Advances in Applied Mathematics 162 (2025) 102778

Contents lists available at ScienceDirect

Advances in Applied Mathematics

journal homepage: www.elsevier.com/locate/yaama

Representation theorems for simplicial complexes and matroidal-like properties of minimal partitioners

C. Bisi [a], F.G. Infusino [b,*]

[a] *Department of Mathematics and Computer Science, University of Ferrara, Via Machiavelli, n.35, 44121, Ferrara, Italy*
[b] *Department of Mathematics and Computer Science, University of Calabria, Via Pietro Bucci, Cubo 30B, 87036 Arcavacata di Rende (CS), Italy*

ARTICLE INFO

Article history:
Received 19 July 2023
Received in revised form 17 July 2024
Accepted 4 September 2024
Available online xxxx

MSC:
primary 05E45, 05B35, 68R05
secondary 68R10, 68R01

Keywords:
Attractive pairings
Quasi-attractive pairings
Closable finitary simplicial complexes
Matroids
Reducts
Graphs

ABSTRACT

A *pairing* on an arbitrary ground set Ω is a triple $\mathfrak{P} := (U, F, \Lambda)$, with U, Λ two sets and $F : U \times \Omega \longrightarrow \Lambda$ a map. Several properties of pairings arise after considering the Moore set system $\mathcal{M}_{\mathfrak{P}}$ and the abstract simplicial complex $\mathcal{N}_{\mathfrak{P}}$ on Ω, defined by taking the maximum and the minimal elements of the equivalence collections with respect to a specific equivalence relation $\approx_{\mathfrak{P}}$, respectively called *minimal* and *maximum* partitioners.

In the present work we first detect various sufficient conditions allowing us to represent specific subfamilies of abstract simplicial complexes as the family of all the minimal partitioners of some pairing on the same ground set. Next, we classify two suitable subcollections of pairings by using generalized matroidal-like properties of $\mathcal{N}_{\mathfrak{P}}$. More in detail, we first determine a sufficient condition on $\mathfrak{P}$ ensuring that the family $\mathcal{N}_{\mathfrak{P}}$ is a *closable finitary simplicial complex* and call the resulting pairings *attractive*. On an arbitrary ground set Ω, attractiveness, together with a finiteness condition, implies that the minimal members of the equivalence collections of each $X \in \mathcal{M}_{\mathfrak{P}}$ with respect to $\approx_{\mathfrak{P}}$ all have the same cardinality. Nevertheless, the converse does not hold, neither in the finite case. To this regard, we find some counterexamples inducing us to introduce the class of *quasi-attractive pairings*. We car-

---

[*] Corresponding author.
E-mail addresses: bsicnz@unife.it (C. Bisi), federico.infusino@unical.it (F.G. Infusino).

https://doi.org/10.1016/j.aam.2024.102778

ried out a detailed analysis of quasi-attractive pairings: for instance we characterize them from a lattice-theoretic point of view and, on a finite ground set Ω, also in term of exchange properties of suitable set systems.

Finally, by taking the adjacence matrix of a simple undirected graph G as a model of pairing, we show that the Petersen graph induces an attractive pairing, while the *Erdös' friendship graphs* induce a quasi-attractive, but not attractive, one.

---

1. Introduction

1.1. General premise

One of the main ideas of Granular Computing (briefly GrC) and Combinatorial Database Theory consists of the attempt of minimizing the number of attributes needed to produce a given degree of knowledge from a large amount of data, i.e. to induce suitable clusters of objects, according to some similarity criterion.

Very often, data is arranged in a tabular form and attributes label columns, while objects label rows. Therefore, from a formal point of view, the main structure to be used in the scope of GrC to collect data is the so-called *Pawlak's information table* [39,41]. As a matter of fact, a Pawlak's information table is a quadruple $(U, \mathrm{Att}, \mathrm{Val}, F)$, where U is a non-empty finite set of objects, Att is a non-empty finite set of attributes, Val is a non-empty set of values and the map $F : U \times \mathrm{Att} \longrightarrow \mathrm{Val}$ is called *information map*.

Nevertheless, it is possible to keep the underlying idea of Pawlak's axiomatization even when attributes form an infinite set. Though such an assumption loses significance in the context of practical applications from computer sciences or applied sciences, it becomes interesting when approaching with different abstract combinatorial and algebraic structures. Hence, let us provide the generalized notion of a *pairing* on a fixed arbitrary (even infinite) ground set Ω as a triple $\mathfrak{P} := (U, \Lambda, F)$, where U and Λ are two non-empty sets and $F : U \times \Omega \longrightarrow \Lambda$ is a map. As it can be clearly understood from the previous very general definition, pairings have been used as a concrete framework starting from which to determine suitable collections of subsets, set operators and set binary relations whose interrelations yield a good combinatorial potential, with applications in various scopes of research, such as design theory [6,24], discrete dynamical systems [1,2], commutative algebra [13], monoid theory [14] categorical homotopy theory [30] and order theory [5,17,18].

A fundamental and relevant tool of the analysis carried out in the present paper is the so-called relation of *functional dependence*, commonly used in *combinatorial database theory* [18,19,32,41]. Functional dependence is a specific binary relation defined on the powerset of a given ground set Ω. Its combinatorial analysis leads to new and interesting research perspectives and fits within a more general approach of investigation aimed at outlining suitable criteria for defining hierarchical clusterings to acquire information on large amounts of data or on abstract structure [37,38,42]. It is now well established that functional dependence set relations yield a cryptomorphic version of closure operators (see Theorem 2.2). The aforementioned result has been recently formalized through the technical notions of *sub-bijection* and *linking maps* [15], that actually formalize the correspondences between various mathematical structures axiomatized in terms of set systems, binary set relations or set operators, and whose analysis has been recently related to their interactions with set-theoretical operations or already pre-existing algebraic or topological structures assigned on Ω [13–15].

At this point, it is worthwhile noticing that the trait d'union between the theory of functional dependences and pairings comes from a suitable *pairing relation*. To describe it, given a pairing $\mathfrak{P} = (U, \Lambda, F)$ on Ω we first need to introduce the binary relation $\equiv_X$ on U by setting $u \equiv_X u' :\Longleftrightarrow [F(u,x) = F(u',x)\ \forall x \in X]$, for any $u, u' \in U$. Next, we may define the wanted pairing relation $\leftarrowtail_{\mathfrak{P}}$ on the powerset $\wp(\Omega)$ of Ω as follows: for each $X, Y \in \wp(\Omega)$, we set $Y \leftarrowtail_{\mathfrak{P}} X$ if and only if for any $u, u' \in U$ such that $u \equiv_X u'$ it also results that $u \equiv_Y u'$. It may be shown that $\leftarrowtail_{\mathfrak{P}}$ is a dependence set relation on Ω and, furthermore, that any dependence set relation on a given ground set Ω may be described as the pairing relation induced by some pairing $\mathfrak{P}$ on Ω (see Theorem 2.9). Due to the cryptomorphism between dependence set relations, closure operators and Moore set systems (i.e. families closed under arbitrary intersections), the aforementioned representation theorem holds for any closure operator and any Moore set system on Ω. Correspondingly, with any pairing $\mathfrak{P}$ we associate a closure operator $M_{\mathfrak{P}}(X) := \{z \in \Omega \mid \{z\} \leftarrowtail_{\mathfrak{P}} X\}$ and its fixed point set $\mathcal{M}_{\mathfrak{P}}$, whose members coincide with the maximum elements of the equivalence collections of the symmetrization $\approx_{\mathfrak{P}}$ of the preorder $\leftarrowtail_{\mathfrak{P}}$ (whence the term *maximum partitioners*).

The present work is a continuation of the research carried out in [12]. As the underlying idea of Pawlak's theory consists of minimizing the number of subsets yielding a fixed degree of knowledge, the perfect candidates for the role of *knowledge optimizer* are played by the minimal members of each equivalence class with respect to $\approx_{\mathfrak{P}}$. We call them *minimal partitioners* and denote the family of all the minimal partitioners of $\mathfrak{P}$ by the symbol $\mathcal{N}_{\mathfrak{P}}$. The previous family is an example of *abstract simplicial complex*, i.e. a set system closed under taking subsets. This fact is very interesting as it allows to link GrC and the theory of pairings to combinatorial algebraic topology [33] in such a way, for instance, the techniques of the latter field [31,40] may be applied to analyze how the information produced by large amounts varies of data or to give new information on algebraic and combinatorial structures.

Unlike maximum partitioners, it is not always possible to construct a pairing whose family of minimal partitioner coincides with a given abstract simplicial complex (see Example 2.10 for further details). Therefore, a representation theorem for abstract simplicial complex must contain limitations: while waiting for a complete description of abstract simplicial complexes through pairings, we exhibit some conditions (both in the finite case and in the infinite case) that could be useful for future research.

Now, one of the conditions which implies the representability of an abstract simplicial complex on a finite ground set in terms of the family of minimal partitioners of a pairing is the matroidality of the complex itself. In addition, $\mathcal{N}_{\mathfrak{P}}$ satisfies various matroidal-like properties without being a matroid. Therefore, the analysis of the relationship between matroidality and minimal partitioners lends itself to be a good starting point for our research.

As matroidality is a condition holding only on finite ground sets, to generalize our perspective we need to work with a cyptomorphic definition of matroidality, involving the validity of the so-called *MacLane-Steinitz exchange axiom* for a closure operator [26]. Thus, in this work, we study in detail two specific subcollections of pairings defined by adding some extra condition using $\leftarrowtail_{\mathfrak{P}}$ and related to the cryptomorphisms exhibit in Theorem 2.4. To this regard, we call *attractive* those pairings such that $\{x\} \not\leftarrowtail_{\mathfrak{P}} (X \setminus \{x\}) \cup \{y\}$ whenever $X \in \wp(\Omega)$, $y \in \Omega$ and $x \in X$ are such that $\{y\} \not\leftarrowtail_{\mathfrak{P}} X$ and $\{x\} \not\leftarrowtail_{\mathfrak{P}} X \setminus \{x\}$. On the other hand, we call *quasi-attractive* those pairings for which there exists $y_x \in Y$ such that $\{x\} \leftarrowtail_{\mathfrak{P}} (X \setminus \{x\}) \cup \{y_x\}$, whenever $X \approx_{\mathfrak{P}} Y$ and $x \in X$.

After having determined a whole range of technical properties for attractive and quasi-attractive pairings on an arbitrary ground set Ω, we also exhibit some non-trivial examples from graph theory. Here, the relation $\equiv_X$ may be reinterpreted as a local symmetry relation in the sense of Erdös [22] as, for any two non-adjacent vertices v, v' of a graph such that $v \equiv_X v'$, the automorphism group of the subgraph induced by $X \cup \{v, v'\}$ turns out to be non-trivial (for further details see [10]).

In the next subsection we explain the main properties of attractive and quasi-attractive pairings.

1.2. Attractive and quasi-attractive pairings

The results of Section 2 (and mainly Theorems 2.4 and 2.9) allow us to take into account all those pairings whose closure operator $M_{\mathfrak{P}}$ is both *algebraic* and *symmetrizing* (see Section 2 for definitions). The resulting pairings are called *locally finite attractive pairings*. This kind of pairings are important because their minimal partitioner family $\mathcal{N}_{\mathfrak{P}}$ is a non-trivial example of *closable finitary simplicial complex* [15].

