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Integral transforms suitable for solving fractional differential equations

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Abstract The purpose of this article is to obtain appropriate tools for solving fractional differential equations that are flexible enough to adapt to different purposes. We thus look for a very general fractional Fourier transform with a phase function which can be appropriately chosen according to the problem you want to face.

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1 Introduction and preliminaries

In the current literature, there are many different definitions of fractional Fourier transforms, employed for instance in the fields of optics, sound, or signal processing (see [6]). Nevertheless, these definitions are not always suitable for solving fractional differential equations, similarly as for partial differential equations by means of the classical Fourier transform. Hence, the idea of looking for an integral operator of the form

$$\mathscr{F}_f u(\xi) = \int_{-\infty}^{+\infty} u(x) K_f(x, \xi) \mathrm{d}x, \tag{1.1}$$

with a kernel $K_f(x, \xi)$ depending on a function $f : \mathbb{R} \to \mathbb{C}$ appropriately chosen, so that $\mathscr{F}_f$ satisfies properties similar to those of the Fourier or Laplace transforms usually needed for solving partial differential equations. In particular, we require that for all u, v sufficiently regular and $\xi \in \mathbb{R}$

(i) $\mathscr{F}_f(u * v)(\xi) = \mathscr{F}_f u(\xi) \cdot \mathscr{F}_f v(\xi)$,
(ii) $\mathscr{F}_f(\mathscr{D}^n_+ u)(\xi) = f^n(\xi) \cdot \mathscr{F}_f u(\xi), \quad \forall n \in \mathbb{R}^+$,
(iii) $\mathscr{F}_f$ is left-invertible, i.e., there exists $\mathcal{G}_f$, such that $\mathcal{G}_f \circ \mathscr{F}_f u(x) = u(x)$ for all $x \in \mathbb{R}$

where $\mathscr{D}^n_+ u$ is the Riemann–Liouville derivative of fractional order n on the half-line, defined by

Definition 1.1 For all real positive n, the *Riemann–Liouville fractional integral on the half-line* is defined by

$$\mathscr{I}^n_+ u(x) = \frac{1}{\Gamma(n)} \int_{-\infty}^{x} (x - t)^{n-1} u(t) \mathrm{d}t,$$

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where $\Gamma(n) = \int_0^{+\infty} x^{n-1} e^{-x} \mathrm{d}x$ is the Gamma function. Set $\mathscr{I}_+^0 u := u$.

The *Riemann–Liouville fractional derivative on the half-line* is then defined by

$$\mathscr{D}_+^n u = \mathscr{D}^{\lceil n \rceil}(\mathscr{I}_+^{\lceil n \rceil - n} u),$$

where $\mathscr{D}^{\lceil n \rceil}$ is the classical derivative of integer order $\lceil n \rceil$ (here $\lceil n \rceil$ is the smallest integer greater than or equal to n).

We refer for instance to [7] for more details about fractional integrals and derivatives.

To understand how to define the kernel function $K_f(x, \xi)$ and which assumptions to require for the function f or for the space of functions where the operator $\mathscr{F}_f$ acts (assumptions on u in (1.1)), let us first formally impose condition (ii) for $n = 1$, that is

$$(\mathscr{F}_f u')(\xi) = f(\xi) \cdot (\mathscr{F}_f u)(\xi). \tag{1.2}$$

Assuming that u and $K_f(\cdot, \xi)$ are sufficiently regular, so that we can integrate by parts, and assuming, moreover, that $x \mapsto u(x) K_f(x, \xi)$ vanishes for $x \to \pm\infty$, we have that

$$\begin{aligned}
(\mathscr{F}_f u')(\xi) &= \int_{-\infty}^{+\infty} u'(x) K_f(x, \xi) \mathrm{d}x \\
&= [u(x) K_f(x, \xi)]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u(x) \frac{\partial K_f(x, \xi)}{\partial x} \mathrm{d}x \\
&= - \int_{-\infty}^{+\infty} u(x) \frac{\partial K_f(x, \xi)}{\partial x} \mathrm{d}x,
\end{aligned}$$

which is equal to

$$f(\xi)(\mathscr{F}_f u)(\xi) = f(\xi) \int_{-\infty}^{+\infty} u(x) K_f(x, \xi) \mathrm{d}x$$

if the kernel K_f satisfies the differential equation

$$\frac{\partial K_f(x, \xi)}{\partial x} = -f(\xi) K_f(x, \xi),$$

whose general solution is of the form

$$K_f(x, \xi) = g(\xi) \mathrm{e}^{-x f(\xi)},$$

for some function $g \colon \mathbb{R} \to \mathbb{C}$.

