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Separable Cowreaths Over Clifford Algebras

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Abstract. The fundamental notion of separability for commutative algebras was interpreted in categorical setting where also the stronger notion of heavily separability was introduced. These notions were extended to (co)algebras in monoidal categories, in particular to cowreaths. In this paper, we consider the cowreath $(A \otimes H_4^{op}, H_4, \psi)$, where H_4 is the Sweedler 4-dimensional Hopf algebra over a field k and $A = Cl(\alpha, \beta, \gamma)$ is the Clifford algebra generated by two elements G, X with relations $G^2 = \alpha$, $X^2 = \beta$ and $XG + GX = \gamma$, $(\alpha, \beta, \gamma \in k)$ which becomes naturally an H_4-comodule algebra. We show that, when char $(k) \neq 2$, this cowreath is always separable and h-separable as well.

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1. Introduction

The general theory of separable algebras over commutative rings was introduced by Auslander and Goldmann [1] . This fundamental concept has many applications in almost all areas of algebra (cf. [10]). The classical notion of separability for algebras over commutative rings can be interpreted in categorical terms. This was done in [13] where the notion of separable functor was introduced and investigated. Recently, motivated by an example related to the tensor algebra, in [2], a stronger version of separable functor, called heavely separable, was introduced and investigated.

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The notion of separability can be extended to (co)algebras in monoidal categories (see [6]). In [12], we investigated h-coseparable coalgebras in monoidal categories. In particular, we showed that, for a (strict) monoidal category $\mathcal{C}$ with $\otimes$-generator unit object, the forgetful functor $F : \mathcal{C}^C \to \mathcal{C}$ is (heavily) separable if and only if the coalgebra C is (heavily) coseparable i.e. there exists a morphism $B : C \otimes C \to 1$ in $\mathcal{C}$ such that conditions

$$(C \otimes B) \circ (\Delta \otimes C) = (B \otimes C) \circ (C \otimes \Delta) \ \text{(Casimir condition)} \tag{1}$$

$$B \cdot \Delta = \varepsilon, \ \text{(normalized condition)} \tag{2}$$

$$(B \otimes B) \cdot (C \otimes \Delta \otimes C) = B \cdot (C \otimes \varepsilon \otimes C) \ \text{(h-coseparability condition)} \tag{3}$$

hold.

Given an algebra A in a monoidal category $\mathcal{C}$ one can construct a new category $\mathcal{T}_A^{\#}$. The objects of this category are pair (X, ψ), where X is an object in $\mathcal{C}$ and $\psi : X \otimes A \to A \otimes X$ is a morphism in $\mathcal{C}$ compatible with algebra structure of A. A triple (A, X, ψ) is called a cowreath in $\mathcal{C}$ if (X, ψ) a coalgebra in $\mathcal{T}_A^{\#}$. This is a kind of generalized entwined structure (as introduced by Brzezinski in [3] in order to develop Galois theory for coalgebras) and its module category, the category $\mathcal{C}(\psi)_A^X$ of entwined modules over cowreath, include other well-know structures such that relative Hopf modules, Doi–Hopf modules and Yetter-Drinfeld modules. This structure is fundamental for developing a Galois theory for quasi-Hopf algebras (see details in [8]). In [9], separable functors for the category of Doi–Hopf modules were studied and generalizations for cowreath have been given in [4,5] and for Galois cowreath in [7]. Then the h-separability was considered in [12]. We proved ([12, Theorem 3.6]) that for a cowreath (A, X, ψ) in $\mathcal{C}$, the h-separability of the forgetful functor $F : \mathcal{C}^C \to \mathcal{C}$ is equivalent to the h-separability of the coalgebra (X, ψ) in $\mathcal{T}_A^{\#}$.

In this paper, we continue the study of cowreaths constructed on two-sided Hopf modules (see [5]). Namely, let H be a Hopf algebra over a field k and let A be a right H-comodule algebra. Thus $A \otimes H^{op}$ is a right $H \otimes H^{op}$-comodule algebra and H is a $H \otimes H^{op}$-module coalgebra and we can construct a cowreath $(A \otimes H^{op}, H)$ in the category of k-vector space. In [12, Theorem 5.1], we determined the separability and h-separability of a cowreath $(A \otimes H^{op}, H, \psi)$.

In [11], A. Masuoka classified cleft extensions for the $H = H_4$ the Sweedler 4-dimensional Hopf algebra over a field k. Naturally, the k-algebra $A = Cl(\alpha, \beta, \gamma)$ generated by two elements G, X and relations $G^2 = \alpha$, $X^2 = \beta$ and $XG + GX = \gamma$, a generalization of the Clifford algebra appeared (note that the usual 4 -dimensional Clifford algebra is one of this type).

We shown that the cowreath $(A \otimes H^{op}, H, \psi)$ is always separable and h-separable. Namely in Theorem 1 (resp. in Theorem 2) we determine all the conditions the bilinear form B should satisfy.

2. Preliminaries

In the following we will adopt all definitions and notations in [4] except for the unit of an algebra A, that we will denote by $u : \underline{1} \to A$.

Let $\mathcal{C}$ be a (strict) monoidal category and let (A, m, u) be an algebra in $\mathcal{C}$. Recall that a *(right) transfer morphism through* A is a pair (X, ψ) with $X \in \mathcal{C}$ and

$$\psi : X \otimes A \to A \otimes X \text{ in } \mathcal{C}$$

such that

$$\begin{aligned} \psi \circ (X \otimes m) &= (X \otimes m) \circ (A \otimes \psi) \circ (\psi \otimes A) \\ \psi \circ (X \otimes u) &= u \otimes X \end{aligned}$$

The category $\mathcal{T}_A^\#$ has objects the right transfers. A morphism $f : X \to Y$ in $\mathcal{T}_A^\#$ is a morphism $f : X \to A \otimes Y$ in $\mathcal{C}$ such that

$$(m \otimes Y) \circ (A \otimes f) \circ \psi = (m \otimes Y) \circ (A \otimes \psi) \circ (f \otimes A)$$

The composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{T}_A^\#$ is

$$g \odot f = (m \otimes Z) \circ (A \otimes g) \circ f$$

and

$$\mathrm{Id}_{(X,\psi)} = u \otimes X.$$

The tensor product of (X, ψ_X) and (Y, ψ_Y) is

$$X \circledast Y = (X \otimes Y, \psi_X \odot \psi_Y = (\psi_X \otimes Y) \circ (X \otimes \psi_Y))$$

The tensor product of $f : X \to X'$ and $g : Y \to Y'$ is

$$f \circledast g = (m \otimes X' \otimes Y') \circ (A \otimes \psi_{X'} \otimes Y') \circ (f \otimes g)\,.$$

The unit object is

$$\left(\underline{1}, r_A^{-1} \circ l_A : \underline{1} \otimes A \to A \otimes \underline{1}\right)$$

Recall also that a *cowreath in* $\mathcal{C}$ is a triple (A, X, ψ) where A is an algebra in $\mathcal{C}$ and (X, ψ) is a coalgebra in $\mathcal{T}_A^\#$. This means that $(X, \psi) \in \mathcal{T}_A^\#$ and there are morphisms

$$\delta : X \to A \otimes X \otimes X \text{ and } \varepsilon : X \to A$$

such that

$$\begin{aligned} &(m \otimes X \otimes X) \circ (A \otimes \psi \otimes X) \circ (A \otimes X \otimes \psi) \otimes (\delta \otimes A) \\ &\quad = (m \otimes X \otimes X) \circ (A \otimes \delta) \circ \psi \\ &\quad\quad \text{i.e. } \delta \text{ is a morphism in } \mathcal{T}_A^\# \end{aligned} \tag{4}$$

$$\begin{aligned} &(m \otimes X \otimes X \otimes X \otimes X) \circ (A \otimes \delta \otimes X) \circ \delta \\ &\quad = (m \otimes X \otimes X \otimes X) \circ (A \otimes \psi \otimes X \otimes X) \circ (A \otimes X \otimes \delta) \circ \delta \\ &\quad\quad \text{coassociativity} \end{aligned} \tag{5}$$

$$m \circ (A \otimes \varepsilon) \circ \psi = m \circ (\epsilon \otimes A) \ \text{ i.e. } \varepsilon \text{ is a morphism in } \mathcal{T}_A^\# \tag{6}$$

$$(m \otimes X) \circ (A \otimes \varepsilon \otimes X) \circ \delta = u \otimes X \text{ left counit property} \tag{7}$$

$$\begin{aligned} &(m \otimes X) \circ (A \otimes \psi) \circ (A \otimes X \otimes \varepsilon) \circ \delta \\ &\quad = u \otimes X \text{ right counit property} \end{aligned} \tag{8}$$

We also recall that an *entwined module over a cowreath* (A, X, ψ) is a pair $(M, \rho : M \to M \otimes X)$ where $(M, \mu) \in \mathcal{C}_A$ satisfying

$$(\rho \otimes X) \circ \rho = (\mu \otimes X \otimes X) \circ (M \otimes \delta) \circ \rho \text{ coassociativity}$$
$$\mu \circ (M \otimes \varepsilon) \circ \rho = \mathrm{Id}_M \text{ counitality}$$
$$\rho \circ \mu = (\mu \otimes X) \circ (M \otimes \psi) \circ (\rho \otimes A) \text{ A-linearity}$$

A morphism between entwined modules is a A-linear morphism $f : M \to N$ such that $(f \otimes X) \circ \rho = \rho \circ f$.

The category of entwined modules will be denoted by $\mathcal{C}(\psi)_A^X$.

Let H be a Hopf algebra, A a right H-comodule algebra. $A \otimes H^{op}$ is a right $H \otimes H^{op}$-comodule algebra with

$$\rho_{A\otimes H^{op}} : A \otimes H^{op} \to A \otimes H^{op} \otimes H \otimes H^{op},$$
$$\rho_{A\otimes H^{op}}(a \otimes h) = (a_0 \otimes h_1) \otimes (a_1 \otimes h_2).$$

H is a right $H \otimes H^{op}$-module coalgebra via

$$h\left(h' \otimes h''\right) = h'' h h'$$

and H can be seen as a coalgebra in $\mathcal{M}_{H\otimes H^{op}} = {}_H\mathcal{M}_H$. We consider

$$\psi : H \otimes A \otimes H^{op} \to A \otimes H^{op} \otimes H$$
$$h \otimes a \otimes l^{op} \mapsto a_0 \otimes l_1 \otimes l_2 h a_1.$$

Then $(H, \psi) \in \mathcal{T}^{\#}_{A\otimes H^{op}}$ and (H, ψ) is a coalgebra in $\mathcal{T}^{\#}_{A\otimes H^{op}}$ via

$$\begin{aligned}
\delta &: H \to A \otimes H^{op} \otimes H \otimes H \\
\delta(h) &= 1_A \otimes 1_H \otimes h_1 \otimes h_2 \\
\epsilon &: H \to A \otimes H^{op} \\
\epsilon(h) &= \varepsilon_H(h) 1_A \otimes 1_H
\end{aligned}$$

The category of Doi–Hopf modules $\mathcal{M}(H \otimes H^{op})^H_{A\otimes H^{op}}$ is isomorphic to the category ${}_H\mathcal{M}^H_A$ of two-sided (H, A)-bimodules over H.

A **Casimir morphism** consists of a k-linear map $B : H \otimes H \to A \otimes H^{op}$ with the following properties:

1. **Casimir condition**

$$\begin{aligned}
&m_{A\otimes H^{op}} \circ (A \otimes H^{op} \otimes B) \circ (\psi \otimes H) \circ (H \otimes \psi) \\
&\quad = (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes \psi) \circ (A \otimes H^{op} \otimes H \otimes B) \circ (\delta \otimes H)
\end{aligned}$$

2. **Morphism condition**

i.e. B is a morphism in $\mathcal{T}^{\#}_{A\otimes H^{op}}$:

$$\begin{aligned}
&m_{A\otimes H^{op}} \circ (B \otimes A \otimes H^{op}) \\
&\quad = m_{A\otimes H^{op}} \circ (A \otimes H^{op} \otimes B) \circ (\psi \otimes A \otimes H) \circ (H \otimes \psi)
\end{aligned}$$

and we consider also the condition

3. **Normalized condition**

$$m_{A\otimes H^{op}} \circ (A \otimes H^{op} \otimes B) \circ \delta = \epsilon$$

We denote by H_4 the Sweedler 4-dimensional Hopf algebra. It is generated by g, x with the relations $g^2 = 1_H$, $x^2 = 0$ and $xg = -gx$. The coalgebra structure is given by $\Delta(g) = g\otimes g, \Delta(x) = x\otimes g+1_H\otimes x, \epsilon(g) = 1$ and $\epsilon(x) = 0$ and the antipode $S(g) = g, S(x) = gx$. It is well-know that $t = x + gx$ (resp. $r = x - gx$) is left (resp. right) integral in H_4.

We consider the Clifford algebra $A = Cl(\alpha, \beta, \gamma)$ generated by G, X with the relations $G^2 = \alpha$, $X^2 = \beta$ and $XG+GX = \gamma$. This Clifford algebra is a right $H = H_4$-comodule algebra via $1_A \to 1_A \otimes 1_H, G \to G \otimes g, X \to X \otimes g + 1_A \otimes x$.

The comultiplication on H_4 can be written:

$$\Delta(g^p x^q) = \sum_{l=0}^{q} g^p x^{q-l} \otimes g^{p+q-l} x^l$$

Let $H = H_4$ and $A = Cl(\alpha, \beta, \gamma)$. The H-coaction on A is given by:

$$\rho_A(G^p X^q) = \sum_{l=0}^{q} G^p X^{q-l} \otimes g^{p+q-l} x^l$$

The bilinear form $B : H \otimes H \to A \otimes H$ is given $h \otimes h' \to B(h, h')$ where in term of bases

$$B(g^i x^j \otimes g^k x^l) =$$
$$= \sum_{m,n,p,q=0}^{1} B(g^i x^j \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q$$

for $0 \leq i, j, k, l \leq 1$.

3. Normalized Condition

The normalized condition is

$$B(h_1 \otimes h_2) = \epsilon(h) 1_A \otimes 1_H \text{ for all } h \in H_4.$$

Hence

$$B(1_H \otimes 1_H) = 1_A \otimes 1_H, \tag{9}$$
$$B(g \otimes g) = 1_A \otimes 1_H, \tag{10}$$
$$B(x \otimes g) + B(1_H \otimes x) = 0, \tag{11}$$
$$B(gx \otimes 1_H) + B(g \otimes gx) = 0. \tag{12}$$

4. Casimir Condition

The Casimir condition is

$$m_{A\otimes H^{op}} \circ (A \otimes H^{op} \otimes B) \circ (\psi \otimes H) \circ (H \otimes \psi) =$$
$$= (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes \psi) \circ (A \otimes H^{op} \otimes H \otimes B) \circ (\delta \otimes H)$$

Thus we get

$$(m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes \psi) \circ (A \otimes H^{op} \otimes H \otimes B))$$
$$\circ (\delta \otimes H)(g^i x^j \otimes g^k x^l) =$$

$$= (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes \psi) \circ (A \otimes H^{op} \otimes H \otimes B))$$
$$\left(\sum_{a=0}^{j} 1_A \otimes 1_H \otimes g^i x^{j-a} \otimes g^{i+j-a} x^a \otimes g^k x^l\right)$$
$$= \sum_{a=0}^{j} (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes \psi)$$
$$(1_A \otimes 1_H \otimes g^i x^{j-a} \otimes B(g^{i+j-a} x^a \otimes g^k x^l))$$
$$= \sum_{a=0}^{j} \sum_{m,n,p,q=0}^{1} (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes \psi)$$
$$(1_A \otimes 1_H \otimes g^i x^{j-a} \otimes B(g^m x^n \otimes g^p x^q q; G^{i+j-a} X^a, g^k xl)(G^m X^n \otimes g^p x^q))$$

Now we compute $\psi(g^i x^{j-a} \otimes G^m X^n \otimes g^p x^q)$. We have

$$\rho_A(G^m X^n) = \sum_{b=0}^{n} G^m X^{n-b} \otimes g^{m+n-b} x^b$$

and

$$\Delta(g^p x^q) = \sum_{c=0}^{q} g^p x^{q-c} \otimes g^{p+q-c} x^c.$$

Hence

$$\psi(g^i x^{j-a} \otimes G^m X^n \otimes g^p x^q)$$
$$= \sum_{b=0}^{n} \sum_{c=0}^{q} G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c} x^c g^i x^{j-a} g^{m+n-b} x^b$$
$$= \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{ic+(c+j-a)(m+n-b)} G^m X^{n-b} \otimes g^p x^{q-c} \otimes$$
$$\otimes g^{p+q-c+i+m+n-b} x^{c+j-a+b}.$$

Finally the left side of the equation is:

$$\sum_{a=0}^{j} \sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\alpha} B(g^{\overline{i+j-a}} x^a \otimes g^k x^l; G^m X^n, g^p x^q)$$
$$G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c+i+m+n-b} x^{c+j-a+b}$$

where $\alpha = ic + (c + j - a)(m + n - b)$.

The right side of the equation is:

$$(m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes B \otimes H) \circ (\psi \otimes H \otimes H) \circ (H \otimes \delta)$$
$$(g^i x^j \otimes g^k x^l)$$
$$= \sum_{a=0}^{l} (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes B \otimes H)$$
$$\circ (\psi \otimes H \otimes H)(g^i x^j \otimes 1_A \otimes 1_H \otimes g^k x^{l-a} \otimes g^{k+l-a} x^a)$$

$$= \sum_{a=0}^{l} (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes B \otimes H)(\psi(g^i x^j \otimes 1_A \otimes 1_H) \otimes g^k x^{l-a} \otimes g^{k+l-a} x^a)$$

$$= \sum_{a=0}^{l} (m_{A\otimes H^{op}} \otimes H) \circ (A \otimes H^{op} \otimes B \otimes H) (1_A \otimes 1_H \otimes g^i x^j \otimes g^k x^{l-a} \otimes g^{k+l-a} x^a)$$

$$= \sum_{a=0}^{l} (m_{A\otimes H^{op}} \otimes H)(1_A \otimes 1_H \otimes B(g^i x^j \otimes g^k x^{l-a}) \otimes g^{k+l-a} x^a)$$

$$= \sum_{a=0}^{l} \sum_{m,n,p,q}^{1} B(g^m x^n \otimes g^p x^q q; G^i X^j, g^k x^{l-a}) G^m X^n \otimes g^p x^q \otimes g^{k+l-a} x^a$$

The final equation is:

$$\sum_{a=0}^{j} \sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\alpha} B(g^{\overline{i+j-a}} x^a \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c+i+m+n-b} x^{c+j-a+b}$$
$$= \sum_{a=0}^{l} \sum_{m,n,p,q=0}^{1} B(g^i x^j \otimes g^k x^{l-a}; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes g^{k+l-a} x^a$$

where $\alpha = ic + (c+j-a)(m+n-b)$.

Now we are going to consider the sixteen occurrences of (i, j, k, l) which will be subdivided in groups sharing the same i and j.