In locally finite attractive pairings, the minimal partitioners Y of a given $X \in \mathcal{M}_{\mathfrak{P}}$ such that $M_{\mathfrak{P}}(Y) = X$ all have the same cardinality (see [11]). The members of the aforementioned collection are named the $\mathfrak{P}$-*reducts* of X, a fundamental tool of combinatorial database theory and data mining thanks to which it is possible to minimize the attribute subsets inducing a given degree of information [39,41]. Their computation (at least on finite ground sets) has been brought back to the well-known problem of determining the minimal transversals of a suitable hypergraph and has been recently used in the context of simple undirected graphs, where $\mathfrak{P}$-reducts have been completely characterized for specific kinds of graphs from both an algebraic and a geometric point of view in [10]. Restricting to finite ground sets - and so dropping out the redundant assumption of locally finiteness - to require attractiveness implies that the previous collections of minimal partitioners behave as the family of X-*bases* of a matroid. Thus, in Section 3 we analyze attractive pairings on arbitrary - even infinite - ground sets and, also in such a situation, we interpret the $\mathfrak{P}$-reducts of X as a sort of generalized bases.

One of the problems we solved in this work is to fully describe when a locally finite pairing is attractive (see Theorem 3.10). To this regard, we proved that a locally finite pairing is attractive if and only if the equality between the family of the $\mathfrak{P}$-reducts of X and the collection of the maximal members of $\mathcal{N}_{\mathfrak{P}} \cap \wp(X)$ holds for any $X \in \wp(\Omega)$. In particular, the previous set-theoretic equality and attractiveness become equivalent when Ω is finite. Nevertheless, if we drop out local finiteness and still work with infinite ground sets, the previous condition becomes only necessary for attractiveness. Therefore, the study of attractiveness becomes richer as more varied. For instance, in Theorem 3.8 we show that attractiveness is a sufficient condition for the collection of the maximal members of $\mathcal{N}_{\mathfrak{P}} \cap \wp(X)$, for any $X \in \wp(\Omega)$, to satisfy a generalized version of the *exchange axiom* for the bases of a matroid [7,26]. Therefore, our results represent a further attempt of relating matroids to the field of granular computing, rough set theory and combinatorial database theory, as it has been done in [21,29,35,36]

On the other hand, it must be noticed that the uniform cardinality of the family of all the $\mathfrak{P}$-reducts of $X \in \mathcal{M}_{\mathfrak{P}}$ simply gives a necessary condition for attractiveness, even when Ω is a finite set. Indeed, examples of non-attractive pairings for which the $\mathfrak{P}$-reducts of any maximum partitioner have uniform cardinality occur in various scopes. In general, as we also said in the above subsection, we may weaken the condition of being attractive by requiring, for any $x \in X$, the existence of $y_x \in Y$ for which $\{x\} \leftarrowtail_{\mathfrak{P}} (X \setminus \{x\}) \cup \{y_x\}$, whenever $X \approx_{\mathfrak{P}} Y$. We call a pairing satisfying the above property *quasi-attractive* and devoted our attention to them throughout Section 4. With regard to such a new subclass of pairings, we demonstrate that they are indeed a weaker version of attractive ones. Successively, we show that in any quasi-attractive pairing $\mathfrak{P}$ on an arbitrary set Ω (even infinite) satisfying local finiteness, the $\mathfrak{P}$-reducts of $X \in \mathcal{M}_{\mathfrak{P}}$ have the same cardinality. Furthermore, when Ω is a finite set, the collection of all the $\mathfrak{P}$-reducts of any maximum partitioner X satisfies the exchange property characterizing the bases of a matroid on Ω, though $\mathcal{N}_{\mathfrak{P}}$ need not to be a matroid. In this sense, quasi-attractiveness relates to the matroidality of a specific sub-set system of $\mathcal{N}_{\mathfrak{P}}$.

In general, it is not an easy task to find non-trivial families of attractive or quasi-attractive pairings; however, in Sections 5 and 6 we exhibit some specific models of attractive and quasi-attractive pairings from graph theory. Indeed, depending on the structural property we want to highlight, we can represent any simple undirected graph G with a finite vertex set by means of a matrix, i.e. a numerical data table. This is the basic idea of classical *algebraic graph theory* [25], according to which the algebraic properties of the matrices and of related structures associated with a graph translate into geometric and combinatorial properties of the graph itself. As can be easily understood, the definition of the map F assumes a fundamental relevance when choosing the matrix to be associated with the graph and, thus, when assigning an interpretation to a problem where graph occurs.

For instance, the adjacence matrix of a simple undirected graph G has been recently interpreted as a pairing $\mathfrak{P}[G]$ on its vertex set [10]. In such a case, the corresponding pairing relation may be explicitly rewritten as follows: $Y \twoheadleftarrow_G X$ if and only if the condition $(v \sim x \iff v' \sim x \ \forall x \in X)$ implies that $(v \sim y \iff v' \sim y \ \forall y \in Y)$, for each $v, v' \in V(G)$, and where $\sim$ denotes the usual adjacence relation.

From an intuitive standpoint, the relation $\twoheadleftarrow_G$ describes how local symmetry transmits with respect to the variation of the vertex subset inducing it, i.e. whenever two vertices are symmetric with respect to a vertex subset X, then they must also be symmetric with respect to the vertex subset Y. So, one may ask for how the addition or the deletion of vertices from a given vertex subset yields a change in the induced local symmetries. In this sense, the behavior of the minimal vertex subset describing the same information about local symmetry as X becomes fundamental. As a consequence, when the adjacence matrix of a graph induces an attractive pairing, the set system $\mathcal{N}_G$ forms a matroid on $V(G)$ [26,43,44] and this provides another way to associate a matroid with a graph, different from the usual construction of the graphic matroid [34].

In addition, when G is attractive, the reducts of the vertex set $V(G)$ are exactly the bases of the matroid $\mathcal{N}_G$. Let us point out that reducts admit alternative interpretations when changing the matrix associated with the graph: for instance, when we consider the pairing $\mathfrak{P}[G, d]$ induced by the distance matrix of a graph, the $\mathfrak{P}[G, d]$-reducts of $V(G)$ are exactly the *resolvent subsets*, that are fundamental to compute the *metric dimension* of the graph [27].

One non-trivial example of a graph whose adjacence matrix induces an attractive pairing is the Petersen graph. Starting from the previous result, an interesting problem to be considered for further and future research concerns the characterization of the structural properties of all those graphs whose adjacence matrix induces an attractive pairing, in line with what it is usually done when new specific properties, operations or quantities are introduced in graph theory [3,20].

Similarly, as quasi-attractiveness is actually weaker than attractiveness, we exhibit a family of graphs whose adjacence matrices induce a quasi-attractive but not attractive pairing, namely the *Erdös' friendship graph* F_n. These graphs have been introduced in well-known paper by Erdös et al. [23], where the authors investigated further extremal properties in graphs and, next, have been extensively analyzed in [8]. More in detail, Erdös' friendship graphs arise as a solution to a problem which is classically exposed in an informative manner as follows: *if a group of people has the property that every pair of people has exactly one friend in common, there exists a person who is friend to all the others?*

To conclude, again as a consequence of the result of Section 6, we want to emphasize that also in this case the problem of characterizing the structural properties of all those graphs whose adjacence matrix induces a quasi-attractive pairing remains open for successive research.

2. Reviews, notations and some fundamental results

Notations. In what follows, we denote by Ω a given arbitrary (even infinite) set and by $\wp(\Omega)$ [resp. $\wp_{\text{fin}}(\Omega)$] its powerset [resp. the family of its finite subsets]. If $X \in \wp_{\text{fin}}(\Omega)$, we denote by $\|X\|$ the number of elements of X. We call X a *k-subset* of Ω if $\|X\| = k$. In general, for any $k \geq 1$ we denote the family of all k-subsets of Ω by $\wp_k(\Omega)$. If $X, Y \in \wp(\Omega)$, we denote by $X \setminus Y$ the difference between X and Y and by $X \triangle Y$ their symmetric difference. We call the elements of $\mathrm{SS}(\Omega) := \wp(\wp(\Omega))$ *set systems* on Ω and those of $\mathrm{BSR}(\Omega) := \wp(\wp(\Omega) \times \wp(\Omega))$ *binary set relations* on Ω. We say that $\mathcal{F} \in \mathrm{SS}(\Omega)$:

- has *uniform cardinality* if it is non-empty and all its members have the same cardinality;
- is a *Moore system* on Ω if $\Omega \in \mathcal{F}$ and $\bigcap \mathcal{F}' \in \mathcal{F}$ for every $\mathcal{F}' \subseteq \mathcal{F}$. We use the symbol $\mathrm{MSY}(\Omega)$ to denote the collection of all the Moore set systems on Ω;
- is an *abstract simplicial complex* on Ω if $X \in \mathcal{F}$ and $Y \subseteq X \implies Y \in \mathcal{F}$. We use the symbol $\mathrm{ASC}(\Omega)$ on Ω to denote the collection of all the abstract simplicial complexes on Ω;
- is a *finitary simplicial complex* if it is an abstract simplicial complex on Ω such that $X \in \mathcal{F}$ whenever $X \in \wp(\Omega)$ and $F \in \mathcal{F}$ for any $F \in \wp_{\text{fin}}(X)$. A finitary simplicial complex is called *closable* if there exists $Y' \in \wp(Y \setminus X)$ such that $\|X'\| = \|Y'\|$ and $X' \cup (Y \setminus Y') \in \mathrm{Max}(\mathcal{F})$ whenever $X, Y \in \mathrm{Max}(\mathcal{F})$ and $X' \in \wp(X \setminus Y)$. We use the symbol $\mathrm{CFSC}(\Omega)$ to denote the collection of all the closable finitary simplicial complexes on Ω;
- is *exchangeable* if $\forall X, Y \in \mathcal{F}$, $\forall x \in X \setminus Y$ $[\exists y \in Y \setminus X \ (X \triangle \{x, y\} \in \mathcal{F})]$.

When Ω is a finite set, we say that an abstract simplicial complex $\mathcal{F}$ on Ω is a *matroid* on Ω if

- for any $X, Y \in \mathcal{F}$ such that $\|X\| = \|Y\| + 1$, there exists $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{F}$.

A *set operator* on Ω is a map $\sigma : \wp(\Omega) \longrightarrow \wp(\Omega)$, and we denote by $\mathrm{OP}(\Omega)$ the family of all the set operators on Ω. We say that $\sigma \in \mathrm{OP}(\Omega)$ is a *closure operator* if, for any $X, Y \in \wp(\Omega)$, we have that $X \subseteq \sigma(X)$, that $\sigma(\sigma(X)) = \sigma(X)$ and that $Y \subseteq X \implies \sigma(Y) \subseteq \sigma(X)$. We denote by $\mathrm{COP}(\Omega)$ the family of all the closure operators on Ω. We say that $\sigma \in \mathrm{OP}(\Omega)$ is:

- *algebraic* if $\sigma(X) = \bigcup\{\sigma(F) \mid F \in \wp_{\text{fin}}(X)\}$, for any $X \in \wp(\Omega)$;
- *symmetrizing* if whenever $Z \in \wp(\Omega)$, $x, y \in \Omega$ and $x \in \sigma(Z \cup \{y\}) \setminus \sigma(Z)$, then $y \in \sigma(Z \cup x)$.

