambda(t))^{-1}\left(X_0 - e^{tB_\lambda}X\right), X_0 - e^{tB_\lambda}X\rangle} e^{\frac{1}{8}\left\langle K_{\lambda,\infty}^{-1}X_0, X_0\right\rangle} dX_0 \qquad (3.32)\\
&= \frac{e^{-\pi t|\lambda| \operatorname{tr}\sqrt{Q_1}}}{(4\pi)^{\frac{n+n_1}{2}}} \frac{e^{-\frac{1}{4}\left\langle\left(e^{tB_\lambda^*}(tK_\lambda(t))^{-1}e^{tB_\lambda} + \frac{1}{2}K_{\lambda,\infty}^{-1}\right)X, X\right\rangle}}{\sqrt{\det tK_\lambda(t)}} \times\\
&\times e^{\frac{1}{4}\left\langle\left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right)^{-1}(tK_\lambda(t))^{-\frac{1}{2}}e^{tB_\lambda}X, (tK_\lambda(t))^{-\frac{1}{2}}e^{tB_\lambda}X\right\rangle} \times\\
&\times \int_{\mathbb{R}^{n+n_1}} \exp\left\{-\frac{1}{4}\left|\left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right)^{\frac{1}{2}}(tK_\lambda(t))^{-\frac{1}{2}}X_0 + \right.\right.\\
&\left.\left. - \left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right)^{-\frac{1}{2}}(tK_\lambda(t))^{-\frac{1}{2}}e^{tB_\lambda}X\right|^2\right\} dX_0\\
&= e^{-\pi t|\lambda| \operatorname{tr}\sqrt{Q_1}}\left(\det\left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right)\right)^{-\frac{1}{2}} \times\\
&\times e^{-\frac{1}{4}\left\langle\left((tK_\lambda(t))^{-1} + \frac{1}{2}e^{-tB_\lambda^*}K_{\lambda,\infty}^{-1}e^{-tB_\lambda}\right)e^{tB_\lambda}X, e^{tB_\lambda}X\right\rangle} \times\\
&\times e^{\frac{1}{4}\left\langle\left((tK_\lambda(t)) - \frac{1}{2}(tK_\lambda(t))K_{\lambda,\infty}^{-1}(tK_\lambda(t))\right)^{-1}e^{tB_\lambda}X, e^{tB_\lambda}X\right\rangle}
\end{aligned}
$$

where in the last equality we have used the change of variables $X_0 \mapsto Z$, where

$$
\begin{aligned}
Z &= \left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right)^{\frac{1}{2}}(4tK_\lambda(t))^{-\frac{1}{2}}X_0 +\\
&- \left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right)^{-\frac{1}{2}}(4tK_\lambda(t))^{-\frac{1}{2}}e^{tB_\lambda}X,
\end{aligned}
$$

and the fact that $\int_{\mathbb{R}^{n+n_1}} e^{-|Z|^2}dZ = \pi^{\frac{n+n_1}{2}}$. At this point a small miracle happens since from Lemma 3.5 we have, for every $X \in \mathbb{R}^{n+n_1}$, $\lambda \in \mathbb{R}^k$, and $t > 0$, that

$$
\begin{aligned}
&\left\langle\left((tK_\lambda(t))^{-1} + \frac{1}{2}e^{-tB_\lambda^*}K_{\lambda,\infty}^{-1}e^{-tB_\lambda}\right)e^{tB_\lambda}X, e^{tB_\lambda}X\right\rangle\\
&\geq \left\langle\left((tK_\lambda(t)) - \frac{1}{2}(tK_\lambda(t))K_{\lambda,\infty}^{-1}(tK_\lambda(t))\right)^{-1}e^{tB_\lambda}X, e^{tB_\lambda}X\right\rangle.
\end{aligned}
$$

Inserting this information in Eq. 3.32 we obtain

$$
0 \leq I_X(\lambda, t) \leq \frac{e^{-\pi t|\lambda| \operatorname{tr}\sqrt{Q_1}}}{\left(\det\left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right)\right)^{\frac{1}{2}}} \leq 2^{\frac{n+n_1}{2}}e^{-\pi t|\lambda| \operatorname{tr}\sqrt{Q_1}}, \qquad (3.33)
$$

where in the last inequality we have exploited (2.3) to obtain $\left(I - \frac{1}{2}(tK_\lambda(t))^{\frac{1}{2}}K_{\lambda,\infty}^{-1}(tK_\lambda(t))^{\frac{1}{2}}\right) \geq \frac{1}{2}I$. Moreover we also have

$$
\lim_{t \to 0^+} I_X(\lambda, t) = 1 \qquad \forall\, X \in \mathbb{R}^{n+n_1},\ \lambda \in \mathbb{R}^k. \qquad (3.34)
$$

The limiting behaviour in Eq. 3.34 can be checked by using the change of variables $X_0 \mapsto Z$ where $X_0 = e^{tB_\lambda}X + (4tK_\lambda(t))^{\frac{1}{2}}Z$ in the definition of $I_X(\lambda, t)$, as this gives

$$I_X(\lambda, t) = \frac{e^{-\pi t|\lambda| \operatorname{tr}\sqrt{Q_1}}}{\pi^{\frac{n+n_1}{2}}} e^{-\frac{1}{8}\left\langle K_{\lambda,\infty}^{-1} X, X\right\rangle} \times$$
$$\times \int_{\mathbb{R}^{n+n_1}} e^{-|Z|^2+\frac{1}{8}\left\langle K_{\lambda,\infty}^{-1}\left(e^{tB_\lambda}X+(4tK_\lambda(t))^{\frac{1}{2}}Z\right),\left(e^{tB_\lambda}X+(4tK_\lambda(t))^{\frac{1}{2}}Z\right)\right\rangle} dZ.$$

Since $e^{tB_\lambda}X + (4tK_\lambda(t))^{\frac{1}{2}}Z \to X$ as $t \to 0^+$, we easily obtain Eq. 3.34 from the above identity. Hence, by exploiting Eqs. 3.33 and 3.34, we can argue as in the proof of Eq. 3.21 in Theorem 3.4 to deduce that

$$\int_{\mathbb{R}^k} \left| v_\lambda(w, y, t) - \hat{f}_0(w, \lambda, y)\right| d\lambda \underset{t\to 0^+}{\longrightarrow} 0. \tag{3.35}$$

Keeping in mind that $v_\lambda(w, y, t) = \hat{f}(w, \lambda, y, t)$, we are now ready to take the inverse Fourier transform, obtaining

$$f(w, \sigma, y, t) = \int_{\mathbb{R}^k}\int_{\mathbb{R}^n\times\mathbb{R}^{n_1}}\int_{\mathbb{R}^k} e^{2\pi i\langle\sigma,\lambda\rangle} e^{-\pi|\lambda|(\frac{1}{2}\langle\sqrt{Q_1}w,w\rangle+\operatorname{tr}\sqrt{Q_1}t)} p_\lambda((w, y), (w_0, y_0), t)$$
$$\times e^{\frac{1}{2}\pi|\lambda|\langle\sqrt{Q_1}w_0,w_0\rangle} e^{-2\pi i\langle\sigma_0,\lambda\rangle} f_0(w_0, \sigma_0, y_0) d\sigma_0 dw_0 dy_0 d\lambda.$$