Remark 1.2 If $g(\xi) \equiv 1$ and $f(\xi) = i\xi$, we get the kernel of the classical Fourier transform

$$(\mathscr{F} u)(\xi) = \hat{u}(\xi) = \int_{-\infty}^{+\infty} u(x) \mathrm{e}^{-ix\xi} \mathrm{d}x,$$

which is defined for u in $L^1(\mathbb{R})$ and, in particular, in the Schwartz class of rapidly decreasing functions

$$\mathscr{S}(\mathbb{R}) := \left\{ f \in C^\infty \colon \sup_{\mathbb{R}} |x^\alpha \mathscr{D}^\beta u(x)| < +\infty, \ \forall \alpha, \beta \in \mathbb{N} \right\},$$

with standard extensions to other classes of functions such as $L^2(\mathbb{R})$ or the space $\mathscr{S}'(\mathbb{R})$ of tempered distributions.

We shall then assume $u \in \mathscr{S}(\mathbb{R})$ in the following.

Remark 1.3 Condition $g(\xi) \equiv 1$ is necessary to obtain property (i) for $u, v \in \mathscr{S}(\mathbb{R})$ and a phase function of the form $if(\xi)$ for a real-valued function f. Indeed, applying Fubini's Theorem, we have that

$$\begin{aligned}
\mathscr{F}_f(u * v)(\xi) &= \int_{-\infty}^{+\infty} (u * v)(x) g(\xi) \mathrm{e}^{-ixf(\xi)} \mathrm{d}x \\
&= g(\xi) \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} u(t) v(x-t) \mathrm{d}t \right) \mathrm{e}^{-ixf(\xi)} \mathrm{d}x \\
&= g(\xi) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(t) v(x-t) \mathrm{e}^{-i(x-t)f(\xi)} \mathrm{e}^{-itf(\xi)} \mathrm{d}t \mathrm{d}x \\
&= g(\xi) \int_{-\infty}^{+\infty} u(t) \mathrm{e}^{-itf(\xi)} \left(\int_{-\infty}^{+\infty} v(x-t) \mathrm{e}^{-i(x-t)f(\xi)} \mathrm{d}x \right) \mathrm{d}t \\
&= g(\xi) \int_{-\infty}^{+\infty} u(t) \mathrm{e}^{-itf(\xi)} \left(\int_{-\infty}^{+\infty} v(y) \mathrm{e}^{-iyf(\xi)} \mathrm{d}y \right) \mathrm{d}t.
\end{aligned} \tag{1.3}$$

This is equal to

$$(\mathscr{F}_f u)(\xi) \cdot (\mathscr{F}_f v)(\xi) = \int_{-\infty}^{+\infty} u(t) g(\xi) \mathrm{e}^{-itf(\xi)} \mathrm{d}t \cdot \int_{-\infty}^{+\infty} v(y) g(\xi) \mathrm{e}^{-iyf(\xi)} \mathrm{d}y$$

only for $g(\xi) \equiv 1$ (we clearly exclude $g(\xi) \equiv 0$).

Remark 1.4 Iterating (1.2) recursively, we get (it is trivial for $n = 0$)

$$\mathscr{F}_f(\mathscr{D}^n u)(\xi) = f^n(\xi) \cdot (\mathscr{F}_f u)(\xi), \qquad \forall n \in \mathbb{N}. \tag{1.4}$$

2 General fractional Fourier transforms

By all the considerations of Sect. 1, we now define the following:

Definition 2.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with polynomial growth, i.e., such that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ and $c_n > 0$ which satisfies

$$|\mathscr{D}^n f(\xi)| \le c_n (1 + |\xi|)^k \qquad \forall x \in \mathbb{R} \tag{2.1}$$

and

$$\frac{f(\xi)}{\xi |\xi|^{\alpha - 1}} \to +\infty, \qquad \text{as } |\xi| \to +\infty, \tag{2.2}$$

for some $\alpha > 0$.

Then, we consider on $\mathscr{S}(\mathbb{R})$ the integral operator $\mathscr{F}_f$ defined by

$$(\mathscr{F}_f u)(\xi) := \int_{-\infty}^{+\infty} u(x) \mathrm{e}^{-ixf(\xi)} \mathrm{d}x. \tag{2.3}$$

Proposition 2.2 *Let f and $\mathscr{F}_f$ be as in Definition 2.1. Then, $\mathscr{F}_f$ is a well-defined operator*

$$\mathscr{F}_f u : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}).$$

Proof Clearly, $\mathscr{F}_f$ is well defined on $\mathscr{S}(\mathbb{R})$, since $f(\xi)$ is real valued, and hence, the integral (2.3) is convergent for $u \in \mathscr{S}(\mathbb{R})$.