We respectively denote by $\mathrm{ACOP}(\Omega)$ and $\mathrm{ASCOP}(\Omega)$ the collections of all the algebraic closure operators and of all the symmetrizing algebraic closure operators.

Posets and Lattices. A *poset* is a pair $\mathbb{P} = (\Omega, \leq)$, where Ω is a set and $\leq$ is a binary, reflexive, antisymmetric and transitive relation on Ω. Let $\mathbb{P}$ be a given poset and $x, y \in \Omega$. We call *upset* of x the subset $(x)^{\uparrow}_{\Omega} \coloneqq \{z \in \Omega \mid x \leq z\}$. We write $x < y$ if $x \leq y$ and $x \neq y$. Moreover we use the symbol $x \| y$ to say that x and y are two non-comparable elements in $\mathbb{P}$. We say that y *covers* x (or that x is a *co-cover* of y), denoted by $x \lessdot y$, if $x < y$ and there exists no element $z \in \Omega$ such that $x < z < y$. We denote by $(y)^{\downarrow,\lessdot}_{\Omega}$ the family of all the co-covers of y in Ω. We set $\mathcal{I}_{\Omega}(y) \coloneqq \{z \in (y)^{\downarrow,\lessdot}_{\Omega} \mid z' \in \Omega \text{ and } z' < y \implies z' \leq z\}$ and $\mathcal{I}(\Omega) \coloneqq \{x \in \Omega \mid \mathcal{I}_{\Omega}(x) \neq \varnothing\}$. Clearly, if $\mathcal{I}_{\Omega}(x) \neq \varnothing$, it contains only one element. We respectively denote the families of the minimal and of the maximal elements in $\mathbb{P}$ by $\mathrm{Min}(P)$ and $\mathrm{Max}(P)$.

The Fundamental Cryptomorphisms. In the present subsection we give the notion of sub-bijection in order to catch the non-formalized concept of *cryptomorphism* [4] when the involved structures are axiomatized by means of set systems, binary set relations or set operators.

Definition 2.1. Let Ω be a fixed ground set. We call any map $\beta : \mathscr{A} \longrightarrow \mathscr{B}$ a *linking map* if $\mathscr{A}, \mathscr{B} \in \{\mathrm{SS}(\Omega), \mathrm{BSR}(\Omega), \mathrm{OP}(\Omega)\}$. Moreover, if $\beta : \mathscr{A} \longrightarrow \mathscr{B}$ and $\gamma : \mathscr{B} \longrightarrow \mathscr{A}$ are two linking maps, $\mathscr{A}' \subseteq \mathscr{A}$ and $\mathscr{B}' \subseteq \mathscr{B}$, we say that a formal writing of the type

$$\begin{tikzcd}
(\mathscr{A} \mid \mathscr{A}') \arrow[r, shift left, "\beta"] & (\mathscr{B} \mid \mathscr{B}') \arrow[l, shift left, "\gamma"]
\end{tikzcd}$$

is a *sub-bijection* if:

(i) $\beta(\mathcal{A}) \in \mathscr{B}'$ for all $\mathcal{A} \in \mathscr{A}'$ and $\gamma(\mathcal{B}) \in \mathscr{A}'$ for all $\mathcal{B} \in \mathscr{B}'$;
(ii) $\gamma(\beta(\mathcal{A})) = \mathcal{A}$ for all $\mathcal{A} \in \mathscr{A}'$ and $\beta(\gamma(\mathcal{B})) = \mathcal{B}$ for all $\mathcal{B} \in \mathscr{B}'$.

In the next result we use the formalism of sub-bijections to express the classical Birkhoff cryptomorphism between Moore set systems and closure operators [16]. Moreover, closure operators turn out to be cryptomorphic (i.e. in sub-bijection) to the so-called *dependence set relations* [41]. To this end, first recall that a binary set relation $\mathcal{D} \in \mathrm{BSR}(\Omega)$ is said a *dependence set relation* on Ω if:

(D1) $Y \subseteq X \implies (Y, X) \in \mathcal{D}$;
(D2) $(Z, Y), (Y, X) \in \mathcal{D} \implies (Z, X) \in \mathcal{D}$;
(D3) $(Y, X) \in \mathcal{D} \iff \forall y \in Y \ [(\{y\}, X) \in \mathcal{D}]$;

for any $X, Y, Z \in \wp(\Omega)$. We denote by $\mathrm{DR}(\Omega)$ the family of all the dependence set relations on Ω.

Theorem 2.2. *[11] In the diagram below*

$$\begin{tikzcd}
(\mathrm{SS}(\Omega) \mid \mathrm{MSY}(\Omega)) \arrow[r, shift left, "\mathrm{Cl}"] & (\mathrm{OP}(\Omega) \mid \mathrm{COP}(\Omega)) \arrow[l, shift left, "\mathrm{Fix}"] \arrow[r, shift left, "\Phi"] & (\mathrm{BSR}(\Omega) \mid \mathrm{DR}(\Omega))) \arrow[l, shift left, "\Psi"]
\end{tikzcd}$$

we get three corresponding sub-bijections, where:

- $\Phi : \sigma \in \mathrm{OP}(\Omega) \mapsto \Phi(\sigma) := \{(Z, W) \in \wp(\Omega) \times \wp(\Omega) \mid Z \subseteq \sigma(W)\} \in \mathrm{BSR}(\Omega)$*;*
- $\Psi : \mathcal{R} \in \mathrm{BSR}(\Omega) \mapsto \Psi_{\mathcal{R}} \in \mathrm{OP}(\Omega)$*, where* $\Psi_{\mathcal{R}}(X) := \bigcup\{Y \in \wp(\Omega) \mid (Y, X) \in \mathcal{R}\}]$ *for any* $X \in \wp(\Omega)$*;*
- $\mathrm{Cl} : \mathcal{F} \in \mathrm{SS}(\Omega) \mapsto \mathrm{Cl}_{\mathcal{F}} \in \mathrm{OP}(\Omega)$*, where* $\mathrm{Cl}_{\mathcal{F}}(X) := \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}$ *for any* $X \in \wp(\Omega)$*;*
- $\mathrm{Fix} : \sigma \in \mathrm{OP}(\Omega) \mapsto \mathrm{Fix}(\sigma) := \{X \in \wp(\Omega) \mid \sigma(X) = X\} \in \mathrm{SS}(\Omega)$*.*

At this point we exhibit another fundamental sub-bijection involving specific kinds of set operators and set systems. To this regard, we need to introduce the following collections of set operators.

Definition 2.3. We call *finitary simplicial operator* on Ω any $\sigma \in \mathrm{OP}(\Omega)$ such that:

- $\sigma(X) \subseteq X$ for any $X \in \wp(\Omega)$;
- if $X \subseteq Y$, then $\sigma(X) \supseteq X \cap \sigma(Y)$;
- $y \in \Omega \setminus [X \cup \sigma(X \cup \{y\})] \implies \exists F \in \wp_{\mathrm{fin}}(X)\ (y \in \Omega \setminus \sigma(F \cup \{y\}))$;
- if $y \in \Omega \setminus [X \cup \sigma(X \cup \{y\})]$ and $z \in \sigma(X \cup \{z\}) \setminus X$, then $z \in \sigma(X \cup \{y, z\})$;

We denote by $\mathrm{FSO}(\Omega)$ the family of all the finitary simplicial operators on Ω. Moreover, we say that a finitary simplicial operator is *normal* if it also satisfies the following condition:

- if $X \in \wp(\Omega)$ and $y \in \Omega$ are such that $y \in \sigma(X \cup \{y\})$, then $\sigma(X) \cup \{y\} \subseteq \sigma(X \cup \{y\})$.

We denote by $\mathrm{NFSO}(\Omega)$ the family of all the normal finitary simplicial operators on Ω.

Theorem 2.4. *[15] In the diagram below*

$$\begin{tikzcd}
(\mathrm{OP}(\Omega) \mid \mathrm{FSO}(\Omega)) \arrow[r, shift left, "\varphi"] & (\mathrm{OP}(\Omega) \mid \mathrm{ACOP}(\Omega)) \arrow[l, shift left, "\zeta"] & \\
(\mathrm{OP}(\Omega) \mid \mathrm{NFSO}(\Omega)) \arrow[u, hook] \arrow[r, shift left, "\varphi"] & (\mathrm{OP}(\Omega) \mid \mathrm{ASCOP}(\Omega)) \arrow[u, hook] \arrow[l, shift left, "\zeta"] \arrow[r, shift left, "\delta"] & (\mathrm{SS}(\Omega) \mid \mathrm{CFSC}(\Omega)) \arrow[l, shift left, "\xi"]
\end{tikzcd}$$

we get three corresponding sub-bijection, where the vertical arrows are inclusions and:

- $\delta : \sigma \in \mathrm{OP}(\Omega) \mapsto \delta(\sigma) := \{X \in \wp(\Omega) \mid \forall x \in X\, [x \notin \sigma(X \setminus \{x\}])\} \in \mathrm{SS}(\Omega)$*;*
- $\xi : \mathcal{F} \in \mathrm{SS}(\Omega) \mapsto \xi_{\mathcal{F}} \in \mathrm{OP}(\Omega)$*, with*

$$\xi_{\mathcal{F}}(X) := \begin{cases} X \cup \{x \in \Omega \setminus X \mid X \cup \{x\} \notin \mathcal{F}\} & \text{if } X \in \mathcal{F} \\ \\ \bigcup\{\xi_{\mathcal{F}}(Y) \mid Y \in \mathcal{F} \cap \wp(X)\} & \text{if } X \in \wp(\Omega) \setminus \mathcal{F} \end{cases}$$

  for any $X \in \wp(\Omega)$*;*
- $\varphi : \sigma \in \mathrm{OP}(\Omega) \mapsto \varphi_\sigma \in \mathrm{OP}(\Omega)$*, where* $\varphi_\sigma(X) := X \cup \{y \in \Omega \setminus X \mid y \notin \sigma(X \cup \{y\})\}$ *for any* $X \in \wp(\Omega)$*;*
- $\zeta : \sigma \in \mathrm{OP}(\Omega) \mapsto \zeta_\sigma \in \mathrm{OP}(\Omega)$*, where* $\zeta_\sigma(X) := \{x \in X \mid x \notin \sigma(X \setminus \{x\})\}$*, for any* $X \in \wp(\Omega)$*.*

Remark 2.5. In the diagram occurring in the statement of Theorem 2.4, one might expect the existence of some suitable subfamily of abstract simplicial complexes in the upper right corner that is related to $\mathrm{ACOP}(\Omega)$ through a sub-bijection. Actually, such a family cannot be found: to be convinced of this once and for all, assume Ω to be a finite ground set, so that the collection of algebraic closure operators on Ω consists of all the closure operators on Ω. If there would exist a sub-bijection between $\mathrm{ACOP}(\Omega)$ and some subcollection of $\mathrm{ASC}(\Omega)$, then such a sub-bijection should hold for our fixed finite set Ω. Now, notice that any abstract simplicial complex on Ω becomes a Moore set system after adding Ω and, moreover, there also exist Moore set systems that are not of the previous form. In other terms, the number of closure operators on Ω is strictly greater than that of abstract simplicial complexes for large $\|\Omega\|$. This proves the non-existence of a sub-bijection involving closure operators and subcollections of abstract simplicial complexes on a finite ground set Ω.