4.1. Case $i = 1, j = 0$

The left side of the equation is

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\alpha} B(g \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes \otimes g^{m+n+p+q-c-b+1} x^{c+b}$$

where $\alpha = c + c(m+n-b)$. The terms with third component 1_H are in the sum

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\alpha} B(g \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes \otimes g^{m+n+p+q-c-b+1} x^{c+b}$$

with $c + b = 0$ i.e. $c = b = 0$ and $m + n + p + q = 1, 3$. Then $\alpha = 0$ and we obtain

$$\sum_{\substack{m,n,p,q=0,\\ p+q+m+n=1}}^{1} B(g \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^q \otimes 1_H. \qquad (13)$$

The terms with third component g are in the sum

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{m+n+p+q-c-b+1}x^{c+b}$$

with $c+b=0$ and $m+n+p+q=0,2,4$. Then $\alpha=0$ and we obtain

$$\sum_{\substack{m,n,p,q=0,\\ p+q+m+n=0}}^{1} B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^q\otimes g \tag{14}$$

The terms with x as the third component are in the sum

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{m+n+p+q-c-b+1}x^{c+b}$$

with $c+b=1$ and $m+n+p+q=0,2,4$. The sum simplifies to

$$\sum_{\underline{m,n,p,q}=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}\otimes x$$

there are two cases:

- $b=1$, $c=0$, then $n=1$ and the sum $m+p+q=1,3$. Therefore the sum simplifies

$$\sum_{m,p,q=0}^{1} B(g\otimes g^kx^l;G^mX,g^px^q)G^m\otimes g^px^q\otimes x$$
$$=B(g\otimes g^kx^l;GX,1_H)G\otimes 1_H\otimes x+B(g\otimes g^kx^l;X,g)1_A\otimes g\otimes x+$$
$$+B(g\otimes g^kx^l;X,x)1_A\otimes x\otimes x+B(g\otimes g^kx^l;GX,gx)G\otimes gx\otimes x$$

- $b=0$, $c=1$, then $q=1$ and the sum $m+n+p=1,3$. Therefore the sums simplifies

$$\sum_{m,n,p=0}^{1}(-1)^{m+n+1}B(g\otimes g^kx^l;G^mX^n,g^px)G^mX^n\otimes g^p\otimes x$$
$$=B(g\otimes g^kx^l;G,x)G\otimes 1_H\otimes x+B(g\otimes g^kx^l;X,x)X\otimes 1_H\otimes x+$$
$$-B(g\otimes g^kx^l;1_A,gx)1_A\otimes g\otimes x-B(g\otimes g^kx^l;GX,gx)GX\otimes g\otimes x$$

Hence we get

$$\begin{aligned}&\left[B(g\otimes g^kx^l;X,g)-B(g\otimes g^kx^l;1_A,gx)\right]1_A\otimes g\otimes x\\&\quad+B(g\otimes g^kx^l;X,x)1_H\otimes x\otimes x+\\&\quad+\left[B(g\otimes g^kx^l;GX,1_H)+B(g\otimes g^kx^l;G,x)\right]G\otimes 1_H\otimes x\\&\quad+B(g\otimes g^kx^l;GX,gx)G\otimes gx\otimes x+\\&\quad+B(g\otimes g^kx^l;X,x)X\otimes 1_H\otimes x\\&\quad-B(g\otimes g^kx^l;GX,gx)GX\otimes g\otimes x\end{aligned} \tag{15}$$

The sums of the terms with gx in the third component appear with $c+b=1$ and $p+q+m+n=1,3$,

$$\sum_{m,n,p,q=0,\, m+n+p+q=0,2,4}^{1} \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes g^kx^l;G^mX^n,g^px^q) G^mX^{n-b}\otimes g^px^{q-c}\otimes gx$$

There are two cases:

- $b=1$ and $c=0$, then $n=1$ and $\alpha=0$.

$$\sum_{\substack{m,p,q=0,\\ m+p+q=0}}^{1} B(g\otimes g^kx^l;G^mX,g^px^q)G^m\otimes g^px^q\otimes gx$$

we obtain

$$B(g\otimes g^kx^l;X,1_H)1_A\otimes 1_H\otimes gx+B(g\otimes g^kx^l;GX,g)G\otimes g\otimes gx +B(g\otimes g^kx^l;GX,x)G\otimes x\otimes gx+B(g\otimes g^kx^l;X,gx)1_A\otimes gx\otimes gx$$

- $c=1$ and $b=0$, then $q=1$ and $\alpha=1+m+n$.

$$\sum_{\substack{m,n,p,q=0,\\ p+m+n=1,3}}^{1} (-1)^{m+n+1}B(g\otimes g^kx^l;G^mX^n,g^px)G^mX^n\otimes g^p\otimes gx$$

and we get

$$-B(g\otimes g^kx^l;1_A,x)1_A\otimes 1_H\otimes gx-B(g\otimes g^kx^l;GX,x)GX\otimes 1_H\otimes gx +B(g\otimes g^kx^l;G,gx)G\otimes g\otimes gx+B(g\otimes g^kx^l;X,gx)X\otimes g\otimes gx$$

Therefore the terms in gx in the third component are

$$\begin{aligned}&\left[B(g\otimes g^kx^l;X,1_H)-B(g\otimes g^kx^l;1_A,x)\right]1_A\otimes 1_H\otimes gx\\ &+B(g\otimes g^kx^l;X,gx)1_A\otimes gx\otimes gx+\\ &+\left[B(g\otimes g^kx^l;GX,g)+B(g\otimes g^kx^l;G,gx)\right]G\otimes g\otimes gx\\ &+B(g\otimes g^kx^l;GX,x)G\otimes x\otimes gx\\ &+B(g\otimes g^kx^l;X,gx)X\otimes g\otimes gx-B(g\otimes g^kx^l;GX,x)GX\otimes 1_H\otimes gx\end{aligned} \tag{16}$$

4.1.1. $i=1, j=k=l=0$. This corresponds to $B(g\otimes 1_H)$.

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes 1_H;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c} \otimes g^{m+n+p+q-c-b+1}x^{c+b}$$
$$=\sum_{m,n,p,q}^{1} B(g\otimes 1_H;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes 1_H$$

where $\alpha=c+c(m+n-b)$.

By looking at the terms with 1_H in the third component and using (13) we obtain

$$B(g\otimes 1_H;G^mX^n,g^px^q)=0 \text{ when } \overline{p+q+m+n}=0 \tag{17}$$

By considering the terms with g in the third component and using both (14) and (17) we realize that they are already zero. The same holds for terms with x in the third component. For the terms with gx in the third component we get, in view of (16)

$$\begin{aligned}
&B\left(g\otimes 1_H;X,1_H\right)-B\left(g\otimes 1_H;1_A,x\right)=0\\
&B\left(g\otimes 1_H;GX,g\right)+B\left(g\otimes 1_H;G,gx\right)=0\\
&B(g\otimes 1_H;X,gx)=0, B(g\otimes 1_H;GX,x)=0\\
&B(g\otimes 1_H;X,gx)=0, B(g\otimes 1_H;GX,x)=0.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
B(g\otimes 1_H)=&\ B\left(g\otimes 1_H;1_A,x\right)1_A\otimes g+B\left(g\otimes 1_H;1_A,x\right)1_A\otimes x\\
&+B\left(g\otimes 1_H;1_A,x\right)X\otimes 1_H+B\left(g\otimes 1_H;G,1_H\right)G\otimes 1_H\\
&+B\left(g\otimes 1_H;G,gx\right)G\otimes gx-B\left(g\otimes 1_H;G,gx\right)GX\otimes g
\end{aligned} \tag{18}$$

4.1.2. $i=1,j=0,k=0,l=1$. This corresponds to $B(g\otimes x)$.

$$\begin{aligned}
&\sum_{m,n,p,q=0}^{1}\ \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes x;G^mX^n,g^px^q)G^mX^{n-b}\\
&\quad\otimes g^px^{q-c}\otimes g^{m+n+p+q-c-b+1}x^{c+b}\\
&=\sum_{m,n,p,q}^{1}B(g\otimes x;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g\\
&\quad+\sum_{m,n,p,q}^{1}B(g\otimes 1_H;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes x
\end{aligned}$$

where $\alpha=c(1+m+n-b)$. By looking at the terms with 1_H in the third component and using (13) we obtain

$$B(g\otimes x;G^mX^n,g^px^q)=0 \text{ for } \overline{m+n+p+q}=1. \tag{19}$$

Using this relation a straightforward computation, similar to the previous one, gives us:

$$\begin{aligned}
B(g\otimes x)=&\ B(g\otimes x;1_A,1_H)1_A\otimes 1_H+B(g\otimes x;1_A,gx)1_A\otimes gx\\
&+B(g\otimes x;G,g)G\otimes g+B(g\otimes x;G,x)G\otimes x\\
&+B(g\otimes 1_H;1_A,x)X\otimes x+[B(g\otimes x;1_A,gx)\\
&+B(g\otimes 1_H;1_A,g)]\,X\otimes g\\
&+\left[B(g\otimes 1_H;G,1_H)-B\left(g\otimes x;G,x\right)\right]GX\otimes 1_H\\
&+B(g\otimes 1_H;G,gx)GX\otimes gx
\end{aligned} \tag{20}$$

4.1.3. $\boldsymbol{i=1, j=0, k=1, l=0.}$ This correspond to $B(g\otimes g)$ we now that it has to be $1_A\otimes 1_H$ by (10) .

4.1.4. $\boldsymbol{i=1, j=0, k=1, l=1.}$ This correspond to $B(g\otimes gx)$.

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes gx;G^mX^n,g^px^q)G^mX^{n-b}$$
$$\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+b}$$
$$=\sum_{a=0}^{1}\sum_{m,n,p,q}^{1}B(g\otimes gx^{1-a};G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g^{k+l-a}x^a$$

where $\alpha=c(1+m+n-b)$. By considering the terms with g in the third component and using (14) we get

$$B(g\otimes gx;G^mX^n,g^px^q)=0 \text{ for } \overline{m+n+p+q}=0. \tag{21}$$

By using both (13), (21) , (15), (16) and (10), a straightforward computation shows that

$$\begin{aligned}B(g\otimes gx) = {} & B(g\otimes gx;1_A,g)1_A\otimes g+B(g\otimes gx;1_A,x)1_A\otimes x\\ & +B(g\otimes gx;G,1_H)G\otimes 1_H+B(g\otimes gx;G,gx)G\otimes gx\\ & +[B(g\otimes gx;1_A,x)+1]X\otimes 1_H-B(g\otimes gx;G,gx)GX\otimes g\end{aligned} \tag{22}$$

4.2. Case $\boldsymbol{i=1, j=1}$

The left side of the equation is

$$\sum_{a=0}^{1}\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g^{\overline{2-a}}x^a\otimes g^kx^l;G^mX^n,g^px^q)$$
$$G^mX^{n-b}\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+1-a+b}$$

where $\alpha=c+(c+1-a)(m+n-b)$ i.e.

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(1_H\otimes g^kx^l;G^mX^n,g^px^q)$$
$$G^mX^{n-b}\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+1+b}+$$
$$+\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(gx\otimes g^kx^l;G^mX^n,g^px^q)$$
$$G^mX^{n-b}\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+b}$$

where $\alpha=c+(c+1)(m+n-b)$ and $\beta=c+c(m+n-b)$. We consider terms with 1_H in the third component: they are in the sum

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(gx\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}$$
$$\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+b}$$

with $c + b = 0$ and $\overline{p + q - c + 1 + m + n - b} = 0$, i.e. $b = c = 0$ and $\overline{p + q + m + n} = 1$. The sum becomes

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=1}}^{1} B(gx \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes 1_H \tag{23}$$

We consider terms with g in the third component. They are in

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\beta} B(gx \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c+1+m+n-b} x^{c+b}$$

$\beta = c + c(m + n - b)$, with $c + b = 0$ and $\overline{p + q + 1 + m + n} = 1$, i.e. $\overline{p + q + m + n} = 0$, then

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=0}}^{1} B(gx \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes g \tag{24}$$

We consider the terms with x in the third component, we have two summands:

(I) in

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\alpha} B(1_H \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c+1+m+n-b} x^{c+1+b}$$

with $c + b = 0$ and $\overline{p + q + 1 + m + n} = 0$, i, e, $\overline{p + q + m + n} = 1$, i.e.

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=1}}^{1} (-1)^{m+n} B(1_H \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes x$$

(II) and in

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\beta} B(gx \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c+1+m+n-b} x^{c+b}$$

with $c + b = 1$.

Two cases arise:

- $b = 1$, $c = 0$, hence $n = 1$ and $\overline{p + q + m} = 1$, we get

$$\begin{aligned}&\sum_{\substack{m,p,q=0\\ \overline{m+p+q}=1}}^{1} B(gx \otimes g^k x^l; G^m X, g^p x^q) G^m \otimes g^p x^q \otimes x\\ &= B(gx \otimes g^k x^l; GX, 1_H) G \otimes 1_H \otimes x + B(gx \otimes g^k x^l; X, g) 1_A \otimes g \otimes x +\\ &\quad + B(gx \otimes g^k x^l; X, x) 1_A \otimes x \otimes x + B(gx \otimes g^k x^l; GX, gx) G \otimes gx \otimes x\end{aligned}$$

• $b=0$, $c=1$, hence $q=1$ and $\overline{p+m+n}=1$, we get

$$\sum_{\substack{m,n,p=0\\ \overline{m+n+p}=1}}^{1}(-1)^{m+n+1}B(gx\otimes g^kx^l;G^mX^n,g^px)G^mX^n\otimes g^p\otimes x$$
$$=B(gx\otimes g^kx^l;G,x)G\otimes 1_H\otimes x+B(gx\otimes g^kx^l;X,x)X\otimes 1_H\otimes x$$
$$-B(gx\otimes g^kx^l;1_A,gx)1_A\otimes g\otimes x-B(gx\otimes g^kx^l;GX,gx)GX\otimes g\otimes x$$

Thus we obtain from **(II):**

$$(B(gx\otimes g^kx^l;GX,1_H)+B(gx\otimes g^kx^l;G,x))G\otimes 1_H\otimes x+$$
$$+(B(gx\otimes g^kx^l;X,g)-B(gx\otimes g^kx^l;1_A,gx))1_A\otimes g\otimes x$$
$$+B(gx\otimes g^kx^l;X,x)1_A\otimes x\otimes x+B(gx\otimes g^kx^l;GX,gx)G\otimes gx\otimes x$$
$$+B(gx\otimes g^kx^l;X,x)X\otimes 1_H\otimes x-B(gx\otimes g^kx^l;GX,gx)GX\otimes g\otimes x$$

In conclusion, we get

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=1}}^{1}(-1)^{m+n}B(1_H\otimes g^kx^l;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes x+$$
$$(B(gx\otimes g^kx^l;GX,1_H)+B(gx\otimes g^kx^l;G,x))G\otimes 1_H\otimes x+$$
$$+(B(gx\otimes g^kx^l;X,g)-B(gx\otimes g^kx^l;1_A,gx))1_A\otimes g\otimes x$$
$$+B(gx\otimes g^kx^l;X,x)1_A\otimes x\otimes x+B(gx\otimes g^kx^l;GX,gx)G\otimes gx\otimes x$$
$$+B(gx\otimes g^kx^l;X,x)X\otimes 1_H\otimes x-B(gx\otimes g^kx^l;GX,gx)GX\otimes g\otimes x. \tag{25}$$

We consider terms with gx in the third component: they are in the two following summands.

(I) In

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(1_H\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q-c+1+m+n-b}x^{c+1+b}$$

with $c+b=0$ and $\overline{p+q+1+m+n}=1$, i.e. $b=c=0$ and $\overline{p+q+m+n}=0$, we get

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=0}}^{1}(-1)^{m+n}B(1_H\otimes g^kx^l;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes gx$$

(II) and in

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(gx\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q-c+1+m+n-b}x^{c+b}$$

with $c+b=1$ and $\overline{p+q-c+1+m+n-b}=1$, i.e. $\overline{p+q+m+n}=1$. We have two cases

(i) $b=1$ and $c=0$, then $n=1$, we obtain

$$\sum_{\substack{m,p,q=0\\ \overline{p+q+m}=0}}^{1} B(gx\otimes g^k x^l; G^m X, g^p x^q) G^m \otimes g^p x^q \otimes gx$$

$$= B(gx\otimes g^k x^l; X, 1_H) 1_A \otimes 1_H \otimes gx + +B(gx\otimes g^k x^l; X, gx) 1_A \otimes gx \otimes gx$$
$$+B(gx\otimes g^k x^l; GX, g) G\otimes g\otimes gx + B(gx\otimes g^k x^l; GX, x) G\otimes x\otimes gx$$

(ii) $b=0$ and $c=1$, then $q=1$, we obtain

$$\sum_{\substack{m,n,p=0\\ \overline{p+q+m}=0}}^{1} (-1)^{m+n+1} B(gx\otimes g^k x^l; G^m X^n, g^p x) G^m X^n \otimes g^p \otimes gx$$

$$= -B(gx\otimes g^k x^l; 1_A, x) 1_A \otimes 1_H \otimes gx - B(gx\otimes g^k x^l; GX, x) GX\otimes 1_H \otimes gx$$
$$+B(gx\otimes g^k x^l; G, gx) G\otimes g\otimes gx + B(gx\otimes g^k x^l; X, gx) X\otimes g\otimes gx$$

Therefore we obtain

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=0}}^{1} (-1)^{m+n} B(1_H\otimes g^k x^l; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes gx +$$
$$+\left[B(gx\otimes g^k x^l; X, 1_H) - B(gx\otimes g^k x^l; 1_A, x)\right] 1_A\otimes 1_H\otimes gx$$
$$+B(gx\otimes g^k x^l; X, gx) 1_A \otimes gx\otimes gx$$
$$+\left[B(gx\otimes g^k x^l; GX, g) + B(gx\otimes g^k x^l; G, gx)\right] G\otimes g\otimes gx$$
$$+B(gx\otimes g^k x^l; GX, x) G\otimes x\otimes gx$$
$$+B(gx\otimes g^k x^l; X, gx) X\otimes g\otimes gx$$
$$-B(gx\otimes g^k x^l; GX, x) GX\otimes 1_H\otimes gx \tag{26}$$

4.2.1. $\boldsymbol{i=1, j=1, k=0, l=0.}$ This corresponds to $B(gx\otimes 1_H)$.
By (12) we know that $B(gx\otimes 1_H) = -B(g\otimes gx)$

4.2.2. $\boldsymbol{i=1, j=1, k=0, l=1.}$ In this case we are computing $B(gx\otimes x)$.

$$\sum_{a=0}^{1}\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha} B(g^{\overline{2-a}} x^a \otimes x; G^m X^n, g^p x^q)$$
$$G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c+1+m+n-b} x^{c+1-a+b}$$
$$=\sum_{a=0}^{1}\sum_{m,n,p,q}^{1} B(gx\otimes x^{1-a}; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes g^{1-a} x^a$$

i.e.

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha} B(1_H\otimes x; G^m X^n, g^p x^q) G^m X^{n-b}$$
$$\otimes g^p x^{q-c} \otimes g^{p+q-c+1+m+n-b} x^{c+1+b} +$$

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(gx\otimes x;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q-c+1+m+n-b}x^{c+b}$$
$$=\sum_{m,n,p,q}^{1}B(gx\otimes x;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g$$
$$+\sum_{m,n,p,q}^{1}B(gx\otimes 1_H;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes x$$

where $\alpha=c+(c+1)(m+n-b)$ and $\beta=c+c(m+n-b)$.

By considering the terms with 1_H in the third component, in view of (23) we get

$$B(gx\otimes x;G^mX^n,g^px^q)=0 \text{ if } \overline{m+n+p+q}=1. \tag{27}$$

By considering the terms with g in the third component, in view of (24) and (27) we do not get any further relation.

The terms with gx in the third component give us, in view of (26),

$$B(1_H\otimes g;G^mX^n,g^px^q)=0 \text{ for } \overline{m+n+p+q}=0. \tag{28}$$

A straightforward computation shows that

$$\begin{aligned}B(gx\otimes x) = {} & B(gx\otimes x;1_A,1_H)1_A\otimes 1_H+B(gx\otimes x;1_A,gx)1_A\otimes gx\\ & +B(gx\otimes x;G,g)G\otimes g+B(gx\otimes x;G,x)G\otimes x\\ & +[B(gx\otimes 1_H;1_A,g)+B(gx\otimes x;1_A,gx)\\ & -B(1_H\otimes x;1_A,g)]\,X\otimes g\\ & +[B(gx\otimes 1_H;1_A,x)-B(1\otimes x;1_A,x)]\,X\otimes x\\ & +[B(gx\otimes 1_H;G,1_H)-B(gx\otimes x;G,x)\\ & +B(1_H\otimes x;G,1_H)]\,GX\otimes 1_H\\ & +[B(1_H\otimes x;G,gx)+B(gx\otimes 1_H;G,gx)]\,GX\otimes gx\end{aligned} \tag{29}$$

4.2.3. $\boldsymbol{i=1,j=1,k=1,l=0}$. This corresponds to $B(gx\otimes g)$. We have

$$\sum_{a=0}^{1}\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g^{\overline{2-a}}x^a\otimes g;G^mX^n,g^px^q)$$
$$G^mX^{n-b}\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+1-a+b}$$
$$=\sum_{m,n,p,q}^{1}B(gx\otimes g;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g$$

i.e.

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(1_H\otimes g;G^mX^n,g^px^q)$$
$$G^mX^{n-b}\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+1+b}+$$

$$+\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(gx\otimes g;G^mX^n,g^px^q)$$
$$G^mX^{n-b}\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+b}$$
$$=\sum_{m,n,p,q}^{1}B(gx\otimes g;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g$$

where $\alpha = c+(c+1)(m+n-b)$ and $\beta = c+c(m+n-b)$.

