Mathematische Zeitschrift (2025) 310:61
https://doi.org/10.1007/s00209-025-03769-9

Mathematische Zeitschrift

Mori dream singular $K3$ surfaces

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Received: 8 January 2025 / Accepted: 14 April 2025
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Abstract
We take a first step towards the classification of singular Mori dream $K3$ surfaces. We prove that if the Picard lattice of a singular $K3$ surface is Mori dream, then the surface is Mori dream. Moreover, we show that for singular $K3$ surfaces, of Picard rank two, being Mori dream is equivalent to contain two negative curves intersecting each other, and apply this result to study Mori dreamness of $K3$ surfaces with a singular point of type A_n.

Keywords Singular $K3$ surfaces · Mori dream spaces

Mathematics Subject Classification Primary 14J28 14E30; Secondary 11D09

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1 Introduction

In this paper we begin the study of a possible classification of $K3$ surfaces that are Mori dream spaces. This last notion has been introduced by Hu and Keel [5], and is motivated by

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Published online: 16 May 2025

the fact that these spaces are well behaved from the point of view of Mori's minimal model program.

Smooth Mori dream $K3$ surfaces have been classified in terms of Picard lattices [1]. We recall that the Picard group of a $K3$ surface X embeds, as a sublattice L_X, of the so called $K3$ lattice Λ_{K3}. We will say that a sublattice of Λ_{K3} is a Mori dream lattice if it is the sublattice associated to a smooth Mori dream $K3$ surface. Furthermore, a smooth $K3$ surface of Picard rank two is Mori dream if and only if it contains a divisor D of self-intersection $D^2 \in \{-2, 0\}$. For higher Picard rank, there exists a comprehensive list of 178 lattices corresponding to smooth Mori dream $K3$ surfaces [2, Theorem 5.1.5.3].

In the smooth setting we prove, in Theorem 2.10, that under mild hypotheses for smooth $K3$ surfaces of Picard rank two being Mori dream depends only on the discriminant of the intersection form:

Theorem 1.1 *Let X, Y be two smooth $K3$ surfaces of Picard rank two such that the discriminant d_X, d_Y of their intersection forms are equal. Assume that both X and Y contain a class of self-intersection $2a$ with $0 \leqslant 2a \leqslant 6$. Then X is Mori dream if and only if Y is Mori dream.*

We apply Theorem 1.1 to identify which smooth quartic $K3$ surfaces in the Noether-Lefschetz locus, defined by the determinantal components, are Mori dream spaces, as detailed in Corollary 2.13.

However, as we will show, the singular case is much more intricate and difficult to deal with than the smooth one. For instance, as shown in Example 3.5, there are families of smooth non Mori dream $K3$ surfaces specializing to singular Mori dream $K3$ surfaces. In this example, the specialization is constructed as a contraction of a smooth non Mori dream $K3$ surface. More generally, determining which contractions of $K3$ surfaces are Mori dream remains an open problem. Our first result in Theorem 3.4, which generalizes [4, Proposition 2.10], is the following (here with *Picard lattice* we mean the Picard group of the surface equipped with the intersection product):

Theorem 1.2 *Let X be a singular $K3$ surface whose Picard lattice L_X is Mori dream. Then X is a Mori dream surface.*

However, Theorem 1.2 leaves unsolved the question of identifying singular $K3$ surfaces that are Mori dream spaces, even though their Picard lattice is not Mori dream. We provide an answer in the case where the $K3$ surface has Picard rank two and at most one singularity of type A_n. Our main results, in Proposition 4.3 and Theorem 4.10, can be summarized as follows:

Theorem 1.3 *Let X be a possibly singular, projective $K3$ surface of Picard rank two. The following are equivalent:*

(i) *X is a Mori dream space;*
(ii) *there are two effective divisors D_1, D_2 on X with $D_i^2 \leqslant 0$ and $D_1 \cdot D_2 > 0$.*

Furthermore, assume that the singular locus of X consists of a point of type A_n and X contains an irreducible curve of negative self-intersection. If $n \notin \{11, 14, 15\}$ then X is Mori dream.

Most of the arguments used to prove Theorem 1.3 consist of an arithmetic part, using the theory of binary quadratic forms, to prove the existence of certain negative divisors, and a geometric part needed to prove their effectiveness.

Finally, in Sect. 5 we apply Theorem 1.3 to provide explicit examples of singular Mori dream $K3$ surfaces embedded in 3-dimensional weighted projective spaces.

2 Smooth Mori dream $K3$ surfaces

We recall the main definitions and prove that, under mild assumptions, a $K3$ surface X of Picard rank two is Mori dream if and only if any $K3$ surface with the same discriminant as X is Mori dream.

Definition 2.1 A $K3$ *surface* is a normal complex projective surface X satisfying the following conditions:

- The canonical divisor is trivial: $K_X \sim 0$;
- The irregularity vanishes: $h^1(X, \mathcal{O}_X) = 0$;
- The singularities of X, if any, are only rational double points.

If X is smooth, we refer to it simply as a *smooth K3 surface*.

Let X be a smooth K3 surface. Then $H^2(X, \mathbb{Z})$, together with the intersection form provided by the cup product, is isometric to the K3 lattice $E_8(-1)^2 \oplus U^3$, the even unimodular lattice of signature $(3, 19)$. The *Picard lattice* L_X of X is its Picard group together with the intersection product. The Picard lattice L_X embeds isometrically into the K3 lattice and has the *transcendental lattice* T_X as its orthogonal complement.

It follows that $L_X \oplus T_X$ has finite index in Λ_{K3}. This index equals the order of $L_X^*/L_X \simeq T_X^*/T_X$, where $L^* := \mathrm{Hom}(L, \mathbb{Z})$ is the dual lattice of L equipped with the rational quadratic form induced by L, we refer to [6] for further details.

Definition 2.2 A normal projective $\mathbb{Q}$-factorial surface X is called a *Mori dream surface* if the Nef cone $\mathrm{Nef}(X)$ is generated by the classes of finitely many semi-ample divisors.

Let us recall [2, Theorem 5.1.5.1].

Theorem 2.3 *Let X be a smooth $K3$ surface. Then the following are equivalent:*

(i) *X is a Mori dream surface.*
(ii) *The effective cone $\mathrm{Eff}(X) \subset \mathrm{Pic}_{\mathbb{Q}}(X)$ is polyhedral.*
(iii) *The automorphism group of X is finite.*

Moreover, if the Picard rank of X is at least three, then X is a Mori dream surface if and only if it contains only finitely many smooth rational curves. In this case, these curves are (-2)-curves and their classes generate the effective cone. In [2] smooth Mori dream $K3$ surfaces are characterized using the classification of $K3$ surfaces with finite automorphism group.

The following result is a consequence of [12, Theorem 1.12.4] and of the global Torelli theorem.

Theorem 2.4 *For any even lattice Λ of signature $(1, \rho)$ with $\rho \leqslant 10$, there exists a smooth $K3$ surface X such that $L_X \simeq \Lambda$.*

For the rest of the section we focus on smooth $K3$ surfaces of Picard rank two. In this case an improvement of Theorem 2.3 is given by [2, Theorem 5.1.5.3] which we now recall.

Theorem 2.5 *Let X be a smooth $K3$ surface of Picard rank two. Then the following are equivalent:*

(i) *X is a Mori dream surface;*

(ii) $\mathrm{Pic}(X)$ *contains a class* D *with* $D^2 \in \{-2, 0\}$.

Definition 2.6 Let X be a $K3$ surface with Picard lattice L_X. The *discriminant* d_X of X (or of L_X) is the determinant of the intersection matrix of any basis of L_X multiplied by -1.