Moreover, the operator $\mathscr{F}_f$ can be related to the classical Fourier transform by

$$(\mathscr{F}_f u)(\xi) = (\mathscr{F}u)(f(\xi)) = (\hat{u} \circ f)(\xi). \tag{2.4}$$

It is well known that $\mathscr{F}u \in \mathscr{S}(\mathbb{R})$ for $u \in \mathscr{S}(\mathbb{R})$ and therefore $(\mathscr{F}u) \circ f \in \mathscr{S}(\mathbb{R})$ by the assumptions (2.1)–(2.2) on f. □

By (1.3) with $g(\xi) \equiv 1$, we already proved that $\mathscr{F}_f$ satisfies property (i):

Theorem 2.3 *Let $\mathscr{F}_f$ be the integral transform defined in* (2.3). *Then*

$$\mathscr{F}_f(u * v) = (\mathscr{F}_f u) \cdot (\mathscr{F}_f v), \qquad \forall u, v \in \mathscr{S}(\mathbb{R}).$$

Moreover, $\mathscr{F}_f$ satisfies property (ii) for $n \in \mathbb{N}$ by (1.4). To prove (ii) for real positive orders of fractional derivatives, we first need some remarks and lemmas.

Remark 2.4 The Gamma function $\Gamma : \mathbb{C} \to \mathbb{R}$ is defined in general on the half complex plane $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$ by

$$\Gamma(z) := \int_0^{+\infty} x^{z-1} \mathrm{e}^{-x} \mathrm{d}x, \qquad \mathrm{Re} z > 0.$$

The integral is indeed convergent at infinity for all $z \in \mathbb{C}$ because of the factor e^{-x}, and it is convergent for $0 < x < 1$ if $\mathrm{Re} z > 0$, since $|\mathrm{e}^{-x} x^{z-1}| \le x^{\mathrm{Re} z - 1}$.

We shall also need the following (see [7, (7.5)] or [3] for more details):

Lemma 2.5 *Let* $0 < \mathrm{Re} z < 1$ *and* $a \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re} a \ge 0$. *Then*

$$\int_0^{+\infty} t^{z-1} e^{-at} dt = \Gamma(z) a^{-z}. \tag{2.5}$$

Let us now prove:

Lemma 2.6 *If* $0 < n < 1$ *and* $u \in \mathscr{S}(\mathbb{R})$, *then for all* $\xi \in \mathbb{R}$ *with* $f(\xi) \ne 0$, *we have that*

$$\mathscr{F}_f(\mathscr{I}_+^n u)(\xi) = (i f(\xi))^{-n} \cdot (\mathscr{F}_f u)(\xi).$$

Proof Let us first express the Riemann–Liouville integral via a convolution product

$$\begin{aligned} \mathscr{I}_+^n u(x) &= \frac{1}{\Gamma(n)} \int_{-\infty}^{x} (x-t)^{n-1} u(t) \mathrm{d}t \\ &= \frac{1}{\Gamma(n)} \int_{-\infty}^{+\infty} (x-t)^{n-1} \chi_{(0,+\infty)}(x-t) u(t) \mathrm{d}t \\ &= \frac{1}{\Gamma(n)} \left(t^{n-1} \chi_{(0,+\infty)}(t) * u(t) \right)(x), \end{aligned} \tag{2.6}$$

where $\chi_{(0,+\infty)}(t)$ is the characteristic function of the half-line $(0, +\infty)$, that is

$$\chi_{(0,+\infty)}(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0. \end{cases}$$

Let us now remark that to prove (1.3), we did not really need $u, v \in \mathscr{S}(\mathbb{R})$. Property (i) holds also for $u \in \mathscr{S}(\mathbb{R})$ and $v(t) = t^{n-1}\chi_{(0,+\infty)}(t) \in \mathscr{S}'(\mathbb{R})$. We thus obtain

$$\begin{aligned}\mathscr{F}_f(\mathscr{I}_+^n u)(\xi) &= \frac{1}{\Gamma(n)}\mathscr{F}_f(t^{n-1}\chi_{(0,+\infty)}(t) * u(t))(\xi)\\ &= \frac{1}{\Gamma(n)}\mathscr{F}_f(t^{n-1}\chi_{(0,+\infty)}(t))(\xi)\cdot(\mathscr{F}_f u)(\xi)\\ &= \left(\frac{1}{\Gamma(n)}\int_0^{+\infty} t^{n-1}\mathrm{e}^{-itf(\xi)}\mathrm{d}t\right)\cdot(\mathscr{F}_f u)(\xi).\end{aligned}$$

We can now apply Lemma 2.5 with $z = n \in (0, 1)$ and $a = if(\xi)$ (hence, $\mathrm{Re}\,a = 0$) and obtain that

$$\mathscr{F}_f(\mathscr{I}_+^n u)(\xi) = (if(\xi))^{-n}(\mathscr{F}_f u)(\xi).$$