Pairings. We call *pairing* on Ω a triple $\mathfrak{P} = (U, F, \Lambda)$, where U, Λ are non-empty sets and $F : U \times \Omega \longrightarrow \Lambda$ is a map having domain $U \times \Omega$ and codomain Λ. Let $\mathrm{PR}(\Omega)$ denote the family of all pairings on Ω. Fix an arbitrary $\mathfrak{P} = (U, F, \Lambda) \in \mathrm{PR}(\Omega)$. For any $X \in \wp(\Omega)$, we define the equivalence relation $\equiv_X$ on U, that we call X-*symmetry relation* (we refer to [10] for the reasons motivating our terminology), as follows

$$\forall u, u' \in U \; [u \equiv_X u' :\Longleftrightarrow F(u, x) = F(u', x) \;\; \forall\, x \in X]$$

Let $[u]_X$ the equivalence class of u with respect to $\equiv_X$ and $\pi_{\mathfrak{P}}(X) := \{[u]_X \mid u \in U\}$. Using the equivalence relation $\equiv_X$, we get a binary relation $\twoheadleftarrow_{\mathfrak{P}}$ on $\wp(\Omega)$ by setting, for each $X, Y \in \wp(\Omega)$,

$$X \twoheadleftarrow_{\mathfrak{P}} Y :\Longleftrightarrow (\forall u, u' \in U \; [u \equiv_Y u' \implies u \equiv_X u'])$$

We call $\twoheadleftarrow_{\mathfrak{P}}$ a *pairing relation* on Ω and we have $\twoheadleftarrow_{\mathfrak{P}} \in \mathrm{DR}(\Omega)$. Furthermore, for each $X, Y \in \wp(\Omega)$ we set

$$X \approx_{\mathfrak{P}} Y :\iff X \twoheadleftarrow_{\mathfrak{P}} Y \text{ and } Y \twoheadleftarrow_{\mathfrak{P}} X \iff \forall u, u' \in U \ (u \equiv_X u' \iff u \equiv_Y u')$$

The relation $\approx_{\mathfrak{P}}$ is an equivalence relation and we denote by $[X]_{\approx_{\mathfrak{P}}}$ the equivalence class of $X \in \wp(\Omega)$.

At this point, let us provide the notion of $\mathfrak{P}$-reduct of a subset X.

Definition 2.6. Let $Y \subseteq X \in \wp(\Omega)$. We say that Y is a $\mathfrak{P}$-*reduct* of X if $X \twoheadleftarrow_{\mathfrak{P}} Y$ and, for every $y \in Y$ $[X \not\twoheadleftarrow_{\mathfrak{P}} Y \setminus \{y\}]$. We denote by $\mathcal{R}_{\mathfrak{P}}(X)$ the set of all $\mathfrak{P}$-reducts of X. If $X = \Omega$, we write $\mathcal{R}_{\mathfrak{P}}$ instead of $\mathcal{R}_{\mathfrak{P}}(\Omega)$.

In the next result we recall some properties of the relation $\twoheadleftarrow_{\mathfrak{P}}$ and of two associated set systems and of the family of the X-reducts, for each subset $X \in \wp(\Omega)$.

Proposition 2.7. [9] *Let $X \in \wp(\Omega)$. The following conditions hold:*

(i) *the subset family $[X]_{\approx_{\mathfrak{P}}}$ is closed under unions and its maximum $M_{\mathfrak{P}}(X) := \bigcup [X]_{\approx_{\mathfrak{P}}}$ may be equivalently expressed as follows:*

$$\begin{aligned} M_{\mathfrak{P}}(X) &= \{z \in \Omega \mid \{z\} \twoheadleftarrow_{\mathfrak{P}} X\} = \{z \in \Omega \mid X \cup \{z\} \approx_{\mathfrak{P}} X\} \\ &= \{z \in \Omega \mid u \equiv_X u' \implies F(u,z) = F(u',z)\}; \end{aligned}$$

(ii) *the set operator $M_{\mathfrak{P}} : W \in \wp(\Omega) \mapsto M_{\mathfrak{P}}(W) \in \wp(\Omega)$ is a closure operator on Ω, the set system $\mathcal{M}_{\mathfrak{P}} := \{W \in \wp(\Omega) \mid M_{\mathfrak{P}}(W) = W\} \in \mathrm{MSY}(\Omega)$ and $\mathbb{M}(\mathfrak{P}) := (\mathcal{M}_{\mathfrak{P}}, \subseteq)$ is a complete lattice;*

(iii) *the set system $\mathcal{N}_{\mathfrak{P}} := \bigcup \{\mathrm{Min}([X]_{\approx_{\mathfrak{P}}}) \mid X \in \mathcal{M}_{\mathfrak{P}}\} = \{X \in \wp(\Omega) \mid \forall x \in X \ [x \in \Omega \setminus M_{\mathfrak{P}}(X \setminus \{x\})]\}$ is an abstract simplicial complex on Ω;*

(iv) $\mathrm{Min}([X]_{\approx_{\mathfrak{P}}}) = \mathcal{R}_{\mathfrak{P}}(X) \subseteq \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$.

Remark 2.8. (i): Readers may consult results of Sections 3, 6 and 7 of [9] to get proofs for the claims of Proposition 2.7.

(ii): For any $X \in \wp(\Omega)$ and $Y \subseteq X$, it results that $Y \in \mathcal{R}_{\mathfrak{P}}(X)$ if and only if the conditions below hold:

(a) $\pi_{\mathfrak{P}}(X) = \pi_{\mathfrak{P}}(Y)$;

(b) $\pi_{\mathfrak{P}}(X) \neq \pi_{\mathfrak{P}}(Y \setminus \{y\})$ for every $y \in Y$.

We call the members of $\mathcal{M}_{\mathfrak{P}}$ and of $\mathcal{N}_{\mathfrak{P}}$ the *maximum partitioners* and the *minimal partitioners* of $\mathfrak{P}$, respectively [12]. In the next fundamental result, we see that any dependence set relation [and, by Theorem 2.2, resp. any Moore set system, any closure operator] on Ω may be represented as the $\twoheadleftarrow_{\mathfrak{P}}$ [resp. $\mathcal{M}_{\mathfrak{P}}$, $M_{\mathfrak{P}}$] of some pairing $\mathfrak{P}$ on Ω. This result is fundamental as it justifies the investigations of the next sections. As a matter of fact, in Section 3 we analyze all those pairings (on arbitrary - even infinite ground sets) whose closure operator is symmetrizing, while in Section 4 we broaden our analysis to those pairings satisfying a weaker condition.

Theorem 2.9. *[11,12] Let Ω be an arbitrary ground set (even infinite). The following identities hold:*

$$\mathrm{MSY}(\Omega) = \{\mathcal{M}_{\mathfrak{P}} \mid \mathfrak{P} \in \mathrm{PR}(\Omega)\}, \quad \mathrm{CLOP}(\Omega) = \{M_{\mathfrak{P}} \mid \mathfrak{P} \in \mathrm{PR}(\Omega)\},$$
$$\mathrm{DR}(\Omega) = \{\leftarrow_{\mathfrak{P}} \mid \mathfrak{P} \in \mathrm{PR}(\Omega)\}$$

Let us see what happens for simplicial complexes. In general, not every simplicial complex may be represented as the family $\mathcal{N}_{\mathfrak{P}}$ of some pairing $\mathfrak{P}$ on Ω

Example 2.10. Let $\Omega = \{1, 2, 3, 4, 5\}$ and

$$\mathcal{F} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}\} \in \mathrm{ASC}(\Omega)$$

be ordered lexicographically (and expressed in string notation). Then $\mathcal{F}$ cannot be represented as the set system $\mathcal{N}_{\mathfrak{P}}$ of some pairing $\mathfrak{P}$. Indeed, note that

$$\pi_{\mathfrak{P}}(\{i\}) \parallel \pi_{\mathfrak{P}}(\{i+1\}) \tag{1}$$

where the indices are taken modulo 5. In fact, fix two elements, say 1 and 2 (the other allowable cases can be treated in a similar way). If $\pi_{\mathfrak{P}}(\{1\}) = \pi_{\mathfrak{P}}(\{2\})$, then $\{1, 2\} \notin \mathcal{N}_{\mathfrak{P}}$; moreover, if $\pi_{\mathfrak{P}}(\{1\}) \prec \pi_{\mathfrak{P}}(\{2\})$ ($\prec$ denotes the usual partial order between set partitions), we easily get $M_{\mathfrak{P}}(\{2\}) \subsetneq M_{\mathfrak{P}}(\{1\})$ and this is impossible since otherwise the subset $\{1, 2\}$ does not belong to $\mathcal{N}_{\mathfrak{P}}$. The case $\pi_{\mathfrak{P}}(\{2\}) \prec \pi_{\mathfrak{P}}(\{1\})$ may be analyzed in the same way.

Let us see that three distinct points cannot induce the same the set partition. In fact, we will always find two consecutive points within the same class, say i and $i + 1$, so $\{i, i + 1\} \approx_{\mathfrak{P}} \{i\}$ and, hence, $\{i, i + 1\} \notin \mathcal{N}_{\mathfrak{P}}$.

Assume now that two points, say 1 and 3 (the other allowable cases can be treated in a similar way), induce the same set partition, i.e. $\pi_{\mathfrak{P}}(\{1\}) = \pi_{\mathfrak{P}}(\{3\})$. By (1), we have $[\{1\}]_{\approx_{\mathfrak{P}}} = \{\{1\}, \{3\}, \{1, 3\}\}$.

At this point, the following cases may occur:

(*i*) $\{2\} \approx_{\mathfrak{P}} \{4\}$ and $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{5\})$;
(*ii*) $\{2\} \approx_{\mathfrak{P}} \{5\}$ and $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{4\})$;
(*iii*) $\pi_{\mathfrak{P}}(\{2\}) \prec \pi_{\mathfrak{P}}(\{4\})$ (or $\pi_{\mathfrak{P}}(\{4\}) \prec \pi_{\mathfrak{P}}(\{2\})$), $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{4\})$ and $\pi_{\mathfrak{P}}(\{4\}) \| \pi_{\mathfrak{P}}(\{5\})$;
(*iv*) $\pi_{\mathfrak{P}}(\{2\}) \prec \pi_{\mathfrak{P}}(\{5\})$ (or $\pi_{\mathfrak{P}}(\{5\}) \prec \pi_{\mathfrak{P}}(\{2\})$), $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{5\})$ and $\pi_{\mathfrak{P}}(\{4\}) \| \pi_{\mathfrak{P}}(\{5\})$;
(*v*) $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{4\})$, $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{5\})$ and $\pi_{\mathfrak{P}}(\{4\}) \| \pi_{\mathfrak{P}}(\{5\})$.