Remark 2.7 Let X be a $K3$ surface of Picard rank two. The intersection matrix and the discriminant are of the following form:

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad d_X = b^2 - 4ac. \tag{2.8}$$

There is a divisor class D of X such that the condition $D^2 = 0$ holds exactly when d_X is a square. The surface admits a divisor class D such that $D^2 = -2$ when the equation

$$ax^2 + bxy + cy^2 = -1$$

admits an integral solution. Moreover, after possibly applying transformations of the form $(x, y) \mapsto (x, x \pm y)$, we can assume that $-a \leqslant b \leqslant a$. Similarly, after possibly applying transformations of the form $(x, y) \mapsto (x, -y)$, we can assume that $0 \leqslant b \leqslant a$.

Furthermore, we recall the following criterion when X is a smooth quartic surface in $\mathbb{P}^3$.

Proposition 2.9 *Let X be a smooth quartic surface in $\mathbb{P}^3$ of Picard rank two and whose intersection matrix is given by (2.8). Then the following are equivalent:*

(i) $\mathrm{Pic}(X)$ *contains a class* D *with* $D^2 = -2$*;*
(ii) *the following equivalence of quadratic forms holds:*

$$ax^2 + bxy + cy^2 \equiv \begin{cases} -x^2 + \frac{1}{4}d_X y^2 & \text{if } b \text{ is even;} \\ -x^2 + xy + \frac{1}{4}(d_X - 1)y^2 & \text{if } b \text{ is odd.} \end{cases}$$

Proof First of all observe that the equation $D^2 = -2$ is equivalent to $ax^2 + bxy + cy^2 = -1$. If the form is equivalent to one of the two displayed forms, then, by evaluating at $(1, 0)$, it follows that the equation $ax^2 + bxy + cy^2 = -1$ admits an integral solution.

Now, assume that the equation $ax^2 + bxy + cy^2 = -1$ has an integral solution (x_0, y_0). Then $\gcd(x_0, y_0) = 1$. Acting with $GL(2, \mathbb{Z})$ we can map (x_0, y_0) to $(1, 0)$. Then $2x^2 + bxy + cy^2$ is mapped to $-x^2 + b'xy + c'y^2$. Applying sufficiently many linear maps of the form $(x, y) \mapsto (x, x \pm y)$ we obtain a form with the same coefficient of x^2 as the original form, and with the coefficient of xy in $\{0, 1\}$. Finally, the statement follows. □

Theorem 2.10 *Let X, Y be two smooth $K3$ surfaces of Picard rank two such that $d_X = d_Y$. Assume that both X and Y contain a class of self-intersection $2a$ with $0 \leqslant 2a \leqslant 6$. Then X is Mori dream if and only if Y is Mori dream.*

Proof If the discriminant is a square the claim follows by Remark 2.7. So assume this is not the case. The intersection matrices of X and Y are

$$\begin{pmatrix} 2a & b_X \\ b_X & 2c_X \end{pmatrix} \text{ and } \begin{pmatrix} 2a & b_Y \\ b_Y & 2c_Y \end{pmatrix}$$

respectively. The corresponding quadratic forms are thus $ax^2 + b_X xy + c_X y^2$ and $ax^2 + b_Y xy + c_Y y^2$. By Remark 2.7 we can assume $0 \leqslant b_X, b_Y \leqslant a$. On the other hand, the equality $d_X = d_Y$ yields $b_X^2 \equiv b_Y^2 \pmod{4a}$. When $a \leqslant 3$ all these conditions imply $b_X = b_Y$, so that the two intersection forms are equivalent. □

Example 2.11 The assumption $a \leqslant 3$ in Theorem 2.10 is necessary. For instance, when $a = 4$ the following intersection matrices

$$\begin{pmatrix} 8 & 4 \\ 4 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 8 & 0 \\ 0 & -4 \end{pmatrix}$$

define two $K3$ surfaces with Picard group of rank two and discriminant 32. The first one is Mori dream, since it contains a class of self-intersection -2, while the second one is not Mori dream because any class in $\mathrm{Pic}(X)$ has self-intersection which is a multiple of 4.

We present some families of smooth Mori dream $K3$ surfaces in $\mathbb{P}^3$ of Picard rank two. Recall that a degree d surface $X = \{f = 0\} \subset \mathbb{P}^3$ is called determinantal if f is the determinant of a matrix whose entries are homogeneous polynomials of $\mathbb{P}^3$. We recall the following result of [9].

Theorem 2.12 *The smooth determinantal $K3$ surfaces in $\mathbb{P}^3$ are parametrized by five prime divisors $\mathcal{D}_1, \ldots, \mathcal{D}_5 \subset \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$, where each $\mathcal{D}_i$ is the family of quartic surfaces containing the following curves:*

- *$\mathcal{D}_1$, a degree 6 and genus 3 curve given by the resolution $0 \to \mathcal{O}_{\mathbb{P}^3}(-4)^3 \to \mathcal{O}_{\mathbb{P}^3}(-3)^4$;*
- *$\mathcal{D}_2$, a twisted cubic;*
- *$\mathcal{D}_3$, the complete intersection of two quadric surfaces;*
- *$\mathcal{D}_4$, a line;*
- *$\mathcal{D}_5$, a conic.*

Moreover, a very general surface in $\mathcal{D}_i$ has Picard rank two and discriminant 20, 17, 16, 9, 12 *respectively.*

Corollary 2.13 *A very general $K3$ surface in $\mathcal{D}_1$ is not a Mori dream space while a very general $K3$ surface in $\mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ or $\mathcal{D}_5$ is a Mori dream space.*

Proof The discriminant of a very general surface X in $\mathcal{D}_3$ or in $\mathcal{D}_4$ is a square, so that X is Mori dream.

For the remaining divisors we are going to apply Theorem 2.10, which allows us to check Mori dreamness for any smooth quartic of Picard rank two with prescribed discriminant. The intersection matrices of a very general surface in $\mathcal{D}_1, \mathcal{D}_2$ or $\mathcal{D}_5$ are

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$$

respectively. For the first two note that the equation $D^2 = -2$ admits an integral solution, while for the third one there can not be a solution since D^2 is a multiple of 4. □

3 Singular Mori dream $K3$ surfaces

In this section we lay the foundations to study Mori dreamness of singular $K3$ surfaces.

Lemma 3.1 *Let X be a $K3$ surface and $\pi : \widetilde{X} \to X$ its minimal resolution of singularities with exceptional locus supported on the irreducible curves $E_1, \ldots, E_n$. Then the following hold:*

(i) *$\widetilde{X}$ is a smooth $K3$ surface.*

(ii) $\pi^* \operatorname{Pic}(X) = \langle E_1, \dots, E_n \rangle^\perp$.
(iii) $\pi_* \colon \operatorname{Cl}(\widetilde{X}) \to \operatorname{Cl}(X)$ *induces an isomorphism of finite abelian groups*

$$\operatorname{Cl}(\widetilde{X})/\pi^* \operatorname{Pic}(X) \oplus \langle E_1, \dots, E_n \rangle \simeq \operatorname{Cl}(X)/\operatorname{Pic}(X).$$

Proof To prove (i) it suffices to show that $K_{\widetilde{X}} \sim 0$ and $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$. Since X has only rational double points the resolution is crepant, so that $K_X \sim 0$ implies $K_{\widetilde{X}} \sim 0$. The vanishing of $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is a consequence of $R^1\pi_*\mathcal{O}_X = 0$, which holds since X has rational singularities, and satisfies $h^1(X, \mathcal{O}_X) = 0$.

We prove (ii). The inclusion $\pi^* \operatorname{Pic}(X) \subseteq \langle E_1, \dots, E_n \rangle^\perp$ is clear. We show the opposite inclusion. Let D be a divisor in $\widetilde{X}$, whose class is orthogonal to the exceptional locus of π. Since X has rational singularities, D restricts trivially on each effective curve supported along this locus. Moreover, by Artin's contractibility criterion $D \sim D'$, where the support of D' is disjoint from the exceptional locus. Thus D is the pullback of a Cartier divisor on X.