□

Theorem 2.7 *Let $n \in \mathbb{R}^+ \setminus \mathbb{N}$ and $u \in \mathscr{S}(\mathbb{R})$. Then, for all $\xi \in \mathbb{R}$ with $f(\xi) \neq 0$, we have that*

$$\mathscr{F}_f(\mathscr{D}_+^n u)(\xi) = (if(\xi))^n(\mathscr{F}_f u)(\xi).$$

Proof By Definition 1.1, Remark 1.4, and Lemma 2.6, for $0 < \lceil n\rceil - n < 1$, we have that

$$\begin{aligned}\mathscr{F}_f(\mathscr{D}_+^n u)(\xi) &= \mathscr{F}_f(\mathscr{D}^{\lceil n\rceil}(\mathscr{I}_+^{\lceil n\rceil-n}u))(\xi) = (if(\xi))^{\lceil n\rceil}\cdot\mathscr{F}_f(\mathscr{I}_+^{\lceil n\rceil-n}u)(\xi)\\ &= (if(\xi))^{\lceil n\rceil}(if(\xi))^{-\lceil n\rceil+n}(\mathscr{F}_f u)(\xi) = (if(\xi))^n(\mathscr{F}_f u)(\xi).\end{aligned}$$

□

From Remark 1.4 and Theorem 2.7, we finally have property (ii):

Corollary 2.8 *Let $n \in \mathbb{R}^+$ and $u \in \mathscr{S}(\mathbb{R})$. Then, for all $\xi \in \mathbb{R}$ with $f(\xi) \neq 0$, we have that*

$$\mathscr{F}_f(\mathscr{D}_+^n u)(\xi) = (if(\xi))^n(\mathscr{F}_f u)(\xi).$$

Let us now prove property (iii):

Theorem 2.9 *Let f be as in Definition 2.1. Then, $\mathscr{F}_f$ admits a left inverse operator given by*

$$(\mathcal{G}_f v)(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} v(\xi)e^{ixf(\xi)}f'(\xi)d\xi, \qquad v \in \mathscr{S}(\mathbb{R}),$$

i.e., $\mathcal{G}_f(\mathscr{F}_f\, u) = u$ for all $u \in \mathscr{S}(\mathbb{R})$.

Proof Let us first remark that by the assumptions on f in Definition 2.1, the derivative $f'(\xi)$ has polynomial growth, and hence, $v(\xi)f'(\xi) \in L^1(\mathbb{R})$ and $\mathcal{G}_f$ is well defined, being $f(\xi)$ real valued.

Moreover, $\mathscr{F}_f u \in \mathscr{S}(\mathbb{R})$ for $u \in \mathscr{S}(\mathbb{R})$ by Proposition 2.2 and, therefore, $\mathcal{G}_f(\mathscr{F}_f u)$ is also well defined

$$\mathcal{G}_f(\mathscr{F}_f u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} u(x)\mathrm{e}^{-ixf(\xi)}\mathrm{d}x\right)\mathrm{e}^{ixf(\xi)}f'(\xi)\mathrm{d}\xi.$$

By the assumption (2.2) on f, we have that $f(\xi) \to +\infty$ for $\xi \to +\infty$ and $f(\xi) \to -\infty$ for $\xi \to -\infty$. It follows that we can consider the change of variables $y = f(\xi)$ and obtain:

$$\begin{aligned}\mathcal{G}_f(\mathscr{F}_f u)(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} u(x)\mathrm{e}^{-ixy}\mathrm{d}x\right)\mathrm{e}^{ixy}\mathrm{d}y\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{u}(y)\mathrm{e}^{ixy}\mathrm{d}y = u(x),\end{aligned}$$

by the inversion formula for the classical Fourier transform. □

Let us now prove other useful properties of $\mathscr{F}_f$.

Proposition 2.10 *Let $u \in \mathscr{S}(\mathbb{R})$ and $h \in \mathbb{R}$. Then, for all $\xi \in \mathbb{R}$, we have that*

$$(\mathscr{F}_f u(x-h))(\xi) = e^{-ihf(\xi)} \cdot (\mathscr{F}_f u(x))(\xi).$$

Proof By the change of variables $s = x - h$, we have

$$(\mathscr{F}_f u(x-h))(\xi) = \int_{-\infty}^{+\infty} u(x-h)\mathrm{e}^{-ixf(\xi)}\mathrm{d}x = \int_{-\infty}^{+\infty} u(s)\mathrm{e}^{-i(s+h)f(\xi)}\mathrm{d}s = \mathrm{e}^{-ihf(\xi)}(\mathscr{F}_f u)(\xi).$$

□

Proposition 2.11 *Let $u \in \mathscr{S}(\mathbb{R})$ and $\lambda > 0$. Then, for all $\xi \in \mathbb{R}$, we have that*

$$(\mathscr{F}_f u(\lambda x))(\xi) = \frac{1}{\lambda}(\mathscr{F}_{f/\lambda} u(x))(\xi).$$