It can be easily checked that in all the above cases the subset $\{1,4\} \in \mathcal{F}$.
On the other hand, assume that $\pi_{\mathfrak{P}}(\{1\}) \prec \pi_{\mathfrak{P}}(\{3\})$ (the case $\pi_{\mathfrak{P}}(\{3\}) \prec \pi_{\mathfrak{P}}(\{1\})$ is the same). Clearly, we cannot have $\pi_{\mathfrak{P}}(\{1\}) \prec \pi_{\mathfrak{P}}(\{3\}) \prec \pi_{\mathfrak{P}}(\{1,3\})$, otherwise $\{1,3\} \in \mathcal{N}_{\mathfrak{P}}$. Assume now that $\{5\} \approx_{\mathfrak{P}} \{3\}$, then $[\{3\}]_{\approx_{\mathfrak{P}}} = \{\{3\},\{5\},\{3,5\}\}$ and the global symmetry class of $\{1\}$ contains $\{1,5\}$, so $\{1,5\} \notin \mathcal{N}_{\mathfrak{P}}$.
The last case to be investigated is $\pi_{\mathfrak{P}}(\{1\}) \prec \pi_{\mathfrak{P}}(\{3\})$, $[\{3\}]_{\approx_{\mathfrak{P}}} = \{\{3\}\}$ and $[\{1\}]_{\approx_{\mathfrak{P}}} = \{\{1\},\{1,3\}\}$. By (1), $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{3\})$. So, if $[\{2,3\}]_{\approx_{\mathfrak{P}}} = \{\{2,3\}\}$, then $\{1,2,3\} \in \mathcal{N}_{\mathfrak{P}}$. On the other hand, it could happen that $[\{2,3\}]_{\approx_{\mathfrak{P}}} = \{\{1,2\},\{2,3\},\{1,2,3\}\}$. However, in this case, by the above argument, we also have $\pi_{\mathfrak{P}}(\{2\}) \| \pi_{\mathfrak{P}}(\{5\})$, so we would have $\{2,5\} \in \mathcal{N}_{\mathfrak{P}}$, that is again an absurd.
Finally, assume that the set partitions of the singletons are all non comparable. If the global symmetry collections of each element i contain only the subset $\{i\}$, then $\{i,i+1\}$ must cover both $\{i\}$ and $\{i+1\}$. Note that $[\{i,i+1\}]_{\approx_{\mathfrak{P}}} = \{\{i,i+1\}\}$ otherwise, within such an equivalence class, we would always find a minimal element not belonging to $\mathcal{F}$. Furthermore, the case in which the collections are respectively $[\{i\}]_{\approx_{\mathfrak{P}}} = \{\{i\},\{i+2\}\}$ (where the indices are taken modulo 5) cannot occur, otherwise that the set partitions of two singletons are comparable, in contrast with our assumptions.

Let us now prove two representation results for suitable collections of abstract simplicial complexes. First we prove that any closable finitary simplicial complex on Ω may be represented as the collection $\mathcal{N}_{\mathfrak{P}}$ of some pairing on Ω as a consequence of the Representation Theorem for closure operators 2.9.

Theorem 2.11. *Let Ω be an arbitrary ground set. Then the correspondence $\mathfrak{P} \in \mathrm{PR}(\Omega) \mapsto \mathcal{N}_{\mathfrak{P}} \in \mathrm{CFSC}(\Omega)$ is surjective.*

Proof. Let $\mathcal{F} \in \mathrm{CFSC}(\Omega)$. By Theorem 2.4 we can consider the associated algebraic, symmetrizing closure operator $\xi_{\mathcal{F}}$. At this point, Theorem 2.9 ensures the existence of some pairing $\mathfrak{P} \in \mathrm{PR}(\Omega)$ such that $M_{\mathfrak{P}} = \xi_{\mathcal{F}}$. To conclude our proof, we claim that $\mathcal{N}_{\mathfrak{P}} = \mathcal{F}$. By part (*iii*) of Proposition 2.7 and by Theorem 2.4 it results that $\mathcal{N}_{\mathfrak{P}} = \{X \in \wp(\Omega) \mid \forall x \in X\ [x \in \Omega \setminus M_{\mathfrak{P}}(X \setminus \{x\})]\} = \delta(M_{\mathfrak{P}}) = \delta(\xi_{\mathcal{F}}) = \mathcal{F}$. □

In the next result, we provide a sufficient condition on the simplicial complex ensuring its representability in terms of the set system $\mathcal{N}_{\mathfrak{P}}$, for some $\mathfrak{P} \in \mathrm{PR}(\Omega)$. To this regard, set

$$\mathcal{F}^{\top} := \{X \in \wp(\Omega) \setminus \mathcal{F} \mid [\exists \mathcal{G} \subseteq \mathcal{F}\ (X = \bigcup \mathcal{G})] \text{ and } [\forall H \in \mathrm{Max}(\mathcal{F})\ (H \nsubseteq X)]\}$$

Definition 2.12. Let $\mathcal{F} \in \mathrm{ASC}(\Omega)$. We say that $\mathcal{F}$ is *strict* if $\mathcal{F}^{\top} = \varnothing$.

We are now ready to show that strict simplicial complexes on finite ground sets may be represented through pairings.

Theorem 2.13. *Let Ω be a finite set and $\mathcal{F} \in \mathrm{ASC}(\Omega)$ be strict. Then there exists a pairing $\mathfrak{P} \in \mathrm{PR}(\Omega)$ such that $\mathcal{N}_{\mathfrak{P}} = \mathcal{F}$.*

Proof. Let $\mathcal{F} \in \mathrm{ASC}(\Omega)$. We can consider on $\mathcal{F}$ a well-order in such a way that $G_0 := \varnothing$ is the first element and the maximal elements of $\mathcal{F}$ are listed last. Set $U := \mathcal{F} \setminus \mathrm{Max}(\mathcal{F}) = \{G_0, G_1, \dots, G_m\}$ for some integer $m \in \mathbb{N}$, and define the map $F : U \times \Omega \to \mathbb{N}$ as follows:

$$F(G_i, z) := \begin{cases} 0 & \text{if } i = 0 \text{ or } z \notin \bigcup \mathcal{F} \\ F(G_{i-1}, z) & \text{if } z \in G_i, \\ F(G_{i-1}, z) + 1 & \text{if } z \notin G_i. \end{cases} \tag{2}$$

Consider then the pairing $\mathfrak{P} := (U, F, \mathbb{N}) \in \mathrm{PR}(\Omega)$. We will show that $\mathcal{N}_{\mathfrak{P}} = \mathcal{F}$. To this regard, let us first prove that

$$G \in \mathrm{Max}(\mathcal{F}) \text{ and } F \in \mathcal{F} \setminus \mathrm{Max}(\mathcal{F}) \implies \pi_{\mathfrak{P}}(G) \neq \pi_{\mathfrak{P}}(F) \tag{3}$$

We split the proof in the following two parts:

- $\pi_{\mathfrak{P}}(G) = G_1|G_2|\dots|G_m$

Suppose by contradiction the existence of $G_i, G_j \in U$ such that $G_i \equiv_G G_j$. This is equivalent to say that $F(G_i, z) = F(G_j, z)$ for every $z \in G$. Without loss of generality, suppose $i < j$. By (2) we must necessarily have $F(G_j, z) = F(G_{j-1}, z)$, so that $z \in G_j$ for every $z \in G$. This implies that $G \subseteq G_j$, contradicting the maximality of G.

- There exist $G_i, G_j \in U$, with $i \neq j$, such that $G_i \equiv_F G_j$.

Since $F \in \mathcal{F} \setminus \mathrm{Max}(\mathcal{F})$, by our choice of U there must exist an index $j = 0, \dots, m$ for which $G_j = F$. Now, by (2) we get $F(G_{j-1}, z) = F(G_j, z)$ for any $z \in F$. This shows that $G_i \equiv_F G_j$.

As a consequence, we deduce that $\pi_{\mathfrak{P}}(G) \neq \pi_{\mathfrak{P}}(F)$ whenever $G \in \mathrm{Max}(\mathcal{F})$ and $F \in \mathcal{F} \setminus \mathrm{Max}(\mathcal{F})$. This proves (3).

At this point, we can show the equality $\mathcal{N}_{\mathfrak{P}} = \mathcal{F}$. In order to show the inclusion $\mathcal{F} \subseteq \mathcal{N}_{\mathfrak{P}}$, it suffices to show that $\mathrm{Max}(\mathcal{F}) \subseteq \mathcal{N}_{\mathfrak{P}}$. Take then $G \in \mathrm{Max}(\mathcal{F})$ and suppose by contradiction that $G \notin \mathcal{N}_{\mathfrak{P}}$. Then there exists $B \subsetneq G$ such that $\pi_{\mathfrak{P}}(G) = \pi_{\mathfrak{P}}(\mathrm{B})$. As $\mathcal{F}$ is an abstract simplicial complex, it follows that $B \in \mathcal{F} \setminus \mathrm{Max}(\mathcal{F})$. By what we proved above, it results that $\pi_{\mathfrak{P}}(G) \neq \pi_{\mathfrak{P}}(\mathrm{B})$, contradicting our choice of B. Therefore, we deduce that $F \in \mathcal{N}_{\mathfrak{P}}$, whence $\mathrm{Max}(\mathcal{F}) \subseteq \mathcal{N}_{\mathfrak{P}}$.

Conversely, let $A \notin \mathcal{F}$. We claim that $A \notin \mathcal{N}_{\mathfrak{P}}$. Since $\mathcal{F}^{\top} = \varnothing$, we have to distinguish two possible cases.

(a) For every $\mathcal{G} \subseteq \mathcal{F}$ we have $A \neq \bigcup \mathcal{G}$.

	a_1	a_2	a_3	a_4	a_5
G_0	$0 =$	$0 \downarrow$	$0 \downarrow$	$0 \downarrow$	$0 \downarrow$
G_1	$a_1 \in G_1,\ 0 \downarrow$	$a_2 \notin G_1,\ 1 =$	$a_3 \notin G_1,\ \to 1 \downarrow$	$a_4 \notin G_1,\ \to 1 \downarrow$	$a_5 \notin G_1,\ \to 1 \downarrow$
G_2	$a_1 \notin G_2,\ \to 1 \downarrow$	$a_2 \in G_2,\ 1 \downarrow$	$a_3 \notin G_2,\ \to 1+1 =$	$a_4 \notin G_2,\ \to 1+1 \downarrow$	$a_5 \notin G_2,\ \to 1+1 \downarrow$
G_3	$a_1 \notin G_3,\ \to 1+1 \downarrow$	$a_2 \notin G_3,\ \to 1+1 \downarrow$	$a_3 \in G_3,\ 2 \downarrow$	$a_4 \notin G_3,\ \to 2+1 =$	$a_5 \notin G_3,\ \to 2+1 \downarrow$
G_4	$a_1 \notin G_4,\ \to 2+1 \downarrow$	$a_2 \notin G_4,\ \to 2+1 \downarrow$	$a_3 \notin G_4,\ \to 2+1 \downarrow$	$a_4 \in G_4,\ 3 \downarrow$	$a_5 \notin G_4,\ \to 3+1 =$
G_5	$a_1 \notin G_5,\ 4 =$	$a_2 \notin G_5,\ \to 3+1 =$	$a_3 \notin G_5,\ \to 3+1 \downarrow$	$a_4 \notin G_5,\ \to 3+1 \downarrow$	$a_5 \in G_5,\ 4 \downarrow$
G_6	$a_1 \in G_6,\ 4 =$	$a_2 \in G_6,\ 4 \downarrow$	$a_3 \notin G_6,\ \to 4+1 =$	$a_4 \notin G_6,\ \to 4+1 \downarrow$	$a_5 \notin G_6,\ \to 4+1 \downarrow$
G_7	$a_1 \in G_7,\ 4 =$	$a_2 \notin G_7,\ \to 4+1 \downarrow$	$a_3 \in G_7,\ 5 \downarrow$	$a_4 \notin G_7,\ \to 5+1 =$	$a_5 \notin G_7,\ \to 5+1 \downarrow$
G_8	$a_1 \in G_8,\ 4 \downarrow$	$a_2 \notin G_8,\ \to 5+1 =$	$a_3 \notin G_8,\ \to 5+1 =$	$a_4 \in G_8,\ \to 6 \downarrow$	$a_5 \notin G_8,\ \to 6+1 \downarrow$
G_9	$a_1 \notin G_9,\ 5 \downarrow$	$a_2 \in G_9,\ 6 =$	$a_3 \in G_9,\ 6 \downarrow$	$a_4 \notin G_9,\ \to 6+1 =$	$a_5 \notin G_9,\ \to 7+1 \downarrow$
G_{10}	$a_1 \notin G_{10},\ \to 5+1 \downarrow$	$a_2 \in G_{10},\ 6 \downarrow$	$a_3 \notin G_{10},\ \to 7 =$	$a_4 \in G_{10},\ \to 7 =$	$a_5 \notin G_{10},\ \to 8+1 \downarrow$
G_{11}	$a_1 \notin G_{11},\ \to 7$	$a_2 \notin G_{11},\ \to 7$	$a_3 \in G_{11},\ \to 7$	$a_4 \in G_{11},\ \to 7$	$a_5 \notin G_{11},\ \to 10$