Finally, we show (iii). The exact sequence

$$0 \longrightarrow \langle E_1, \dots, E_n \rangle \longrightarrow \pi_*^{-1} \operatorname{Pic}(X) \xrightarrow{\pi_*} \operatorname{Pic}(X) \longrightarrow 0, \qquad \pi^* \colon \operatorname{Pic}(X) \to \pi_*^{-1} \operatorname{Pic}(X)$$

admits a splitting given by the pullback. Thus the central term is the direct sum of the other two and the statement follows. □

Remark 3.2 Let X be a $K3$ surface and $\pi \colon \widetilde{X} \to X$ its minimal resolution of singularities with exceptional locus supported on the irreducible curves $E_1, \dots, E_n$. The map $\operatorname{Pic}(\widetilde{X}) \to \langle E_1, \dots, E_n \rangle^*$, given by $D \mapsto \langle D, - \rangle$ is not necessarily surjective. For instance, if $\widetilde{X}$ has Picard group of rank two with intersection matrix

$$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix},$$

then $\widetilde{X}$ is the resolution of a singular quartic surface X of $\mathbb{P}^3$ whose singular locus consists of an ordinary double point. In this case $\operatorname{Pic}(X) = \langle H \rangle$, where $H^2 = 4$ and the exceptional locus consists of just one (-2)-curve E coming from the resolution of the singularity. Thus

$$\pi^* \operatorname{Pic}(X) \oplus \langle E \rangle = \operatorname{Pic}(\widetilde{X})$$

since the two lattices have the same rank and intersection matrix. Thus $\operatorname{Cl}(X) = \operatorname{Pic}(X)$, by Lemma 3.1.

On the other hand $\langle E \rangle^*$ contains a linear form whose value on E is 1. Such a linear form can not come from $\operatorname{Pic}(\widetilde{X})$ since the intersection product with E of any element of this lattice is an even number.

Definition 3.3 A sublattice $L \subseteq \Lambda_{K3}$ is a *Mori dream lattice* if it belongs to the list given in [2, Theorem 5.1.5.3].

The following result generalizes [4, Proposition 2.10]. Indeed we have no assumption on how the (-2)-curves of the minimal resolution intersect the exceptional locus.

Theorem 3.4 *A $K3$ surface whose Picard lattice is Mori dream is a Mori dream surface.*

Proof Let X be a K3 surface whose Picard lattice L_X is Mori dream. Let $\Delta \subset \mathbb{C}$ be a disc around the origin. Choose a 1-parameter family of period $\omega_t \in T_X \otimes \mathbb{C}$, with $t \in \Delta$, such that

$$\omega_t^\perp \cap \Lambda_{K3} = L_X,$$

for a very general $t \in \Delta$, and

$$\omega_0^{\perp} \cap \Lambda_{K3} = L_{\widetilde{X}}.$$

By the surjectivity of the global Torelli map this corresponds to a family of $K3$ surfaces over Δ whose very general member has Picard lattice isometric to L_X and whose special fiber $\widetilde{X}$ over $0 \in \Delta$ has Picard lattice isometric to $L_{\widetilde{X}}$.

The choice of an ample class $H \in L_X$ produces a polarized family $\mathcal{X} \to \Delta$ whose very general member is a smooth projective $K3$ surface $\mathcal{X}_t$ with Picard lattice isometric to L_X and whose special fiber is isomorphic to the singular $K3$ surface X. By hypothesis the very general member is a Mori dream surface, so that its effective cone is polyhedral.

By [8, Proposition 1.3] it follows that the effective cone of the central fiber $\mathcal{X}_0 \simeq X$ is polyhedral as well. On the other hand, if D is a nef divisor class on X, then D is semiample by the base point free theorem, so X is a Mori dream surface. □

The proof of Theorem 3.4 yields that if a singular $K3$ surface X is a specialization of smooth Mori dream $K3$ surfaces then X is Mori dream. Hence, it allows us to produce singular Mori dream $K3$ surfaces as limits of smooth Mori dream $K3$ surfaces.

On the other hand, not all singular $K3$ surfaces which are Mori dream can be obtained in this way, as shown by the following example.

Example 3.5 Let $\widetilde{X}$ be a smooth $K3$ surface whose Picard lattice is isometric to the following

$$\begin{pmatrix} 4 & 2 & 0 & 0 \\ 2 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{pmatrix}.$$

Such a surface exists by [6, Corollary 3.1]. Denote by (H, E_1, E_2, E_3) the corresponding basis of the Picard lattice. Up to isometry we can assume H to be nef and the remaining three curves to be (-2)-curves. According to [10, Cor. p. 11], the divisor $3H$ defines a birational morphism $\pi : \widetilde{X} \to X$ that contracts precisely the (-2)-curves orthogonal to H. Since $H^{\perp} = \langle H - 2E_1, E_2, E_3 \rangle$, and the equation $(a(H - 2E_1) + bE_2 + cE_3)^2 = -2$ defines an ellipsoid containing exactly two integral points, we conclude that the only such curves are E_2 and E_3. Thus, X has exactly two simple double points, obtained by contracting these curves, and no other singularities.

By Lemma 3.1 the Picard lattice of X is $\langle H, -H + 2E_1 + E_2 + E_3 \rangle = \langle E_2, E_3 \rangle^{\perp}$, and it has the following intersection matrix

$$\begin{pmatrix} 4 & 0 \\ 0 & -8 \end{pmatrix}.$$

Any smooth element $\mathcal{X}_t$ of the family $\mathcal{X} \to \Delta$ is a smooth $K3$ surface with Picard lattice given by the above 2×2 matrix. By Theorem 2.5 such a surface is not Mori dream since it does not contain classes of self-intersection 0 or -2.

On the other hand, the classes E_1 and $H - E_1 - E_2 - E_3$ are effective since they are of self-intersection -2 and their intersection with H is $2 > 0$. Moreover, both are irreducible due to the last intersection value. An easy calculation shows that the images of both curves in X have self-intersection -1. Thus X is a Mori dream space as we will see in Proposition 4.3.

4 Mori dream $K3$ surfaces of Picard rank two

In this section we address the case of $K3$ surfaces of Picard rank two whose singular locus consists of an A_n-point.

Lemma 4.1 *Let X be a $\mathbb{Q}$-factorial projective variety and $f : Y \to X$ a resolution of singularities. If every nef class on Y is semiample then the same holds for X.*

Proof Let D be a nef divisor on X. Then f^*D is also nef, and by assumption it is semiample. Hence, after possibly replacing f^*D with a sufficiently large multiple, we get a divisor D' that is disjoint from the exceptional locus of f. Consequently, $D = f_*f^*D$ is semiample. □

Remark 4.2 The case of Picard number $\varrho(X) = 2$ is not covered in the statement of [8, Proposition 1.2], which assumes $\varrho(X) \geqslant 3$. However, the same argument applies verbatim to yield the following: let X be a normal $\mathbb{Q}$-factorial algebraic surface with $\varrho(X) = 2$ and positive light cone $Q \subseteq N^1(X)_{\mathbb{R}}$. Let C_1 and C_2 be irreducible curves on X. Then the following are equivalent:

(1) $Q \subseteq \mathrm{Cone}([C_1], [C_2])$;
(2) $\overline{\mathrm{Eff}}(X) = \mathrm{Cone}([C_1], [C_2])$.

We will make extensive use of the following result.

Theorem 4.3 *Let X be a $K3$ surface of Picard rank two. Then the following are equivalent:*

(i) X is a Mori dream space.
(ii) There are two effective divisors D_1, D_2 of X such that $D_i^2 \leqslant 0$ and $D_1 \cdot D_2 > 0$.