By considering the terms with 1_H in the third component, in view of (23) we get

$$B(gx\otimes g;G^mX^n,g^px^q) = 0 \text{ for } \overline{p+q+m+n} = 1. \tag{30}$$

A straightforward computation then shows

$$\begin{aligned} B(gx\otimes g) = {} & B(gx\otimes g;1_A,1_H)1_A\otimes 1_H + B(gx\otimes g;1_A,gx)1_A\otimes gx \\ & +B(gx\otimes g;G,g)G\otimes g + B(gx\otimes g;G,x)G\otimes x \\ & +[B(gx\otimes g;1_A,gx) - B(1_H\otimes g;1_A,g)]\,X\otimes g - B(1_H\otimes g;1_A,x)X\otimes x \\ & +B(1_H\otimes g;G,gx)GX\otimes gx + [-B(gx\otimes g;G,x) \\ & +B(1_H\otimes g;G,1_h)]\,GX\otimes 1_H \end{aligned} \tag{31}$$

4.2.4. $i=1, j=1, k=1, l=1$. This case corresponds to $B(gx\otimes gx)$.

$$\sum_{a=0}^{1}\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g^{\overline{2-a}}x^a\otimes g^kx^l;G^mX^n,g^px^q)$$
$$G^mX^{n-b}\otimes g^px^{q-c}\otimes g^{p+q-c+1+m+n-b}x^{c+1-a+b}$$
$$=\sum_{a=0}^{1}\sum_{m,n,p,q}^{1}B(gx\otimes gx^{1-a};G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g^{2-a}x^a$$

i.e.

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(1_H\otimes gx;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q-c+1+m+n-b}x^{c+1+b}+$$
$$+\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(gx\otimes gx;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q-c+1+m+n-b}x^{c+b}$$
$$=\sum_{m,n,p,q=0}^{1}B(gx\otimes gx;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes 1$$
$$+\sum_{m,n,p,q=0}^{1}B(gx\otimes g;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes gx$$

where $\beta = c+c(m+n-b)$ and $\alpha = c+(c+1)(m+n-b)$.

We consider the terms with g in the third component and by (24) we get

$$B(gx \otimes gx; G^m X^n, g^p x^q) = 0 \text{ for } \overline{p+q+m+n} = 0. \tag{32}$$

As far as the terms with x in the third component, in view of (25), we get

$$B(1_H \otimes gx; G^m X^n, g^p x^q) = 0 \text{ for } \overline{m+n+p+q} = 1. \tag{33}$$

By using (23), (32) and (26), a straightforward computation shows

$$\begin{aligned} B(gx \otimes gx) &= B(gx \otimes gx; 1_A, g)1_A \otimes g + B(gx \otimes gx; 1_A, x)1_A \otimes x \\ &+B(gx \otimes gx; G, 1_H)G \otimes 1_H + B(gx \otimes gx; G, gx)G \otimes gx \\ &+ [-B(1_H \otimes gx; 1_A, 1_H) + B(gx \otimes gx; 1_A, x) + B(gx \otimes g; 1_A, 1_H)]\, X \otimes 1_H \\ &[-B(1_H \otimes gx; 1_A, gx) + B(gx \otimes g; 1_A, gx)]\, X \otimes gx \\ &[B(gx \otimes g; G, g) + B(1_H \otimes gx; G, g) - B(gx \otimes gx; G, gx)]\, GX \otimes g \\ &[B(gx \otimes g; G, x) + B(1_H \otimes gx; G, x)]\, GX \otimes x \end{aligned} \tag{34}$$

4.3. Case $i = 0, j = 0$

The left side of the equality is

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\alpha} B(1_H \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c} \otimes g^{p+q-c+m+n-b} x^{c+b}.$$

The terms with 1_H in third position appear as

$$\sum_{\substack{m,n,p,q=0 \\ \overline{m+n+p+q}=0}}^{1} B(1_H \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes 1_H, \tag{35}$$

The terms with g in third position are

$$\sum_{\substack{m,n,p,q=0 \\ \overline{m+n+p+q}=1}}^{1} B(1_H \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes g \tag{36}$$

Now we consider the terms with x in third position. In this case $c + b = 1$, and $p + q + m + n = 1, 3$. We have two subclasses

- $c = 1$ and $b = 0$ and $q = 1$. The sum is:

$$\begin{aligned} &\sum_{m,n,p=0}^{1} (-1)^{m+n} B(1_H \otimes g^k x^l; G^m X^n, g^p x) G^m X^n \otimes g^p \otimes x \\ &= B(1_H \otimes g^k x^l; 1_A, x)1_A \otimes 1_H \otimes x - B(1_H \otimes g^k x^l; X, gx) X \otimes g \otimes x \\ &\quad -B(1_H \otimes g^k x^l; G, gx) G \otimes g \otimes x + B(1_H \otimes g^k x^l; GX, x) GX \otimes 1_A \otimes x \end{aligned} \tag{37}$$

- $b = 1$ and $c = 0$, so $n = 1$. The sum is:

$$\sum_{r,t,u=0}^{1} B(1_H \otimes g^k x^l; G^r X, g^t x^u) G^r \otimes g^t x^u \otimes x$$

$$= B(1_H \otimes g^k x^l; X, 1_H)1 \otimes 1 \otimes x + B(1_H \otimes g^k x^l; GX, g)G \otimes g \otimes x + \\ +B(1_H \otimes g^k x^l; GX, x)G \otimes x \otimes x + B(1_H \otimes g^k x^l; X, gx)1 \otimes gx \otimes x \tag{38}$$

In conclusion the term with x in third position are

$$\begin{aligned}&\left[B(1_H \otimes g^k x^l; 1_A, x) + B(1_H \otimes g^k x^l; X, 1_H)\right] 1_A \otimes 1_H \otimes x\\ &+B(1_H \otimes g^k x^l; X, gx)1_A \otimes gx \otimes x\\ &+\left[B(1_H \otimes g^k x^l; GX, g) - B(1_H \otimes g^k x^l; G, gx)\right] G \otimes g \otimes x\\ &+B(1_H \otimes g^k x^l; GX, x)G \otimes x \otimes x\\ &-B(1_H \otimes g^k x^l; X, gx)X \otimes g \otimes x + B(1_H \otimes g^k x^l; GX, x)GX \otimes 1 \otimes x\end{aligned} \tag{39}$$

Now, consider the terms with gx in third position. Then

$$\sum_{\substack{m,n,p,q=0\\ p+q+m+n\in\{0,2,4\}\\ c+b=1}}^{1} \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha} B(1_H \otimes g^k x^l; G^m X^n, g^p x^q)G^m X^{n-b} \\ \otimes g^p x^{q-c} \otimes gx$$

Thus $c + b = 1$ i.e. $b = 1$ and $c = 0$ or $c = 1$ and $b = 0$.

If $b = 1$ and $c = 0$, then $n = 1$ and $m + p + q = 1, 3$, the sum is: $B(1_H \otimes g^k x^l; GX, 1_H)G \otimes 1_H \otimes gx + B(1_H \otimes g^k x^l; X, g)1_A \otimes g \otimes gx + B(1_H \otimes g^k x^l; X, x)1_A \otimes x \otimes gx + B(1_H \otimes g^k x^l; GX, gx)G \otimes gx \otimes gx$.
If $c = 1$ then $q = 1$, $m + n + q = 1, 3$, the sum is
$-B(1_H \otimes g^k x^l; G, x)G \otimes 1_H \otimes gx - B(1_H \otimes g^k x^l; X, x)X \otimes 1_H \otimes gx + B(1_H \otimes g^k x^l; 1_A, gx)1_A \otimes g \otimes gx + B(1_H \otimes g^k x^l; GX, gx)GX \otimes g \otimes gx$.

Thus we get

$$\begin{aligned}&\left[B(1_H \otimes g^k x^l; 1_A, gx) + B(1_H \otimes g^k x^l; X, g)\right] 1_A \otimes g \otimes gx\\ &\left[B(1_H \otimes g^k x^l; GX, 1_H) - B(1_H \otimes g^k x^l; G, x)\right] G \otimes 1_H \otimes gx\\ &+B(1_H \otimes g^k x^l; X, x)1_A \otimes x \otimes gx + B(1_H \otimes g^k x^l; GX, gx)G \otimes gx \otimes gx\\ &-B(1_H \otimes g^k x^l; X, x)X \otimes 1_H \otimes gx + B(1_H \otimes g^k x^l; GX, gx)GX \otimes g \otimes gx.\end{aligned} \tag{40}$$

4.3.1. $i = 0, j = 0, k = 0, l = 0$. This corresponds to $B(1_H, 1_H)$ which is $1_A \otimes 1_H$ by (9) .

4.3.2. $i = 0, j = 0, k = 0, l = 1$. This correspond to $B(1_H \otimes x)$. The equation is:

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha} B(1_H \otimes x; G^m X^n, g^p x^q)G^m X^{n-b} \otimes g^p x^{q-c} \\ \otimes g^{p+q-c+m+n-b} x^{c+b} \\ = \sum_{m,n,p,q}^{1} B(1_H \otimes x; G^m X^n, g^p x^q)G^m X^n \otimes g^p x^q \otimes g$$

$$+\sum_{m,n,p,q}^{1} B(1_H\otimes 1_H; G^mX^n, g^px^q)G^mX^n\otimes g^px^q\otimes x$$

since, by (9) $B(1_H\otimes 1_H; G^mX^n, g^px^q) = 0$ except for $B(1_H\otimes 1_H; 1_A, 1_H) = 1$. We get

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(1_H\otimes x; G^mX^n, g^px^q)G^mX^{n-b}$$
$$\otimes g^px^{q-c}\otimes g^{p+q-c+m+n-b}x^{c+b}$$
$$=\sum_{m,n,p,q}^{1} B(1_H\otimes x; G^mX^n, g^px^q)G^mX^n\otimes g^px^q\otimes g$$
$$+1_A\otimes 1_H\otimes x \text{ where } \alpha = c(m+n-b).$$

By using (35) we get

$$B(1_H\otimes x; G^mX^n, g^px^q) = 0 \text{ and } \overline{m+n+p+q=0} \tag{41}$$

By using (36), (41), (41) and (39) a straightforward computation shows

$$\begin{aligned}
B(1_H\otimes x) &= B(1_H\otimes x; 1_A, g)1_A\otimes g + B(1_H\otimes x; 1_A, x)1_A\otimes x\\
&\quad +B(1_H\otimes x; G, 1_H)G\otimes 1_H\\
&(1-B(1_H\otimes x; 1_A, x))X\otimes 1_H + B(1_H\otimes x; GX, g)G\otimes gx\\
&\quad +B(1_H\otimes x; GX, g)GX\otimes g
\end{aligned} \tag{42}$$

4.3.3. $\boldsymbol{i=0, j=0, k=1, l=0.}$ This correspond to $B(1_H\otimes g)$. The equation is:

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(1_H\otimes g; G^mX^n, g^px^q)G^mX^{n-b}$$
$$\otimes g^px^{q-c}\otimes g^{p+q-c+m+n-b}x^{c+b}$$
$$=\sum_{m,n,p,q}^{1} B(1_H\otimes g; G^mX^n, g^px^q)G^mX^n\otimes g^px^q\otimes g \text{ where } \alpha$$
$$= c(m+n-b).$$

By using (35), (28) and (39) a straightforward computation shows

$$\begin{aligned}
B(1_H\otimes g) = {}& B(1_H\otimes g; 1_A, g)1_A\otimes g + B(1_H\otimes g; 1_A, x)1_A\otimes x\\
&+B(1_H\otimes g; G, 1_H)G\otimes 1_H\\
&-B(1_H\otimes g; 1_A, x)X\otimes 1_H + B(1_H\otimes g; G, gx)G\otimes gx\\
&+B(1_H\otimes g; G, gx)GX\otimes g
\end{aligned} \tag{43}$$

4.3.4. $\boldsymbol{i=0, j=0, k=1, l=1.}$ We compute $B(1_H\otimes gx)$. We have to solve the following equation:

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(1_H\otimes gx; G^mX^n, g^px^q)G^mX^{n-b}$$
$$\otimes g^px^{q-c}\otimes g^{p+q+m+n-b-c}x^{c+b}$$

$$= \sum_{m,n,p,q}^{1} B(1_H \otimes gx; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes 1_H$$
$$+ \sum_{m,n,p,q}^{1} B(1_H \otimes g; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes gx$$

where $\alpha = c(m+n-b)$.

By using (33), (40) and (28) a straightforward computation shows

$$\begin{aligned} B(1_H \otimes gx) = & B(1_H \otimes gx; 1_A, 1_H) 1_A \otimes 1_H + B(1_H \otimes gx; 1_A, gx) 1_A \otimes gx \\ & + B(1_H \otimes gx; G, g) G \otimes g + B(1_H \otimes gx; G, x) G \otimes x \\ & + [B(1_H \otimes gx; G, x) + B(1_H \otimes g; G, 1_H)] GX \otimes 1_H \\ & + [B(1_H \otimes g; 1_A, g) - B(1_H \otimes gx; 1_A, gx)] X \otimes g \\ & + B(1_H \otimes g; 1_A, x) X \otimes x + B(1_H \otimes g; G, gx) GX \otimes gx \end{aligned} \quad (44)$$

4.4. Case $i = 0, j = 1$

The left side of the equality is

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\alpha} B(g \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b}$$
$$\otimes g^p x^{q-c} \otimes g^{p+q+m+n-b-c} x^{c+1+b}$$
$$+ \sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\beta} B(x \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b}$$
$$\otimes g^p x^{q-c} \otimes g^{p+q+m+n-b-c} x^{c+b}$$

where $\alpha = (c+1)(m+n-b)$ and $\beta = c(m+n-b)$. The terms with third component 1_H are in

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\beta} B(x \otimes x; G^m X^n, g^p x^q) G^m X^{n-b} \otimes g^p x^{q-c}$$
$$\otimes g^{p+q+m+n-b-c} x^{c+b}$$

with $c + b = 0$ (i.e. $c = b = 0$) and $p + q + m + n = 0, 2, 4$. Then it simplifies to:

$$\sum_{\substack{m,n,p,q=0 \\ m+n+p+q=0}}^{1} B(x \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^n \otimes g^p x^q \otimes 1_H \quad (45)$$

The terms with third component g are in

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n} \sum_{c=0}^{q} (-1)^{\beta} B(x \otimes g^k x^l; G^m X^n, g^p x^q) G^m X^{n-b}$$
$$\otimes g^p x^{q-c} \otimes g^{p+q+m+n-b-c} x^{c+b}$$

with $c+b=0$ (i.e. $c=b=0$) and $p+q+m+n=1,3$. Then it simplifies to:

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=1}}^{1} B(x\otimes g^kx^l;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g \tag{46}$$

The terms in the left side with third component x are either in

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}$$
$$\otimes g^px^{q-c}\otimes g^{p+q+m+n-b-c}x^{c+1+b}$$

with $c+b=0$ (i.e. $c=b=0$) and $p+q+m+n=0,2,4$ and we get

$$\sum_{\substack{m,n,p,q=0\\ p+q+m+n=0,2,4}}^{1} (-1)^{m+n}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes x$$

or in

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(x\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q+m+n-b-c}x^{c+b}$$

with $c+b=1$ and $p+q+m+n=1,3$, i.e. we have again two cases:

- $b=1$ and $c=0$ so $n=1$, and $m+p+q=0,2$ and since $\beta=0$ the sum is

$$\sum_{\substack{m,p,q=0\\ m+p+q=0,2}}^{1} B(x\otimes g^kx^l;G^mX,g^px^q)G^m\otimes g^px^q\otimes x$$
$$= B(x\otimes g^kx^l;X,1_H)1_A\otimes 1_H\otimes x + B(x\otimes g^kx^l;GX,g)G\otimes g\otimes x$$
$$+B(x\otimes g^kx^l;GX,x)G\otimes x\otimes x + B(x\otimes g^kx^l;X,gx)1_A\otimes gx\otimes x$$

- $c=1$ and $b=0$ so $q=1$ and $m+p+q=0,2$ and the sum is

$$\sum\nolimits^{1}_{\substack{m,n,p=0\\ m+n+p=0,2}}(-1)^{m+n}B(x\otimes g^kx^l;G^mX^n,g^px)G^mX^n\otimes g^p\otimes x$$
$$= B(x\otimes g^kx^l;1_A,x)1_A\otimes 1_H\otimes x + B(x\otimes g^kx^l;GX,x)GX\otimes 1_H\otimes x$$
$$-B(x\otimes g^kx^l;G,gx)G\otimes g\otimes x - B(x\otimes g^kx^l;X,gx)X\otimes g\otimes x$$

Thus, we obtain

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=0}}^{1} (-1)^{m+n}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes x$$
$$+B(x\otimes g^kx^l;X,1_H)1_A\otimes 1_H\otimes x + \left[B(x\otimes g^kx^l;X,1_H)\right.$$
$$\left.+B(x\otimes g^kx^l;1_A,x)\right]1_A\otimes 1_H\otimes x + B(x\otimes g^kx^l;X,gx)1_A\otimes gx\otimes x$$
$$+\left[B(x\otimes g^kx^l;GX,g) - B(x\otimes g^kx^l;G,gx)\right]G\otimes g\otimes x$$
$$+B(x\otimes g^kx^l;GX,x)G\otimes x\otimes x - B(x\otimes g^kx^l;X,gx)X\otimes g\otimes x$$

$$+B(x\otimes g^kx^l;GX,x)GX\otimes 1_H\otimes x \tag{47}$$

The terms with third component gx are either in

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q+m+n-b-c}x^{c+1+b}$$

with $c+b=0$ and $p+q+m+n=1,3$. Thus in

$$\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=1}}^{1}(-1)^{m+n}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes gx,$$

or in

$$\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(x\otimes x;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}$$
$$\otimes g^{p+q+m+n-b-c}x^{c+b}$$

with $c+b=1$ and $p+q+m+n=0,2,4$. Two cases arise:

- $b=1$ and $c=0$, then $n=1$. We get

$$\sum_{\substack{m,p,q=0,\\ m+p+q=0,2}}^{1}\sum_{c=0}^{q}B(x\otimes g^kx^l;G^mX,g^px^q)G^m\otimes g^px^q\otimes gx$$
$$=B(x\otimes g^kx^l;GX,1_H)G\otimes 1_H\otimes gx+B(x\otimes g^kx^l;X,g)1_A\otimes g\otimes gx$$
$$+B(x\otimes g^kx^l;X,x)1_A\otimes x\otimes gx+B(x\otimes g^kx^l;GX,gx)G\otimes gx\otimes gx$$

- $b=0$ and $c=1$, then $q=1$. We get

$$\sum_{\substack{m,n,p=0\\ m+p+n=0,2}}^{1}(-1)^{m+n}B(x\otimes g^kx^l;G^mX^n,g^px)G^mX^n\otimes g^p\otimes gx$$
$$=-B(x\otimes g^kx^l;G,x)G\otimes 1_H\otimes gx-B(x\otimes g^kx^l;X,x)X\otimes 1_H\otimes gx$$
$$+B(x\otimes g^kx^l;1_A,gx)1_A\otimes g\otimes gx+B(x\otimes g^kx^l;GX,gx)GX\otimes g\otimes gx$$

Thus we get

$$\begin{aligned}&\sum_{\substack{m,n,p,q=0\\ \overline{p+q+m+n}=1}}^{1}(-1)^{m+n}B(g\otimes g^kx^l;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes gx+\\ &+\left[B(x\otimes g^kx^l;X,g)+B(x\otimes g^kx^l;1_A,gx)\right]1_A\otimes g\otimes gx+\\ &+\left[B(x\otimes g^kx^l;GX,1_H)-B(x\otimes g^kx^l;G,x)\right]G\otimes 1_H\otimes gx+\\ &+B(x\otimes g^kx^l;X,x)1_A\otimes x\otimes gx+B(x\otimes g^kx^l;GX,gx)G\otimes gx\otimes gx\\ &-B(x\otimes g^kx^l;X,x)X\otimes 1_H\otimes gx+B(x\otimes g^kx^l;GX,gx)GX\otimes g\otimes gx\end{aligned} \tag{48}$$

4.4.1. $i = 0, j = 1, k = 0, l = 0$. This corresponds to $B(x \otimes 1_H)$

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes 1_H;G^mX^n,g^px^q)G^mX^{n-b} \otimes g^px^{q-c}\otimes g^{p+q-c+m+n-b}x^{c+1+b} + \sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(x\otimes 1_H;G^mX^n,g^px^q)G^mX^{n-b} \otimes g^px^{q-c}\otimes g^{p+q-c+m+n-b}x^{c+b} = \sum_{m,n,p,q}^{1} B(x\otimes 1_H;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes 1_H$$

where $\alpha = (c+1)(m+n-b)$ and $\beta = c(m+n-b)$. In view (46) we obtain

$$B(x\otimes 1_H;G^mX^n,g^px^q) = 0 \text{ for } \overline{m+n+p+q} = 1 \tag{49}$$