Fig. 1. A visualization of the recursive construction of the map $F_{\mathfrak{P}}$ for the abstract simplicial complex of Example 2.14.

In view of our assumption, there exists $z \in A \setminus \bigcup \mathcal{F}$. By (2) we get $\pi_{\mathfrak{P}}(\{z\}) = \pi_{\mathfrak{P}}(\emptyset) = G_0G_1 \dots G_m$, i.e. $G_i \equiv_{\{z\}} G_j$ for every $G_i, G_j \in U$. Let now $\widehat{A} := A \setminus \{z\}$. Then $\pi_{\mathfrak{P}}(\widehat{A}) = \pi_{\mathfrak{P}}(\mathrm{A})$, so that $A \notin \mathrm{Min}([\widehat{A}]_{\approx_{\mathfrak{P}}})$, that is $A \notin \mathcal{N}_{\mathfrak{P}}$.

(b) There exists $H \in \mathrm{Max}(\mathcal{F})$ such that $H \subsetneq A$.

By (2) we easily get the equality $\pi_{\mathfrak{P}}(\mathrm{A}) = \pi_{\mathfrak{P}}(H)$, whence $A \notin \mathrm{Min}([H]_{\approx_{\mathfrak{P}}})$. This shows also in the present case that $A \notin \mathcal{N}_{\mathfrak{P}}$ and the proof concludes here. □

Example 2.14. Let $\Omega = \{a_1, a_2, a_3, a_4, a_5\}$ and

$$\mathcal{F} = \{\emptyset, 1, 2, 3, 4, 5, 12, 13, 14, 23, 24, 34, 15, 25, 35, 45, 123, 124, 134, 234\} \in \mathrm{ASC}(\Omega)$$

be ordered lexicographically (and expressed in string notation on the indices of the elements of Ω) in such a way that its maximal members are listed last. It is quite easy to see that $\mathcal{F}^{\top} = \emptyset$, i.e. $\mathcal{F}$ is strict. Following the construction proposed in the proof of Theorem 2.13, we now construct a suitable pairing $\mathfrak{P}$ representing $\mathcal{F}$. The reader may see in Figs. 1 and 2 the detailed construction and the resulting pairing. Clearly, our construction gives rise to one of the pairings representing $\mathcal{F}$. The same abstract simplicial complex is represented by the *Erdos' friendship graph* F_2 (see Section 6 for details).

2.1. Graphs and pairings

We refer the reader to [20] for any general notion concerning graph theory. Let $G = (V(G), E(G))$ be a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v, v'\} \in E(G)$ and $v \nsim v'$ otherwise. We call the set $N_G(v) := \{w \in V(G) \mid v \sim w\}$ the *neighborhood* of v in G and we set

	a_1	a_2	a_3	a_4	a_5
G_0	0	0	0	0	0
G_1	0	1	1	1	1
G_2	1	1	2	2	2
G_3	2	2	2	3	3
G_4	3	3	3	3	4
G_5	4	4	4	4	4
G_6	4	4	5	5	5
G_7	4	5	5	6	6
G_8	4	6	6	6	7
G_9	5	6	6	7	8
G_{10}	6	6	7	7	9
G_{11}	7	7	7	7	10

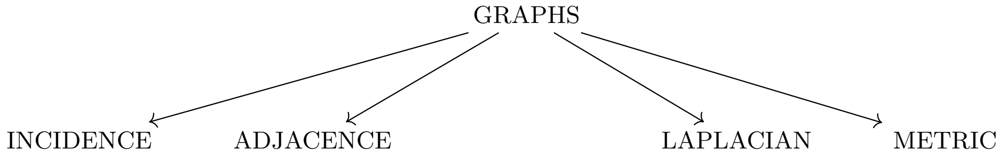
Fig. 2. The pairing $\mathfrak{P}$ representing the abstract simplicial complex of Example 2.14.

$N_G(\mathrm{A}) := \bigcup\{N_G(v) \mid v \in A\}$ for any $A \in \wp(V(G))$. We say that a graph G is *regular* if $\|N_G(v)\| = \|N_G(w)\|$ for each $v, w \in V(G)$ and, in particular, we say that G is *k-regular* if $\|N_G(v)\| = k$ for each $v \in V(G)$. Moreover, we say that G is *strongly regular with parameters* (n, k, λ, ν) if it is a k-regular graph on n vertices, every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have ν common neighbors.

Two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are said *isomorphic*, denoted by $G \cong H$, if there exists a bijection $\phi : V(G) \longrightarrow V(H)$ such that for all $v, v' \in V(G)$ it results that $\{v, v'\} \in E(G) \iff \{\phi(v), \phi(v')\} \in E(H)$. We say that $H = (V(H), E(H))$ is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $X \subseteq V(G)$, the *generated subgraph* by X in G, denoted by $G[X]$, is the graph having X as vertex set and such that, for any $v, v' \in X$, $\{v, v'\} \in E(G[X])$ if and only if $\{v, v'\} \in E(G)$.
If v and w are two distinct vertices of G and $k \geq 1$ is an integer, a *k-path* (or sometimes simply a *path*) between v and w is a graph $P_k = (V(P_k), E(P_k))$, where $V(P_k) = \{v_0, \dots, v_k\}$, $E(P_k) = \{\{, v_1\}, \dots, \{v_{k-1}, v_k\}\}$, $v_1 = v$ and $v_k = w$. In such a case, the number $k - 1$ is called the *length* of the path. In particular, any 1-path is a single vertex. We denote by $d(v, w)$ the distance between v and w, i.e. the length of any shortest path between v and w. By convention, we also assume that $d(u, u) = 0$ for any $u \in V(G)$. We say that a graph G is *connected* if for any two distinct vertices $v, w \in V(G)$ there exists a path between them. If a graph is not connected, we call the maximal connected subgraphs of G the *connected components* of G. We will denote by $P_{k_1,\dots,k_s}$ any disjoint union of s paths $P_{k_1}, \dots, P_{k_s}$, where P_{k_i} is a k_i-path, for $i = 1, \dots, s$. Below we give some classical examples of graphs with vertex set $\{v_1, \dots, v_n\}$:

- *Complete graph on n vertices.* It is denoted by K_n and $E(K_n) = \wp_2(V(K_n))$.
- *n-cycle.* It is denoted by C_n and $E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$.

There are various ways of associating a pairing with a graph. All of these assignments depend on the existence of various ways of associating a matrix with a graph. Below, we summarize some possible examples of the matrices used to investigate a graph from an algebraic perspective:



Given a graph G, for each of the previous tipologies of matrices we can consider a suitable associated pairing. For instance, we write $\mathfrak{P}[G, \mathrm{adj}]$ (or simply G) when considering the pairing induced by the adjacence matrix (that we call *adjacence pairing*), i.e. $\mathfrak{P}[G] := (V(G), F, \{0,1\}) \in \mathrm{PR}(V(G))$ of G, where

$$F(u,v) := \begin{cases} 1 & \text{if } \ u \sim v \\ 0 & \text{otherwise} \end{cases}$$

For any $X \in \wp(V(G))$, the relation $\equiv_X$ translates as

$$\forall v, v' \in V(G) \ [v \equiv_X v' \iff N_G(v) \cap X = N_G(v') \cap X]$$

On the other hand, we write $\mathfrak{P}[G, \mathrm{met}]$ when considering the pairing induced by the distance matrix (that we call *distance pairing*), i.e. $\mathfrak{P}[G, \mathrm{met}] := (V(G), d, \mathbb{N}) \in \mathrm{PR}(V(G))$ of G, where $d : V(G) \times V(G) \longrightarrow \mathbb{N}$ denotes the usual distance between vertices of G.

3. Attractive pairings

In the present section we provide the notion of attractive pairing on Ω and next analyze some basic properties of such structures. The previous notions generalize to the arbitrary case the analysis of the matroidal-like properties of the minimal partitioners of a pairing. In the first subsection, we exhibit a sufficient condition ensuring the matroidality of $\mathcal{N}_{\mathfrak{P}}$. In the second subsection, we deal with attractive pairings in general. Such a collection of pairings has been introduced in [11] and various examples coming from algebra and topology have been exhibited: for instance, the pairing $\mathfrak{P} = (\mathbb{R}^2, d, \mathbb{R}_{\geq 0})$, where $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}_{\geq 0}$ is the *post-office metric* defined by setting

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \\ \|x\| + \|y\| & \text{otherwise.} \end{cases}$$

turns out to be an attractive pairing.

In this section we refine and improve the analysis carried out in [11]. There, it has been shown that $\mathcal{R}_{\mathfrak{P}}(X)$ admits uniform cardinality when $\mathfrak{P}$ is a locally finite attractive pairing and $X \in \mathcal{M}_{\mathfrak{P}}$. This is a matroidal-like property that allows us to think of $\mathcal{R}_{\mathfrak{P}}(X)$ as a sort of bases. Thus, here we better investigate the matroidal-like properties of $\mathcal{R}_{\mathfrak{P}}(X)$, by showing that $\mathcal{R}_{\mathfrak{P}}(X)$ coincides with $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ and forms an exchangeable set system when $\mathfrak{P}$ is attractive, while when $\mathfrak{P}$ is locally finite the equality of the previous two set systems characterizes attractiveness. In particular, when $\mathfrak{P}$ is both locally finite and attractive, then the set system $\mathcal{N}_{\mathfrak{P}}$ is a closable finitary simplicial complex and, when Ω is finite, it turns out to be a matroid. We use the last result to exhibit two families of graphs inducing attractive pairings.