Since $f_0 \in \mathscr{S}$ and thanks to Eq. 3.33, $f(w, \sigma, y, t)$ is a well-defined and smooth function and it coincides with the expression stated in Eq. 3.28. Proceeding verbatim as in the proof of Theorem 3.4 and using Eq. 3.35, we also see that f is solution of the Cauchy problem Eq. 3.1. We conclude that the function $h(\cdot,\cdot,\cdot)$ defined by Eq. 3.27 does provide the heat kernel. We stress that, for any $(w, \sigma, y), (w_0, \sigma_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{n_1}$ and $t > 0$, $h((w, \sigma, y), (w_0, \sigma_0, y_0), t)$ is well-defined (and smooth) since, by arguing as in Eq. 3.32-Eq. 3.33, we have

$$\begin{aligned}
&|h((w, \sigma, y), (w_0, \sigma_0, y_0), t)| \\
&\le \int_{\mathbb{R}^k} e^{-\pi|\lambda|(\frac{1}{2}\langle\sqrt{Q_1}w,w\rangle+\operatorname{tr}\sqrt{Q_1}t)} p_\lambda((w, y), (w_0, y_0), t) e^{\frac{1}{2}\pi|\lambda|\langle\sqrt{Q_1}w_0,w_0\rangle} d\lambda \\
&\le (2\pi)^{-\frac{n+n_1}{2}} \int_{\mathbb{R}^k} \frac{e^{-\pi t|\lambda|\operatorname{tr}\sqrt{Q_1}}}{\sqrt{\det tK_\lambda(t)}} d\lambda < \infty,
\end{aligned}$$

where in the last inequality we have used that $\operatorname{tr}\sqrt{Q_1} > 0$ and the fact that $(\det tK_\lambda(t))^{-1}$ grows at most polynomially with respect to λ (see the explicit form of $tK_\lambda(t)$ in Example 2.3 with $D_1 = \pi|\lambda|\sqrt{Q_1}$). This finishes the proof of the theorem. □

If we set $n = n_1 \in \mathbb{N}$ and $Q_1 = B = I_n$, the PDE in Eq. 3.1 becomes the hybrid equation highlighted in Eq. 1.7. In this special situation, for $\lambda \in \mathbb{R}^k$ with $\lambda \neq 0$ and $t > 0$, we have

$$e^{tB_\lambda} = \begin{pmatrix} e^{-2\pi t|\lambda|} I_n & O_{n\times n} \\ \frac{1-e^{-2\pi t|\lambda|}}{2\pi|\lambda|} I_n & I_n \end{pmatrix}$$

and

$$tK_\lambda(t) = e^{-2\pi t|\lambda|} \begin{pmatrix} \frac{\sinh(2\pi t|\lambda|)}{2\pi|\lambda|} I_n & \frac{\cosh(2\pi t|\lambda|)-1}{4\pi^2|\lambda|^2} I_n \\ \frac{\cosh(2\pi t|\lambda|)-1}{4\pi^2|\lambda|^2} I_n & \frac{e^{2\pi t|\lambda|}(2\pi t|\lambda|-\sinh(2\pi t|\lambda|))+(\cosh(2\pi t|\lambda|)-1)(e^{2\pi t|\lambda|}-1)}{8\pi^3|\lambda|^3} I_n \end{pmatrix},$$

which yields the explicit formula Eq. 1.10 for the heat kernel.

4 A Class of Heat Kernels from Conformal CR Geometry

In this section we construct the heat kernel of a class of hybrid evolution equations that play an important role in conformal CR geometry. In Section 1 we have already discussed the extension problem Eq. 1.2 in the Heisenberg group $\mathbb{H}^n$ in the seminal work of Frank et al. [19]. More in general, we now consider a Lie group of Heisenberg type $\mathbb{G}$ with logarithmic coordinates (z, σ), where $z \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}^k$ (see Section 4.2). If $\mathscr{L} = \Delta_z + \frac{|z|^2}{4}\Delta_\sigma + \sum_{\ell=1}^k \partial_{\sigma_\ell}\Theta_\ell$, denotes a horizontal Laplacian in $\mathbb{G}$ (see Eq. 4.33 below), then in this more general framework the parabolic counterpart of the extension problem Eq. 1.2 is as follows: given a function $u \in C_0^\infty(\mathbb{G}\times\mathbb{R})$, find a function $U \in C^\infty(\mathbb{G}\times\mathbb{R}\times\mathbb{R}_y^+)$ such that

$$\begin{cases} \partial_{yy}U + \frac{1-2s}{y}\partial_y U + \frac{y^2}{4}\Delta_\sigma U + \mathscr{L}U - \partial_t U = 0, & \text{in } \mathbb{G}\times\mathbb{R}\times\mathbb{R}_y^+, \\ U(g,t,0) = u(g,t), & (g,t)\in\mathbb{G}\times\mathbb{R}. \end{cases} \tag{4.1}$$

Our present objective is the computation of the heat kernel of the evolution PDE in Eq. 4.1, i.e. of the equation defined in $\mathbb{G}\times\mathbb{R}\times\mathbb{R}_y^+$ as

$$\mathfrak{L}_{(s)}U - \partial_t U \overset{def}{=} \partial_{yy}U + \frac{1-2s}{y}\partial_y U + \frac{y^2}{4}\Delta_\sigma U + \mathscr{L}U - \partial_t U = 0 \tag{4.2}$$

The following is our main result.

Theorem 4.1 *Let $\mathbb{G}$ be a group of Heisenberg type. For every $0 < s < 1$ the heat kernel with pole at the origin of the operator $\mathfrak{L}_{(s)} - \partial_t$ in* Eq. 4.2 *is given by*

$$\mathfrak{q}_{(s)}((z,\sigma),t,y) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k+1-s}}\int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle}\left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2+y^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}}d\lambda. \tag{4.3}$$

The relevance of the heat kernel (4.3), and its intertwined counterpart obtained by replacing s with $-s$, is in the fact that such functions encapsulate properties of conformal invariance of the time independent pseudo-differential operator $\mathscr{L}_s$ already mentioned in the introduction. These aspects have played a central role in our recent works [25] and [26] and also in the earlier papers by Roncal and Thangavelu [43, 44] which contain a heuristic ansatz of the expression Eq. 4.3.

As in the case of $\mathbb{H}^n$ (which constitutes the $k = 1$ case of our treatment), a key observation here is that the PDE in Eq. 4.2 is the restriction of the equation

$$\Delta_w U + \frac{|w|^2}{4}\Delta_\sigma U + \mathscr{L}U - \partial_t U = 0 \tag{4.4}$$

to functions depending on the variable $y = |w|$, where w belongs to the space with fractal dimension $\mathbb{R}^{2(1-s)}$. The link between Eq. 4.4 and the PDE in Eq. 4.2 is readily seen by observing that, if $y = |w|$, then on a function $u(w) = \psi(y)$ we have $\Delta_w u = \partial_{yy}\psi + \frac{1-2s}{y}\partial_y\psi$. Another remark is that Eq. 4.4 is of hybrid type since the variable σ appears in both equations

$$\Delta_w U + \frac{|w|^2}{4}\Delta_\sigma U - \partial_t U = 0,$$

and

$$\mathscr{L}U - \partial_t U = 0,$$

see Eq. 4.33 below for the expression of $\mathscr{L}$. Also observe that, similarly to the situation of the operator in Eq. 3.1 in Section 3, the PDE in Eq. 4.4 contains the limiting case in which the fractal dimension $n_1 = 2(1-s) = 0$ of the variable w vanishes, which is equivalent to letting $s \nearrow 1$. In such case the PDEs Eqs. 4.2, 4.4 formally become

$$\mathscr{L} U - \partial_t U = 0, \tag{4.5}$$

the heat equation in $\mathbb{G}$ associated with the horizontal Laplacian $\mathscr{L}$. For the latter the heat kernel is well-known and it is given by

$$q((z,\sigma),(\zeta,\tau),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}} e^{\frac{i}{t}(\langle \tau-\sigma,\lambda\rangle + \frac{1}{2}\langle J(\lambda)\zeta, z\rangle)} e^{-\frac{|z-\zeta|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda. \tag{4.6}$$

In the special case of the Heisenberg group $\mathbb{H}^n$ one has $m = 2n$, $k = 1$, and Eq. 4.6 gives back the famous formula independently found by Hulanicki [32] and Gaveau [27]. We mention here that in [18] Folland proved the existence of the heat kernel in any Carnot group, but of course in such generality one does not have an explicit representation such as Eq. 4.6.