By using (49), (17), (49) and (48), a straightforward computation shows

$$\begin{aligned} B(x\otimes 1_H) = {} & B(x\otimes 1_H;1_A,1_H)1_A\otimes 1_H + B(x\otimes 1_H;1_A,gx)1_A\otimes gx \\ & +B(x\otimes 1_H,G,g)G\otimes g + B(x\otimes 1_H;G,x)G\otimes x \\ & +[-B(x\otimes 1_H;1_A,gx) - B(g\otimes 1_H;1_A,g)]\,X\otimes g \\ & -B(g\otimes 1_H;1_A,X)X\otimes x + [B(x\otimes 1_H;G,x) \\ & +B(g\otimes 1_H;G,1_H)]\,GX\otimes 1_H + B(g\otimes 1_H;G,gx)GX\otimes gx \end{aligned} \tag{50}$$

4.4.2. $i = 0, j = 1, k = 0, l = 1$. We compute $B(x \otimes x)$.

$$\sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes x;G^mX^n,g^px^q)G^mX^{n-b} \otimes g^px^{q-c} \otimes g^{p+q+m+n-b-c}x^{c+1+b} + \sum_{m,n,p,q=0}^{1} \sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(x\otimes x;G^mX^n,g^px^q)G^mX^{n-b} \otimes g^px^{q-c} \otimes g^{p+q+m+n-b-c}x^{c+b} = \sum_{m,n,p,q}^{1} B(x\otimes x;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes g + \sum_{m,n,p,q}^{1} B(x\otimes 1_H;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes x$$

where $\alpha = (c+1)(m+n-b)$ and $\beta = c(m+n-b)$. Concerning the terms with third component 1_H, in view of (45), we get

$$B(x\otimes x;G^mX^n,g^px^q) = 0 \text{ for } \overline{m+n+p+q=0} \tag{51}$$

By using (51), (19) and (47), a straightforward computation shows

$$B(x\otimes x) = B(x\otimes x;1_A,g)1_A\otimes g + B(x\otimes x;1_A,x)1_A\otimes x$$

$$
\begin{aligned}
&+B(x\otimes x;G,1_H)G\otimes 1_H\\
&+B(x\otimes x;G,gx)G\otimes gx+[B(x\otimes 1_H;1_A,1_H)-B(x\otimes x;1_A,x)\\
&-B(g\otimes x;1_A,1_H)]\,X\otimes 1_H\\
&+[B(x,\otimes 1_H;1_A,gx)-B(g\otimes x;1_A,gx)]\,X\otimes gx\\
&+[B(x\otimes x;G,gx)+B(x\otimes 1_H;G,g)+B(g\otimes x;G,g)]\,GX\otimes g\\
&+[B(x\otimes 1_H;G,x)+B(g\otimes x;G,x)]\,GX\otimes x
\end{aligned} \tag{52}
$$

4.4.3. $i=0, j=1, k=1, l=0$. This corresponds to $B(x\otimes g)\overset{(11)}{=}-B(1_H\otimes x)$. See (42) .

4.4.4. $i=0, j=1, k=1, l=1$. This corresponds to $B\,(x\otimes gx)$.

$$
\begin{aligned}
&\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\alpha}B(g\otimes gx;G^mX^n,g^px^q)G^mX^{n-b}\\
&\quad\otimes g^px^{q-c}\otimes g^{p+q-c+m+n-b}x^{c+1+b}\\
&\quad+\sum_{m,n,p,q=0}^{1}\sum_{b=0}^{n}\sum_{c=0}^{q}(-1)^{\beta}B(x\otimes gx;G^mX^n,g^px^q)G^mX^{n-b}\otimes g^px^{q-c}\\
&\otimes g^{p+q-c+m+n-b}x^{c+b}\\
&=\sum_{m,n,p,q}^{1}B(x\otimes gx;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes 1_H\\
&\quad+\sum_{m,n,p,q}^{1}B(x\otimes g;G^mX^n,g^px^q)G^mX^n\otimes g^px^q\otimes gx
\end{aligned}
$$

where $\alpha=(c+1)(m+n-b)$ and $\beta=c(m+n-b)$. In view of (46), we get

$$B(x\otimes gx;G^mX^n,g^px^q)=0 \text{ for } \overline{m+n+p+q}=1 \tag{53}$$

By using (53), (21) and (48), a straightforward computation shows

$$
\begin{aligned}
B(x\otimes gx)&=B(x\otimes gx;1_A,1_H)1_A\otimes 1_H+B(x\otimes gx;1_A,gx)(1_A\otimes gx)\\
&+B(x\otimes gx;G,g)G\otimes g+B(x,gx;G,x)G\otimes x\\
&+[B(x\otimes g;1_A,g)-B(x\otimes gx;1_A,gx)-B(g\otimes gx;1_A,g)]\,X\otimes g\\
&+[B(x\otimes g;1_A,x)-B(g\otimes gx;1_A,x)]\,X\otimes x\\
&+[B(x\otimes g;G,gx)+B(g\otimes gx;G,gx)]\,GX\otimes gx\\
&+[B(x\otimes g;G,1_H)+B(x\otimes gx;G,x)+B(g\otimes gx;G,1_H)]\,GX\otimes 1_H
\end{aligned} \tag{54}
$$

Proposition 1. *Let $B: H\otimes H\to A\otimes H^{op}$ a bilinear form. Then B is normalized and satisfies Casimir condition if and only if the equalities* (9), (43), (42), (44) , (18), (10), (20), (10) , (22), (50), (11), (52) , (54), (12), (31) *and* (29) *hold.*

5. Morphism Condition

Now, we investigate when $B : H \otimes H \to A \otimes H^{op}$ is a morphism in $\mathcal{T}^{\sharp}_{A\otimes H^{op}}$. i.e.

$$m_{A\otimes H^{op}} \circ (B \otimes A \otimes H^{op}) = m_{A\otimes H^{op}} \circ (A \otimes B) \circ (\phi \otimes H) \circ (H \otimes \phi)$$

This can be written as

$$B(h \otimes h') \cdot (a \otimes h'') = (a_0 \otimes h''_1) \cdot B(h''_2 h a_1 \otimes h''_3 h' a_2)$$

This equality can be split in

$$B(h \otimes h')(a \otimes 1_H) = (a_0 \otimes 1_H)B(ha_1 \otimes h'a_2) \tag{55}$$

$$B(h \otimes h')(1_A \otimes h'') = (1_A \otimes h''_1)B(h''_2 h \otimes h''_3 h') \tag{56}$$

In fact assume that (55) and (56) hold. Then we have

$$\begin{aligned} B(h \otimes h') \cdot (a \otimes h'') &= B(h \otimes h') \cdot (a \otimes 1_H) \cdot (1_A \otimes h'') \\ &= (a_0 \otimes 1_H) \cdot B(ha_1 \otimes h'a_2) \cdot (1_A \otimes h'') \\ &= (a_0 \otimes 1_H) \cdot (1_A \otimes h''_1) \cdot B(h''_2 h a_1 \otimes h''_3 h' a_2) \end{aligned}$$

Now it is easy to show that if the equality (55) is true for $a, b \in A$, then it is true for the product. Similarly if the equality (56) is true for $s, t \in H$, then it is true for the product. Therefore we only need to compute equality (56) for $1_A \otimes g$ e $1_A \otimes x$ and equality (55) for $G \otimes 1_H$ and $X \otimes 1_H$.

5.1. Case $1_A \otimes g$

We have

$$B(h \otimes h')\ (1_A \otimes g) = (1_A \otimes g)B(gh \otimes gh') \tag{57}$$

that is equivalen to

$$B(gh \otimes gh') = (1_A \otimes g)B(h \otimes h')(1_A \otimes g). \tag{58}$$

Now (58) can be rewritten as

$$\begin{aligned} &\textstyle\sum_{i,j,k,l=0}^{1} B(gh \otimes gh', G^i X^j, g^k x^l) G^i X^j \otimes g^k x^l \\ &\quad = \textstyle\sum_{i,j,k,l=0}^{1} B(h \otimes h', G^i X^j, g^k x^l) G^i X^j \otimes g g^k x^l g) \\ &\quad = \textstyle\sum_{i,j,k,l=0}^{1} (-1)^l B(h \otimes gh', G^i X^j, g^k x^l) G^i X^j \otimes g^k x^l. \end{aligned}$$

Thus, equation (57) is equivalent to

$$B(gh \otimes gh'; a, f) = B(h \otimes h'; a, f) \quad a = 1_A, G, X, GX \text{ and } f = 1_H, g \tag{59}$$

$$B(gh \otimes gh'; a, f) = -B(h \otimes h'; a, f) \quad a = 1_A, G, X, GX \text{ and } f = x, gx \tag{60}$$

Now, using the forms of the elements given in Proposition 1 it is straightforward to prove that these conditions hold.

5.2. Case $1_A \otimes x$

We have

$$B(h\otimes h')(1_A\otimes x)=(1_A\otimes x)B(gh\otimes gh')+B(xh\otimes gh')+B(h\otimes xh'). \tag{61}$$

From (61) we get

$$\begin{aligned}&B(h\otimes h')(1_A\otimes x)\\&\quad=(1_A\otimes x)B(gh\otimes gh')+B(xh\otimes gh')+B(h\otimes xh')\\&\quad\overset{(58)}{=}(1_A\otimes x)\,(1_A\otimes g)\,B(h\otimes h')\,(1_A\otimes g)+B(xh\otimes gh')+B(h\otimes xh')\\&\qquad(1_A\otimes gx)B(h\otimes h')(1_A\otimes g)+B(xh\otimes gh')+B(h\otimes xh')\end{aligned}$$

Thus

$$\begin{aligned}B(h\otimes h')(1_A\otimes x)=&(1_A\otimes gx)B(h\otimes h')(1_A\otimes g)\\&+B(xh\otimes gh')+B(h\otimes xh')\end{aligned} \tag{62}$$

The left hand side of (62) is

$$\begin{aligned}&(\textstyle\sum_{i,j,k,l=0}^{1}B(h\otimes h',G^iX^j,g^kx^l)G^iX^j\otimes g^kx^l)(1_A\otimes x)\\&\quad=\textstyle\sum_{i,j,k=0}^{1}(-1)^kB(h\otimes h',G^iX^j,g^k)G^iX^j\otimes g^kx\end{aligned}$$

The first summand of the right hand side of (62) is

$$(1_A\otimes gx)B(h\otimes h')(1_A\otimes g)=\textstyle\sum_{i,j,k=0}^{1}B(h\otimes h',G^iX^j,g^k)G^iX^j\otimes g^kx$$

Thus we get

$$\begin{aligned}&\textstyle\sum_{i,j,k=0}^{1}(-1)^kB(h\otimes h',G^iX^j,g^k)G^iX^j\otimes g^kx\\&\quad=\textstyle\sum_{i,j,k=0}^{1}B(h\otimes h',G^iX^j,g^k)G^iX^j\otimes g^kx+\\&\qquad+\textstyle\sum_{i,j,k,l}B(xh\otimes gh',G^iX^j,g^kx^l)G^iX^j\otimes g^kx^l\\&\qquad+\textstyle\sum_{i,j,k,l=0}^{1}B(h\otimes xh',G^iX^j,g^kx^l)G^iX^j\otimes g^kx^l\end{aligned}$$

which is equivalent to

$$2B(h\otimes h',a,g)+B(xh\otimes gh',a,gx)+B(h\otimes xh',a,gx)=0 \text{ for every } a \tag{63}$$

and

$$B(xh\otimes gh',a,f)+B(h\otimes xh',a,f)=0 \text{ for } f\neq gx \text{ and any } a \tag{64}$$

Note that if these equalities hold for some h,h' they do hold also for gh and gh'. In fact we have

For $f=1,g$

$$\begin{aligned}&B(xgh\otimes ggh',a,f)+B(gh\otimes xgh',a,f)\\&\quad=-B(gxh\otimes ggh',a,f)-B(gh\otimes gxh',a,f)\\&\overset{(59)}{=}-B(xh\otimes gh',a,f)-B(h\otimes xh',a,f)\overset{(64)}{=}0 \text{ for any } a\end{aligned}$$

for $f=x$

$$\begin{aligned}&B(xgh\otimes ggh',a,f)+B(gh\otimes xgh',a,f)\\&\quad=-B(gxh\otimes ggh',a,f)-B(gh\otimes gxh',a,f)\end{aligned}$$

$$\overset{(60)}{=} B(xh \otimes gh', a, f)B(h \otimes xh', a, f) \overset{(64)}{=} 0 \text{ for any a },$$
$$\text{for } f = gx$$
$$2B(gh \otimes gh', a, g) + B(xgh \otimes ggh', a, gx) + B(gh \otimes xgh', a, gx) =$$
$$2B(gh \otimes gh', a, g) - B(gxh \otimes ggh', a, gx) - B(gh \otimes gxh', a, gx)$$
$$\overset{(59)(60)}{=} 2B(h \otimes h', a, g) + B(xh \otimes h', a, gx)$$
$$+B(h \otimes xh', a, gx) \overset{(63)}{=} 0 \text{ for any a }.$$

Therefore we have to check the previous equality for the six elements: $B(1_H \otimes g), B(1_H \otimes x), B(1_H \otimes gx), B(x \otimes 1_H), B(x \otimes x)$ and $B(x \otimes gx)$ mentioned before.

5.2.1. (a) $h = 1_H, h' = g$. In this case (63) and (64) become:

$$B(x \otimes 1_H; a, gx) = B(1_H \otimes gx; a, gx) - 2B(1_H \otimes g, a, g) \text{for all } a \tag{65}$$
$$B(x \otimes 1_H; a, f) = B(1_H \otimes gx; a, f) \text{ for } f \neq gx \text{and any} a \tag{66}$$

By using (50), (44), (66) , (65) and (59), we obtain

$$\begin{aligned} B(x \otimes 1_H) = B(1_H \otimes gx; 1_A, 1_H)1_A \otimes 1_H \\ + [B(1_H \otimes gx; 1_A, gx) - 2B(1_H \otimes g, 1_A, g)]\, 1_H \otimes gx \\ +B(1_H \otimes gx, G, g)G \otimes g + B(1_H \otimes gx; G, x)G \otimes x \\ + [-B(1_H \otimes gx; 1_A, gx) + B(1_H \otimes g, 1_A, g)]\, X \otimes g \\ +B(1_H \otimes g; 1_A, x)X \otimes x \\ +([B(1_H \otimes gx; G, x) + B(1_H \otimes g; G, 1_H)])GX \otimes 1_H \\ +B(1_H \otimes g, G, gx)GX \otimes gx \end{aligned} \tag{67}$$

Now

$$\begin{aligned} B(1_H \otimes g, G, gx) \overset{(50)}{=} B(x \otimes 1_H; GX, gx) \overset{(65)}{=} B(1_H \otimes gx; GX, gx) \\ -2B(1_H \otimes g, GX, g) \overset{(44)(43)}{=} \\ = B(1_H \otimes g; G, gx) - 2B(1_H \otimes g; G, gx) \end{aligned}$$

so that we get

$$2B(1_H \otimes g, G, gx) = 0 \tag{68}$$

5.2.2. (b) $h = 1_H, h' = x$. In this case (63) and (64) become:

$$B(x \otimes gx, a, gx) = -2B(1_H \otimes x, a, g) \text{ for all } a. \tag{69}$$
$$B(x \otimes gx, a, f) = 0 \text{ for } f \neq gx \text{ and any } a. \tag{70}$$

By using (54), (69), (70), (59), (60) and (11) we obtain

$$B(x \otimes gx) = -2B(1_H \otimes x\,; 1_A, g)(1_A \otimes gx) - 2B(1_H \otimes x; G, gx)GX \otimes gx \tag{71}$$

5.2.3. (c) $h = 1_H, h' = gx$. In this case (63) and (64) become:

$$B(x\otimes x, a, gx) = -2B(1_H \otimes gx, a, g) \text{ for all } a \tag{72}$$

$$B(x\otimes x, a, f) = 0 \text{ for } f \neq gx \text{ and any } a \tag{73}$$

By using (52), (73), (72) , (67), (59) and (60) we obtain

$$B(x\otimes x) = -2B(1_H \otimes gx, G, g)G\otimes gx + [2B(1_H\otimes gx; 1_A, gx) \\ -2B(1_H\otimes g; 1_A, g)]\, X\otimes gx. \tag{74}$$

Cases $h = x, h' = 1_H$; $h = x, h' = x$ and $h = x$, $h' = gx$ give us no more information. In conclusion we get the new following form of the last three elements (67), (71) and (74)

5.3. Case $G \otimes 1_H$

We have

$$B(h\otimes h')(G\otimes 1_A) = (G\otimes 1_A)B(hg\otimes h'g) \tag{75}$$

Now, by computing the left side of (75), we get

$$\begin{aligned} &B(h\otimes h')(G\otimes 1_H)\\ &= \sum_{k,l=0}^{1}\left[B(h\otimes h', G, g^kx^l)\alpha + B(h\otimes h', X, g^kx^l)\gamma\right](1_A\otimes g^kx^l) + \\ &\quad + \sum_{k,l=0}^{1}\left[(B(h\otimes h', 1_A, g^kx^l) + B(h\otimes h', GX, g^kx^l)\gamma\right](G\otimes g^kx^l) + \\ &\quad - \sum_{k,l=0}^{1}\alpha B(h\otimes h', GX, g^kx^l)\left(X\otimes g^kx^l\right)\\ &\quad - \sum_{k,l=0}^{1} B(h\otimes h', X, g^kx^l)\left(GX\otimes g^kx^l\right)\end{aligned}$$

Now we compute the right side of (75)

$$\begin{aligned}(G\otimes 1_H)B(hg\otimes h'g) &= (G\otimes 1_H)\textstyle\sum_{i,j,k,l=0}^{1} B(hg\otimes h'g, G^iX^j, g^kx^l)(G^iX^j\otimes g^kx^l)\\ &= \textstyle\sum_{i,j,k,l=0}^{1} B(hg\otimes h'g, G^iX^j, g^kx^l)(G^{i+1}X^j\otimes g^kx^l)\\ &= \textstyle\sum_{j,k,l=0}^{1} B(hg\otimes h'g, X^j, g^kx^l)(GX^j\otimes g^kx^l)\\ &\quad + \textstyle\sum_{j,k,l=0}^{1} \alpha B(hg\otimes h'g, GX^j, g^kx^l)(X^j\otimes g^kx^l)\end{aligned}$$

Comparing both sides we obtain for any f in the basis of H.

$$\alpha B(h\otimes h', G, f) + \gamma B(h\otimes h', X, f) = \alpha B(hg\otimes h'g, G, f) \tag{76}$$

$$B(h\otimes h', 1_A, f) + \gamma B(h\otimes h', GX, f) = B(hg\otimes h'g, 1_A, f) \tag{77}$$

$$-\alpha B(h\otimes h', GX, f) = \alpha B(hg\otimes h'g, GX, f) \tag{78}$$

$$-B(h\otimes h', X, f) = B(hg\otimes h'g, X, f) \tag{79}$$

Assume that (76) holds for some h, h'. Let us prove it holds also for gh and gh'.

$$\begin{aligned}\text{For } f = 1_H, g\alpha B(gh\otimes gh', G, f) + \gamma B(gh\otimes gh', X, f) - \alpha B(ghg\otimes gh'g, G, f)\\ \overset{(59)}{=} \alpha B(h\otimes h', G, f) + \gamma B(h\otimes h', X, f) - \alpha B(hg\otimes h'g, G, f) \overset{(76)}{=} 0\end{aligned}$$

$$\text{For } f=x,gx\ \alpha B(gh\otimes gh',G,f)+\gamma B(gh\otimes gh',X,f)-\alpha B(ghg\otimes gh'g,G,f)$$
$$\overset{(60)}{=}-\alpha B(h\otimes h',G,f)-\gamma B(h\otimes h',X,f)+\alpha B(hg\otimes h'g,G,f)\overset{(76)}{=}0$$

Assume that (77) holds for some h,h'. Let us prove it holds also for gh and gh'.

$$\text{For } f=1_H,g\ B(gh\otimes gh',1_A,f)+\gamma B(gh\otimes gh',GX,f)-B(ghg\otimes gh'g,1_A,f)$$
$$\overset{(59)}{=}B(h\otimes h',1_A,f)+\gamma B(h\otimes h',GX,f)-B(hg\otimes h'g,1_A,f)\overset{(77).}{=}0$$
$$\text{For } f=x,gxB(gh\otimes gh',1_A,f)+\gamma B(gh\otimes gh',GX,f)-B(ghg\otimes gh'g,1_A,f)$$
$$\overset{(60)}{=}-B(h\otimes h',1_A,f)-\gamma B(h\otimes h',GX,f)+B(hg\otimes h'g,1_A,f)\overset{(76)}{=}0$$

Assume that (78) holds for some h,h'. Let us prove it holds also for gh and gh'.

$$\text{For } f=1_H,g\ \alpha B(gh\otimes gh',GX,f)+\alpha B(ghg\otimes gh'g,GX,f)$$
$$\overset{(59)}{=}\alpha B(h\otimes h',GX,f)+\alpha B(hg\otimes h'g,GX,f)\overset{(78)}{=}0$$
$$\text{For } f=x,gx\ \alpha B(gh\otimes gh',GX,f)+\alpha B(ghg\otimes gh'g,GX,f)$$
$$\overset{(60)}{=}-\alpha B(h\otimes h',GX,f)-\alpha B(hg\otimes h'g,GX,f)\overset{(78)}{=}0$$

Assume that (79) holds for some h,h'. Let us prove it holds also for gh and gh'.

$$\text{For } f=1_H,g\ B(gh\otimes gh',X,f)+B(ghg\otimes gh'g,X,f)$$
$$\overset{(59)}{=}B(h\otimes h',X,f)+B(hg\otimes h'g,X,f)\overset{(79)}{=}0$$
$$\text{For } f=x,gx\ B(gh\otimes gh',X,f)+B(ghg\otimes gh'g,X,f)$$
$$\overset{(60)}{=}-B(h\otimes h',X,f)-B(hg\otimes h'g,X,f)\overset{(79)}{=}0$$

Thus we have to consider only the usual six cases.