Given an attractive pairing $\mathfrak{P}$, we will see that the family of all $\mathfrak{P}$-reducts of each subset $X \in \wp(\Omega)$ coincides with the family of the maximal members of $\mathcal{N}_{\mathfrak{P}}$ contained in X and form an exchangeable set system. We finally demonstrate that when Ω is a finite set, the coincidence between $\mathcal{R}_{\mathfrak{P}}(X)$ and $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ becomes also a sufficient condition for attractiveness.

3.1. Matroidality on finite ground sets

Throughout the present subsection we assume Ω to be a finite set. We will detect a sufficient condition ensuring the matroidality of the family of the minimal partitioners of a pairing on Ω. To this regard, let us define the function $\mathrm{rk}_{\mathfrak{P}} : \wp(\Omega) \longrightarrow \mathbb{N}$ as follows:

$$\forall X \in \wp(\Omega)\ [\mathrm{rk}_{\mathfrak{P}}(X) := \max\{\|Z\| \mid Z \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))\}]$$

In the next result, we state some basic properties of the function $\mathrm{rk}_{\mathfrak{P}}$.

Proposition 3.1. *The function* $\mathrm{rk}_{\mathfrak{P}}$ *satisfies the following properties:*

- *(i)* $\mathrm{rk}_{\mathfrak{P}}(\varnothing) = 0$;
- *(ii)* $\mathrm{rk}_{\mathfrak{P}}(X) \leq \mathrm{rk}_{\mathfrak{P}}(Y)$ *whenever* $X \subseteq Y \in \wp(\Omega)$;
- *(iii)* $0 \leq \mathrm{rk}_{\mathfrak{P}}(X) \leq \|X\|$ *for any* $X \in \wp(\Omega)$;
- *(iv)* $\mathrm{rk}_{\mathfrak{P}}(X) \leq \mathrm{rk}_{\mathfrak{P}}(X \cup \{z\}) \leq \mathrm{rk}_{\mathfrak{P}}(X) + 1$ *for any* $X \in \wp(\Omega)$ *and* $z \in \Omega$;
- *(v)* $\mathcal{N}_{\mathfrak{P}} = \{X \in \wp(\Omega) \mid \mathrm{rk}_{\mathfrak{P}}(X) = \|X\|\}$.

Let us observe that the map $\mathrm{rk}_{\mathfrak{P}}$ behaves as the rank function of a matroid whose independent set coincides with $\mathcal{N}_{\mathfrak{P}}$ and whose bases are the elements of $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}})$. However, in general, $\mathcal{N}_{\mathfrak{P}}$ is not a matroid, even if it has many matroidal-like properties.

At this point, let us introduce the set operator $\mathrm{Drk}_{\mathfrak{P}} : \wp(\Omega) \longrightarrow \wp(\Omega)$ defined as follows:

$$\mathrm{Drk}_{\mathfrak{P}}(X) := \{z \in \Omega \mid \mathrm{rk}_{\mathfrak{P}}(X \cup \{z\}) = \mathrm{rk}_{\mathfrak{P}}(X)\}$$

In the next result we show that $\mathcal{N}_{\mathfrak{P}}$ is a matroid on Ω when the operators $\mathrm{Drk}_{\mathfrak{P}}$ and $M_{\mathfrak{P}}$ coincide.

Theorem 3.2. *If* $\mathrm{Drk}_{\mathfrak{P}}(X) = M_{\mathfrak{P}}(X)$ *for any* $X \in \wp(\Omega)$, *then* $\mathcal{N}_{\mathfrak{P}}$ *is a matroid on* Ω.

Proof. Assume that $\mathrm{Drk}_{\mathfrak{P}}(X) = M_{\mathfrak{P}}(X)$ for any $X \in \wp(\Omega)$. We now prove that $\mathrm{rk}_{\mathfrak{P}}$ is the rank function of a matroid (see Theorem 2.15 of [26]). In view of parts (i) and (iv) of Proposition 3.1, we must only prove that

$$\forall X \in \wp(\Omega),\ z, y \in \Omega \smallsetminus X\ [\mathrm{rk}_{\mathfrak{P}}(X \cup \{z\}) = \mathrm{rk}_{\mathfrak{P}}(X \cup \{y\}) \\ = \mathrm{rk}_{\mathfrak{P}}(X) \implies \mathrm{rk}_{\mathfrak{P}}(X \cup \{z\} \cup \{y\}) = \mathrm{rk}_{\mathfrak{P}}(X)]$$

Assume that $\mathrm{rk}_{\mathfrak{P}}(X \cup \{z\}) = \mathrm{rk}_{\mathfrak{P}}(X \cup \{y\}) = \mathrm{rk}_{\mathfrak{P}}(X)$, so that $z, y \in \mathrm{Drk}_{\mathfrak{P}}(X) = M_{\mathfrak{P}}(X)$. Hence, $X \cup \{z\} \cup \{y\} \approx_{\mathfrak{P}} X$. Moreover, it clearly results that $M_{\mathfrak{P}}(X \cup \{z\}) = \mathrm{Drk}_{\mathfrak{P}}(X \cup \{z\}) = \mathrm{Drk}_{\mathfrak{P}}(X) = M_{\mathfrak{P}}(X)$ and $M_{\mathfrak{P}}(X \cup \{y\}) = \mathrm{Drk}_{\mathfrak{P}}(X \cup \{y\}) = \mathrm{Drk}_{\mathfrak{P}}(X) = M_{\mathfrak{P}}(X)$. Thus, $y \in M_{\mathfrak{P}}(X \cup \{z\}) = \mathrm{Drk}_{\mathfrak{P}}(X \cup \{z\})$, therefore we conclude that $\mathrm{rk}_{\mathfrak{P}}(X \cup \{z\} \cup \{y\}) = \mathrm{rk}_{\mathfrak{P}}(X \cup \{z\}) = \mathrm{rk}_{\mathfrak{P}}(X)$. This shows our claim and, by part (v) of Proposition 3.1, we have that $\mathrm{rk}_{\mathfrak{P}}$ is the rank function of the matroid $\mathcal{N}_{\mathfrak{P}}$. □

3.2. Attractive pairings

In the present subsection we provide the notions of *locally finite* and *attractive* pairings on Ω and, next, we will exhibit various results on them.

Definition 3.3. We say that a pairing $\mathfrak{P} \in \mathrm{PR}(\Omega)$ is *locally finite* if

$$\forall X, Y \in \wp(\Omega), \forall y \in Y\ [Y \leftarrowtail_{\mathfrak{P}} X \implies \exists X_y \in \wp_{\mathrm{fin}}(X)\ (\{y\} \leftarrowtail_{\mathfrak{P}} X_y)$$

We denote by $\mathrm{PR}_{\mathrm{lf}}(\Omega)$ the set of all locally finite pairings on Ω. Moreover, we say that $\mathfrak{P}$ is *attractive* if

$$\forall X \in \wp(\Omega), \forall y \in \Omega, \forall x \in X\ [\{y\} \not\leftarrowtail_{\mathfrak{P}} X \text{ and } \{x\} \not\leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\} \implies \{x\} \not\leftarrowtail_{\mathfrak{P}} X \triangle \{x, y\}]$$

We denote by $\mathrm{PR}_{\mathrm{a}}(\Omega)$ the collection of all attractive pairings on Ω and by $\mathrm{PR}_{\mathrm{lfa}}(\Omega) := \mathrm{PR}_{\mathrm{a}}(\Omega) \cap \mathrm{PR}_{\mathrm{lf}}(\Omega)$ to denote the collection of all locally finite, attractive pairings on Ω.

It is evident that when Ω is a finite set, then a pairing $\mathfrak{P} \in \mathrm{PR}(\Omega)$ is locally finite. Let us now alternatively characterize the condition for a pairing of being attractive.

Proposition 3.4. *The following two conditions are equivalent:*

(i) $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$*;*

(ii) $\forall X \in \wp(\Omega)$, $\forall y \in \Omega \smallsetminus X$, $\forall x \in X$ $[\{y\} \leftarrowtail_{\mathfrak{P}} X$ *and* $\{y\} \not\leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\} \implies \{x\} \leftarrowtail_{\mathfrak{P}} X \triangle \{x, y\}]$.

Proof. (i) $\implies$ (ii): Let $X \in \wp(\Omega)$, $y \in \Omega \smallsetminus X$ and $x \in X$ be such that $\{y\} \leftarrowtail_{\mathfrak{P}} X$ and $\{y\} \not\leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\}$. Suppose by contradiction that $\{x\} \not\leftarrowtail_{\mathfrak{P}} X \triangle \{x, y\}$. If we set $A := X \triangle \{x, y\} = (X \smallsetminus \{x\}) \cup \{y\}$, then we can restate the last two assumptions as

$$\{x\} \not\leftarrowtail_{\mathfrak{P}} A \text{ and } \{y\} \not\leftarrowtail_{\mathfrak{P}} A \smallsetminus \{y\} = X \smallsetminus \{x\},$$

so that, since $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$, we conclude that $\{y\} \not\leftarrowtail_{\mathfrak{P}} A \triangle \{x, y\} = X$, absurd.

$(ii) \implies (i)$: Let $X \in \wp(\Omega)$, $y \in \Omega$ and $x \in X$ be such that $\{y\} \not\leftarrowtail_{\mathfrak{P}} X$ and $\{x\} \not\leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\}$. Suppose by contradiction that $\{x\} \leftarrowtail_{\mathfrak{P}} X \triangle \{x, y\}$. If we set $A := X \triangle \{x, y\} = (X \smallsetminus \{x\}) \cup \{y\}$, then we can restate the last two assumptions as

$$\{x\} \leftarrowtail_{\mathfrak{P}} A \text{ and } \{x\} \not\leftarrowtail_{\mathfrak{P}} A \smallsetminus \{y\} = X \smallsetminus \{x\},$$

so that by our hypothesis we conclude that $\{y\} \leftarrowtail_{\mathfrak{P}} A \triangle \{x, y\} = X$, absurd. □

Remark 3.5. We refer the reader to Proposition 6.7 of [11] to find an equivalent proof of Proposition 3.4 involving a third condition for defining attractiveness, that here we do not use.

In the next result we provide a property satisfied by any attractive pairing.

Proposition 3.6. *Let $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$, $X \in \wp(\Omega)$ and $y, z \in \Omega$ be such that*

$$M_{\mathfrak{P}}(\{z\}) \cap M_{\mathfrak{P}}(X) = \varnothing \text{ and } M_{\mathfrak{P}}(\{z\}) \cap M_{\mathfrak{P}}(X \cup \{y\}) \neq \varnothing \tag{4}$$

Then $M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{z\}) \neq \varnothing$.