Proof By the change of variables $s = \lambda x$, we have

$$(\mathscr{F}_f u(\lambda x))(\xi) = \int_{-\infty}^{+\infty} u(\lambda x)\mathrm{e}^{-ixf(\xi)}\mathrm{d}x = \frac{1}{\lambda}\int_{-\infty}^{+\infty} u(s)\mathrm{e}^{-i\frac{s}{\lambda}f(\xi)}\mathrm{d}s = \frac{1}{\lambda}(\mathscr{F}_{f/\lambda} u)(\xi).$$

□

Proposition 2.12 *Let $u \in \mathscr{S}(\mathbb{R})$ and $n \in \mathbb{N} \setminus \{0\}$. Then, for all $\xi \in \mathbb{R}$, we have that*

$$\mathscr{D}^n(\mathscr{F}_f u(x))(\xi) = n!\sum_{k=1}^{n} h(\xi)(\mathscr{F}_f(-ix)^k u(x))(\xi),$$

where

$$h(\xi) = \frac{1}{k!}\sum_{\substack{\sigma_1+\ldots\sigma_k=n\\ \sigma_j\geq 1}} \frac{\mathscr{D}^{\sigma_1}f(\xi)}{\sigma_1!}\cdots\frac{\mathscr{D}^{\sigma_k}f(\xi)}{\sigma_k!}. \tag{2.7}$$

Proof Since $(\mathscr{F}_f u)(\xi) = (\mathscr{F}u)(f(\xi)) = ((\mathscr{F}u)\circ f)(\xi)$, the proof follows from the analogous one for the classical Fourier transform $\mathscr{F}$:

$$\mathscr{D}_\xi^n(\mathscr{F}_f u)(\xi) = \mathscr{D}_\xi^n\int_{-\infty}^{+\infty} u(x)\mathrm{e}^{-ixf(\xi)}\mathrm{d}x = \int_{-\infty}^{+\infty} u(x)\mathscr{D}_\xi^n\mathrm{e}^{-ixf(\xi)}\mathrm{d}x. \tag{2.8}$$

Note that we could differentiate under the integral sign, since $u \in \mathscr{S}(\mathbb{R})$.

Let us now compute the nth derivative of the composite function $\mathrm{e}^{-ixf(\xi)}$ by means of the Faà di Bruno formula

$$\mathscr{D}^n(\varphi\circ f)(\xi) = n!\sum_{k=1}^{n}\frac{(\mathscr{D}^k\varphi)(f(\xi))}{k!}\sum_{\substack{\sigma_1+\ldots\sigma_k=n\\ \sigma_j\geq 1}}\frac{\mathscr{D}^{\sigma_1}f(\xi)}{\sigma_1!}\cdots\frac{\mathscr{D}^{\sigma_k}f(\xi)}{\sigma_k!}$$

with $\varphi(y) = \mathrm{e}^{-ixy}$

$$\mathscr{D}^n_\xi(\mathrm{e}^{-ixf(\xi)}) = n!\sum_{k=1}^n \frac{\mathscr{D}^k_y \mathrm{e}^{-ixy}\Big|_{y=f(\xi)}}{k!} \sum_{\substack{\sigma_1+\ldots\sigma_k=n\\ \sigma_j\geq 1}} \frac{\mathscr{D}^{\sigma_1}_\xi f(\xi)}{\sigma_1!}\cdots\frac{\mathscr{D}^{\sigma_k}_\xi f(\xi)}{\sigma_k!}$$

$$= n!\sum_{k=1}^n (-ix)^k \frac{\mathrm{e}^{-ixf(\xi)}}{k!} \sum_{\substack{\sigma_1+\ldots\sigma_k=n\\ \sigma_j\geq 1}} \frac{\mathscr{D}^{\sigma_1}_\xi f(\xi)}{\sigma_1!}\cdots\frac{\mathscr{D}^{\sigma_k}_\xi f(\xi)}{\sigma_k!}$$

$$= \mathrm{e}^{-ixf(\xi)} n!\sum_{k=1}^n \frac{(-ix)^k}{k!} \sum_{\substack{\sigma_1+\ldots\sigma_k=n\\ \sigma_j\geq 1}} \frac{\mathscr{D}^{\sigma_1}_\xi f(\xi)}{\sigma_1!}\cdots\frac{\mathscr{D}^{\sigma_k}_\xi f(\xi)}{\sigma_k!}.$$

Substituting into (2.8), we get

$$\mathscr{D}^n_\xi(\mathscr{F}_f u)(\xi) = n!\sum_{k=1}^n \frac{1}{k!} \sum_{\substack{\sigma_1+\ldots\sigma_k=n\\ \sigma_j\geq 1}} \frac{\mathscr{D}^{\sigma_1} f(\xi)}{\sigma_1!}\cdots\frac{\mathscr{D}^{\sigma_k} f(\xi)}{\sigma_k!} \int_{-\infty}^{+\infty} u(x)(-ix)^k \mathrm{e}^{-ixf(\xi)}\mathrm{d}x$$

$$= n!\sum_{k=1}^n h(\xi)(\mathscr{F}_f(-ix)^k u(x))(\xi),$$

for $h(\xi)$ defined as in (2.7). □

Remark 2.13 The assumptions considered in this section of rapidly decreasing functions u in the domain of $\mathscr{F}_f$ for a function f with polynomial growth may be modified in such a way that all the integrals converge.