Remark 4.2 The reader should note that if in Eq. 4.3 we formally set $s = 1$ and $y = 0$ we perfectly recover the Hulanicki-Gaveau formula Eq. 4.6 when the pole $(\zeta, \tau) = (0, 0)$!

Similarly to what we did in Section 3, in the present section we first provide in Theorem 4.6 a totally self-contained and elementary proof of the construction of the heat kernel for the limiting case Eq. 4.5. We do this not just for groups of Heisenberg type, but in the more general framework of a Carnot group $\mathbb{G}$ of step two. Of course the result per se is not new, as Cygan established it in [16], but our proof is. Although the relevant PDE Eq. 4.5 is not hybrid in the sense specified in the opening of this paper, the motivation for including here the construction of its heat kernel is twofold: (i) on one hand it allows to present our approach to Theorem 4.1 in a significant, yet simplified setting; (ii) on the other hand we feel that our self-contained proof will be of interest to workers in analysis and PDEs who are not directly familiar with those important and deep tools, such as e.g. group representation theory, Laguerre calculus, complex Hamiltonians or a priori ansatz, which in one form or another have entered the previous related works such as [4, 6, 8, 14, 16, 27, 32, 34, 37, 38, 42].

4.1 The Generalised Harmonic Oscillator with a Complex Drift

In what follows given a number $n \in \mathbb{N}$ we denote by $S \in M_{n\times n}(\mathbb{R})$ a skew-symmetric matrix, i.e., we assume $S^\star = -S$. We intend to solve the Cauchy problem for the following generalised harmonic oscillator with a complex drift

$$\begin{cases} \Delta_z \tilde{v} - |Sz|^2\, \tilde{v} + 2i\langle Sz, \nabla_z \tilde{v}\rangle - \partial_t \tilde{v} = 0, \\ \tilde{v}(z,0) = \tilde{v}_0(z) \qquad\qquad z \in \mathbb{R}^n,\ t > 0, \end{cases} \tag{4.7}$$

where $\tilde{v}_0$ is suitably chosen. We will need the following lemma that allows to eliminate the complex drift from Eq. 4.7.

Lemma 4.3 *Suppose that v and $\tilde{v}$ are connected by the relation*

$$v(z,t) = \tilde{v}(e^{-2itS} z, t). \tag{4.8}$$

Then, $\tilde{v}$ is a solution to the PDE in Eq. 4.7 *if and only if v is a solution to the equation*

$$\Delta_z v - |Sz|^2 v - \partial_t v = 0.$$

Proof Let $\tilde{v}$ be a solution to the PDE in Eq. 4.7. We note that by the skew-symmetry of S we know that

$$e^{-2itS^\star}e^{-2itS} = I_n.$$

Using this observation the reader can verify by a direct computation that

$$\Delta v(z,t) = \Delta\tilde{v}(e^{-2itS}z,t).$$

It is also clear from Eq. 4.8 that

$$\partial_t v(z,t) = -2i\langle Sz, \nabla\tilde{v}(e^{-2itS}z,t)\rangle + \partial_t\tilde{v}(e^{-2itS}z,t).$$

Combining the latter two equations we easily reach the desired conclusion. □

Returning to the Cauchy problem Eq. 4.7 the following is the main result of this subsection.

Proposition 4.4 *Let*

$$\mathscr{Q}(z,\zeta,t) = \frac{e^{i\langle Sz,\zeta\rangle}}{(4\pi t)^{\frac{n}{2}}}\left(\det j\left(2t\sqrt{S^\star S}\right)\right)^{\frac{1}{2}} e^{-\frac{1}{4t}\langle j\left(2t\sqrt{S^\star S}\right)\cosh\left(2t\sqrt{S^\star S}\right)(z-\zeta),z-\zeta\rangle} \tag{4.9}$$

Then, for any $\tilde{v}_0 \in C(\mathbb{R}^n)$ *such that*

$$e^{-\frac{1}{2}\left\langle\sqrt{S^\star S}z,z\right\rangle}\tilde{v}_0(z) \in L^\infty(\mathbb{R}^n),$$

the function

$$\tilde{v}(z,t) = \int_{\mathbb{R}^n}\mathscr{Q}(z,\zeta,t)\tilde{v}_0(\zeta)d\zeta \tag{4.10}$$

solves Eq. 4.7.

Proof It is clear that using Lemma 4.3 the Cauchy problem Eq. 4.7 is transformed into

$$\begin{cases}\Delta_z v - |Sz|^2\, v - \partial_t v = 0,\\ v(z,0) = \tilde{v}_0(z) \qquad\qquad z\in\mathbb{R}^n,\ t>0,\end{cases} \tag{4.11}$$

for the function v defined by Eq. 4.8. If we now define $D = \sqrt{S^\star S} = \sqrt{-S^2}$, then clearly $D \geq 0$, $D^\star = D$, and $|Sz|^2 = |Dz|^2$. We can thus re-write Eq. 4.11 as follows

$$\begin{cases}\Delta_z v - |Dz|^2\, v - \partial_t v = 0,\\ v(z,0) = \tilde{v}_0(z) \qquad\qquad z\in\mathbb{R}^n,\ t>0.\end{cases}$$

According to Proposition 3.2 the function

$$v(z,t) = \int_{\mathbb{R}^n}\mathscr{M}(z,\zeta,t)\tilde{v}_0(\zeta)d\zeta,$$

with $D = \sqrt{-S^2}$ and $\mathscr{M}(z,\zeta,t)$ as in Eq. 3.7, solves the latter problem. Undoing Eq. 4.8 we have proved that the function defined by

$$\tilde{v}(z,t) = \int_{\mathbb{R}^n}\mathscr{M}(e^{2itS}z,\zeta,t)\tilde{v}_0(\zeta)d\zeta \tag{4.12}$$

solves Eq. 4.7. We next want to further simplify the expression Eq. 4.12. Keeping in mind the explicit expression Eq. 3.7 for $\mathscr{M}$, we have

$$\begin{aligned}&\mathscr{M}(e^{2itS}z,\zeta,t)\\ &= (4\pi t)^{-\frac{n}{2}}\sqrt{\det j(2tD)}\exp\left\{-\frac{1}{4t}\Big(\langle j(2tD)\cosh(2tD)e^{2itS}z, e^{2itS}z\rangle\right.\\ &\left.+\langle j(2tD)\cosh(2tD)\zeta,\zeta\rangle - 2\langle j(2tD)\, e^{2itS}z,\zeta\rangle\Big)\right\},\end{aligned} \tag{4.13}$$

We now observe that, since S commutes with any even analytic function of $D=\sqrt{-S^2}$, we have in particular

$$\langle j(2tD)\cosh(2tD)e^{2itS}z, e^{2itS}z\rangle = \langle j(2tD)\cosh(2tD)z,z\rangle, \tag{4.14}$$

as well as

$$\begin{aligned}e^{2itS} &= \sum_{k=0}^{\infty}\frac{(2itS)^{2k}}{(2k)!}+\sum_{k=0}^{\infty}\frac{(2itS)^{2k+1}}{(2k+1)!}\\ &= \sum_{k=0}^{\infty}\frac{(-4t^2S^2)^k}{(2k)!}+2itS\sum_{k=0}^{\infty}\frac{(-4t^2S^2)^k}{(2k+1)!}\\ &= \sum_{k=0}^{\infty}\frac{(2tD)^{2k}}{(2k)!}+2itS\sum_{k=0}^{\infty}\frac{(2tD)^{2k}}{(2k+1)!}\\ &= \cosh(2tD)+2it\,(j(2tD))^{-1}\,S.\end{aligned} \tag{4.15}$$