Proof We prove (i)⇒(ii). If X is Mori dream then there are two irreducible curves C_1, C_2 whose classes generate the effective cone of X. Thus $C_i^2 \leqslant 0$ for $i = 1, 2$. By the Hodge index theorem the intersection form on $C_1^{\perp}$ is not negative definite.

It follows that $C_1^{\perp}$ intersects the interior of the light cone Q, so that the class of C_2 lies in the positive half-plane defined by C_1. Thus $C_1 \cdot C_2 > 0$.

We prove (ii)⇒(i). The hypothesis implies that the cone generated by the classes of D_1 and D_2 contains the light cone Q. Indeed if $D = uD_1 + vD_2$, then

$$D^2 = u^2D_1^2 + 2uvD_1 \cdot D_2 + v^2D_2^2$$

can be positive only if $uv > 0$ (if both u and v are negative then D can not be effective). As a consequence two irreducible components C_1 and C_2 in the support of D_1 and D_2 generate a cone which contains Q.

Thus, Remark 4.2 yields that the effective cone of X is polyhedral. Since the resolution of X is a smooth $K3$ surface, by Lemma 4.1, every nef divisor on X is semiample and hence X is Mori dream. □

The following result will help us to find divisors D_1, D_2 on a singular $K3$ surface X satisfying (ii) in Proposition 4.3.

Lemma 4.4 *Let X be a $K3$ surface of Picard rank two. Assume that X does not contain divisors of self-intersection zero. Let D_1 be a divisor on X with negative self-intersection. Then there exists a divisor D_2, not linearly equivalent to $-D_1$, such that*

$$D_2^2 = D_1^2 \quad \text{and} \quad D_1 \cdot D_2 > 0.$$

Proof Set $\alpha := D_1^2$, and write the class of D_1 with respect to a basis of the divisor class group of X. Then, after clearing denominators, the equation $D^2 = \alpha$ takes the form $Q(x, y) = N$, where

$$Q(x, y) = ax^2 + bxy + cy^2$$

is a binary quadratic form and $a, b, c, N \in \mathbb{Z}$. By the Hodge index theorem the discriminant $\Delta := b^2 - 4ac$ is positive. Moreover, the hypothesis that X does not contain divisors with self-intersection zero implies that Δ is not a square. It follows that if the equation $Q(x, y) = N$ has an integral solution then it has infinitely many solutions [15, Corollary 11.14].

On the other hand, given $\alpha, \beta \in \mathbb{Q}$, there are only a finite number of classes D satisfying

$$D^2 = \alpha \text{ and } D \cdot D_1 = \beta,$$

since the second equation implies that D lies on a line in the affine plane with coordinates x, y, and the first equation can have at most two roots on such a line. Hence, there exists a D_2 such that $D_2^2 = \alpha$, $D_2 \notin \{D_1, -D_1\}$, and $D_1 \cdot D_2 \neq 0$. After possibly replacing D_2 with $-D_2$, we can assume $D_1 \cdot D_2 > 0$, proving the statement. □

We now have to address the effectiveness of the divisor D_2 whose existence is guaranteed by the previous lemma. Let D_1 and D_2 be divisors as in Lemma 4.4, in many cases we can prove that D_2 is effective provided that D_1 is.

Let X be a singular $K3$ surface and $\pi : \widetilde{X} \to X$ a resolution. Given a Weil divisor D on X we denote by $\bar{D}$ its strict transform and by $\pi^* D$ its pullback, as defined in [11]. Set

$$E_D := \pi^* D - \bar{D} \qquad \text{and} \qquad \{E_D\} := E_D - \lfloor E_D \rfloor.$$

Proposition 4.5 *With the above notation, if $D^2 + \{E_D\}^2 > -4$ then either D or $-D$ is linearly equivalent to an effective divisor.*

Proof The equality $\pi^* D - \{E_D\} = \bar{D} + \lfloor E_D \rfloor$ implies that

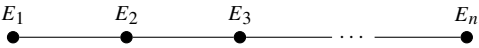
$$D^2 + \{E_D\}^2 = \pi^* D^2 + \{E_D\}^2 = (\pi^* D - \{E_D\})^2$$

is an even integer. Thus this number must be at least -2. On the other hand, it equals the self-intersection of the divisor $D' := \bar{D} + \lfloor E_D \rfloor$, and thus either D' or $-D'$ is linearly equivalent to an effective divisor. Since $\pi_* D' = D$ the statement follows. □

In what follows with a slight abuse of notation, we will use $\mathrm{Exc}(\pi)$ to denote both the set of curves in the exceptional locus of π and the subgroup of the Picard group they generate. From the definition of $\{E_D\}$ it follows that this $\mathbb{Q}$-divisor is effective with rational coefficients smaller than one. Moreover, for any irreducible curve C in $\mathrm{Exc}(\pi)$, we have

$$\{E_D\} \cdot C \in \mathbb{Z}.$$

This implies that $\{E_D\}$ belongs to the dual lattice $\mathrm{Hom}(\mathrm{Exc}(\pi), \mathbb{Z})$. Recall that a singularity of type A_n has a minimal resolution consisting of a chain of (-2)-curves, labeled as shown in the following graph.



In this case the discriminant group is cyclic of order $n + 1$:

$$\mathrm{Hom}(\mathrm{Exc}(\pi), \mathbb{Z}) / \mathrm{Exc}(\pi) \simeq \left\langle \frac{1}{n+1} \sum_{i=1}^{n} i E_i \right\rangle.$$

Thus $\{E_D\}$ is either the generator or the fractional part of a multiple of this.

Proposition 4.6 *With the above notation, assume that the singular locus of X consists of a single point of type A_n with $n \leqslant 7$, and define*

$$E := \frac{1}{n+1} \sum_{i=1}^{n} i E_i.$$

Then, for any $k \in \{1, \ldots, n\}$, the following congruence holds:

$$\{kE\}^2 = k^2\{E\}^2 \pmod 2 - 2.$$

Proof This can be directly verified by computing the Cartan matrix of A_n for $n \leqslant 7$. A Magma program performing this calculation is available at:

https://github.com/alaface/singk3/blob/main/Proposition%204.6

□

Remark 4.7 For $n \geqslant 8$, the statement of Proposition 4.6 does not hold. For example, when $n = 8$ and $k = 4$, we obtain

$$\{kE\}^2 = -\frac{20}{9}, \quad \text{while} \quad k^2\{E\}^2 \pmod 2 - 2 = -\frac{2}{9}.$$

To continue with our analysis we need some preliminary results on Cartan matrices of A_n singularities.

Remark 4.8 Let C^n be the Cartan matrix of the A_n singularity, that is the opposite of the intersection matrix of the curves $E_1, \ldots, E_n$ which appear in the minimal resolution of the singularity. This is the $n \times n$ matrix whose diagonal entries are 2, the entries on the superdiagonal and on the subdiagonal are -1, and the remaining entries are 0:

$$C^n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

We will denote by $C^n_{i,j}$ the submatrix of C^n obtained by removing the row indexed by i and the column indexed by j. The determinant of C^n is given by $\det(C^n) = n + 1$. Let $(C^n)^{-1}$ be the inverse of C^n, and $d_{i,i}$ the i-th diagonal entry of $(C^n)^{-1}$. Then

$$d_{i,i} = \frac{i(n-i+1)}{n+1}$$

for $i = 1, \ldots, n$. In particular,

$$d_{i,i} = d_{n-i+1,n-i+1}$$

for $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$. Furthermore, if $n \geqslant 8$ then $d_{i,i} \geqslant 2$ for $i = 3, \ldots, n-2$. We referee to [13, Page 295] for details on these claims.

We are now ready to proceed with the study of the A_n case.