3 An example of application

In this section, we shall apply the results of Sect. 2 to solve a linear partial differential equation of fractional order (similar to a diffusion-type fractional differential equation considered in [4, Sect. 5]), where the Riemann–Liouville fractional derivatives and the Caputo fractional derivatives are involved. We shall work again in the Schwartz class $\mathscr{S}$ for the sake of simplicity, but more general spaces may be considered.

Let us first introduce the Caputo fractional derivatives. These are related to the Riemann–Liouville fractional integrals and derivatives on an interval $[a, b]$ for $n \in \mathbb{R}^+$

$$\mathscr{I}^n_a f(x) := \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f(t)\mathrm{d}t, \qquad x \in [a, b],$$

$$\mathscr{D}^n_a f(x) = \mathscr{D}^{\lceil n\rceil}\mathscr{I}^{\lceil n\rceil - n}_a f(x), \qquad x \in [a, b].$$

The Caputo fractional derivative of order $n \in \mathbb{R}^+$ can be defined by (there are other equivalent formulations, see [2, Thm. 3.1])

$$\mathscr{D}^n_{*a} f := \mathscr{I}^{\lceil n\rceil - n}_a \mathscr{D}^{\lceil n\rceil} f.$$

If $a = 0$, we denote $\mathscr{I}^n_0 = \mathscr{I}^n$ and $\mathscr{D}^n_{*0} = \mathscr{D}^n_*$.

For a function of two variables $u(x, t)$, we denote by $\mathscr{D}^\alpha_{+,x} u(x, t)$ its Riemann–Liouville fractional derivative of order $\alpha \in \mathbb{R}^+$ on the half-line with respect to the x-variable and by $\mathscr{D}^\beta_{*,t} u(x, t)$ its Caputo fractional derivative of order $\beta \in \mathbb{R}^+$ (for $a = 0$) with respect to the t-variable.

Let us then consider the following fractional Cauchy problem, for $\alpha, \beta \in \mathbb{R}^+$:

$$\begin{cases} \mathscr{D}^\alpha_{+,x} u(x, t) = \mathscr{D}^\beta_{*,t} u(x, t), & x \in \mathbb{R}, t > 0 \\ u(x, 0) = g_0(x) & \\ \frac{\partial^k u(x,0)}{\partial x^k} = g_k(x), & k = 1, \ldots, \lceil\beta\rceil - 1,\ x \in \mathbb{R} \end{cases} \tag{3.1}$$

for given $g_j \in \mathscr{S}(\mathbb{R})$, $0 \le j \le \lceil \beta \rceil - 1$.

To solve the Cauchy problem (3.1), we shall also make use of the Laplace operator, defined for $u \in \mathscr{S}(\mathbb{R})$ by

$$(\mathscr{L}u)(s) := \int_0^{+\infty} u(t)\mathrm{e}^{-st}\mathrm{d}t, \qquad s \in \mathbb{C},\ \mathrm{Re}s > 0.$$

We refer, for instance, to [1] or [8] for the Laplace transform on more general spaces of functions, or for further details (the Laplace transform will be then defined for $\mathrm{Re}s > \alpha$, where α is the abscissa of convergence and depends on the function u). We recall here some basic properties of the Laplace transform. If $u, v \in \mathscr{S}(\mathbb{R})$ with support in $[0, +\infty)$, the convolution operator takes the form

$$(u * v)(t) = \int_0^t u(\tau)v(t-\tau)\mathrm{d}\tau$$

and for all $s \in \mathbb{C}$ with $\mathrm{Re}s > 0$

$$\mathscr{L}(u * v)(s) = (\mathscr{L}u)(s) \cdot (\mathscr{L}v)(s). \tag{3.2}$$

Moreover, the following relation between integer-order differentiation and Laplace transform is well known, for all $s \in \mathbb{C}$ with $\mathrm{Re}s > 0$ (cf. [1,8]):

$$\begin{aligned}(\mathscr{L}\mathscr{D}^n u)(s) &= s^n(\mathscr{L}u)(s) - \sum_{k=0}^{n-1} s^k u^{(n-k-1)}(0)\\ &= s^n(\mathscr{L}u)(s) - \sum_{k=0}^{n-1} s^{n-k-1}u^{(k)}(0), \qquad n \in \mathbb{N}.\end{aligned} \tag{3.3}$$