We thus find from Eq. 4.15

$$-2\langle j(2tD)\,e^{2itS}z,\zeta\rangle = -2\langle j(2tD)\ \cosh(2tD)z,\zeta\rangle - 4it\langle Sz,\zeta\rangle. \tag{4.16}$$

Replacing now Eq. 4.14 and Eq. 4.16 in Eq. 4.13, we finally obtain

$$\begin{aligned}\mathscr{M}(e^{2itS}z,\zeta,t) &= (4\pi t)^{-\frac{n}{2}}\sqrt{\det j(2tD)}e^{i\langle Sz,\zeta\rangle}\exp\left\{-\frac{1}{4t}\Big(\langle j(2tD)\cosh(2tD)z,z\rangle\right.\\ &\left.+\ \langle j(2tD)\cosh(2tD)\zeta,\zeta\rangle - 2\langle j(2tD)\ \cosh(2tD)z,\zeta\rangle\Big)\right\}\\ &= \frac{e^{i\langle Sz,\zeta\rangle}}{(4\pi t)^{\frac{n}{2}}}\sqrt{\det j(2tD)}\exp\left\{-\frac{1}{4t}\langle j(2tD)\cosh(2tD)(z-\zeta),z-\zeta\rangle\right\}\\ &= \mathscr{Q}(z,\zeta,t).\end{aligned}$$

This concludes the proof. □

It is clear from the previous proof that the kernel $\mathscr{Q}(z,\zeta,t)$ is equal to $\mathscr{M}(e^{2itS}z,\zeta,t)$ with $\mathscr{M}$ given by Eq. 3.7 and with $D=\sqrt{S^\star S}$. Using the commutation property in Eq. 4.14 we then deduce from Eq. 3.16 that

$$\begin{aligned}&\int_{\mathbb{R}^n}\mathscr{Q}(z,\zeta,t)d\zeta\\ &= \frac{1}{\sqrt{\det\cosh 2t\sqrt{S^\star S}}}\exp\left\{-\frac{1}{4t}\langle j(2t\sqrt{S^\star S})\Big(\cosh 2t\sqrt{S^\star S}-\cosh^{-1}2t\sqrt{S^\star S}\Big)\,z,z\rangle\right\}.\end{aligned} \tag{4.17}$$

This equation will be used in the proof of Theorem 4.6 below.

4.2 Gaveau, Hulanicki and Cygan met Ornstein and Uhlenbeck

Henceforth, we denote by $\mathbb{G}$ a Carnot group of step two and we let $\mathfrak{g} = V_1 \oplus V_2$ indicate its Lie algebra, with inner product $\langle\cdot,\cdot\rangle$. Recall that the step two assumption means that $[V_1, V_1] = V_2$ and that $[V_1, V_2] = \{0\}$. We let $m = \dim(V_1)$, $k = \dim(V_2)$, and we fix orthonormal basis $\{e_1, ..., e_m\}$ and $\{\varepsilon_1, ..., \varepsilon_k\}$ for V_1 and V_2 respectively. For points $z \in V_1$ and $\sigma \in V_2$ we will use either one of the representations $z = \sum_{j=1}^m z_j e_j$, $\sigma = \sum_{\ell=1}^k \sigma_\ell \varepsilon_\ell$, or also $z = (z_1, ..., z_m)$, $\sigma = (\sigma_1, ..., \sigma_k)$. Whenever convenient, we routinely identify $V_1 \cong \mathbb{R}^m$ and $V_2 \cong \mathbb{R}^k$ and the points $g, g' \in \mathbb{G}$ with their logarithmic coordinates $(z, \sigma), (\zeta, \tau) \in \mathbb{R}^{m+k}$ respectively. Recall that the Kaplan mapping $J : V_2 \to \mathrm{End}(V_1)$ is defined by

$$\langle J(\sigma)z, \zeta\rangle = \langle [z, \zeta], \sigma\rangle = -\langle J(\sigma)\zeta, z\rangle. \tag{4.18}$$

Clearly, $J(\sigma)^\star = -J(\sigma)$, and one has $\langle J(\sigma)z, z\rangle = 0$. Moreover, by the bracket generating assumption $[V_1, V_1] = V_2$, we know that the map J is injective. Via the Baker-Campbell-Hausdorff formula

$$\exp(z+\sigma)\exp(\zeta+\tau) = \exp\left(z+\zeta+\sigma+\tau+\frac{1}{2}[z,\zeta]\right) \tag{4.19}$$

the map Eq. 4.18 identifies the non-Abelian multiplication law in $\mathbb{G}$

$$g \circ g' = (z+\zeta, \sigma+\tau+\frac{1}{2}\sum_{\ell=1}^k \langle J(\varepsilon_\ell)z, \zeta\rangle \varepsilon_\ell).$$

For future use we observe that

$$(g')^{-1} \circ g = (z-\zeta, \sigma-\tau+\frac{1}{2}\sum_{\ell=1}^k \langle J(\varepsilon_\ell)z, \zeta\rangle \varepsilon_\ell). \tag{4.20}$$

The map Eq. 4.18 also induces a complex geometry in the Lie group. This becomes particularly transparent in the special case of groups of Heisenberg type for which $J(\sigma)^2 = -|\sigma|^2 I_{V_1}$, see the seminal works [33] and [15]. In general, for the mapping

$$A(\sigma) \overset{def}{=} J^\star(\sigma)J(\sigma) = -J(\sigma)^2, \tag{4.21}$$

we have $\langle A(\sigma)z, z\rangle = |J(\sigma)z|^2 \geq 0$, and therefore it defines a symmetric nonnegative element of $\mathrm{End}(V_1)$ for every $\sigma \in V_2$. Consequently, the matrix $\sqrt{A(\sigma)}$ is well-defined.

Remark 4.5 Notice that $\mathrm{Ker}\, A(\sigma) = \mathrm{Ker}\, J(\sigma)$ can have positive dimension. Nevertheless, the injectivity of J ensures that $A(\sigma)$ is not the null endomorphism for every $\sigma \neq 0$. Moreover, being $J(\sigma)$ skew-symmetric, the dimension of the range of $A(\sigma)$ has to be at least two. Since the linearity of J allows us to write $\sqrt{A(\sigma)} = |\sigma|\sqrt{A(\sigma/|\sigma|)}$, one can deduce that there exists $k_0 > 0$ such that $\sqrt{A(\sigma)}$ has at least two eigenvalues bigger than $k_0|\sigma|$. This implies that

$$\det j(\sqrt{A(\sigma)}) \leq (j(k_0|\sigma|))^2 = \left(\frac{2k_0|\sigma|}{e^{k_0|\sigma|} - e^{-k_0|\sigma|}}\right)^2. \tag{4.22}$$

If for $j = 1, ..., m$ we define left-invariant vector fields by the Lie rule $X_j u(g) = \frac{d}{ds} u(g \circ \exp s e_j)\Big|_{s=0}$, then by Eq. 4.19 one obtains in the logarithmic coordinates (z, σ)

$$X_j = \partial_{z_j} + \frac{1}{2}\sum_{\ell=1}^{k} \langle J(\varepsilon_\ell) z, e_j\rangle \partial_{\sigma_\ell} = \partial_{z_j} + \frac{1}{2}\sum_{\ell=1}^{k}\sum_{i=1}^{m} z_i \langle J(\varepsilon_\ell) e_i, e_j\rangle \partial_{\sigma_\ell}. \tag{4.23}$$