5.3.1. (a) $h=1_H,h'=g$. In view of (43) we proceed as follows.

Equality (77) for $f=g$ and $f=x$ gives us

1) $B(1_H\otimes g,1_A,g)+\gamma B(1_H\otimes g,GX,g)=B(g\otimes 1_H,1_A,g)\overset{(59)}{=}B(1_H\otimes g,1_A,g)$ Hence

$$\gamma B(1_H\otimes g,G,gx)=0 \tag{80}$$

2) $B(1_H\otimes g,1_A,x)+\gamma B(1_H\otimes g,GX,x)=B(g\otimes 1_H,1_A,x)\overset{(60)}{=}-B(1_H\otimes g,1_A,x)$ since $B(1_H\otimes g,GX,x)\overset{(43)}{=}0$, we get

$$2B(1_H\otimes g,1_A,x)=0. \tag{81}$$

Now (76) for $f=1_H$ gives us

3) $\alpha B(1_H\otimes g,G,1_H)+\gamma B(1_H\otimes g,X,1_H)=\alpha B(g\otimes 1_H,G,1_H)\overset{(59)}{=}\alpha B(1_H\otimes g,G,1_H)$ Thus we get, in view of (43),

$$\gamma B(1_H\otimes g,1_A,x)=\gamma B(1_H\otimes g,X,1_H)=0. \tag{82}$$

Consider now (79) for $f=1_H$.

4) $-B(1_H \otimes g, X, 1_H) = B(g \otimes 1_H, X, 1_H) \overset{(59)}{=} B(1_H \otimes g, X, 1_H)$ so that $2B(1_H \otimes g, X, 1_H) = 0$ which means, in view of (43), $2B(1_H \otimes g, 1_A, x) = 0$ which is true by (81) . And now (76) for $f = gx$

5) $\alpha B(1_H \otimes g, G, gx) + \gamma B(1_H \otimes g, X, gx) = \alpha B(g \otimes 1_H, G, gx) \overset{(60)}{=} -\alpha B(1_H \otimes g, G, gx)$ Thus $2\alpha B(1_H \otimes g, G, gx) + \gamma B(1_H \otimes g, X, gx) = 0$, but $B(1_H \otimes g, X, gx) = 0$ therefore

$$2\alpha B(1_H \otimes g, G, gx) = 0. \tag{83}$$

From (78) for $f = g$ we obtain 6) $-\alpha B(1_H \otimes g, GX, g) = \alpha B(g \otimes 1_H, GX, g) \overset{(59)}{=} \alpha B(1_H \otimes g, GX, g)$ Thus $2\alpha B(1_H \otimes g, GX, g) = 0$ by the form of the element $2\alpha B(1_H \otimes g, G, gx) = 0$. This is 5).

Following a procedure analogous to that of case a) we get

5.3.2. (b) $h = 1_H, h' = x$. We get

(1) $B(1_H \otimes x, 1_A, x) + \gamma B(1_H \otimes x, GX, x) = -B(g \otimes gx, 1_A, x) \overset{(60)}{=} B(1_H \otimes x, 1_A, x)$ i.e. $\gamma B(1_H \otimes x, GX, x) = 0$ this is already known in view of (42) .

(2) $B(1_H \otimes x, 1_A, g) + \gamma B(1_H \otimes x, GX, g) = -B(g \otimes gx, 1_A, g) \overset{(60)}{=} -B(1_H \otimes x, 1_A, g)$

$$2B(1_H \otimes x, 1_A, g) + \gamma B(1_H \otimes x, GX, g) = 0 \tag{84}$$

(3) $\alpha B(1_H \otimes x, G, 1_H) + \gamma B(1_H \otimes x, X, 1_H) = -\alpha B(g \otimes gx, G, 1_H)$ $\alpha B(1_H \otimes x, G, 1_H) + \gamma B(1_H \otimes x, X, 1_H) = -\alpha B(1_H \otimes x, G, 1_H)$ $2\alpha B(1_H \otimes x, G, 1_H) + \gamma B(1_H \otimes x, X, 1_H) = 0$ $2\alpha B(1_H \otimes x, G, 1_H) + \gamma[1 - B(1_H \otimes x; 1_A, x)] = 0$

$$2\alpha B(1_H \otimes x, G, 1_H) - \gamma B(1_H \otimes x; 1_A, x) + \gamma = 0 \tag{85}$$

The following equalities are already known.

(4) $\alpha B(1_H \otimes x, G, gx) + \gamma B(1_H \otimes x, X, gx) = -\alpha B(g \otimes gx, G, gx)$ $\alpha B(1_H \otimes x, G, gx) + \gamma B(1_H \otimes x, X, gx) = \alpha B(1_H \otimes x, G, gx)$

$\gamma B(1_H \otimes x, X, gx) = 0$

(5) $B(1_H \otimes x, X, 1_H) = B(g \otimes gx, X, 1_H)$, $B(1_H \otimes x, X, 1_H) = B(1_H \otimes x, X, 1_H)$

(6) $\alpha B(1_H \otimes x, GX, g) = \alpha B(g \otimes gx, GX, g)$

5.3.3. (c) $h = 1_H, h' = gx$. We get

(1) $B(1_H \otimes gx, 1_A, 1_H) + \gamma B(1_H \otimes gx, GX, 1_H) = -B(g \otimes x, 1_A, 1_H) \overset{(60)}{=} -B(1_H \otimes gx, 1_A, 1_H)$ so that $2B(1_H \otimes gx, 1_A, 1_H) + \gamma B(1_H \otimes gx, GX, 1_H) = 0$. By considering (44), we get

$$2B(1_H \otimes gx, 1_A, 1_H) + \gamma[B(1_H \otimes gx; G, x) + B(1_H \otimes g; G, 1_H)] = 0 \tag{86}$$

(2) $B(1_H \otimes gx, 1_A, gx) + \gamma B(1_H \otimes gx, GX, gx) = -B(g \otimes x, 1_A, gx) \overset{(60)}{=} B(1_H \otimes gx, 1_A, gx)$ Thus $\gamma B(1_H \otimes gx, GX, gx) = 0$ i.e.$\gamma B(1_H \otimes g; G, gx) = 0$ This is (80).

(3) $\alpha B(1_H \otimes gx, G, g) + \gamma B(1_H \otimes gx, X, g) = -\alpha B(g \otimes x, G, g) \overset{(60)}{=} -\alpha B(1_H \otimes gx, G, g)$

$2\alpha B(1_H \otimes gx, G, g) + \gamma B(1_H \otimes gx, X, g) = 0$

$$2\alpha B(1_H \otimes gx, G, g) + \gamma[B(1_H \otimes g; 1_A, g) - B(1_H \otimes gx; 1_A, gx)] = 0 \tag{87}$$

(4) $\alpha B(1_H \otimes gx, G, x) + \gamma B(1_H \otimes gx, X, x) = -\alpha B(g \otimes x, G, x) \overset{(60)}{=} \alpha B(1_H \otimes gx, G, x)$

so that we get $\gamma B(1_H \otimes gx, X, x) = 0$ i.e. $\gamma B(1_H \otimes g; 1_A, x) = 0$. This is true by (82).

(5) $-\alpha B(1_H \otimes gx, GX, 1_H) \overset{(78)}{=} \alpha B\left(g \otimes gxg, GX, 1_H\right) = -\alpha B\left(g \otimes x, GX, 1_H\right) \overset{(60)}{=} -\alpha B\left(1_H \otimes gx, GX, 1_H\right)$

This is trivial.

(6) $-B(1_H \otimes gx, X, g) = B(g \otimes gxg, X, g) = -B\left(g \otimes x, X, g\right) \overset{(60)}{=} -B(1_H \otimes gx, X, g)$. Thus we obtain

$-B(1_H \otimes gx, X, g) = -B(g \otimes x, X, g)$. This follows from (59).

(7) $-B(1_H \otimes gx, X, x) = -B(g \otimes x, X, x)$. This is true in view of (44) and (20) .

(8) $-\alpha B(1_H \otimes gx, GX, gx) = \alpha B(g \otimes gxg, GX, gx) = -\alpha B(g \otimes x, GX, gx) \overset{(60)}{=} \alpha B(1_H \otimes gx, GX, gx)$

so that we get $2\alpha B(1_H \otimes gx, GX, gx) = 0$. Now, in view of (44), the coefficient of $GX \otimes gx$ is $B(1_H \otimes g; G, gx)$ and hence we obtain $2\alpha B(1_H \otimes g, G, gx) = 0$. This is (83).

5.3.4. (d) $\boldsymbol{h = x, h' = 1_H}$. By considering (50) we obtain the following.

(1) $B(x \otimes 1_H, 1_A, 1_H) + \gamma B(x \otimes 1_H, GX, 1_H) \overset{(77)}{=} -B(gx \otimes g, 1_A, 1_H) \overset{(60)}{=} -B(x \otimes 1_H, 1_A, 1_H)$

Hence we get $2B(x \otimes 1_H, 1_A, 1_H) = -\gamma B(x \otimes 1_H, GX, 1_H)$ and, in view of (50) we obtain

$2B(x \otimes 1_H, 1_A, 1_H) = -\gamma(B(x \otimes 1_H; G, x) + B(g \otimes 1_H; G, 1_H)) \overset{(60)}{=} B(1_H \otimes g; G, 1_H)$. Using (67) we obtain (86) again.

(2) $B(x \otimes 1_H, 1_A, gx) + \gamma B(x \otimes 1_H, GX, gx) \overset{(77)}{=} -B(gx \otimes g, 1_A, gx) \overset{(60)}{=} B(x \otimes 1_H, 1_A, gx)$

Hence by (50), we obtain $\gamma B(1_H \otimes g; G, gx) = 0$.This is (80)

(3) $\alpha B(x \otimes 1_H, G, g) + \gamma B(x \otimes 1_H, X, g) \overset{(76)}{=} -\alpha B(gx \otimes g, G, g) \overset{(60)}{=} -\alpha B(x \otimes 1_H, G, g)$

so that we get, by (50), $2\alpha B(x \otimes 1_H, G, g) + \gamma(-B(x \otimes 1_H; 1_A, gx) - B(g \otimes 1_H; 1_A, g)) = 0$. Since $B(g \otimes 1_H; 1_A, g) \overset{(60)}{=} B(1_H \otimes g; 1_A, g)$ we obtain

$$2\alpha B(x \otimes 1_H, G, g) + \gamma(-B(x \otimes 1_H; 1_A, gx) - B(1_H \otimes g; 1_A, g)) = 0 .$$

using (67) this becomes the already known (87) .

(4) $\alpha B(x \otimes 1_H, G, x) + \gamma B(x \otimes 1_H, X, x) \overset{(76)}{=} -\alpha B(gx \otimes g, G, x) \overset{(60)}{=} \alpha B(x \otimes 1_H, G, x)$.

Thus we get $\gamma B(x \otimes 1_H, X, x) = 0$ and hence, using (50) , $\gamma B(g \otimes 1_H; 1_A, x) = 0$ and since $\gamma B(1_H \otimes g; 1_A, x) \overset{(60)}{=} -\gamma B(g \otimes 1_H; 1_A, x)$ we get $\gamma B(1_H \otimes g; 1_A, x) = 0$ which is true by (82).

(5) $-B(x \otimes 1_H, X, x) \overset{(79)}{=} -B(gx \otimes g, X, x) \overset{(60)}{=} B(x \otimes 1_H, X, x)$

and hence $B(x\otimes 1_H, X, x) = 0$ so that we get, using (50) , $B(g\otimes 1_H; 1_A, x) = 0$ and hence $B(1_H\otimes g; 1_A, x) \overset{(60)}{=} -B(g\otimes 1_H; 1_A, x) = 0$ which is true by (81).

(6) $-\alpha B(x\otimes 1_H, GX, gx) \overset{(78)}{=} -\alpha B(gx\otimes g, GX, gx) \overset{(60)}{=} \alpha B(x\otimes 1_H, GX, gx)$ so that we get

$\alpha B(x\otimes 1_H, GX, gx) = 0$ and hence, using (50) , $\alpha B(g\otimes 1_H; G, gx) = 0$ which holds by (83) .

(7) $-\alpha B(x\otimes 1_H, GX, 1_H) \overset{(78)}{=} -\alpha B(gx\otimes g, GX, 1_H) \overset{(60)}{=} -\alpha B(x\otimes 1_H, GX, 1_H)$ which is trivial.

(8) $-B(x\otimes 1_H, X, g) \overset{(79)}{=} -B(gx\otimes g, X, g) \overset{(60)}{=} -B(x\otimes 1_H, X, g)$.This is trivial.

5.3.5. (e) $\boldsymbol{h = x, h' = x.}$ By using (74) we obtain the following.

(1) $B(x\otimes x, 1_A, x) + \gamma B(x\otimes x, GX, x) \overset{(77)}{=} B(xg\otimes xg, 1_A, x) = B(gx\otimes gx, 1, x) \overset{(60)}{=} -B(x\otimes x, 1_A, x)$

so that we get $2B(x\otimes x, 1_A, x) + \gamma B(x\otimes x, GX, x) = 0$ so that we get, by (74),

$$2B(x\otimes x, 1_A, x) + \gamma\left[B(x\otimes 1_H; G, x) + B(g\otimes x; G, x)\right] = 0.$$

By using (67) and (60) and (74), this becomes

$$0 + \gamma\left[B(1_H\otimes gx; G, x) - B(1_H\otimes gx; G, x)\right] = 0$$

which is trivial.

(2) $B(x\otimes x, 1_A, g) + \gamma B(x\otimes x, GX, g) \overset{(77)}{=} B(xg\otimes xg, 1_A, g) = B(gx\otimes gx, 1_A, g) \overset{(60)}{=} B(x\otimes x, 1_A, g)$

$B(x\otimes x, 1_A, g) + \gamma B(x\otimes x, GX, g) = B(x\otimes x, 1_A, g)$

and hence

$$\gamma B(x\otimes x, GX, g) = 0$$

which holds by (74).

(3) $\alpha B(x\otimes x, G, 1_H) + \gamma B(x\otimes x, X, 1_H) \overset{(76)}{=} \alpha B(xg\otimes xg, G, 1_H) = \alpha B(gx\otimes gx, G, 1_H) \overset{(59)}{=} \alpha B(x\otimes x, G, 1_H)$

We obtain $\gamma B(x\otimes x, X, 1_H) = 0$. This follows from (74).

(4) $\alpha B(x\otimes x, G, gx) + \gamma B(x\otimes x, X, gx) \overset{(76)}{=} \alpha B(xg\otimes xg, G, gx) = \alpha B(gx\otimes gx, G, gx) \overset{(60)}{=} -\alpha B(x\otimes x, G, gx)$

so that we get

$2\alpha B(x\otimes x, G, gx) + \gamma B(x\otimes x, X, gx) = 0$ and hence, by (74),

$$-4\alpha B(1_H\otimes gx, G, g) + 2\gamma\left[B(1_H\otimes gx; 1_A, gx) - B(1_H\otimes g, 1_A, g)\right] = 0 \tag{88}$$

(5) $-B(x\otimes x, X, 1_H) \overset{(79)}{=} B(xg, xg, X, 1_H) = B(gx\otimes gx, X, 1_H) \overset{(60)}{=} B(x\otimes x, X, 1_H)$

so that we get $2B(x\otimes x, X, 1_H)=0$ true by (74).

(6) $-B(x\otimes x, X, gx) \overset{(79)}{=} B(xg\otimes xg, X, gx) = B(gx\otimes gx, X, gx) \overset{(60)}{=} -B(x\otimes x, X, gx)$

Trivial.

(7) $-\alpha B(x\otimes x, GX, g) \overset{(78)}{=} \alpha B(gx\otimes gx, GX, g) \overset{(60)}{=} \alpha B(x\otimes x, GX, g)$

so that we get $2\alpha B(x\otimes x, GX, g)=0$ this is true by (74).

(8) $-\alpha B(x\otimes x, GX, x) \overset{(78)}{=} \alpha B(gx\otimes gx, GX, x) \overset{(60)}{=} -\alpha B(x\otimes x, GX, x)$

This is trivial.