A similar formula holds also for the Caputo fractional derivative (cf. [2])

Proposition 3.1 *Let $n \in \mathbb{R}^+$ and $u \in \mathscr{S}(\mathbb{R})$. Then, for all $s \in \mathbb{C}$ with $\mathrm{Re}s > 0$*

$$(\mathscr{L}\mathscr{D}^n_* u)(s) = s^n(\mathscr{L}u)(s) - \sum_{k=0}^{\lceil n \rceil - 1} s^{n-k-1}\mathscr{D}^k u(0).$$

Proof From (2.6), (3.2), (2.5), and (3.3), we get for $n \in \mathbb{R}^+ \setminus \mathbb{N}$

$$\begin{aligned}\mathscr{L}(\mathscr{D}^n_* u)(s) &= \mathscr{L}(\mathscr{I}^{\lceil n \rceil - n}\mathscr{D}^{\lceil n \rceil}u)(s)\\ &= \mathscr{L}\left(\frac{1}{\Gamma(\lceil n \rceil - n)} t^{\lceil n \rceil - n - 1}\chi_{(0,+\infty)}(t) * \mathscr{D}^{\lceil n \rceil}u(t)\right)(s)\\ &= \frac{1}{\Gamma(\lceil n \rceil - n)}\mathscr{L}\big(t^{\lceil n \rceil - n - 1}\chi_{(0,+\infty)}(t)\big)(s) \cdot (\mathscr{L}\mathscr{D}^{\lceil n \rceil}u)(s)\\ &= \frac{1}{\Gamma(\lceil n \rceil - n)}\left(\int_0^{+\infty} t^{\lceil n \rceil - n - 1}\mathrm{e}^{-st}\mathrm{d}t\right) \cdot (\mathscr{L}\mathscr{D}^{\lceil n \rceil}u)(s)\\ &= s^{-\lceil n \rceil + n}\mathscr{L}(\mathscr{D}^{\lceil n \rceil}u)(s)\\ &= s^{n - \lceil n \rceil}\left(s^{\lceil n \rceil}(\mathscr{L}u)(s) - \sum_{k=0}^{\lceil n \rceil - 1} s^{\lceil n \rceil - k - 1}u^{(k)}(0)\right)\\ &= s^n(\mathscr{L}u)(s) - \sum_{k=0}^{\lceil n \rceil - 1} s^{n-k-1}\mathscr{D}^k u(0).\end{aligned}$$

□

We are now ready to solve the Cauchy problem (3.1). Assume that $u \in \mathscr{S}(\mathbb{R}^2)$ and choose any f as in Definition 2.1, so that $\mathscr{F}_f$ satisfies all properties (i)–(iii) of Sect. 2. Applying $\mathscr{F}_f$ to both sides of the given equation, by Theorem 2.7, we have

$$(if(\xi))^\alpha(\mathscr{F}_f u(\cdot,t))(\xi,t) = \mathscr{D}^\beta_{*,t}(\mathscr{F}_f u)(\xi,t). \tag{3.4}$$

Applying now the Laplace transform to both sides of (3.4), we have, for $s \in \mathbb{C}$ with $\mathrm{Re}s > 0$

$$\begin{aligned}(if(\xi))^\alpha \cdot (\mathscr{L}\mathscr{F}_f u)(\xi,s) &= \mathscr{L}(\mathscr{D}^\beta_{*,t}(\mathscr{F}_f u))(\xi,s)\\ &= s^\beta \mathscr{L}(\mathscr{F}_f u)(\xi,s) - \sum_{k=0}^{\lceil\beta\rceil-1} s^{\beta-k-1}\mathscr{D}^k_t(\mathscr{F}_f u)(\xi,0)\\ &= s^\beta \mathscr{L}(\mathscr{F}_f u)(\xi,s) - \sum_{k=0}^{\lceil\beta\rceil-1} s^{\beta-k-1}(\mathscr{F}_f g_k)(\xi)\end{aligned}$$

by Proposition 3.1. Therefore

$$\mathscr{L}(\mathscr{F}_f u)(\xi,s) = \sum_{k=0}^{\lceil\beta\rceil-1} \frac{s^{\beta-k-1}}{s^\beta - (if(\xi))^\alpha}(\mathscr{F}_f g_k)(\xi). \tag{3.5}$$