Proposition 4.9 *Assume that the singular locus of X consists of a single point of type A_n with $n \geqslant 8$. Let $\pi\colon \widetilde{X} \to X$ be a minimal resolution of singularities with exceptional locus supported on $E_1, \ldots, E_n$, and let D be an irreducible curve in X with $D^2 < 0$. Then $\bar{D} \cdot E_i \neq 0$ for at most one value of $i \in \{1, 2, n-1, n\}$, and in this case we have*

$$D^2 = \begin{cases} -\frac{n+2}{n+1} & \text{if} \quad i \in \{1, n\}; \\ -\frac{4}{n+1} & \text{if} \quad i \in \{2, n-1\}. \end{cases}$$

Proof Recall that

$$D^2 = \pi^* D^2 = \pi^* D \cdot (\bar{D} + E_D) = \pi^* D \cdot \bar{D} = (\bar{D} + E_D) \cdot \bar{D}$$

since $\pi^* D$ is orthogonal to E_D. Moreover, since D is irreducible, $\bar{D}$ must be a (-2)-curve. On the other hand

$$0 = (\bar{D} + E_D) \cdot E_D$$

yields $E_D \cdot \bar{D} = -E_D^2$, so that

$$D^2 = -2 - E_D^2.$$

Let $E_D = \sum_{k=1}^{n} \alpha_k E_k$. Then

$$\beta_i := D \cdot E_i = -E_D \cdot E_i = -(\sum_{k=1}^{n} \alpha_k E_k) \cdot E_i.$$

Set $\alpha := (\alpha_1, \ldots, \alpha_n)$, $\beta := (\beta_1, \ldots, \beta_n)$, and let $M := (E_i \cdot E_j)$ be the intersection matrix of $E_1, \ldots, E_n$. Then $\beta = -\alpha \cdot M$ or equivalently $\alpha = -\beta \cdot M^{-1}$. Thus

$$D^2 = -2 - E_D^2 = -2 - \alpha \cdot M \cdot \alpha^t = -2 - \beta \cdot M^{-1} \cdot \beta^t.$$

By definition the vector β has non-negative integral entries. If it contains only one non-zero entry, in position i, then

$$-\beta \cdot M^{-1} \cdot \beta^t = -M^{-1}_{i,i},$$

the opposite of the i-th diagonal entry of the negative definite matrix M^{-1}. When $n \geqslant 8$ this entry is at least 2 as soon as $3 \leqslant i \leqslant n-2$ by Remark 4.8. If $\beta = (1, 0, \ldots, 0)$ or $\beta = (0, \ldots, 0, 1)$ then, by Remark 4.8, we have

$$D^2 = -2 - M^{-1}_{1,1} = -2 + \frac{n}{n+1} = -\frac{n+2}{n+1}.$$

If either $\beta = (0, 1, 0 \ldots, 0)$ or $\beta = (0, \ldots, 0, 1, 0)$ then, by Remark 4.8, we have

$$D^2 = -2 - M^{-1}_{2,2} = -2 + \frac{2n-2}{n+1} = -\frac{4}{n+1}$$

proving the statement. □

We now prove the main result of this section.

Theorem 4.10 *Let X be a $K3$ surface of Picard rank two whose singular locus consists of a point of type A_n. Assume that X contains an irreducible curve with negative self-intersection. If $n \notin \{11, 14, 15\}$, then X is a Mori dream space.*

Proof Let $D_1 \subset X$ be an irreducible curve with negative self-intersection. Observe that $-2 \leqslant D_1^2 < 0$. By Lemma 4.4, there exists a divisor D_2 on X that is not linearly equivalent to $-D_1$ and satisfies $D_1 \cdot D_2 > 0$ and $D_2^2 = D_1^2$. We claim that D_2 is linearly equivalent to an effective divisor, so that the statement follows from Theorem 4.3. We now prove the claim. Since $\{E_{D_2}\}$ belongs to the discriminant group of A_n, which is cyclically generated by $\{E\}$, we can write

$$\{E_{D_2}\} = \{kE\}$$

for some integer $0 \leqslant k \leqslant n$. If $k = 0$, then $\{E_{D_2}\} = 0$ and $D_2^2 = -2$, so that Proposition 4.5 implies the claim. Now assume $k > 0$. If $n \leqslant 7$, Proposition 4.6 gives

$$\{E_{D_2}\}^2 = \{kE\}^2 = k^2\{E\}^2 \pmod 2 - 2.$$

The right-hand side implies $-2 \leqslant \{E_{D_2}\}^2 < 0$, so that the claim follows again from Proposition 4.5. For $n \geqslant 8$, since D_2^2 equals the self-intersection of an irreducible negative curve, Proposition 4.9 implies that

$$D_2^2 \in \left\{-\frac{n+2}{n+1}, -\frac{4}{n+1}\right\}.$$

On the other hand, we know that

$$D_2^2 + \{E_{D_2}\}^2 \equiv 0 \pmod 2.$$

In most cases, this equation uniquely determines $\{E_{D_2}\}^2$, but there are exceptions. When $1 \leqslant n \leqslant 18$, the following cases arise:

where E is the generator of the discriminant group. In each of these cases, the sum $D_2^2 + \{E_{D_2}\}^2$ equals -2 for the first choice of $\{E_{D_2}\}$ and -4 for the second choice. In all other cases, the claim follows from Proposition 4.5. □

n	$\{E\}^2$	$\{kE\}^2$
11	$-\frac{11}{12}$	$\{5E\}^2 = -\frac{35}{12}$
14	$-\frac{14}{15}$	$\{4E\}^2 = -\frac{44}{15}$
14	$\{2E\}^2 = -\frac{26}{15}$	$\{7E\}^2 = -\frac{56}{15}$
15	$\{2E\}^2 = -\frac{7}{4}$	$\{6E\}^2 = -\frac{15}{4}$

5 Codimension one $K3$ surfaces with one A_n singularity

Let $\mathbb{P} := \mathbb{P}(a, b, c, d)$ be a weighted projective space, and let $X \subseteq \mathbb{P}$ be a quasismooth anticanonical surface of degree $a + b + c + d$. Then X is a $K3$ surface whose singularities are only rational double points of type A_n. There are 95 such families of $K3$ surfaces [14]. Among these, the ones that contain at most one singular point coming from the ambient singularities belong to the following 7 families (here $\widetilde{X} \to X$ is a minimal resolution of singularities).

$\deg(X)$	Type of singularity	Ambient space $\mathbb{P}(d_1, d_2, d_3, d_4)$	$\mathrm{Pic}(\widetilde{X})$
21	A_9	$\mathbb{P}(1, 3, 7, 10)$	$E_8 \oplus U$
12	A_1	$\mathbb{P}(1, 1, 4, 6)$	U
10	A_2	$\mathbb{P}(1, 1, 3, 5)$	$A_1 \oplus U$
9	A_3	$\mathbb{P}(1, 1, 3, 4)$	$A_2 \oplus U$
6	—	$\mathbb{P}(1, 1, 1, 3)$	(2)
5	A_1	$\mathbb{P}(1, 1, 1, 2)$	$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$
4	—	$\mathbb{P}(1, 1, 1, 1)$	(4)

Theorem 5.1 *Let $X \subseteq \mathbb{P}(d_1, d_2, d_3, d_4)$ be one of the K3 surfaces appearing in the cases listed in the above table, i.e., quasismooth anticanonical hypersurfaces in a weighted projective 3-space with at most one singular point inherited from the ambient space. Then, the $K3$ surfaces X that contain an irreducible curve of non-positive self-intersection and have Picard number $\rho(X) = 2$ are those listed in the following table.*

In each case, we exhibit the intersection numbers between the ample generator H of $\mathrm{Pic}(X)$ induced by the ambient space, and the negative curve $D_1 \subset X$.