Although we will not make an explicit use of this fact, we note that Eq. 4.23 implies the following commutation relation

$$[X_i, X_j] = \sum_{\ell=1}^{k} \langle J(\varepsilon_\ell) e_i, e_j\rangle \partial_{\sigma_\ell}.$$

Given a function $f \in C^1$ we will indicate by $\nabla_H f = (X_1 f, ..., X_m f)$ its horizontal gradient, and set $|\nabla_H f| = (\sum_{j=1}^m (X_j f)^2)^{1/2}$. The horizontal Laplacian generated by the orthonormal basis $\{e_1, ..., e_m\}$ of V_1 is the second-order differential operator on $\mathbb{G}$ defined by

$$\mathscr{L} f = \sum_{j=1}^{m} X_j^2 f,$$

where $X_1, ..., X_m$ are given by Eq. 4.23. A computation gives

$$\mathscr{L} = \Delta_z + \frac{1}{4}\sum_{\ell,\ell'=1}^{k} \langle J(\varepsilon_\ell) z, J(\varepsilon_{\ell'}) z\rangle \partial_{\sigma_\ell}\partial_{\sigma_{\ell'}} + \sum_{\ell=1}^{k} \Theta_\ell \partial_{\sigma_\ell}, \tag{4.24}$$

where Δ_z represents the standard Laplacian in the variable $z = (z_1, ..., z_m)$, and

$$\Theta_\ell = \sum_{s=1}^{m} \langle J(\varepsilon_\ell) z, e_s\rangle \partial_{z_s}. \tag{4.25}$$

For $f \in \mathscr{S}$ (recall that we are thinking of $\mathbb{G} \cong \mathbb{R}^m \times \mathbb{R}^k$) we now consider the Cauchy problem

$$\begin{cases} \mathscr{L} u - \partial_t u = 0 & \text{in } \mathbb{G} \times (0, \infty), \\ u(g, 0) = f(g) & g \in \mathbb{G}. \end{cases} \tag{4.26}$$

The aim of this section is to provide a new simple proof of the following classical result.

Theorem 4.6 (Gaveau-Hulanicki-Cygan) *The heat kernel in $\mathbb{G} \times (0, \infty)$ is given by the formula*

$$q(g, g', t) = 2^k (4\pi t)^{-(\frac{m}{2}+k)} \int_{\mathbb{R}^k} e^{\frac{i}{t}\left(\langle \tau - \sigma, \lambda\rangle + \frac{1}{2}\langle J(\lambda)\zeta, z\rangle\right)} \left(\det j(\sqrt{A(\lambda)})\right)^{1/2} \tag{4.27}$$
$$\times \exp\left\{ - \frac{1}{4t}\langle j(\sqrt{A(\lambda)}) \cosh \sqrt{A(\lambda)}(z - \zeta), z - \zeta\rangle\right\} d\lambda,$$

where $g = (z, \sigma)$ and $g' = (\zeta, \tau)$.

It is appropriate to mention that although the integral representations in the above cited literature appear in different forms with respect to Eq. 4.27, they are in fact equivalent to it. The advantage of our presentation is that it is particularly transparent and is easily applicable for instance to delicate questions in geometric measure theory treated in our work [24].

Proof of Theorem 4.6 Fix $f \in \mathscr{S}$. To solve Eq. 4.26 we start from the expression Eq. 4.24 of the horizontal Laplacian, keeping also Eq. 4.25 in mind. We identify $\mathbb{G}$ with $\mathbb{R}^m \times \mathbb{R}^k$ and, denoting for $\lambda \in \mathbb{R}^k$

$$\hat{u}(z, \lambda, t) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \lambda, \sigma \rangle} u(z, \sigma, t) d\sigma,$$

the partial Fourier transform of u with respect to the central variable $\sigma \in \mathbb{R}^k$, we obtain from Eq. 4.24

$$\begin{cases} \Delta_z \hat{u} - \pi^2 \sum_{\ell, \ell'=1}^k \lambda_\ell \lambda_{\ell'} \langle J(\varepsilon_\ell) z, J(\varepsilon_{\ell'}) z \rangle \hat{u} + 2\pi i \sum_{\ell=1}^k \lambda_\ell \Theta_\ell \hat{u} - \partial_t \hat{u} = 0, \\ \hat{u}(z, \lambda, 0) = \hat{f}(z, \lambda) \qquad z \in \mathbb{R}^m, \ \lambda \in \mathbb{R}^k, \ t > 0. \end{cases} \tag{4.28}$$

Now, it is not difficult to recognise that

$$\sum_{\ell, \ell'=1}^k \lambda_\ell \lambda_{\ell'} \langle J(\varepsilon_\ell) z, J(\varepsilon_{\ell'}) z \rangle = |J(\lambda) z|^2. \tag{4.29}$$

Furthermore, we obtain from Eq. 4.25

$$\sum_{\ell=1}^k \lambda_\ell \Theta_\ell \hat{u} = \sum_{\ell=1}^k \lambda_\ell \sum_{s=1}^m \langle J(\varepsilon_\ell) z, e_s \rangle \partial_{z_s} \hat{u} = \sum_{s=1}^m \langle J(\lambda) z, e_s \rangle \partial_{z_s} \hat{u} = \langle J(\lambda) z, \nabla_z \hat{u} \rangle. \tag{4.30}$$

Using the identities Eqs. 4.29, 4.30 we can thus write Eq. 4.28 in the form

$$\begin{cases} \Delta_z \tilde{v} - \pi^2 |J(\lambda) z|^2 \ \tilde{v} + 2\pi i \langle J(\lambda) z, \nabla_z \tilde{v} \rangle - \partial_t \tilde{v} = 0, \\ \tilde{v}(z, 0) = \hat{f}(z, \lambda) \qquad z \in \mathbb{R}^m, \ t > 0, \end{cases} \tag{4.31}$$

where for $\lambda \in \mathbb{R}^k$ fixed, we have let $\tilde{v}(z, t) = \hat{u}(z, \lambda, t)$. To solve problem Eq. 4.31 we now apply Proposition 4.4 with the choice of the skew-symmetric matrix $S = \pi J(\lambda)$, so that the symmetric matrix $D = \sqrt{-S^2}$ is presently given by $D = \pi \sqrt{A(\lambda)} = \pi \sqrt{-J(\lambda)^2}$ (we warn the reader that the role of the dimension n in Proposition 4.4 is now taken by the dimension m of the first layer V_1). Formula Eq. 4.10 allows to conclude that

$$\hat{u}(z, \lambda, t) = \int_{\mathbb{R}^m} \mathscr{Q}(z, \zeta, t) \hat{f}(\zeta, \lambda) d\zeta$$

with the kernel $\mathscr{Q}$ as in Eq. 4.9. We can now argue as in the proof of Theorem 3.4. First, by exploiting Eq. 4.17, we can ensure that $\hat{u}(z, \lambda, t) \to \hat{f}(z, \lambda)$ in $L^1(\mathbb{R}^k, d\lambda)$ as $t \to 0^+$. Then, we can conclude that the function

$$u(z, \sigma, t) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} e^{2\pi i \langle \sigma - \tau, \lambda \rangle} \mathscr{Q}(z, \zeta, t) f(\zeta, \tau) d\lambda d\zeta d\tau$$

solves the Cauchy problem Eq. 4.26. The desired heat kernel on the group $\mathbb{G}$ is thus given by