5.3.6. (f) $h=x, h'=gx$. By considering (71) we obtain

$$4B(1_H\otimes x, 1_A, g) + 2\gamma B(1_H\otimes x; G, gx) = 0 \tag{89}$$

5.4. Case $X\otimes 1_H$

We have

$$B(h\otimes h')(X\otimes 1_H) = (X\otimes 1_H)B(hg\otimes h'g) + B(hx\otimes h'g) + B(h\otimes h'x). \tag{90}$$

Now we compute the left side of (90).

$$\begin{aligned}&B(h\otimes h')(X\otimes 1_H)\\ &\quad=\textstyle\sum_{k,l=0}^{1}\beta B(h\otimes h', X, g^kx^l)1_A\otimes g^kx^l + \beta B(h\otimes h', GX, g^kx^l)G\otimes g^kx^l\\ &\qquad+B(h\otimes h', 1_A, g^kx^l)X\otimes g^kx^l + B(h\otimes h', G, g^kx^l)GX\otimes g^kx^l\end{aligned}$$

Now, by computing first summand of the right side (90), we get

$$\begin{aligned}&(X\otimes 1_H)B(hg\otimes h'g)\\ &\quad=\textstyle\sum_{k,l=0}^{1}[\beta B(gh\otimes gh', X, g^kx^l) + \gamma B(gh\otimes gh', G, g^kx^l)]1_A\otimes g^kx^l\\ &\qquad-\beta B(gh\otimes gh', GX, g^kx^l)G\otimes g^kx^l\\ &\qquad+[B(gh\otimes gh', 1_A, g^kx^l) + \gamma B(gh\otimes gh', GX, g^kx^l)]X\otimes g^kx^l\\ &\qquad-B(gh\otimes gh', G, g^kx^l)GX\otimes g^kx^l\end{aligned}$$

We can summarize the equality (90) in the following form:

$$\begin{aligned}B(h\otimes h', 1_A, f) &= B(hg\otimes h'g, 1_A, f) + \gamma B(hg\otimes h'g, GX, f)\\ &\quad+B(hx\otimes h'g; X, f) + B(h\otimes h'x; X, f) \text{ for all } f\in H.\end{aligned} \tag{91}$$

$$\begin{aligned}B(h\otimes h', G, f) &= -B(hg\otimes h'g, G, f) + B(hx\otimes h'g; GX, f)\\ &\quad+B(h\otimes h'x; GX, f) \text{ for all } f\in H.\end{aligned} \tag{92}$$

$$\begin{aligned}\beta B(h\otimes h', X, f) &= \beta B(hg\otimes h'g, X, f) + \gamma B(hg\otimes h'g, G, f)\\ &\quad+B(hx\otimes h'g; 1_A, f) + B(h\otimes h'x; 1_A, f)\end{aligned} \tag{93}$$

for all $f\in H$

$$\begin{aligned}\beta B(h\otimes h', GX, f) &= -\beta B(hg\otimes h'g, GX, f) + B(hx\otimes h'g; G, f)\\ &\quad+B(h\otimes h'x; G, f)\\ &\quad\text{for all } f\in H\end{aligned} \tag{94}$$

Assume that (91) holds for some h, h'. Let us prove it holds also for gh and gh'.

$$\begin{aligned}
\text{For } f = 1_H, g, \quad & B(ghg \otimes gh'g, 1_A, f) + \gamma B(ghg \otimes gh'g, GX, f) \\
& + B(ghx \otimes gh'g; X, f) \\
& + B(gh \otimes gh'x; X, f) - B(gh \otimes gh', 1_A, f) \\
& \overset{(59)}{=} B(hg \otimes h'g, 1_A, f) + \gamma B(hg \otimes h'g, GX, f) + B(hx \otimes h'g; X, f) \\
& + B(h \otimes h'x; X, f) - B(h \otimes h', 1_A, f) \overset{(91).}{=} 0 \\
\text{For } f = x, gx, \quad & B(ghg \otimes gh'g, 1_A, f) + \gamma B(ghg \otimes gh'g, GX, f) \\
& + B(ghx \otimes gh'g; X, f) + B(gh \otimes gh'x; X, f) - B(gh \otimes gh', 1_A, f) \\
& \overset{(60)}{=} -B(hg \otimes h'g, 1_A, f) - \gamma B(hg \otimes h'g, GX, f) \\
& - B(hx \otimes h'g; X, f) - B(h \otimes h'x; X, f) + B(h \otimes h', 1_A, f) \overset{(91)}{=} 0
\end{aligned}$$

Assume that (92) holds for some h, h'. Let us prove it holds also for gh and gh'.

$$\begin{aligned}
\text{For } f = 1_H, g, \quad & -B(ghg \otimes gh'g, G, f) + B(ghx \otimes gh'g; GX, f) \\
& + B(gh \otimes gh'x; GX, f) - B(gh \otimes gh', G, f) \\
& \overset{(59)}{=} -B(hg \otimes h'g, G, f) + B(hx \otimes h'g; GX, f) \\
& + B(h \otimes h'x; GX, f) - B(h \otimes h', G, f) \overset{(92)}{=} 0 \\
\text{For } f = x, gx, \quad & -B(ghg \otimes gh'g, G, f) + B(ghx \otimes gh'g; GX, f) \\
& + B(gh \otimes gh'x; GX, f) - B(gh \otimes gh', G, f) \\
& \overset{(60)}{=} B(hg \otimes h'g, G, f) - B(hx \otimes h'g; GX, f) \\
& - B(h \otimes h'x; GX, f) + B(h \otimes h', G, f) \overset{(92)}{=} 0
\end{aligned}$$

Assume that (93) holds for some h, h'. Let us prove it holds also for gh and gh'.

$$\begin{aligned}
\text{For } f = 1_H, g, \quad & \beta B(ghg \otimes gh'g, X, f) + \gamma B(ghg \otimes gh'g, G, f) \\
& + B(ghx \otimes gh'g; 1_A, f) + B(gh \otimes gh'x; 1_A, f) - \beta B(gh \otimes gh', X, f) \\
& \overset{(59)}{=} \beta B(hg \otimes h'g, X, f) + \gamma B(hg \otimes h'g, G, f) \\
& + B(hx \otimes h'g; 1_A, f) + B(h \otimes h'x; 1_A, f) - \beta B(h \otimes h', X, f) \overset{(93)}{=} 0 \\
\text{For } f = x, gx, \quad & \beta B(ghg \otimes gh'g, X, f) + \gamma B(ghg \otimes gh'g, G, f) \\
& + B(ghx \otimes gh'g; 1_A, f) + B(gh \otimes gh'x; 1_A, f) - \beta B(gh \otimes gh', X, f) \\
& \overset{(60)}{=} -\beta B(hg \otimes h'g, X, f) - \gamma B(hg \otimes h'g, G, f) \\
& - B(hx \otimes h'g; 1_A, f) - B(h \otimes h'x; 1_A, f) + \beta B(h \otimes h', X, f) \overset{(93)}{=} 0
\end{aligned}$$

Assume that (94) holds for some h, h'. Let us prove it holds also for gh and gh'.

$$\begin{aligned} \text{For } f = 1_H, g \quad & -\beta B(ghg \otimes gh'g, GX, f) + B(ghx \otimes gh'g; G, f) \\ & + B(gh \otimes gh'x; G, f) - \beta B(gh \otimes gh', GX, f) \\ & \overset{(59)}{=} -\beta B(hg \otimes h'g, GX, f) + B(hx \otimes h'g; G, f) \\ & + B(h \otimes h'x; G, f) - \beta B(h \otimes h', GX, f) \overset{(94)}{=} 0 \\ \text{For } f = x, gx \quad & -\beta B(ghg \otimes gh'g, GX, f) + B(ghx \otimes gh'g; G, f) \\ & + B(gh \otimes gh'x; G, f) - \beta B(gh \otimes gh', GX, f \\ & \overset{(60)}{=} \beta B(hg \otimes h'g, GX, f) - B(hx \otimes h'g; G, f) \\ & - B(h \otimes h'x; G, f) + \beta B(h \otimes h', GX, f) \overset{(94)}{=} 0 \end{aligned}$$

Therefore we have to check the previous equality for the usual six elements.

5.4.1. (a) $h = 1_H, h' = g$. $B(1_H \otimes g, 1_A, f) \overset{(91)}{=} B(g \otimes 1_H, 1_A, f) + \gamma B(g \otimes 1_H, GX, f) + B(x \otimes 1_H; X, f) + B(1_H \otimes gx; X, f)$

$\beta B(1_H \otimes g, X, f) \overset{(93)}{=} \beta B(g \otimes 1_H, X, f) + \gamma B(g \otimes 1_H, G, f) + B(x \otimes 1_H; 1_A, f) + B(1_H \otimes gx; 1_A, f)$

$\beta B(1_H \otimes g, GX, f) \overset{(94)}{=} -\beta B(g \otimes 1_H, GX, f) + B(x \otimes 1_H; G, f) + B(1_H \otimes gx; G, f)$

$B(1_H \otimes g, G, f) \overset{(92)}{=} -B(g \otimes 1_H, G, f) + B(x \otimes 1_H; GX, f) + B(1_H \otimes gx; GX, f)$

Now using (59) and (60) we get

$$\begin{aligned} B(1_H \otimes g, 1_A, f) & \overset{(91)(59)}{=} B(1_H \otimes g, 1_A, f) + \gamma B(1_H \otimes g, GX, f) \\ & + B(x \otimes 1_H; X, f) + B(1_H \otimes gx; X, f) \quad \text{and} \quad f = 1_H, g \end{aligned} \tag{95}$$

$$\begin{aligned} B(1_H \otimes g, 1_A, f) & \overset{(91)(60)}{=} -B(1_H \otimes g, 1_A, f) - \gamma B(1_H \otimes g, GX, f) \\ & + B(x \otimes 1_H; X, f) + B(1_H \otimes gx; X, f) \quad \text{and} \quad f = x, gx \end{aligned} \tag{96}$$

$$\begin{aligned} B(1_H \otimes g, G, f) & \overset{(92)(59)}{=} -B(1_H \otimes g, G, f) \\ & + B(x \otimes 1_H; GX, f) + B(1_H \otimes gx; GX, f) \text{ and } f = 1_H, g \end{aligned} \tag{97}$$

$$\begin{aligned} B(1_H \otimes g, G, f) & \overset{(92)(60)}{=} B(1_H \otimes g, G, f) \\ & + B(x \otimes 1_H; GX, f) + B(1_H \otimes gx; GX, f) \text{ and } f = x, gx \end{aligned} \tag{98}$$

$$\begin{aligned} \beta B(1_H \otimes g, X, f) & \overset{(93)(59)}{=} \beta B(1_H \otimes g, X, f) + \gamma B(1_H \otimes g, G, f) \\ & + B(x \otimes 1_H; 1_A, f) + B(1_H \otimes gx; 1_A, f) \quad \text{and} \quad f = 1_H, g \end{aligned} \tag{99}$$

$$\begin{aligned} \beta B(1_H \otimes g, GX, f) & \overset{(94)(59)}{=} -\beta B(1_H \otimes g, GX, f) + B(x \otimes 1_H; G, f) \\ & + B(1_H \otimes gx; G, f) \quad \text{and} \quad f = 1_H, g \end{aligned} \tag{100}$$

Now, in view of (43), we obtain the following.

(1) $B(1_H\otimes g,1_A,g)\overset{(95)}{=}B(1_H\otimes g,1_A,g)+\gamma B(1\otimes g,GX,g)+B(x\otimes 1;X,g)+B(1\otimes gx;X,g)$
so that, by using (67) and (44), we get

$$\gamma B(1_H\otimes g,G,gx)+2\left[-B(1_H\otimes gx;1_A,gx)+B(1_H\otimes g,1_A,g)\right]=0 \tag{101}$$

(2) $B(1_H\otimes g,1_A,x)=B(g\otimes 1_H,1_A,x)+\gamma B(g\otimes 1_H,GX,x)+B(x\otimes 1_H;X,x)+B(1_H\otimes gx;X,x)$

By means of (96), we get

$$2B(1_H\otimes g,1_A,x)=B(x\otimes 1_H;X,x)+B(1_H\otimes gx;X,x). \tag{102}$$

By using this and (67) and (44) we get $2B(1_H\otimes g,1_A,x)=B(1_H\otimes g;1_A,x)+B(1_H\otimes g;1_A,x)$ which is trivial.

(3) $B(1_H\otimes g,G,1_H)\overset{(97)}{=}-B(1_H\otimes g,G,1_H)+B(x\otimes 1_H;GX,1_H)+B(1_H\otimes gx;GX,1_H)$

so that, by means of (67) and (44), we obtain

$$2B(1_H\otimes gx;G,x)=0 \tag{103}$$

(4) $B(1_H\otimes g,G,gx)\overset{(98)}{=}B(1_H\otimes g,G,gx)+B(x\otimes 1_H;GX,gx)+B(1_H\otimes gx;GX,gx)$

so that $B(x\otimes 1_H;GX,gx)+B(1_H\otimes gx;GX,gx)=0$. By using (67) and (44), we obtain $2B(1_H\otimes g,G,gx)=0$ which is (68).

(5) $\beta B(1_H\otimes g,X,1_H)\overset{(99)}{=}\beta B(1_H\otimes g,X,1_H)+\gamma B(1_H\otimes g,G,1_H)+B(x\otimes 1_H;1_A,1_H)+B(1_H\otimes gx;1_A,1_H)$

so that, by using (67), we get

$$\gamma B(1_H\otimes g,G,1_H)+2B(1_H\otimes gx;1_A,1_H)=0 \tag{104}$$

(6) $\beta B(1_H\otimes g,GX,g)\overset{(99)}{=}-\beta B(1_H\otimes g,GX,g)+B(x\otimes 1_H;G,g)+B(1_H\otimes gx;G,g)$

so that we get $2\beta B(1_H\otimes g,GX,g)=B(x\otimes 1_H;G,g)+B(1_H\otimes gx;G,g)$.

Since $B(1_H\otimes g,GX,g)\overset{(43)}{=}B(1_H\otimes g,G,gx)$, we obtain, by means of (67),

$$2\beta B(1_H\otimes g,G,gx)=2B(1_H\otimes gx;G,g) \tag{105}$$

5.4.2. (b) $\boldsymbol{h=1_H,h'=x}$**.** By using (91) we get

$$B(1_H\otimes x,1_A,f)=-B(g\otimes gx,1_A,f)-\gamma B(g\otimes gx,GX,f)-B(x\otimes gx;X,f) \tag{106}$$

Now, by using (59) and (60), we obtain

$$\begin{aligned}&2B(1_H\otimes x,1_A,f)\overset{(106)(59)}{=}-\gamma B(1_H\otimes x,GX,f)\\&-B(gx\otimes x;X,f)\text{ and } f=1_H,g \end{aligned}\tag{107}$$

$$\begin{aligned}&B(1_H\otimes x,1_H,f)\overset{(106)(60)}{=}B(1_H\otimes x,1_A,f)+\gamma B(1_H\otimes x,GX,f)\\&+B(gx\otimes x;X,f)\quad\text{and}\quad f=x,gx.\end{aligned}\tag{108}$$

By means of (92), we obtain

$$B(1_H \otimes x, G, f) = B(g \otimes gx, G, f) - B(x \otimes gx; GX, f) \tag{109}$$

Now, by (59), we have

$$B(1_H \otimes x, G, f)^{(109)(59)} \\ = B(1_H \otimes x, G, f) - B(gx \otimes x; GX, f) \text{ and } f = 1_H, g \tag{110}$$

$$B(1_H \otimes x, G, f)^{(109)(60)} \\ = -B(1_H \otimes x, G, f) + B(gx \otimes x; GX, f) \text{ and } f = x, gx \tag{111}$$

In view of (42) we consider the following cases.

(1)

$2B(1_H \otimes x, 1_A, g) \overset{(107)}{=} -\gamma B(1_H \otimes x, GX, g) - B(x \otimes gx; X, g)$By using (71) we get $2B(1_H \otimes x, 1_A, g) = -\gamma B(1_H \otimes x, GX, g)$. This is (84).

(2)

$B(1_H \otimes x, 1_A, x) \overset{(108)}{=} B(1_H \otimes x, 1_A, x) + \gamma B(1_H \otimes x, GX, x) - B(x \otimes gx; X, x)$ using (71) we obtain $\gamma B(1_H \otimes x, GX, x) = 0$ which is trivial by (42).

(3)

$B(1_H \otimes x, G, 1_H) \overset{(110)}{=} B(1_H \otimes x, G, 1_H) - B(gx \otimes x; GX, 1_H) = B(1_H \otimes x, G, 1_H) - B(gx \otimes x; GX, 1_H) \overset{(60)}{=}$

$= B(1_H \otimes x, G, 1_H) - B(x \otimes gx; GX, 1_H) \overset{(71)}{=} B(1_H \otimes x, G, 1_H)$ This is trivial.

(4) $B(1_H \otimes x, G, gx) \overset{(111)}{=} -B(1_H \otimes x, G, gx) + B(gx \otimes x; GX, gx)$. Therefore we get

$$2B(1_H \otimes x, G, gx) = B(gx \otimes x; GX, gx) \overset{(60)}{=} -B(x \otimes gx; GX, gx) \\ \overset{(71)}{=} 2B(1_H \otimes x; G, gx). \text{ This is trivial.}$$

Now, by using (93), we get

$$\beta B(1_H \otimes x, X, f) \\ = -\beta B(g \otimes gx, X, f) - \gamma B(g \otimes gx, G, f) - B(x \otimes gx; 1_A, f) \tag{112}$$

(5) $\beta B(1_H \otimes x, X, 1_H) \overset{(112)(60)}{=} -\beta B(1_H \otimes x, X, 1_H) - \gamma B(1_H \otimes x, G, 1_H) - B(x \otimes gx; 1_A, 1_H)$

$2\beta B(1_H \otimes x, X, 1_H) = -\gamma B(1_H \otimes x, G, 1_H) - B(x \otimes gx; 1_A, 1_H)$ and by (71) and (42) we get

$$2\beta[1 - B(1_H \otimes x; 1_A, x)] = -\gamma B(1_H \otimes x, G, 1_H) \tag{113}$$

(6) $\beta B(1_H \otimes x, GX, f) \overset{(94)}{=} -\beta B(g \otimes xg, GX, f) + B(x \otimes xg; G, f) + B(1_H \otimes xx; G, f) = \beta B(g \otimes gx, GX, f) - B(x \otimes gx; G, f)$
so that

$$\beta B(1_H \otimes x, GX, f) = \beta B(g \otimes gx, GX, f) - B(x \otimes gx; G, f) \text{ for all } f \tag{114}$$

and hence $\beta B(1_H \otimes x, GX, g) \overset{(114)}{=} \beta B(g \otimes gx, GX, g) - B(x \otimes gx; G, g) \overset{(60)}{=} \beta B(1_H \otimes x, GXc, g) - B(gx \otimes x; G, g)$ so that we get $B(gx \otimes x; G, g) = B(x \otimes gx; G, g) = 0$ which is already known by (71).

5.4.3. (c) $h = 1_H, h' = gx$. By means of (91), we get

$$B(1_H \otimes gx, 1_A, f) = -B(g \otimes x, 1_A, f) - \gamma B(g \otimes x, GX, f) - B(x \otimes x; X, f). \tag{115}$$

(1) By using (115), (59) , (74) and (44) we get

$2B(1_H \otimes gx, 1_A, 1_H) + \gamma[B(1_H \otimes g; G, 1_H) + B(1_H \otimes gx; G, x)]$ this is (86).

(2) By using (115), (60) , (74) and, (44) we obtain

$$\gamma B(1_H \otimes g, G, gx) - 2\left[B(1_H \otimes gx; 1_A, gx) - B(1_H \otimes g, 1_A, g)\right] = 0. \tag{116}$$

In view of (92) we get

$$B(1_H \otimes gx, G, f) = B(g \otimes x, G, f) - B(x \otimes x; GX, f). \tag{117}$$

(3) $B(1_H \otimes gx, G, g) \overset{(117)}{=} B(g \otimes x, G, g) - B(x \otimes x; GX, g) \overset{(60)}{=} B(1_H \otimes gx, G, g) - B(x \otimes x; GX, g)$

and hence we get $B(x \otimes x; GX, g) = 0$ which is true in view of (74)

(4) $B(1_H \otimes gx, G, x) \overset{(117)}{=} B(g \otimes x, G, x) - B(x \otimes x; GX, x) \overset{(60)}{=} -B(1_H \otimes gx, G, x) - B(x \otimes x; GX, x)$

so that we get in view of (74)

$$2B(1_H \otimes gx, G, x) = 0 \tag{118}$$

$\beta B(1_H \otimes gx, X, f) \overset{(93)}{=} = \beta B(g \otimes gxg, X, f) + \gamma B(g \otimes gxg, G, f) + B(x \otimes gxg; 1_A, f) + B(1_H \otimes gxx; 1_A, f)$

$= -\beta B(g \otimes x, X, f) - \gamma B(g \otimes x, G, f) - B(x \otimes x; 1_A, f)$ so that we get

$$\beta B(1_H \otimes gx, X, f) = -\beta B(g \otimes x, X, f) - \gamma B(g \otimes x, G, f) - B(x \otimes x; 1_A, f). \tag{119}$$

(5) In view of (119) (59) (44) and(74), we get

$$2[B(1_H \otimes g, 1_A, g) - B(1_H \otimes gx, 1_A, gx)] = -\gamma B(1_H \otimes gx, G, g). \tag{120}$$

(6) By using (119), (60) and (74) we get

$$\gamma B(1_H \otimes gx, G, x) = 0. \tag{121}$$

Now, in view of (94), we have

$$\beta B(1_H \otimes gx, GX, f) = \beta B(g \otimes x, GX, f) - B(x \otimes x; G, f). \tag{122}$$

(7) $\beta B(1_H \otimes gx, GX, 1_H) \overset{(122)}{=} \beta B(g \otimes x, GX, 1_H) - B(x \otimes x; G, 1_H) \overset{(60)}{=} \beta B(1_H \otimes gx, GX, 1_H) - B(x \otimes x; G, 1_H)$

so that we get $B(x \otimes x; G, 1_H) = 0$ which is true in view of (74)

(8) By using (122), (60), (74) and (44), we get

$$2\beta B(1_H \otimes g, G, gx) = 2B(1_H \otimes gx, G, g) \tag{123}$$

5.4.4. (d) $h = x, h' = 1_H$. By means of (91) we obtain

$$B(x\otimes 1_H, 1_A, f) = -B(gx\otimes g, 1_A, f) - \gamma B(gx \otimes g, GX, f) + B(x \otimes x; X, f). \quad (124)$$

(1) By using (124), (59) , (74) and (67), we get

$$2B(1_H \otimes gx; 1_A, 1_H) = -\gamma B(1_H \otimes g; G, 1_H)\ . \quad (125)$$

(2) In view of (124), (60), (67) and (74) we obtain

$$\gamma B(1_H \otimes g, G, gx) + 2B(1_H \otimes gx, 1_A, gx) - 2B(1_H \otimes g, 1_A, g) = 0. \quad (126)$$

Now by using (92), (59) and (60) we get

$$B(x \otimes x; GX, f) = 0 \text{ this is know in view of (74) for } f = 1_H, g$$

and

$$2B(x \otimes 1_H, G, f) = B(x \otimes x; GX, f) \text{ for } f = x, gx$$

in particular
(3)$B(x \otimes x; GX, g) = 0$ which is known.
(4)$2B(x \otimes 1_H, G, x) = B(x \otimes x; GX, x)$ By (67) and (74) we get $2B(1_H \otimes gx, G, x) = 0$ which is (118). Now, by using (93) we get

$$\beta B(x \otimes 1_H, X, f) \overset{(95)}{=} -\beta B(x \otimes 1_H, X, f) - \gamma B(x \otimes 1_H, G, f) + B(x \otimes x; 1_A, f) \text{ for } f = 1_H, g.$$

$$\beta B(x \otimes 1_H, X, f) \overset{(96)}{=} \beta B(x \otimes 1_H, X, f) + \gamma B(x \otimes 1_H, G, f) + B(x \otimes x; 1_A, f) \text{ for } f = x, gx.$$

In particular
(5) $2\beta B(x \otimes 1_H, X, g) = -\gamma B(x \otimes 1_H, G, g) + B(x \otimes x; 1_A, g)$ By means of (67) and (74), we get

$$2\beta[-B(1_H \otimes gx, 1_A, gx) + B(1_H \otimes g, 1_A, g)] = -\gamma B(1_H \otimes gx, G, g). \quad (127)$$

(6) $\beta B(x\otimes 1_H, X, x) = \beta B(x\otimes 1_H, X, x) + \gamma B(x\otimes 1_H, G, x) + B(x\otimes x; 1_A, x)$ i.e.