Degree of X	Ambient space	H^2	$H \cdot D_1$	D_1^2
21	$\mathbb{P}(1, 3, 7, 10)$	$\frac{1}{10}$	n	$-2, 0$
		$\frac{1}{10}$	$n + \frac{3}{10}, n + \frac{7}{10}$	$-\frac{11}{10}$
		$\frac{1}{10}$	$n + \frac{3}{5}, n + \frac{7}{5}$	$-\frac{2}{5}$
12	$\mathbb{P}(1, 1, 4, 6)$	$\frac{1}{2}$	n	$-2, 0$
		$\frac{1}{2}$	$n + \frac{1}{2}$	$-\frac{3}{2}$
10	$\mathbb{P}(1, 1, 3, 5)$	$\frac{2}{3}$	n	$-2, 0$
		$\frac{2}{3}$	$n + \frac{2}{3}, n + \frac{4}{3}$	$-\frac{4}{3}$
9	$\mathbb{P}(1, 1, 3, 4)$	$\frac{3}{4}$	n	$-2, 0$
		$\frac{3}{4}$	$n + \frac{3}{4}, n + \frac{5}{4}$	$-\frac{5}{4}$
		$\frac{3}{4}$	$n + \frac{3}{2}$	-1
6	$\mathbb{P}(1, 1, 1, 3)$	2	n	$-2, 0$
5	$\mathbb{P}(1, 1, 1, 2)$	$\frac{5}{2}$	n	$-2, 0$
		$\frac{5}{2}$	$n + \frac{1}{2}$	$-\frac{3}{2}$
4	$\mathbb{P}(1, 1, 1, 1)$	4	n	$-2, 0$

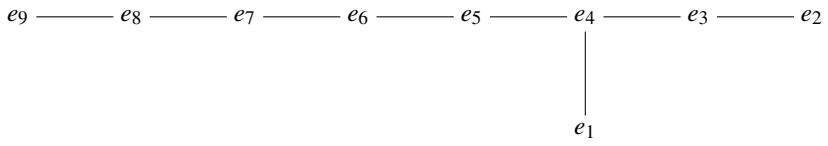
Proof We illustrate the argument in the case of the surface $X_{21} \subseteq \mathbb{P}(1, 3, 7, 10)$. The analysis for the remaining cases is analogous.

Let $\pi : \widetilde{X} \to X$ be the minimal resolution. Then $\widetilde{X}$ is a smooth $K3$ surface whose Picard group is isometric to the lattice $E_8 \oplus U$ (see [3]). Denote by $e_1, \ldots, e_8$ the generators of the E_8 part, and by f_1, f_2 those of the hyperbolic plane U. We may assume that f_1 is nef, and hence it defines an elliptic fibration on $\widetilde{X}$.

In this fibration, the curves corresponding to the classes $e_1, \ldots, e_8$ form an E_8-fiber component. To complete the fiber, one must add the curve whose class is given by

$$e_9 := f_1 - (3e_1 - 2e_2 - 4e_3 - 6e_4 - 5e_5 - 4e_6 - 3e_7 - 2e_8),$$

so that the configuration of classes $e_1, \ldots, e_9$ reflects the extended Dynkin diagram of type $\widetilde{E}_8$:



The class $f_2 - f_1$ defines a section of the elliptic fibration. This section intersects only the curve e_9 in a single point. Therefore, the chain of classes

$$e_2, e_3, \ldots, e_9, f_2 - f_1$$

corresponds to a configuration of type A_9, which is contracted by π. After contracting this chain, the image of the class e_1 defines a divisor class $H \in \mathrm{Cl}(X)$.

Now, consider a specialization of $\widetilde{X}$ in this family such that the Picard number increases to $\rho(\widetilde{X}) = 11$, and suppose there exists a new irreducible curve $\widetilde{D}_1 \subset \widetilde{X}$ with self-intersection $\widetilde{D}_1^2 = -2$ or 0. Let

$$n := \widetilde{D}_1 \cdot e_1.$$

Depending on the position of $\widetilde{D}_1$ relative to the A_9 configuration, we obtain different intersection matrices after contracting the A_9 chain:

- If $\widetilde{D}_1$ is disjoint from all the curves in the A_9 configuration, then its image in X is a curve D_1 disjoint from the exceptional locus of π, so that $D_1^2 \in \{-2, 0\}$.
- If $\widetilde{D}_1$ is a (-2)-curve that intersects only one of the extremal curves of the A_9 configuration - namely e_2 or $f_2 - f_1$ - then $D_1^2 = -\frac{11}{10}$.
- If $\widetilde{D}_1$ is a (-2)-curve intersecting one of the intermediate nodes of the A_9 configuration, such as e_3 or e_8, then $D_1^2 = -\frac{2}{5}$.

□

In what follows, we provide an explicit example in each case and describe its geometry. Let E be the exceptional locus of the minimal resolution $\pi : \widetilde{X} \to X$, and denote by $\Lambda_E \subseteq \mathrm{Pic}(\widetilde{X})$ the sublattice generated by the classes of the irreducible components of E. We denote by $\mathcal{L} \subseteq |-K_{\mathbb{P}}|$ the sublinear system of the anticanonical linear system of $\mathbb{P}$ consisting of hypersurfaces that contain a fixed curve D_1.

Proposition 5.2 *Let X be a very general element of $\mathcal{L}$, and let $\mathrm{Aut}_{D_1}(\mathbb{P}) \subseteq \mathrm{Aut}(\mathbb{P})$ be the subgroup of automorphisms of $\mathbb{P}$ that preserve the curve D_1. If*

$$\dim \mathcal{L} - \dim \mathrm{Aut}_{D_1}(\mathbb{P}) = 18 - \mathrm{rk}(\Lambda_E),$$

then $\mathrm{rk}(\mathrm{Pic}(X)) = 2$.

Proof By construction, the Picard group $\mathrm{Pic}(\widetilde{X})$ contains the sublattice $\Lambda_E \oplus \langle H, \widetilde{D}_1 \rangle$, where H is the pullback of the ample generator of the divisor class group of $\mathbb{P}$, and $\widetilde{D}_1 \subseteq \widetilde{X}$ is the strict transform of D_1.

For any surface $X \in \mathcal{L}$, the set of surfaces $Y \in \mathcal{L}$ isomorphic to X via an automorphism in $\mathrm{Aut}_{D_1}(\mathbb{P})$ lies in at most a countable union of orbits. Indeed, any isomorphism $\phi\colon Y \to X$ induces an embedding of X into $\mathbb{P}$ via the linear system $|\phi_*(H_{|Y})|$, and by assumption this embedding must preserve the curve D_1.

Since the Picard group of a $K3$ surface is countable, only countably many such embeddings (up to the action of $\mathrm{Aut}_{D_1}(\mathbb{P})$) can occur. It follows that the dimension of the moduli space of (possibly singular) $K3$ surfaces in $\mathcal{L}$ is given by

$$\dim \mathcal{L} - \dim \mathrm{Aut}_{D_1}(\mathbb{P}).$$

On the other hand, by the global Torelli theorem, the dimension of the moduli space of $K3$ surfaces with Picard number r is $20 - r$. Hence,

$$\dim \mathcal{L} - \dim \mathrm{Aut}_{D_1}(\mathbb{P}) = 20 - \mathrm{rk}(\mathrm{Pic}(\widetilde{X})) \leqslant 18 - \mathrm{rk}(\Lambda_E).$$

If equality holds, then we must have

$$\mathrm{rk}(\mathrm{Pic}(\widetilde{X})) = \mathrm{rk}(\Lambda_E) + 2,$$

and since H and $\widetilde{D}_1$ are linearly independent from Λ_E, it follows that $\mathrm{rk}(\mathrm{Pic}(X)) = 2$, as claimed. □

Next we will show that Proposition 5.2 applies to the surfaces $X_{21}, X_{12}, X_{10}, X_9, X_5$. We list the relevant numbers in the following table:

Degree of X	Ambient space	$\dim \mathcal{L}$	$\dim \mathrm{Aut}_{D_1}(X)$	$\mathrm{rk}\,\Lambda_E$
21	$\mathbb{P}(1, 3, 7, 10)$	10	1	9
12	$\mathbb{P}(1, 1, 4, 6)$	25	8	1
10	$\mathbb{P}(1, 1, 3, 5)$	24	8	2
9	$\mathbb{P}(1, 1, 3, 4)$	22	7	3
5	$\mathbb{P}(1, 1, 1, 2)$	27	10	1

For each of these type of surfaces we will compute explicit examples, we will extensively use the following simple result.