Here, we follow [4] (cf. also [2]) to invert the Laplace transform: the Mittag–Leffler function

$$E_{\gamma,\delta}(z) := \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\gamma n+\delta)}, \qquad z\in\mathbb{C},\ \gamma>0,\ \delta\in\mathbb{R}$$

is an entire function of z, related to the Laplace transform by

$$(\mathscr{L}(t^{\delta-1}E_{\gamma,\delta}(\lambda t^\gamma))(s) = \frac{s^{\gamma-\delta}}{s^\gamma-\lambda}, \qquad \gamma,\delta>0,\ \lambda, s\in\mathbb{C},\ \mathrm{Re}s>0,\ \left|\frac{\lambda}{s^\gamma}\right|<1.$$

Substituting in (3.5) for $\gamma=\beta$, $\delta=k+1$, and $\lambda=(if(\xi))^\alpha$, we have, for $x\in\mathbb{R}$ and $t>0$ (cf. [1])

$$(\mathscr{F}_f u)(\xi,t) = \sum_{k=0}^{\lceil\beta\rceil-1} t^k E_{\beta,k+1}((if(\xi))^\alpha t^\beta)\cdot(\mathscr{F}_f g_k)(\xi). \tag{3.6}$$

Applying $\mathcal{G}_f$ to both sides of (3.6), by Theorem 2.9, we finally obtain the solution of the Cauchy problem (3.1)

$$u(x,t) = \frac{1}{2\pi}\sum_{k=0}^{\lceil\beta\rceil-1} t^k \int_{-\infty}^{+\infty} \mathrm{e}^{ixf(\xi)} f'(\xi)E_{\beta,k+1}((if(\xi))^\alpha t^\beta)(\mathscr{F}_f g_k)(\xi)\mathrm{d}\xi, \tag{3.7}$$

for $x\in\mathbb{R}$ and $t>0$.

Example 3.2 Let us consider the fractional Cauchy problem (3.1) for $\alpha=\frac{1}{3}$ and $\beta=\frac{1}{2}$

$$\begin{cases}\mathscr{D}^{1/3}_{+,x}u(x,t) = \mathscr{D}^{1/2}_{*,t}u(x,t), & x\in\mathbb{R}, t>0\\ u(x,0)=g_0(x).\end{cases} \tag{3.8}$$

Note that $\lceil\beta\rceil=1$, and from (3.7), we get the solution

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{ixf(\xi)} f'(\xi)E_{\frac{1}{2},1}((if(\xi))^{1/3}t^{1/2})(\mathscr{F}_f g_0)(\xi)\mathrm{d}\xi. \tag{3.9}$$

Choosing $f(\xi) = \xi^3$, we have that

$$E_{\frac{1}{2},1}((if(\xi))^{1/3}t^{1/2}) = E_{\frac{1}{2},1}(-i\xi\sqrt{t}) = \mathrm{e}^{-\xi^2 t}\mathrm{erfc}(i\xi\sqrt{t}), \tag{3.10}$$

where $\mathrm{erfc}(z)$ is the *complementary error function* defined by

$$\mathrm{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^{+\infty} \mathrm{e}^{-t^2} \mathrm{d}t.$$

Taking for instance

$$g_0(x) = \mathrm{e}^{-x^2/2}, \tag{3.11}$$

we have that $\hat{g}_0(\xi) = \sqrt{2\pi}e^{-\xi^2/2}$, and hence, $\mathscr{F}_f(g_0)(\xi) = \sqrt{2\pi}e^{-\frac{1}{2}f^2(\xi)} = \sqrt{2\pi}e^{-\frac{1}{2}\xi^6}$. Substituting it into (3.9), together with (3.10), we finally get the solution

$$u(x,t) = \frac{3}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \xi^2 \mathrm{e}^{ix\xi^3 - \frac{1}{2}\xi^6 - \xi^2 t} \mathrm{erfc}(i\xi\sqrt{t}) \mathrm{d}\xi. \tag{3.12}$$

4 Conclusions

We have considered a very general fractional Fourier transform

$$(\mathscr{F}_f u)(\xi) := \int_{-\infty}^{+\infty} u(x)\mathrm{e}^{-ixf(\xi)}\mathrm{d}x \tag{4.1}$$

for any smooth real phase function $f(\xi)$ with polynomial growth satisfying

$$\frac{f(\xi)}{\xi|\xi|^{\alpha-1}} \to +\infty, \qquad \text{as } |\xi| \to +\infty,$$

for some $\alpha > 0$, and studied its properties useful for applications to fractional differential equations, such as its left inverse transform or its behavior with respect to convolution, translations, dilatations, and the Riemann–Liouville fractional derivatives and integrals.

This approach has the advantage of the large freedom in the choice of the phase function $f(\xi)$, easily adaptable to different problems. Example 3.2 shows that for a fractional differential equation as in (3.8)–(3.11), the choice of $f(\xi) = \xi^3$ gives rise to the solution (3.12) by a straightforward calculation.

Further developments could be achieved by looking for an integral transform similar to (4.1) suitable for the Grünwald–Letnikov fractional derivative, in view of numerical approximations useful for applications.

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