$\gamma B(x \otimes 1_H, G, x) + B(x \otimes x; 1_A, x) = 0$
By (67) and (74) , we get $\gamma B(1_H \otimes gx, G, x) = 0$ which is (121).
Now, by using (94), we get

$$\beta B(x \otimes 1_H, GX, f) \overset{(95)}{=} \beta B(x \otimes 1_H, GX, f) + B(x \otimes x; G, f) \text{ for } f = 1_H, g$$

and

$$\beta B(x \otimes 1_H, GX, f) \overset{(96)}{=} -\beta B(x \otimes 1_H, GX, f) + B(x \otimes x; G, f) \text{ for } f = x, gx.$$

In particular
(7) For $f = 1_H$we get $B(x \otimes x; G, 1_H) = 0$ which is already known by (74) and

(8) for $f = gx$ and by using (67) and (74) we get

$$2\beta B(1_H \otimes g, G, gx) = -2B(1_H \otimes gx; G, g). \tag{128}$$

5.4.5. (e) $h = h' = x$.. By using (91), (59) and (60) we obtain

$$\gamma B(x \otimes x, GX, f) = 0 \text{ for } f = 1_H, g \tag{129}$$

and

$$2B(x \otimes x, 1_A, f) = -\gamma B(x \otimes x, GX, f) \text{ for } f = x, gx \tag{130}$$

so that we get
(1) $\gamma B(x \otimes x, GX, g) = 0$ which is true in view of (74)
(2) $2B(x \otimes x, 1_A, x) = -\gamma B(x \otimes x, GX, x)$ which is true in view of (74).

Now by using (92) we get

$$B(x \otimes x, G, f) \overset{(59)}{=} -B(x \otimes x, G, f) \text{ for } f = 1_H, g$$

and

$$B(x \otimes x, G, f) \overset{(59)}{=} B(x \otimes x, G, f) \text{ for } f = x, gx$$

in particular
(3) $2B(x \otimes x, G, 1_H) = 0$ which is true in view of (74).
(4)$B(x \otimes x, G, gx) = B(x \otimes x, G, gx)$ which is trivial. Now, in view of (93), (59) and (60) we get

$$\gamma B(x \otimes x, G, f) \text{ for } f = 1_H, g$$

and

$$2\beta B(x \otimes x, X, f) = -\gamma B(x \otimes x, G, f) \text{ for } f = x, gx$$

In particular
(5) $\gamma B(x \otimes x, G, 1_H) = 0$. This is true in view of (74).
(6) $2\beta B(x \otimes x, X, gx) = -\gamma B(x \otimes x, G, gx)$ By (74) we get $2\beta[2B(1_H \otimes gx, 1_A, gx) - 2B(1_H \otimes g, 1_A, g)] = -\gamma [-2B(1_H \otimes gx, G, g)]$ i.e.

$$4\beta[B(1_H \otimes gx, 1_A, gx) - B(1_H \otimes g, 1_A, g)] - 2\gamma [B(1_H \otimes gx, G, g)] = 0 \tag{131}$$

Now, by using (94), (59) and (60), we get

$$2\beta B(x \otimes x, GX, f) = 0 \text{ for } f = 1_H, g$$

and

$$\beta B(x \otimes x, GX, f) = \beta B(x \otimes x, GX, f) \text{ for } f = x, gx.$$

In particular
(7) $\beta B(x \otimes x, GX, g) = -\beta B(x \otimes x, GX, g)$which is true by (74) and
(8) $\beta B(x \otimes x, GX, x) = \beta B(x \otimes x, GX, x)$ which is trivial.

5.4.6. (f) $h = x\ h' = gx$. By using (91), (59) and (60) , we get

$$\gamma B(x \otimes gx, GX, f) = 0 \text{ for } f = 1_H, g$$

and

$$2B(x \otimes gx, 1_A, f) = -\gamma B(x \otimes gx, GX, f) \text{ for } f = x, gx$$

In particular
(1) for $f = 1_H$ we get $\gamma B(x \otimes gx, GX, 1_H) = 0$ which is true in view of (71).
(2) for $f = gx$ we get, by using (71), we get

$$-4B(1_H \otimes x, 1_A, g) = 2\gamma B(1_H \otimes x; G, gx) \tag{132}$$

Now, in view of (94) and (60) we get $\beta B(x \otimes gx, GX, f) = \beta B(x \otimes gx, GX, f)$ for $f = x, gx$, which is trivial.

5.5. The List of Equalities

Thus we obtained the following equalities: (80), (81), (82), (83) , (83), (84), (85) , (86), (87), (88) , (89), (101), (103), (68), (104), (105) , (113), (116), (118) , (120), (121), (123) , (125), (126), (127) , (128), (131) and (132) .

5.6. Simplifications for Char(k) $\neq$ 2

The reader can check that, by assuming char$(k) \neq 2$ we obtain the following list which we relabel as follows.

$$B(1_H \otimes g, 1_A, x) = 0 \tag{133}$$
$$B(1_H \otimes g, G, gx) = 0 \tag{134}$$
$$B(1_H \otimes gx; G, g) = 0 \tag{135}$$
$$B(1_H \otimes gx, G, x) = 0 \tag{136}$$
$$B(1_H \otimes g, 1_A, g) = B(1_H \otimes gx; 1_A, gx) \tag{137}$$
$$2B(1_H \otimes x, 1_A, g) + \gamma B(1_H \otimes x, GX, g) = 0 \tag{138}$$
$$2\alpha B(1_H \otimes x, G, 1_H) - \gamma B(1_H \otimes x; 1_A, x) + \gamma = 0 \tag{139}$$
$$2B(1_H \otimes gx, 1_A, 1_H) + \gamma B(1_H \otimes g; G, 1_H)] = 0 \tag{140}$$
$$2\beta[1 - B(1_H \otimes x; 1_A, x)] = -\gamma B(1_H \otimes x, G, 1_H) \tag{141}$$

5.7. The New Form of The Six Elements Using Last List

In the following, we set: $\mathbf{A} = B(1_H \otimes g; 1_A, g)$, $\mathbf{B} = B(1_H \otimes g; G, 1_H)$, $\mathbf{C} = B(1_H \otimes x; G, 1_H)$, $\mathbf{D} = B(1_H \otimes x; 1_A, x)$, $\mathbf{E} = B(1_H \otimes x, GX, g)$. By using the above equalities we get the new form of the six elements.

$$B(1_H \otimes g) = \mathbf{A}\,(1_A \otimes g) + \mathbf{B}\,(G \otimes 1_H) \tag{142}$$
$$\begin{aligned} B(1_H \otimes x) = {} & -\frac{\gamma}{2}\mathbf{E}\,(1_A \otimes g) + \mathbf{D}\,(1_A \otimes x) + \mathbf{C}\,(G \otimes 1_H) + \mathbf{E}\,(G \otimes gx) \\ & [1_A - \mathbf{D}]\,(X \otimes 1_H) + \mathbf{E}\,(GX \otimes g) \end{aligned} \tag{143}$$
$$B(1_H \otimes gx) = -\frac{\gamma}{2}\mathbf{B}\,(1_A \otimes 1_H) + \mathbf{A}\,(1_A \otimes gx) + \mathbf{B}\,(GX \otimes 1_H) \tag{144}$$
$$B(x \otimes 1_H) = -\frac{\gamma}{2}\mathbf{B}\,(1_A \otimes 1_H) - \mathbf{A}\,(1_A \otimes gx) + \mathbf{B}\,(GX \otimes 1_H) \tag{145}$$
$$B(x \otimes x) = 0 \tag{146}$$
$$B(x \otimes gx) = \gamma \mathbf{E}(1_A \otimes gx) - 2\mathbf{E}\,(GX \otimes gx) \tag{147}$$

6. The Separability Result

Theorem 1. *Let* $A = Cl(\alpha,\beta,\gamma)$ *and* $H = H_4$. *Assume that* $\mathrm{char}\,(k) \neq 2$. *Then the cowreath* $(A\otimes H^{op}, H)$ *is separable with respect to any bilinear form satisfying* (142), (143), (144), (145), (146) *and* (147) *and* (139), (141).

Proof. In view of [4, Proposition 7.4], we have to find a bilinear form

$$B : H\otimes H \to A\otimes H^{op}$$

which is a Casimir morphism satisfying also the normalized condition. In view of Proposition (1) and of the morphism condition, since the last form of the six elements (142), (143), (144) , (145), (146) and (147) was obtained by using all the equalities in 5.6 except (139) and (141) we rewrite these two equalities remaining equalities.

$$2\alpha\mathbf{C} - \gamma\mathbf{D} = -\gamma \tag{148}$$

$$\gamma\mathbf{C} - 2\beta\mathbf{D} = -2\beta \tag{149}$$

Note that this system has always a solution, namely $\mathbf{C} = 0$ and $\mathbf{D} = 1$.

Thus by means of any k-linear map $B : H\otimes H \to A\otimes H^{op}$ satisfying all these equalities, the cowreath $(A\otimes H^{op}, H)$ is separable. □

7. h-Separability

Now, we are going to investigate the h-separability as introduced in [12, Theorem 5.1]. We still assume $\mathrm{char}\,(k) \neq 2$. Equation [12, (3)] in our case reduces to

$$B(h\otimes h'_1)B(h'_2\otimes h'') = \epsilon(h')B(h\otimes h'') \tag{150}$$

for all $h, h', h'' \in H_4$.

Thus possible values of h' in the bases we obtain four equations:

$$B(h\otimes 1_H)B(1_H\otimes h'') = B(h\otimes h'') \tag{151}$$

$$B(h\otimes g)B(g\otimes h'') = B(h\otimes h'') \tag{152}$$

$$B(h\otimes x)B(g\otimes h'') + B(h\otimes 1_H)B(x\otimes h'') = 0 \tag{153}$$

$$B(h\otimes gx)B(1_H\otimes h'') + B(h\otimes g)B(gx\otimes h'') = 0 \tag{154}$$

We claim that (152) and (154) follows from (151) and (153). Multiplying equation (151) by $(1_A\otimes g)$ on the left and on the right we get

$$(1_A\otimes g)B(h\otimes 1_H)(1_A\otimes g)(1_A\otimes g)B(1_H\otimes h'')(1_A\otimes g) = (1_A\otimes g)\\ B(h\otimes h'')(1_A\otimes g).$$

By equation (58) we obtain

$$B(gh\otimes g)B(g\otimes gh'') = B(gh\otimes gh'').$$

Since any $h\in H$ can be written as $h = g\,(gh)$, we deduce that (152) can be obtained from (151).

Similarly, multiplying equation (153) by $(1_A\otimes g)$ on the left and on the right we get

$$(1_A\otimes g)B(h\otimes x)(1_A\otimes g)(1_A\otimes g)B(g\otimes h'')(1_A\otimes g) + (1_A\otimes g)B(h\otimes 1_H)$$

$$(1_A \otimes g)(1_A \otimes g)B(x \otimes h'')(1_A \otimes g) = 0$$

so that, by equation (58), we obtain

$$B(gh \otimes gx)B(1_H \otimes gh'') + B(gh \otimes g)B(gx \otimes gh'') = 0$$

and we get (154).

7.1. The First Equation

Here we will analyze the several occurrences of equation (151).

Whenever either $h = 1_H$ or $h'' = 1_H$, in view of (9), the equation is trivially satisfied.

7.1.1. $\boldsymbol{h = g, h'' = g}$**.** By using (58), (142) and (10)

$$\left[\left(\mathbf{A}^2 + \mathbf{B}^2\alpha\right)(1_A \otimes 1_H) + 2\mathbf{AB}\,(G \otimes g)\right] = 1_A \otimes 1_H.$$

Therefore we obtain

$$\mathbf{A}^2 + \mathbf{B}^2\alpha = 1 \tag{155}$$

$$\mathbf{AB} = 0 \tag{156}$$

7.1.2. $\boldsymbol{h = g, h'' = x}$**.** Proceeding as in the previous case we get, by using (142) , (143) we obtain the first side

$$\begin{aligned}
&(1_A \otimes g)\left[\mathbf{A}\,(1_A \otimes g) + \mathbf{B}\,(G \otimes 1_H)\right](1_A \otimes g) \\
&\Big[-\frac{\gamma}{2}\mathbf{E}\,(1_A \otimes g) + \mathbf{D}\,(1_A \otimes x) + \mathbf{C}\,(G \otimes 1_H) + \mathbf{E}\,(G \otimes gx) \\
&\qquad + \left[1_A - \mathbf{D}\right](X \otimes 1_H) + \mathbf{E}\,(GX \otimes g)\Big] \\
&\quad = \left[\mathbf{A}\,(1_A \otimes g) + \mathbf{B}\,(G \otimes 1_H)\right] \\
&\left[\begin{array}{c} -\frac{\gamma}{2}\mathbf{E}\,(1_A \otimes g) + \mathbf{D}\,(1_A \otimes x) + \mathbf{C}\,(G \otimes 1_H) \\ +\mathbf{E}\,(G \otimes gx) + \left[1_A - \mathbf{D}\right](X \otimes 1_H) + \mathbf{E}\,(GX \otimes g) \end{array}\right] \\
&\quad = -\frac{\gamma}{2}\mathbf{AE}\,(1_A \otimes 1_H) - \mathbf{AD}\,(1_A \otimes gx) + \mathbf{AC}\,(G \otimes g) - \mathbf{AE}\,(G \otimes x) \\
&\qquad + \mathbf{A}\,(1_A - \mathbf{D})\,(X \otimes g) + \mathbf{AE}\,(GX \otimes 1_H) + \\
&\qquad - \frac{\gamma}{2}\mathbf{BE}\,(G \otimes g) + \mathbf{BD}\,(G \otimes x) + \alpha\mathbf{BC}\,(1_A \otimes 1_H) + \alpha\mathbf{BE}\,(1_A \otimes gx) \\
&\qquad + \mathbf{B}\,(1_A - \mathbf{D})\,(GX \otimes 1_H) + \alpha\mathbf{BE}\,(X \otimes g) \\
&\quad = \left(-\frac{\gamma}{2}\mathbf{AE} + \alpha\mathbf{BC}\right)(1_A \otimes 1_H) + (-\mathbf{AD} + \alpha\mathbf{BE})\,(1_A \otimes gx) \\
&\qquad + \left(\mathbf{AC} - \frac{\gamma}{2}\mathbf{BE}\right)(G \otimes g) + (\mathbf{BD} - \mathbf{AE})\,(G \otimes x) \\
&\qquad + \left[\alpha\mathbf{BE} + \mathbf{A}\,(1_A - \mathbf{D})\right](X \otimes g) + \left[\mathbf{AE} + \mathbf{B}\,(1_A - \mathbf{D})\right](GX \otimes 1_H)
\end{aligned}$$

and by using (144) we get the second side

$$\begin{aligned}
&(1_A \otimes g)\left[-\frac{\gamma}{2}\mathbf{B}\,(1_A \otimes 1_H) + \mathbf{A}\,(1_A \otimes gx) + \mathbf{B}\,(GX \otimes 1_H)\right](1_A \otimes g) \\
&\quad = -\frac{\gamma}{2}\mathbf{B}\,(1_A \otimes 1_H) - \mathbf{A}\,(1_A \otimes gx) + \mathbf{B}\,(GX \otimes 1_H).
\end{aligned}$$

In conclusion we get the following equality

$$\left(-\frac{\gamma}{2}\mathbf{AE} + \alpha\mathbf{BC}\right)(1_A \otimes 1_H) + (-\mathbf{AD} + \alpha\mathbf{BE})\,(1_A \otimes gx)$$

$$+\left(\mathbf{AC}-\frac{\gamma}{2}\mathbf{BE}\right)(G\otimes g)++(\mathbf{BD}-\mathbf{AE})(G\otimes x)$$
$$+\left[\alpha\mathbf{BE}+\mathbf{A}(1_A-\mathbf{D})\right](X\otimes g)+\left[\mathbf{AE}+\mathbf{B}(1_A-\mathbf{D})\right](GX\otimes 1_H)$$
$$=-\frac{\gamma}{2}\mathbf{B}(1_A\otimes 1_H)-\mathbf{A}(1_A\otimes gx)+\mathbf{B}(GX\otimes 1_H)$$

which gives us

$$-\frac{\gamma}{2}\mathbf{AE}+\alpha\mathbf{BC}=-\frac{\gamma}{2}\mathbf{B} \tag{157}$$
$$-\mathbf{AD}+\alpha\mathbf{BE}=-\mathbf{A} \tag{158}$$
$$\mathbf{AE}=\mathbf{BD} \tag{159}$$
$$\mathbf{AC}=\frac{\gamma}{2}\mathbf{BE} \tag{160}$$

7.1.3. $\boldsymbol{h=g,\ h''=gx}$**.** Proceeding as in the previous case, by using (142) , (144) we obtain the first side of the equation

$$(1_A\otimes g)\left[\mathbf{A}(1_A\otimes g)+\mathbf{B}(G\otimes 1_H)\right](1_A\otimes g)$$
$$\left[-\frac{\gamma}{2}\mathbf{B}(1_A\otimes 1_H)+\mathbf{A}(1_A\otimes gx)+\mathbf{B}(GX\otimes 1_H)\right]$$
$$=\left[\mathbf{A}(1_A\otimes g)+\mathbf{B}(G\otimes 1_H)\right]$$
$$\left[-\frac{\gamma}{2}\mathbf{B}(1_A\otimes 1_H)+\mathbf{A}(1_A\otimes gx)+\mathbf{B}(GX\otimes 1_H)\right]$$
$$=-\frac{\gamma}{2}\mathbf{AB}(1_A\otimes g)-\mathbf{A}^2(1_A\otimes x)$$
$$+\mathbf{AB}(GX\otimes g)-\frac{\gamma}{2}\mathbf{B}^2(G\otimes 1_H)+\mathbf{AB}(G\otimes gx)+\alpha\mathbf{B}^2(X\otimes 1_H)$$

and by using (143) we get the second side

$$-\frac{\gamma}{2}\mathbf{E}(1_A\otimes g)-\mathbf{D}(1_A\otimes x)+\mathbf{C}(G\otimes 1_H)-\mathbf{E}(G\otimes gx)$$
$$+(1_A-\mathbf{D})(X\otimes 1_H)+\mathbf{E}(GX\otimes g).$$

In conclusion we get the following equality

$$-\frac{\gamma}{2}\mathbf{AB}(1_A\otimes g)-\mathbf{A}^2(1_A\otimes x)$$
$$+\mathbf{AB}(GX\otimes g)-\frac{\gamma}{2}\mathbf{B}^2(G\otimes 1_H)+\mathbf{AB}(G\otimes gx)+\alpha\mathbf{B}^2(X\otimes 1_H)$$
$$=-\frac{\gamma}{2}\mathbf{E}(1_A\otimes g)-\mathbf{D}(1_A\otimes x)+\mathbf{C}(G\otimes 1_H)-\mathbf{E}(G\otimes gx)$$
$$+(1_A-\mathbf{D})(X\otimes 1_H)+\mathbf{E}(GX\otimes g)$$

so that, in view of (156) gives us

$$\gamma\mathbf{E}=0 \tag{161}$$
$$\mathbf{A}^2=\mathbf{D} \tag{162}$$
$$-\frac{\gamma}{2}\mathbf{B}^2=\mathbf{C} \tag{163}$$
$$\mathbf{AB}=\mathbf{E} \tag{164}$$

and hence, by (156), we get

$$\mathbf{E}=0 \tag{165}$$

$$\alpha\mathbf{B}^2 = 1_A - \mathbf{D} \tag{166}$$

7.1.4. $h = x, h'' = g$.

$$B(x\otimes 1_H)B(1_H\otimes g) = B(x\otimes g) \overset{(11)}{=} -B(1_H\otimes x)$$

so that we get the first side

$$\begin{aligned}
&\left[-\frac{\gamma}{2}\mathbf{B}\left(1_A\otimes 1_H\right) - \mathbf{A}\left(1_A\otimes gx\right) + \mathbf{B}\left(GX\otimes 1_H\right)\right]\\
&\quad\left[\mathbf{A}\left(1_A\otimes g\right) + \mathbf{B}\left(G\otimes 1_H\right)\right]\\
&= -\frac{\gamma}{2}\mathbf{AB}\left(1_A\otimes g\right) - \mathbf{A}^2\left(1_A\otimes x\right) + \mathbf{AB}\left(GX\otimes g\right) +\\
&\quad -\frac{\gamma}{2}\mathbf{B}^2\left(G\otimes 1_H\right) - \mathbf{AB}\left(G\otimes gx\right) + \gamma\mathbf{B}^2 G\otimes 1_H - \alpha\mathbf{B}^2 X\otimes 1_H\\
&\overset{(156)}{=} -\mathbf{A}^2\left(1_A\otimes x\right) + \frac{\gamma}{2}\mathbf{B}^2\left(G\otimes 1_H\right) - \alpha\mathbf{B}^2 X\otimes 1_H
\end{aligned}$$

and by using (143) we get

$$\begin{aligned}
&-\mathbf{A}^2\left(1_A\otimes x\right) + \frac{\gamma}{2}\mathbf{B}^2\left(G\otimes 1_H\right) - \alpha\mathbf{B}^2 X\otimes 1_H\\
&= \frac{\gamma}{2}\mathbf{E}\left(1_A\otimes g\right) - \mathbf{E}\left(G\otimes gx\right) - \mathbf{E}\left(GX\otimes g\right) - \mathbf{D}\left(1_A\otimes x\right) - \mathbf{C}\left(G\otimes 1_H\right)\\
&\quad + \left(\mathbf{D} - 1_A\right)\left(X\otimes 1_H\right)
\end{aligned}$$

so that in view of (162), (163), (165) and (166) we get no new information.