Lemma 5.3 *Let $H_{d_1}, \ldots, H_{d_n} \subset \mathbb{P}(a_0, \ldots, a_n)$ be hypersurfaces of degree $d_1, \ldots, d_n$ respectively. Then the intersection number of the H_{d_i} is given by*

$$H_{d_1} \cdots H_{d_n} = \frac{\prod_{i=1}^n d_i}{\prod_{i=0}^n a_i}.$$

Proof See [7, Corollary 1.1]. □

As we said all intersection numbers in the following examples are computed using Lemma 5.3.

5.3.1 $X_{21} \subset \mathbb{P}(1, 3, 7, 10)$. Consider the map

$$\begin{array}{rccc} \overline{\gamma}: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}(1, 3, 7, 10) \\ & [u : v] & \mapsto & [u^3 : u^4v^5 : uv^{20} : v^{30}] \end{array}$$

and let Γ_1 be its image. Let G be the group of automorphisms of $\mathbb{P}(1,3,7,10)$ stabilizing Γ_1. An automorphism $\alpha \in G$ fixes $[0:0:0:1]$, and therefore yields an automorphism $\overline{\alpha}$ of $\mathbb{P}^1$ fixing $[0:1]$, that is of the form

$$\begin{array}{rccc} \overline{\alpha}: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ & [u:v] & \mapsto & [au:bu+v]. \end{array}$$

Now, via $\overline{\gamma}$, such automorphism induces the following transformation

$$x = a^3u^3,\ y = a^4u^4(bu+v)^5,\ z = au(bu+v)^{20},\ w = (bu+v)^{30}.$$

If $b = 0$ this transformation is the restriction to Γ_1 of the following automorphism of $\mathbb{P}(1,3,7,10)$:

$$\begin{array}{rccc} \alpha: & \mathbb{P}(1,3,7,10) & \longrightarrow & \mathbb{P}(1,3,7,10) \\ & [x:y:z:w] & \mapsto & [a^3x:a^4y:az:w]. \end{array}$$

On the other hand, if $b \neq 0$ there is no way to obtain for instance the monomial $u^4u^4v = u^8v$ as a product of $x = a^3u^3$, $y = a^4u^4(bu+v)^5$, $z = au(bu+v)^{20}$, $w = (bu+v)^{30}$. Therefore, $\overline{\alpha}$ does not come from an automorphism of $\mathbb{P}(1,3,7,10)$, and hence $G \cong \mathbb{C}^*$.

Consider the surface $X_{21} = \{f = 0\}$ where

$$\begin{aligned} f = {} & x^{14}z + x^{11}yz + 2x^{11}w - x^9y^4 - x^8y^2z + 2x^8yw + x^7z^2 - x^6y^5 - x^5y^3z + \\ & x^4yz^2 + x^4zw - x^3y^6 - x^2y^4z - x^2y^3w - xy^2z^2 + xw^2 - y^7 - z^3. \end{aligned}$$

Then X_{21} is a $K3$ surface whose singular locus consists of a $\frac{1}{10}(3,7)$ singularity at $[0:0:0:1]$.

Now, let $S = \{x^5z - y^4 = 0\} \subset \mathbb{P}(1,3,7,10)$. The intersection $X_{21} \cap S$ has two irreducible components

$$X_{21} \cap S = \Gamma_1 \cap \Gamma_2$$

where Γ_2 is the curve whose ideal is generated by the following four polynomials

$$\begin{array}{l} x^{11}w - x^8y^2z + x^8yw - x^5y^3z + \frac{1}{2}(x^4yz^2 + x^4zw - x^2y^3w - xy^2z^2 + xw^2 - z^3); \\ x^{10}y^2 + x^7y^3 - \frac{1}{2}(x^6yz - x^3y^2z - x^2z^2 - y^2w); \\ x^{13} + x^{10}y + \frac{1}{2}(x^6z - x^4y^3 + x^3w + y^2z); \\ x^5z - y^4. \end{array}$$

Consider the projection $\pi : \mathbb{A}^4 \setminus \{(0,0,0,0)\} \to \mathbb{P}(1,3,7,10)$. The intersection $\pi^{-1}(\Gamma_1) \cap \pi^{-1}(\Gamma_2)$ has a reduced component that gets mapped by π to six simple points and a component of multiplicity 16 that gets mapped by π to $[0:0:0:1]$. Hence, $\Gamma_1 \cdot \Gamma_2 = 6 + \frac{6}{10} = \frac{38}{5}$. Moreover, $\Gamma_1 \cdot H = \frac{6}{10}$. Since $\Gamma_1 + \Gamma_2$ is cut out in X_{21} by a surface of degree 12 we have

$$12\,H = \Gamma_1 + \Gamma_2$$

and intersecting with Γ_1 we get

$$\frac{36}{5} = 12\,H \cdot \Gamma_1 = \Gamma_1^2 + \Gamma_1 \cdot \Gamma_2 = \Gamma_1^2 + \frac{38}{5}.$$

So, $\Gamma_1^2 = -\frac{2}{5}$. Now, from

$$\frac{72}{5} = 144\,H^2 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_2 + \Gamma_2^2 = -\frac{2}{5} + \frac{76}{5} + \Gamma_2^2$$

so that $\Gamma_2^2 = -\frac{2}{5}$.

Finally, we found two effective curves such that

$$\Gamma_1^2 = \Gamma_2^2 = -\frac{2}{5},\ \Gamma_1 \cdot \Gamma_2 = \frac{38}{5}$$

and Proposition 4.3 yields that a very general member of the linear system of $K3$ surfaces containing Γ_1 is a Mori dream space.

5.3 $X_{2+a+b} \subset \mathbb{P}(1, 1, a, b)$

In the following three examples we will always be in the following situation:

$$X_{2+a+b} \subset \mathbb{P}(1, 1, a, b)$$

will be an anticanonical surface, hence of degree $2 + a + b$, general among those containing a component Γ_1 of the reducible curve

$$\Gamma_1 \cup \Gamma_2 = \{z = 0\} \cap X_{2+a+b} \subset \mathbb{P}(1, 1, a, b),$$

where $[x : y : z : w]$ are homogeneous coordinates on $\mathbb{P}(1, 1, a, b)$. Moreover, one of the following conditions will hold:

- $\gcd(a, b) = 1$ and a divides $2 + a + b$;
- $\gcd(a, b) > 1$ and both a and b divide $2 + a + b$.

The singular locus of X_{2+a+b} will consist of a single A_k point $p \in X_{2+a+b}$, located at $p := [0 : 0 : 0 : 1]$ in the first case and along the singular ambient curve $x = y = 0$ in the second case.

5.3.1 $X_{12} \subset \mathbb{P}(1, 1, 4, 6)$

Consider the $K3$ surface

$$X_{12} = \{zA_8 + wB_6 = 0\} \subset \mathbb{P}(1, 1, 4, 6)$$

with $A_8 \in k[x, y, z, w]_8$ and $B_6 \in k[x, y, z, w]_6$. The singular locus of X_{12} consists of a $\frac{1}{2}(1, 1)$ singularity at $[0 : 0 : 1 : 0]$. The curve $\Gamma_1 \cup \Gamma_2$ is cut out by $\{z = 0\}$. So

$$8 = 16\, H^2 = (\Gamma_1 + \Gamma_2)^2 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_2 + \Gamma_2^2.$$

Since $\Gamma_1 \cap \Gamma_2 = \{z = w = B_6 = 0\}$ we have $\Gamma_1 \cdot \Gamma_2 = 6$. From $4H = \Gamma_1 + \Gamma_2$ we get

$$4\, H \cdot \Gamma_1 = \Gamma_1^2 + \Gamma_1 \cdot \Gamma_2$$

and hence $\Gamma_1^2 = \Gamma_2^2 = -2$. Finally, we found two effective curves such that

$$\Gamma_1^2 = \Gamma_2^2 = -2,\ \Gamma_1 \cdot \Gamma_2 = 6$$

and Proposition 4.3 yields that a very general member of the linear system of $K3$ surfaces containing Γ_1 is a Mori dream space.