$$\begin{aligned} q((z, \sigma), (\zeta, \tau), t) &= \int_{\mathbb{R}^k} e^{2\pi i \langle \sigma - \tau, \lambda \rangle} \mathscr{Q}(z, \zeta, t) d\lambda \\ &= (4\pi t)^{-\frac{m}{2}} \int_{\mathbb{R}^k} e^{2\pi i \langle \sigma - \tau, \lambda \rangle} e^{i\pi \langle J(\lambda) z, \zeta \rangle} \left(\det j(2\pi t \sqrt{A(\lambda)}) \right)^{1/2} \\ &\quad \times \exp\left\{ -\frac{1}{4t} \langle j(2\pi t \sqrt{A(\lambda)}) \cosh\left(2\pi t \sqrt{A(\lambda)} \right) (z - \zeta), z - \zeta \rangle \right\} d\lambda \end{aligned}$$

where we have used the expression Eq. 4.9. Making the change of variable $(-2\pi t\lambda) \to \lambda$ and exploiting the linearity and the skew-symmetry of the mapping $J(\cdot)$, we finally obtain

$$q((z,\sigma),(\zeta,\tau),t) = 2^k(4\pi t)^{-(\frac{m}{2}+k)} \int_{\mathbb{R}^k} e^{\frac{i}{t}\left(\langle \tau-\sigma,\lambda\rangle + \frac{1}{2}\langle J(\lambda)\zeta,z\rangle\right)} \left(\det j(\sqrt{A(\lambda)})\right)^{1/2}$$
$$\times \exp\left\{-\frac{1}{4t}\langle j(\sqrt{A(\lambda)})\cosh\sqrt{A(\lambda)}(z-\zeta), z-\zeta\rangle\right\} d\lambda. \tag{4.32}$$

From Eq. 4.22 it is easy to verify that the integral in Eq. 4.32 is finite, and the kernel $q((z,\sigma),(\zeta,\tau),t)$ is a smooth function. This completes the proof of Theorem 4.6. □

For the sake of completeness we close this brief subsection by showing how to recover the Gaveau-Hulanicki formula Eq. 4.6 from Theorem 4.6. We recall that a Carnot group of step two $\mathbb{G}$ is said of Heisenberg type if for every $\lambda \in V_2$ the mapping $A(\lambda)$ in Eq. 4.21 satisfies $A(\lambda) = -J(\lambda)^2 = |\lambda|^2 I_m$. Therefore, $\sqrt{A(\lambda)} = |\lambda| I_m$, and we have

$$j(\sqrt{A(\lambda)}) = j(|\lambda| I_m) = j(|\lambda|)I_m,$$

and also

$$j(\sqrt{A(\lambda)})\cosh\sqrt{A(\lambda)} = j(|\lambda|)\cosh|\lambda| I_m = \frac{|\lambda|}{\tanh|\lambda|} I_m.$$

We thus obtain from Eq. 4.27

$$q((z,\sigma),(\zeta,\tau),t) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k}} \int_{\mathbb{R}^k} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}} e^{\frac{i}{t}(\langle\tau-\sigma,\lambda\rangle + \frac{1}{2}\langle J(\lambda)\zeta,z\rangle)} e^{-\frac{|z-\zeta|^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda$$

which coincides with the expression recalled in Eq. 4.6.

4.3 Proof of Theorem 4.1

In this section we finally turn to the proof of Theorem 4.1. As we have already observed, we begin from the crucial observation that, in a group of Heisenberg type $\mathbb{G}$, the relevant heat equation associated with Eq. 4.2 is Eq. 4.4, with the variable w running in the space with fractal dimension $\mathbb{R}^{2(1-s)}$. Henceforth, to continue the analysis we proceed formally and treat the number $2(1-s)$ as if it were an integer. In this way we will arrive to a specific candidate for a heat kernel in the form Eq. 4.3. Only at that point we will rigorously justify our computations and complete the proof.

We begin by observing that the assumption $J(\lambda)^2 = -|\lambda|^2 I_m$ implies in particular that $\langle J(\varepsilon_\ell)z, J(\varepsilon_{\ell'})z\rangle = |z|^2\delta_{\ell\ell'}$ for every $z \in V_1$ and every $\ell, \ell' \in \{1, ..., k\}$ (see, e.g, [21, Prop. 2.9]). Inserting this information in Eq. 4.24 we conclude that in a group of Heisenberg type the horizontal Laplacian is given by

$$\mathscr{L} = \Delta_z + \frac{|z|^2}{4}\Delta_\sigma + \sum_{\ell=1}^k \Theta_\ell \partial_{\sigma_\ell}. \tag{4.33}$$

Combining Eq. 4.4 and Eq. 4.33, it is clear that we presently want to solve the Cauchy problem in $\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^{2(1-s)} \times (0,\infty)$

$$\begin{cases} \Delta_z U + \Delta_w U + \frac{|z|^2+|w|^2}{4}\Delta_\sigma U + \sum_{\ell=1}^k \Theta_\ell \partial_{\sigma_\ell} U - \partial_t U = 0, \\ U(z,\sigma,w,0) = u_0(z,\sigma,w), \end{cases} \tag{4.34}$$

where the function u_0 is assigned in $\mathscr{S}\left(\mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^{2(1-s)}\right)$. Proceeding as in the Section 4.2 we now take a partial Fourier transform of the problem Eq. 4.34 with respect to the variable $\sigma \in \mathbb{R}^k$. Denoting by $\hat{U}(z,\lambda,w,t) = \int_{\mathbb{R}^k} e^{-2\pi i\langle\lambda,\sigma\rangle} U(z,\sigma,w,t)d\sigma$, and using the identity Eq. 4.30, we obtain from Eq. 4.34

$$\begin{cases} (\Delta_z + \Delta_w)\hat{U} - \pi^2|\lambda|^2(|z|^2+|w|^2)\hat{U} + 2\pi i\langle J(\lambda)z, \nabla_z\hat{U}\rangle - \partial_t\hat{U} = 0, \\ \hat{U}(z,\lambda,w,0) = \hat{u}_0(z,\lambda,w) \qquad (z,\lambda,w) \in \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^{2(1-s)},\ t>0. \end{cases}$$

At this point we make critical use of the elimination of the complex drift transformation introduced in Lemma 4.3. For $\lambda \in \mathbb{R}^k$, we define

$$V(z,w,t) = \hat{U}(e^{-2\pi i t J(\lambda)}z, \lambda, w, t).$$

We leave it to the reader to verify that, similarly to Lemma 4.3, the function V now solves the problem

$$\begin{cases} (\Delta_z + \Delta_w)V - \pi^2|\lambda|^2(|z|^2+|w|^2)V - \partial_t V = 0, \\ V(z,w,0) = \tilde{U}(z,w,0). \end{cases}$$

According to Proposition 3.2, if we denote by $\mathscr{M}$ the kernel in Eq. 3.7 with the choice of $D = \pi|\lambda| I_{m+2(1-s)}$, we deduce that V is given by

$$V(z,w,t) = \int_{\mathbb{R}^m\times\mathbb{R}^{2(1-s)}} \mathscr{M}((z,w),(\zeta,w'),t)\tilde{U}(\zeta,w',0)d\zeta dw'$$

which we can rewrite as

$$\hat{U}(z,\lambda,w,t) = \int_{\mathbb{R}^m\times\mathbb{R}^{2(1-s)}} \mathscr{M}((e^{2\pi i t J(\lambda)}z, w),(\zeta,w'),t)\hat{u}_0(\zeta,\lambda,w')d\zeta dw'.$$

By taking the inverse Fourier transform we then have

$$U(z,\sigma,w,t) = \tag{4.35}$$
$$= \int_{\mathbb{R}^m\times\mathbb{R}^k\times\mathbb{R}^{2(1-s)}} \int_{\mathbb{R}^k} e^{2\pi i\langle\sigma-\tau,\lambda\rangle} \mathscr{M}((e^{2\pi i t J(\lambda)}z, w),(\zeta,w'),t)u_0(\zeta,\tau,w')d\lambda d\zeta d\tau dw'.$$