7.1.5. $h = x, h'' = x$.

$$B(x\otimes 1_H)B(1_H\otimes x) = B(x\otimes x)$$

We get, by using (143), (145) and (165), the first side

$$\begin{aligned}
&\left[-\frac{\gamma}{2}\mathbf{B}\left(1_A\otimes 1_H\right) - \mathbf{A}\left(1_A\otimes gx\right) + \mathbf{B}\left(GX\otimes 1_H\right)\right]\left[\mathbf{D}\left(1_A\otimes x\right)\right]\\
&\quad + \mathbf{C}\left(G\otimes 1_H\right) + \left(1_A - \mathbf{D}\right)\left(X\otimes 1_H\right)\\
&= -\frac{\gamma}{2}\mathbf{BD}\left(1_A\otimes x\right) + -\frac{\gamma}{2}\mathbf{BC}\left(G\otimes 1_H\right) + \frac{\gamma}{2}\mathbf{B}\left(\mathbf{D} - 1_A\right)\left(X\otimes 1_H\right)\\
&\quad -\mathbf{AC}\left(G\otimes gx\right) + \left(\mathbf{AD} - \mathbf{A}\right)\left(X\otimes gx\right)\\
&\quad +\mathbf{BD}\left(GX\otimes x\right) + \mathbf{BC}\left(\left[\gamma G - \alpha X\right]\otimes 1_H\right) + \beta\mathbf{B}\left(1_A - \mathbf{D}\right)\left(G\otimes 1_H\right)\\
&= -\frac{\gamma}{2}\mathbf{BD}\left(1_A\otimes x\right) + \left[\frac{\gamma}{2}\mathbf{BC} + \beta\mathbf{B}\left(1_A - \mathbf{D}\right)\right]\left(G\otimes 1_H\right) +\\
&\quad + \left[\frac{\gamma}{2}\mathbf{B}\left(\mathbf{D} - 1_A\right) - \alpha\mathbf{BC}\right]\left(X\otimes 1_H\right) - \mathbf{AC}\left(G\otimes gx\right)\\
&\quad + \left(\mathbf{AD} - \mathbf{A}\right)\left(X\otimes gx\right) + \mathbf{BD}\left(GX\otimes x\right)
\end{aligned}$$

so that, in view of (146), (159), (160) , (158), (157) (165) we get

$$\frac{\gamma}{2}\mathbf{BC}+\beta\mathbf{B} = 0(167). \tag{167}$$

7.1.6. $h = x, h'' = gx$.

$$B(x \otimes 1_H)B(1_H \otimes gx) = B(x \otimes gx)$$

In view of (145) and (144) the first side is

$$\begin{aligned}
&\left[-\frac{\gamma}{2}\mathbf{B}\left(1_A \otimes 1_H\right) - \mathbf{A}\left(1_A \otimes gx\right) + \mathbf{B}\left(GX \otimes 1_H\right)\right] \\
&\left[-\frac{\gamma}{2}\mathbf{B}\left(1_A \otimes 1_H\right) + \mathbf{A}\left(1_A \otimes gx\right) + \mathbf{B}\left(GX \otimes 1_H\right)\right] \\
= &\frac{\gamma^2}{4}\mathbf{B}^2\left(1_A \otimes 1_H\right) - \frac{\gamma}{2}\mathbf{AB}\left(1_A \otimes gx\right) - \frac{\gamma}{2}\mathbf{B}^2\left(GX \otimes 1_H\right) \\
&+\frac{\gamma}{2}\mathbf{AB}\left(1_A \otimes gx\right) - \mathbf{AB}\left(GX \otimes gx\right) + \\
&-\frac{\gamma}{2}\mathbf{B}^2\left(GX \otimes 1_H\right) + \mathbf{AB}\left(GX \otimes gx\right) + \mathbf{B}^2\gamma GX \otimes 1_H + -\alpha\beta\mathbf{B}^2\left(1_A \otimes 1_H\right) \\
&\overset{(156)}{=} \left(\frac{\gamma^2}{4} - \alpha\beta\right)\mathbf{B}^2\left(1_A \otimes 1_H\right)
\end{aligned}$$

so that, by using (147) and (165) we get

$$\left(\frac{\gamma^2}{4} - \alpha\beta\right)\mathbf{B}^2 = 0. \tag{168}$$

7.1.7. $h = gx, h'' = g$.

$$B(gx \otimes 1_H)B(1_H \otimes g) = B(gx \otimes g)$$

By (58) we get

$$\left(1_A \otimes g\right) B(x \otimes g)\left(1_A \otimes g\right) B(1_H \otimes g) = \left(1_A \otimes g\right) B(x \otimes 1_H)\left(1_A \otimes g\right)$$

so that by (11) we get

$$-\left(1_A \otimes g\right) B(1_H \otimes x)\left(1_A \otimes g\right) B(1_H \otimes g) = \left(1_A \otimes g\right) B(x \otimes 1_H)\left(1_A \otimes g\right)$$

and hence the first side is, by using (165) and (159),

$$\begin{aligned}
&\left[\mathbf{D}\left(1_A \otimes x\right) - \mathbf{C}\left(G \otimes 1_H\right) + \left(\mathbf{D} - 1_A\right)\left(X \otimes 1_H\right)\right] \\
&\left[\mathbf{A}\left(1_A \otimes g\right) + \mathbf{B}\left(G \otimes 1_H\right)\right] \\
= &\mathbf{AD}\left(1_A \otimes gx\right) - \mathbf{AC}\left(G \otimes g\right) + \mathbf{A}\left(\mathbf{D} - 1_A\right)\left(X \otimes g\right) + \\
&+\mathbf{BD}\left(G \otimes x\right) - \alpha\mathbf{BC}\left(1_A \otimes 1_H\right) + \gamma\left(\mathbf{BD} - \mathbf{B}\right)\left(1_A \otimes 1_H\right) \\
&\quad + \left(\mathbf{B} - \mathbf{BD}\right)\left(GX \otimes 1_H\right) \\
= &\mathbf{AD}\left(1_A \otimes gx\right) - \mathbf{AC}\left(G \otimes g\right) + \mathbf{A}\left(\mathbf{D} - 1_A\right)\left(X \otimes g\right) \\
&\left[-\alpha\mathbf{BC} - \gamma\mathbf{B}\right]\left(1_A \otimes 1_H\right) + \mathbf{B}\left(GX \otimes 1_H\right)
\end{aligned}$$

and the second side is

$$-\frac{\gamma}{2}\mathbf{B}\left(1_A \otimes 1_H\right) + \mathbf{A}\left(1_A \otimes gx\right) + \mathbf{B}\left(GX \otimes 1_H\right)$$

so that we get in view of (158), (160), (157) and (165) we do not get anything new.

7.1.8. $h = gx, h'' = x$.

$$B(gx \otimes 1_H)B(1_H \otimes x) = B(gx \otimes x)$$

By (58) we get

$$-\left(1_A \otimes g\right) B(1_H \otimes x) \left(1_A \otimes g\right) B(1_H \otimes x) = \left(1_A \otimes g\right) B(x \otimes gx) \left(1_A \otimes g\right)$$

and hence, by using (165) and (159), the first side is

$$\begin{aligned}
&\left[-\mathbf{D}\left(1_A \otimes xg\right) - \mathbf{C}\left(G \otimes g\right) + \left(\mathbf{D} - 1_A\right)\left(X \otimes g\right)\right] \\
&\left[\mathbf{D}\left(1_A \otimes xg\right) + \mathbf{C}\left(G \otimes g\right) + \left(1_A - \mathbf{D}\right)\left(X \otimes g\right)\right] \\
&= \mathbf{DC}\left(G \otimes x\right) + \left(\mathbf{D} - \mathbf{D}^2\right)\left(X \otimes x\right) - \mathbf{CD}\left(G \otimes x\right) - \alpha \mathbf{C}^2\left(1_A \otimes 1_H\right) \\
&\quad + \left(-\mathbf{C} + \mathbf{CD}\right)\left(GX \otimes 1_H\right) + \left(\mathbf{D} - 1_A\right)\mathbf{D}\left(X \otimes x\right) + \mathbf{C}\left(\mathbf{D} - 1_A\right) \\
&\quad \left(\left(\gamma - GX\right) \otimes 1_H\right) - \beta\left(1_A - \mathbf{D}\right)^2\left(1_A \otimes 1_H\right) \\
&= \left[-\alpha \mathbf{C}^2 + \gamma \mathbf{C}\left(\mathbf{D} - 1_A\right) - \beta\left(1_A - \mathbf{D}\right)^2\right]\left(1_A \otimes 1_H\right).
\end{aligned}$$

In view of (147) and (165) second side is 0 so that we obtain

$$-\alpha \mathbf{C}^2 + \gamma \mathbf{C}\left(\mathbf{D} - 1\right) - \beta\left(1 - \mathbf{D}\right)^2 = 0 \tag{169}$$

7.1.9. $h = gx, h'' = gx$.

$$B(gx \otimes 1_H)B(1_H \otimes gx) = B(gx \otimes gx)$$

By using (58), (11) and (146) we obtain

$$\left(1_A \otimes g\right) B(1_H \otimes x) \left(1_A \otimes g\right) B(1_H \otimes gx) = 0$$

In view of (143), (144) and (165) we obtain

$$\begin{aligned}
&\left(1_A \otimes g\right)\left[\mathbf{D}\left(1_A \otimes x\right) + \mathbf{C}\left(G \otimes 1_H\right) + \left(1_A - \mathbf{D}\right)\left(X \otimes 1_H\right)\right]\left(1_A \otimes g\right) \\
&\left[-\frac{\gamma}{2}\mathbf{B}\left(1_A \otimes 1_H\right) + \mathbf{A}\left(1_A \otimes gx\right) + \mathbf{B}\left(GX \otimes 1_H\right)\right] \\
&= \left[-\mathbf{D}\left(1_A \otimes x\right) + \mathbf{C}\left(G \otimes 1_H\right) + \left(1_A - \mathbf{D}\right)\left(X \otimes 1_H\right)\right] \\
&\quad \left[-\frac{\gamma}{2}\mathbf{B}\left(1_A \otimes 1_H\right) + \mathbf{A}\left(1_A \otimes gx\right) + \mathbf{B}\left(GX \otimes 1_H\right)\right] \\
&= \frac{\gamma}{2}\mathbf{BD}\left(1_A \otimes x\right) - \mathbf{BD}\left(GX \otimes x\right) + \\
&\quad -\frac{\gamma}{2}\mathbf{BC}\left(G \otimes 1_H\right) + \mathbf{AC}\left(G \otimes gx\right) + \alpha \mathbf{BC}\left(X \otimes 1_H\right) + \\
&\quad -\frac{\gamma}{2}\left(1_A - \mathbf{D}\right)\mathbf{B}\left(X \otimes 1_H\right) + \mathbf{A}\left(1_A - \mathbf{D}\right)\left(X \otimes gx\right) + \mathbf{B}\left(1_A - \mathbf{D}\right) \\
&\quad \left(\left(\gamma X - \beta G\right) \otimes 1_H\right) \\
&\overset{(159)(165)}{=} -\frac{\gamma}{2}\mathbf{BC}\left(G \otimes 1_H\right) + \mathbf{AC}\left(G \otimes gx\right) + \alpha \mathbf{BC}\left(X \otimes 1_H\right) + \\
&\quad -\frac{\gamma}{2}\mathbf{B}\left(X \otimes 1_H\right) + \mathbf{A}\left(1_A - \mathbf{D}\right)\left(X \otimes gx\right) + \mathbf{B}\left(\left(\gamma X - \beta G\right) \otimes 1_H\right) \\
&= \left[-\frac{\gamma}{2}\mathbf{BC} - \beta \mathbf{B}\right]\left(G \otimes 1_H\right) + \mathbf{AC}\left(G \otimes gx\right) + \\
&\quad + \left[\alpha \mathbf{BC} - \frac{\gamma}{2}\mathbf{B} + \gamma \mathbf{B}\right]\left(X \otimes 1_H\right) + +\mathbf{A}\left(1_A - \mathbf{D}\right)\left(X \otimes gx\right).
\end{aligned}$$

Hence, by using (160), (158), (167) and (165), we obtain $\alpha \mathbf{BC} + \frac{\gamma}{2}\mathbf{B} = 0$ which follows from (157) and (165).

Remark 1. By using (165) we get the new form of the following elements.

$$B(1_H \otimes x) = \mathbf{D}\,(1_A \otimes x) + \mathbf{C}\,(G \otimes 1_H) + (1_A - \mathbf{D})\,(X \otimes 1_H) \quad (170)$$

$$B(x \otimes gx) = 0(147) \quad (171)$$

By using (159), (160), (158) and (165) we obtain

$$\mathbf{BD} = 0 \quad (172)$$

$$\mathbf{AC} = 0 \quad (173)$$

$$\mathbf{AD} = \mathbf{A}. \quad (174)$$

7.2. The Third Equation

Here, by using the results above, we show how (153) can be deduced from (151) . By using (151) , we get

$$\begin{aligned} &B(h \otimes x)B(g \otimes h'') + B(h \otimes 1_H)B(x \otimes h'') \\ &= B(h \otimes 1_H)B(1_H \otimes x)B(g \otimes 1_H)B(1_H \otimes h'') \\ &\quad + B(h \otimes 1_H)B(x \otimes 1_H)B(1_H \otimes h'') \end{aligned}$$

now we compute

$$\begin{aligned} &B(1_H \otimes x)B(g \otimes 1_H) \overset{(58)}{=} B(1_H \otimes x)(1_H \otimes g)B(1_H \otimes g)(1_H \otimes g) \\ &\overset{(170)(142)}{=} [\mathbf{D}\,(1_A \otimes x) + \mathbf{C}\,(G \otimes 1_H) + (1_A - \mathbf{D})\,(X \otimes 1_H)] \\ &(1_A \otimes g)\,[\mathbf{A}\,(1_A \otimes g) + \mathbf{B}\,(G \otimes 1_H)]\,(1_A \otimes g) \\ &= [\mathbf{D}\,(1_A \otimes x) + \mathbf{C}\,(G \otimes 1_H) + (1_A - \mathbf{D})\,(X \otimes 1_H)] \\ &[\mathbf{A}\,(1_A \otimes g) + \mathbf{B}\,(G \otimes 1_H)] \\ &\overset{(172)(173)(174)}{=} \mathbf{A}\,(1_A \otimes gx) + \alpha\mathbf{BC}\,(1_A \otimes 1_H) + \mathbf{B}\,((\gamma - GX) \otimes 1_H) \\ &= \mathbf{A}\,(1_A \otimes gx) + [\alpha\mathbf{BC} + \mathbf{B}\gamma]\,(1_A \otimes 1_H) - \mathbf{B}\,(GX \otimes 1_H) \\ &\overset{(157),(165))}{=} \mathbf{A}\,(1_A \otimes gx) + \frac{\gamma}{2}\mathbf{B}\,(1_A \otimes 1_H) - \mathbf{B}\,(GX \otimes 1_H) \\ &\overset{(145)}{=} -B(x \otimes 1_H) \end{aligned}$$

From this we deduce (153).

7.3. The Main Result

Theorem 2. *Let $A = Cl(\alpha, \beta, \gamma)$ and $H = H_4$. Assume that* $\operatorname{char}(k) \neq 2$. *Then the cowreath $(A \otimes H^{op}, H)$ is always h-separable. This happens whenever the bilinear form $B : H \otimes H \to A \otimes H^{op}$ satisfies (142), (143), (144), (145), (146) and (147) and (139), (141).*

Proof. We have already seen that (150) is equivalent to (151) . In view of (165), we know that $\mathbf{E} = 0$, by taking in account of this we collect here all the equalities we have obtained in (7.1) .

$$\begin{aligned} \mathbf{A}^2 + \mathbf{B}^2\alpha &= 1(155) \\ \mathbf{AB} &= 0(156) \\ \alpha\mathbf{BC} &= -\frac{\gamma}{2}\mathbf{B}(157) \end{aligned}$$

$$\begin{aligned}
\mathbf{AD} &= \mathbf{A}(158)\\
\mathbf{BD} &=\mathbf{0}(159)\\
\mathbf{AC} &= 0(160)\\
\mathbf{A}^2 &= \mathbf{D}(162)\\
-\frac{\gamma}{2}\mathbf{B}^2 &= \mathbf{C}(163)\\
\mathbf{E} &= 0(165)\\
\alpha\mathbf{B}^2 &= 1_A-\mathbf{D}(166)\\
-\alpha\mathbf{C}^2+\gamma\mathbf{C}\left(\mathbf{D}-1\right)-\beta\left(1-\mathbf{D}\right)^2 &= 0(169).
\end{aligned}$$

Moreover, we have to add the two separability equalities

$$\gamma\left(1-\mathbf{D}\right)=-2\alpha\mathbf{C}(148)$$

and

$$2\beta[1-\mathbf{D}]=-\gamma\mathbf{C}(149).$$

Assume $\mathbf{B}=0$. In this case we get that all the equalities above reduce to the following.

$$\mathbf{C}=0(163),\mathbf{E}=0165,\mathbf{A}^2=1(155),\mathbf{D}=1(166)$$

and we have a solution. □

Remark 2. In contrast note that if $\mathbf{A}=0$, we get

$$\mathbf{E}=0(165),0=\mathbf{D}(162),\gamma=-2\alpha\mathbf{C}(139),2\beta=-\gamma\mathbf{C}(141),\mathbf{B}^2\alpha=1(155).$$

Hence $\alpha\in\left(k^{\times}\right)^2$ and $\mathbf{B}\neq 0$ so that from

$$\alpha\mathbf{BC}=-\frac{\gamma}{2}\mathbf{B}(157)$$

we get

$$\mathbf{C}=-\frac{\gamma}{2\alpha}$$

so that (163) can be deduced from (155) . By substituting $\mathbf{C}$ inside (169), we obtain $4\alpha\beta-\gamma^2=0$. Therefore we are exactly in the situation of [12, Theorem 6.1].

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