5.3.2 $X_{10} \subset \mathbb{P}(1, 1, 3, 5)$

Consider the $K3$ surface

$$X_{10} = \{zA_7 + wB_5 = 0\} \subset \mathbb{P}(1, 1, 3, 5)$$

where $A_7 \in k[x, y, z, w]_7$, $B_5 \in k[x, y, z, w]_5$. Note that X_{10} has a unique singular point at $[0 : 0 : 1 : 0]$ which is a singularity of type $\frac{1}{3}(1, 2)$. The curve $\Gamma_1 \cup \Gamma_2$ is cut out by $\{z = 0\}$, so

$$6 = (\Gamma_1 + \Gamma_2)^2 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_2 + \Gamma_2^2.$$

Moreover, $\Gamma_1 \cap \Gamma_2 = \{z = w = B_5 = 0\}$ yields $\Gamma_1 \cdot \Gamma_2 = 5$. Set $\Gamma_3 = \{w = A_7 = 0\}$. Then $\Gamma_1 \cup \Gamma_3$ is cut out by $\{w = 0\}$. Hence

$$\frac{50}{3} = (\Gamma_1 + \Gamma_3)^2 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_3 + \Gamma_3^2$$

and $\Gamma_1 \cap \Gamma_3 = \{z = w = A_7 = 0\}$ yields $\Gamma_1 \cdot \Gamma_3 = 7$. We have $5H = \Gamma_1 + \Gamma_3$ and $5H \cdot \Gamma_3 = \Gamma_1 \cdot \Gamma_3 + \Gamma_3^2 = \frac{35}{3}$. So

$$\Gamma_3^2 = \frac{35}{3} - \Gamma_1 \cdot \Gamma_3 = \frac{14}{3}.$$

Now

$$\frac{50}{3} = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_3 + \Gamma_3^2 = \Gamma_1^2 + \frac{56}{3}$$

yields $\Gamma_1^2 = -2$ and from

$$6 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_2 + \Gamma_2^2 = 8 + \Gamma_2^2$$

we get $\Gamma_2^2 = -2$. Finally, we got two effective curves such that

$$\Gamma_1^2 = \Gamma_2^2 = -2, \ \Gamma_1 \cdot \Gamma_2 = 5$$

and Proposition 4.3 yields that a very general member of the linear system of $K3$ surfaces containing Γ_1 is a Mori dream space.

5.3.3 $X_9 \subset \mathbb{P}(1, 1, 3, 4)$

Consider the $K3$ surface

$$X_9 = \{zA_6 + wB_5 = 0\} \subset \mathbb{P}(1, 1, 3, 4)$$

where $A_6 \in k[x, y, z, w]_6$, $B_5 \in k[x, y, z, w]_5$. We have

$$\Gamma_1 = \{z = w = 0\} \text{ and } \Gamma_2 = \{z = B_5 = 0\}.$$

Note that X_9 has a unique singular point at $[0 : 0 : 0 : 1]$ which is a singularity of type $\frac{1}{4}(1, 3)$. The curve $\Gamma_1 \cup \Gamma_2$ is cut out by $\{z = 0\}$, so

$$\frac{27}{4} = (\Gamma_1 + \Gamma_2)^2 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_2 + \Gamma_2^2.$$

Moreover, $\Gamma_1 \cap \Gamma_2 = \{z = w = B_5 = 0\}$ yields $\Gamma_1 \cdot \Gamma_2 = 5$. Set $\Gamma_3 = \{w = A_6 = 0\}$. Then $\Gamma_1 \cup \Gamma_3$ is cut out by $\{w = 0\}$. Hence

$$12 = (\Gamma_1 + \Gamma_3)^2 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_3 + \Gamma_3^2$$

and $\Gamma_1 \cap \Gamma_3 = \{z = w = A_6 = 0\}$ yields $\Gamma_1 \cdot \Gamma_3 = 6$. We have $4H = \Gamma_1 + \Gamma_3$ and $4H \cdot \Gamma_3 = \Gamma_1 \cdot \Gamma_3 + \Gamma_3^2 = 8$. So

$$\Gamma_3^2 = 8 - \Gamma_1 \cdot \Gamma_3 = 2.$$

Now

$$12 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_3 + \Gamma_3^2 = \Gamma_1^2 + 14$$

yields $\Gamma_1^2 = -2$ and from

$$\frac{27}{4} = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_2 + \Gamma_2^2$$

we get $\Gamma_2^2 = -\frac{5}{4}$. Finally, we got two effective curves such that

$$\Gamma_1^2 = -2,\ \Gamma_2^2 = -\frac{5}{4},\ \Gamma_1 \cdot \Gamma_2 = 5$$

and Proposition 4.3 yields that a very general member of the linear system of $K3$ surfaces containing Γ_1 is a Mori dream space.

5.3.4 $X_5 \subset \mathbb{P}(1, 1, 1, 2)$

Consider the $K3$ surface

$$X_5 = \{yA_4 + wB_3 = 0\} \subset \mathbb{P}(1, 1, 1, 2)$$

where $A_4 \in k[x, y, z, w]_4$, $B_3 \in k[x, y, z, w]_3$. We have

$$\Gamma_1 = \{y = w = 0\} \text{ and } \Gamma_2 = \{y = B_3 = 0\}.$$

Note that X_5 has a unique singular point at $[0 : 0 : 0 : 1]$ which is a singularity of type $\frac{1}{2}(1, 1)$. Since $\Gamma_1 \cdot \Gamma_2 = 3$ We have

$$1 = H \cdot \Gamma_1 = \Gamma_1^2 + \Gamma_1 \cdot \Gamma_2 = \Gamma_1^2 + 3.$$

So $\Gamma_1^2 = -2$. Furthermore,

$$\frac{5}{2} = H^2 = \Gamma_1^2 + 2\Gamma_1 \cdot \Gamma_2 + \Gamma_2^2 = -2 + 6 + \Gamma_2^2$$

yields $\Gamma_2^2 = -\frac{3}{2}$. Summing-up we found two effective curves such that

$$\Gamma_1^2 = -2,\ \Gamma_2^2 = -\frac{3}{2},\ \Gamma_1 \cdot \Gamma_2 = 3$$

and Proposition 4.3 yields that a very general member of the linear system of $K3$ surfaces containing Γ_1 is a Mori dream space.

5.3 Magma scripts

The Magma code used in this paper is available at

https://github.com/alaface/singk3

Acknowledgements The first named author has been partially supported by Proyecto FONDECYT Regular n. 1230287. The second named author has been supported by the PRIN 2022 project-20223B5S8L "Birational geometry of moduli spaces and special varieties", and is a member of the Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni of the Istituto Nazionale di Alta Matematica "F. Severi" (GNSAGA-INDAM). The third named author has been supported by the FAPESP postdoctoral grant 2019/23499-7 and BEPE Research Internship Abroad grant 2023/01360-2, and thanks for the hospitality the University of Ferrara and the University of Concepción where part of this work was developed. We thank the referee for the comments that allowed us to improve the paper.

Funding Open access funding provided by Università degli Studi di Ferrara within the CRUI-CARE Agreement.

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