From Eq. 4.35 we can extract the following heat kernel for the problem Eq. 4.34

$$Q((z,\sigma,w),(\zeta,\tau,w'),t) = \int_{\mathbb{R}^k} e^{2\pi i\langle\sigma-\tau,\lambda\rangle} \mathscr{M}((e^{2\pi i t J(\lambda)}z, w),(\zeta,w'),t)d\lambda.$$

By arguing as in Eq. 4.13-Eq. 4.16 (where we have deduced the expression of $\mathscr{Q}$) and similarly to Eq. 4.32, we obtain from the explicit expression of $\mathscr{M}$ that

$$\begin{aligned} &Q((z,\sigma,w),(\zeta,\tau,w'),t) \\ &= \frac{2^k}{(4\pi t)^{\frac{m}{2}+k+1-s}} \int_{\mathbb{R}^k} e^{\frac{i}{t}(\langle\tau-\sigma,\lambda\rangle + \frac{1}{2}\langle J(\lambda)\zeta,z\rangle)} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} \\ &\times\ e^{-\frac{1}{4t}\left(\frac{|\lambda|}{\tanh|\lambda|}(|z-\zeta|^2+|w|^2+|w'|^2) - 2\frac{|\lambda|}{\sinh|\lambda|}\langle w,w'\rangle\right)} d\lambda. \end{aligned} \tag{4.36}$$

If we set $w' = 0$ into Eq. 4.36, and we keep Eq. 4.20 in mind, as well as the definition of $\mathfrak{q}_{(s)}(\cdot,\cdot,\cdot)$ in Eq. 4.3, we easily realize that

$$\begin{aligned} Q((z,\sigma,w),(\zeta,\tau,0),t) &= Q(((\zeta,\tau)^{-1}\circ(z,\sigma),w),((0,0),0),t) \\ &= \mathfrak{q}_{(s)}((\zeta,\tau)^{-1}\circ(z,\sigma),t,|w|). \end{aligned}$$

The previous identities, together with the already discussed relation between the Eqs. 4.4 and 4.2, make the function

$$\mathfrak{q}_{(s)}((z,\sigma),t,y)$$

the candidate heat kernel of $\mathfrak{L}_{(s)} - \partial_t$ with pole at the origin for any value of the fractional parameter $s \in (0, 1)$. We now rigorously prove this fact, thus establishing Theorem 4.1.

On the one hand, a straightforward computation shows that $\mathfrak{q}_{(s)}((z, \sigma), t, y)$ does solve the Eq. 4.2 in $\{t > 0\}$. We are thus left with understanding in which sense the kernel $\mathfrak{q}_{(s)}((z, \sigma), t, y)$ approaches the delta-function $\delta_{(0,0,0)}$ as $t \to 0^+$. With this in mind we introduce the measure

$$d\mu \overset{def}{=} \frac{2\pi^{1-s}}{\Gamma(1-s)} y^{1-2s} dy d\sigma dz.$$

This is a natural measure for the operator $\mathfrak{L}_{(s)}$ since it is easy to check that $\mathfrak{L}_{(s)}$ is symmetric (i.e. formally self-adjoint) with respect to $d\mu$. Moreover, the renormalising constant $\frac{2\pi^{1-s}}{\Gamma(1-s)}$ has been chosen in such a way that the following lemma holds true.

Lemma 4.7 *For every $t > 0$ we have*

$$\int_{\mathbb{R}^m \times \mathbb{R}^k \times (0,\infty)} \mathfrak{q}_{(s)}((z, \sigma), t, y) d\mu = 1.$$

Proof For $M > 0$ denote

$$K_M = \left\{(z, \sigma, y) \in \mathbb{R}^m \times \mathbb{R}^k \times (0, \infty) \ : \ |\sigma| \le M\right\}.$$

We want to show that

$$\exists \lim_{M \to +\infty} \int_{K_M} \mathfrak{q}_{(s)}((z, \sigma), t, y) d\mu = 1. \tag{4.37}$$

It is easy to see that, for any fixed $t > 0$,

$$\begin{aligned} &\int_{K_M} \left|\mathfrak{q}_{(s)}((z, \sigma), t, y)\right| d\mu \\ &\le \frac{2^k}{(4\pi t)^{\frac{m}{2}+k+1-s}} \int_{K_M} \left(\int_{\mathbb{R}^k} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} d\lambda\right) e^{-\frac{|z|^2+y^2}{4t}} d\mu < \infty. \end{aligned}$$

By Fubini's theorem we are then free to choose the order of integration. With this in mind, we notice that

$$\begin{aligned} &\int_{\mathbb{R}^m} \int_0^\infty \mathfrak{q}_{(s)}((z, \sigma), t, y) \left(\frac{2\pi^{1-s}}{\Gamma(1-s)} y^{1-2s}\right) dy dz \\ &= \frac{2^k}{(4\pi t)^k} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle\sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} \left(\frac{\tanh|\lambda|}{|\lambda|}\right)^{\frac{m}{2}+1-s} d\lambda \\ &= \int_{\mathbb{R}^k} e^{-2\pi i\langle\sigma,\lambda\rangle} \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s} d\lambda, \end{aligned}$$

which implies

$$\int_{K_M} \mathfrak{q}_{(s)}((z, \sigma), t, y) d\mu = \int_{\{\sigma \in \mathbb{R}^k \,:\, |\sigma| \le M\}} \int_{\mathbb{R}^k} e^{-2\pi i\langle\sigma,\lambda\rangle} \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s} d\lambda d\sigma.$$

We now stress that, being the function $f_t(\lambda) := \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s}$ in the Schwartz class $\mathscr{S}\left(\mathbb{R}^k\right)$, its Fourier transform $\hat{f}_t$ is still in $\mathscr{S}\left(\mathbb{R}^k\right)$ and in particular in $L^1\left(\mathbb{R}^k\right)$. Hence we have

$$\exists \lim_{M\to+\infty}\int_{K_M} \mathfrak{q}_{(s)}((z,\sigma),t,y)d\mu = \lim_{M\to+\infty}\int_{\{|\sigma|\le M\}} \hat{f}_t(\sigma)d\sigma = \int_{\mathbb{R}^k} \hat{f}_t(\sigma)d\sigma = 1,$$

where in the last step we have applied the inversion theorem for the Fourier transform and the fact that $f_t(0) = 1$. This concludes the proof of Eq. 4.37. □

In order to complete the proof of Theorem 4.1 we need to show the validity of the following limiting relation:

$$\begin{aligned}&\forall \phi \in \mathscr{S}\left(\mathbb{R}^m \times \mathbb{R}^k \times [0,\infty)\right) \quad \text{we have}\\ &\lim_{t\to 0^+}\int_{\mathbb{R}^m\times\mathbb{R}^k\times(0,\infty)} \mathfrak{q}_{(s)}((z,\sigma),t,y)\phi(z,\sigma,y)d\mu = \phi(0,0,0).\end{aligned} \tag{4.38}$$

For any fixed ϕ as in Eq. 4.38, let us introduce the functions $g_0,\ g_t : \mathbb{R}^k \to \mathbb{R}$ defined, for any $t > 0$, by

$$g_0(\lambda) = \hat{\phi}(0,\lambda,0)$$

and

$$\begin{aligned}g_t(\lambda) \;=\; &\frac{2\pi^{1-s}}{\Gamma(1-s)(4\pi t)^{\frac{m}{2}+1-s}}\left(\frac{2\pi t|\lambda|}{\sinh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s}\\ &\times\int_{\mathbb{R}^m\times(0,\infty)} e^{-\frac{|z|^2+y^2}{4t}\frac{2\pi t|\lambda|}{\tanh 2\pi t|\lambda|}}\hat{\phi}(z,\lambda,y)y^{1-2s}dydz.\end{aligned}$$

We notice that $g_0,\ g_t \in \mathscr{S}\left(\mathbb{R}^k\right)$. We want to show that

$$g_t \to g_0 \quad \text{in } L^1\left(\mathbb{R}^k\right) \quad \text{as } t \to 0^+. \tag{4.39}$$

In order to prove Eq. 4.39 we first notice that, by exploiting the change of variables

$$z \mapsto \xi = \frac{z}{\sqrt{4t}}\sqrt{\frac{2\pi t|\lambda|}{\tanh 2\pi t|\lambda|}} \quad \text{and} \quad y \mapsto \eta = \frac{y^2}{4t}\frac{2\pi t|\lambda|}{\tanh 2\pi t|\lambda|},$$

we have

$$\begin{aligned}g_t(\lambda) \;=\; &\frac{1}{\Gamma(1-s)\pi^{\frac{m}{2}}}\left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s}\\ &\times\int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s}e^{-|\xi|^2-\eta}\hat{\phi}\left(\xi\sqrt{4t\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}},\lambda,\sqrt{4t\eta\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}\right)d\eta d\xi.\end{aligned}$$

Therefore, using also the identity $\int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s} e^{-|\xi|^2-\eta} d\eta d\xi = \Gamma(1-s)\pi^{\frac{m}{2}}$ and recalling that $g_0(\lambda) = \hat{\phi}(0, \lambda, 0)$, we can write

$$\begin{aligned}
&\int_{\mathbb{R}^k} |g_t(\lambda) - g_0(\lambda)|\, d\lambda = \frac{1}{\Gamma(1-s)\pi^{\frac{m}{2}}} \int_{\mathbb{R}^k} \Bigg| \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s} \int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s} e^{-|\xi|^2-\eta} \\
&\times \hat{\phi}\left(\xi\sqrt{4t\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}, \lambda, \sqrt{4t\eta\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}\right) d\eta d\xi + \\
&- g_0(\lambda) \int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s} e^{-|\xi|^2-\eta} d\eta d\xi \left(\left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s} + 1 - \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s}\right) \Bigg| d\lambda \\
&\le \frac{1}{\Gamma(1-s)\pi^{\frac{m}{2}}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s} e^{-|\xi|^2-\eta} \times \\
&\times \left| \hat{\phi}\left(\xi\sqrt{4t\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}, \lambda, \sqrt{4t\eta\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}\right) - \hat{\phi}(0, \lambda, 0) \right| d\eta d\xi d\lambda + \\
&+ \frac{1}{\Gamma(1-s)\pi^{\frac{m}{2}}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s} e^{-|\xi|^2-\eta} \hat{\phi}(0, \lambda, 0) \left(1 - \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s}\right) d\eta d\xi d\lambda.
\end{aligned}$$

Since $\phi \in \mathscr{S}\left(\mathbb{R}^m \times \mathbb{R}^k \times [0, \infty)\right)$, we know that there exists $C_\phi > 0$ such that

$$\left|\hat{\phi}(z, \lambda, y)\right| \le \frac{C_\phi}{1 + |\lambda|^{k+1}} \quad \text{for all } (z, \lambda, y) \in \mathbb{R}^m \times \mathbb{R}^k \times (0, \infty)$$

and that moreover

$\hat{\phi}(z, \lambda, y)$ is continuous at $(z, y) = (0, 0)$ (the continuity is actually uniform in $\lambda \in \mathbb{R}^k$).

The last two properties allow to exploit the dominated convergence theorem and infer that:

$$\lim_{t\to 0^+} \int_{\mathbb{R}^k} \int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s} e^{-|\xi|^2-\eta} \hat{\phi}(0, \lambda, 0) \left(1 - \left(\frac{1}{\cosh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s}\right) d\eta d\xi d\lambda = 0,$$

and

$$\begin{aligned}
&\lim_{t\to 0^+} \int_{\mathbb{R}^k} \int_{\mathbb{R}^m\times(0,\infty)} \eta^{-s} e^{-|\xi|^2-\eta} \times \\
&\qquad \times \left| \hat{\phi}\left(\xi\sqrt{4t\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}, \lambda, \sqrt{4t\eta\frac{\tanh 2\pi t|\lambda|}{2\pi t|\lambda|}}\right) - \hat{\phi}(0, \lambda, 0) \right| d\eta d\xi d\lambda = 0.
\end{aligned}$$

In turn, these limiting relations imply that

$$\int_{\mathbb{R}^k} |g_t(\lambda) - g_0(\lambda)|\, d\lambda \underset{t\to 0^+}{\longrightarrow} 0.$$

This concludes the proof of Eq. 4.39. From Eq. 4.39 and the continuity of the Fourier transform from L^1 to L^∞, we deduce that

$$\check{g}_t \to \check{g}_0 \quad \text{in } L^\infty\left(\mathbb{R}^k\right) \quad \text{as } t \to 0^+, \tag{4.40}$$

where we have used the notation $\check{g}$ for the inverse Fourier transform $\check{g}(\sigma) = \int_{\mathbb{R}^k} e^{2\pi i\langle\lambda,\sigma\rangle} g(\lambda) d\lambda$. Moreover, for any $\sigma_0 \in \mathbb{R}^k$, we have $\check{g}_0(\sigma_0) = \phi(0, \sigma_0, 0)$ and, by an application of Fubini's theorem, also

$$\begin{aligned}\check{g}_t(\sigma_0) &= \frac{2\pi^{1-s}}{\Gamma(1-s)(4\pi t)^{\frac{m}{2}+1-s}} \\ &\times \int_{\mathbb{R}^k}\int_{\mathbb{R}^m\times(0,\infty)} e^{2\pi i\langle\sigma_0,\lambda\rangle}\left(\frac{2\pi t|\lambda|}{\sinh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2+y^2}{4t}\frac{2\pi t|\lambda|}{\tanh 2\pi t|\lambda|}}\hat{\phi}(z,\lambda,y)y^{1-2s}dydzd\lambda \\ &= \frac{1}{(4\pi t)^{\frac{m}{2}+1-s}}\int_{\mathbb{R}^m\times\mathbb{R}^k\times(0,\infty)}\int_{\mathbb{R}^k} e^{-2\pi i\langle\sigma-\sigma_0,\lambda\rangle}\left(\frac{2\pi t|\lambda|}{\sinh 2\pi t|\lambda|}\right)^{\frac{m}{2}+1-s} \\ &\times e^{-\frac{|z|^2+y^2}{4t}\frac{2\pi t|\lambda|}{\tanh 2\pi t|\lambda|}}\phi(z,\sigma,y)d\lambda d\mu \\ &= \int_{\mathbb{R}^m\times\mathbb{R}^k\times(0,\infty)} \mathfrak{q}_{(s)}((z,\sigma-\sigma_0),t,y)\phi(z,\sigma,y)d\mu.\end{aligned}$$

Hence, Eq. 4.40 implies that

$$\int_{\mathbb{R}^m\times\mathbb{R}^k\times(0,\infty)} \mathfrak{q}_{(s)}((z,\sigma-\sigma_0),t,y)\phi(z,\sigma,y)d\mu \underset{t\to 0^+}{\longrightarrow} \phi(0,\sigma_0,0) \quad \text{uniformly for } \sigma_0 \in \mathbb{R}^k.$$

In particular, when applied to $\sigma_0 = 0$, this completes the proof of Eq. 4.38 and finishes the proof of Theorem 4.1.

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Declarations

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