Proof. By part (i) of Proposition 2.7 there exists $u \in \Omega$ such that

$$\{u\} \leftarrowtail_{\mathfrak{P}} \{z\} \text{ and } \{u\} \leftarrowtail_{\mathfrak{P}} X \cup \{y\} \tag{5}$$

As $\{u\} \leftarrowtail_{\mathfrak{P}} \{z\}$, the first condition in (4) implies that

$$\{u\} \not\leftarrowtail_{\mathfrak{P}} X \tag{6}$$

If $u = y$, then we clearly have $M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{z\}) \neq \varnothing$. Thus suppose $u \neq y$ and assume by contradiction that

$$M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{z\}) = \varnothing \tag{7}$$

Hence, by part (i) of Proposition 2.7 and by (D1) and (D2), we easily deduce that $\{y\} \not\leftarrowtail_{\mathfrak{P}} X$. Now, by (5) and (6), using part (ii) of Proposition 3.4 applied to $X \cup \{y\}$, $u \in \Omega \smallsetminus (X \cup \{y\})$ and to y, we get the condition $\{y\} \leftarrowtail_{\mathfrak{P}} X \cup \{u\}$, whence $X \cup \{y\} \leftarrowtail_{\mathfrak{P}} X \cup \{u\}$. At this point, from the second condition of (5) we deduce that $X \cup \{y\} \approx_{\mathfrak{P}} X \cup \{u\}$. So, using the first condition of (5), we easily check that $X \cup \{y\} \leftarrowtail_{\mathfrak{P}} X \cup \{z\}$. Thus, by (D1), it results that $\{y\} \leftarrowtail_{\mathfrak{P}} X \cup \{z\}$, whence $y \in M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{z\})$, in contrast with (7). This shows that $M_{\mathfrak{P}}(\{y\}) \cap M_{\mathfrak{P}}(X \cup \{z\}) \neq \varnothing$. □

In [11] it has been proved that the collection of all the $\mathfrak{P}$-reduct of a maximum partitioner X is always non-empty set when the pairing is both locally finite and attractive. Let us recall such a result.

Theorem 3.7. *[11]. Let* $\mathfrak{P} \in \mathrm{PR}_{\mathrm{lfa}}(\Omega)$ *and* $X \in \mathcal{M}_{\mathfrak{P}}$. *Then* $\mathcal{R}_{\mathfrak{P}}(X)$ *has uniform cardinality.*

At this point, we will establish two specific properties of attractive pairings on arbitrary ground sets, namely the fact that the X-reducts coincide with the maximal members of $\mathcal{N}_{\mathfrak{P}}$ contained in X and form an exchangeable set system.

Theorem 3.8. *Let* $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$. *Then, for any* $X \in \wp(\Omega)$ *the following conditions hold:*

(i) $\mathcal{R}_{\mathfrak{P}}(X) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$;
(ii) $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ *is an exchangeable set system on* Ω.

Proof. (i): Let $X \in \wp(\Omega)$. We claim that $\mathcal{R}_{\mathfrak{P}}(X) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$. To this regard, it suffices to show the inclusion $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X)) \subseteq \mathcal{R}_{\mathfrak{P}}(X)$ since, by part (iv) of Proposition 2.7, the reverse inclusion is always true. Take therefore $Y \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$. We will first demonstrate that $X \twoheadleftarrow_{\mathfrak{P}} Y$. Assume by contradiction the existence of $x \in X$ such that

$$\{x\} \not\twoheadleftarrow_{\mathfrak{P}} Y \tag{8}$$

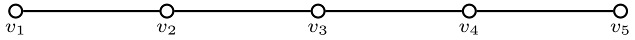
Hence $x \in \Omega \setminus M_{\mathfrak{P}}(Y)$ by part (i) of Proposition 2.7. Moreover, as $Y \in \mathcal{N}_{\mathfrak{P}}$, it results that $\{y\} \not\twoheadleftarrow_{\mathfrak{P}} Y \setminus \{y\}$ for any $y \in Y$. Using (8) and the fact that $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$, it follows that the condition

$$\{y\} \not\twoheadleftarrow_{\mathfrak{P}} Y \triangle \{x, y\} \tag{9}$$

holds for any $y \in Y$. Consider now the subset $Y \cup \{x\}$. In view of (8) and of (9), it easily follows that $Y \cup \{x\} \in \mathcal{N}_{\mathfrak{P}} \cap \wp(X)$, contradicting the fact that $Y \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ and proving that $X \twoheadleftarrow_{\mathfrak{P}} Y$.
Now, fix some $y \in Y$. We have $X \not\twoheadleftarrow_{\mathfrak{P}} Y \setminus \{y\}$ since, otherwise, we would get $\{y\} \twoheadleftarrow_{\mathfrak{P}} Y \twoheadleftarrow_{\mathfrak{P}} X \twoheadleftarrow_{\mathfrak{P}} Y \setminus \{y\}$, which contradicts the assumption that $Y \in \mathcal{N}_{\mathfrak{P}}$. So $Y \in \mathcal{R}_{\mathfrak{P}}(X)$.
(ii): Let $B, C \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ and $x \in B \setminus C$. In view of the previous part (i), we have that $\mathcal{R}_{\mathfrak{P}}(X) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$. Thus $B \approx_{\mathfrak{P}} C$. Now, as $B \in \mathcal{N}_{\mathfrak{P}}$, it results that $\{x\} \not\twoheadleftarrow_{\mathfrak{P}} B \setminus \{x\}$. Therefore $B \not\twoheadleftarrow_{\mathfrak{P}} B \setminus \{x\}$ and we deduce that $C \not\twoheadleftarrow_{\mathfrak{P}} B \setminus \{x\}$. In particular, there exists $y \in C$ such that

$$\{y\} \not\twoheadleftarrow_{\mathfrak{P}} B \setminus \{x\}. \tag{10}$$

This forces $y \in C \setminus B$ and $\{y\} \twoheadleftarrow_{\mathfrak{P}} B$. Therefore, in view of part (ii) of Proposition 3.4, we have $\{x\} \twoheadleftarrow_{\mathfrak{P}} B \triangle \{x, y\}$. Hence



	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	0	0
v_2	1	0	1	0	0
v_3	0	1	0	1	0
v_4	0	0	1	0	1
v_5	0	0	0	1	0

Fig. 3. The Graph P_5 of Example 3.9 and its Adjacency Matrix.

$$B \vartriangle \{x, y\} \approx_{\mathfrak{P}} B \cup \{y\} \approx_{\mathfrak{P}} B. \tag{11}$$

Set $B' := B \vartriangle \{x, y\}$. Clearly, $B' \in \wp(X)$. We claim that $B' \in \mathcal{N}_{\mathfrak{P}}$ or, equivalently, that

$$\{b'\} \not\leftarrowtail_{\mathfrak{P}} B' \smallsetminus \{b'\} \tag{12}$$

for each $b' \in B'$. In view of (10), the claim holds when we take $b' := y$. So, take $b' \in B \smallsetminus \{x\}$ and assume by contradiction that (12) does not hold. Thus

$$\{b'\} \leftarrowtail_{\mathfrak{P}} B' \smallsetminus \{b'\} = (B \smallsetminus \{x\}) \vartriangle \{y, b'\}. \tag{13}$$

Notice that

$$\{b'\} \not\leftarrowtail_{\mathfrak{P}} B \smallsetminus \{x, b'\}, \tag{14}$$

otherwise $\{b'\} \leftarrowtail_{\mathfrak{P}} B \smallsetminus \{b'\}$, contradicting the fact that $B \in \mathcal{N}_{\mathfrak{P}}$. Now, using part (ii) of Proposition 3.4 on the conditions (13) and (14), we get $\{y\} \leftarrowtail_{\mathfrak{P}} (B \smallsetminus \{x, b'\}) \cup \{b'\} = B \smallsetminus \{x\}$, which contradicts (10). Therefore, we conclude that (12) holds for each $b' \in B'$. This shows that $B' \in \mathcal{N}_{\mathfrak{P}}$.
Finally, let $B'' \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ be such that $B' \subsetneq B''$. As $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X)) = \mathcal{R}_{\mathfrak{P}}(X)$, we get $B'' \in \mathcal{R}_{\mathfrak{P}}(X)$, contradicting (11). So, $B' \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ and $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ is exchangeable. □

In the following example, we see that if $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ is exchangeable for any $X \in \wp(\Omega)$ (but not necessarily equal to $\mathcal{R}_{\mathfrak{P}}(X)$ for some $X \in \wp(\Omega)$), then the corresponding pairing should not be attractive.

Example 3.9. Let us consider the pairing $\mathfrak{P} := \mathfrak{P}[P_5, \mathrm{adj}]$ induced by the adjacency matrix of the graph P_5, where $V := V(P_5)$. In Fig. 3 it has been explicited the adjacency matrix of P_5 and its geometric representation.

By using part (i) of Proposition 2.7 we can easily check that:

$$\begin{aligned}\mathcal{M}_{\mathfrak{P}} = \{&\varnothing, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_5\},\\ &\{v_3, v_4\}, \{v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_3, v_4, v_5\}, V\},\end{aligned}$$

while in view of part (iii) of the same Proposition we get

$$\begin{aligned}\mathcal{N}_{\mathfrak{P}} = \{\varnothing, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1,v_2\}, \{v_1,v_3\}, \{v_1,v_4\}, \{v_1,v_5\}, \{v_2,v_3\},\\ \{v_2,v_4\}, \{v_2,v_5\}, \{v_3,v_4\}, \{v_3,v_5\}, \{v_4,v_5\}, \{v_1,v_2,v_3\}, \{v_1,v_2,v_4\}, \{v_1,v_2,v_5\},\\ \{v_1,v_3,v_4\}, \{v_1,v_4,v_5\}, \{v_2,v_3,v_5\}, \{v_2,v_4,v_5\}, \{v_3,v_4,v_5\}\},\end{aligned}$$

so that, through part (*iv*) of Proposition 2.7 we can easily show that $\mathcal{R}_{\mathfrak{P}}(V) = \{\{v_1,v_2,v_4\},\{v_2,v_4,v_5\}\}$. However, we refer the reader to [9,10] for an alternative and quite interesting technique useful to the determination of $\mathcal{R}_{\mathfrak{P}}(V)$ (and in general of $\mathcal{R}_{\mathfrak{P}}(X)$ for each $X \in \wp(V)$) that involves the computation of the minimal transversals of the so-called *hypergraph of the dissymetry neighborhoods* of pairs of vertices of a graph G (equivalently *hypergraph of the discernibility neighborhoods* according to the terminology stated in [9] for an arbitrary pairing). In addition, we have:

$$\begin{aligned}\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(V)) = \{\{v_1,v_2,v_3\}, \{v_1,v_2,v_4\}, \{v_1,v_2,v_5\}, \{v_1,v_3,v_4\},\\ \{v_1,v_4,v_5\}, \{v_2,v_3,v_5\}, \{v_2,v_4,v_5\}, \{v_3,v_4,v_5\}\},\\ \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(\{v_1,v_2,v_3,v_4\})) = \{\{v_1,v_2,v_3\}, \{v_1,v_2,v_4\}, \{v_1,v_3,v_4\}\},\\ \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(\{v_1,v_2,v_3,v_5\})) = \{\{v_1,v_2,v_3\}, \{v_1,v_2,v_5\}, \{v_2,v_3,v_5\}\},\\ \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(\{v_1,v_2,v_4,v_5\})) = \{\{v_1,v_2,v_4\}, \{v_1,v_2,v_5\}, \{v_1,v_4,v_5\}, \{v_2,v_4,v_5\}\},\\ \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(\{v_1,v_3,v_4,v_5\})) = \{\{v_1,v_3,v_4\}, \{v_1,v_4,v_5\}, \{v_3,v_4,v_5\}\},\\ \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(\{v_2,v_3,v_4,v_5\})) = \{\{v_2,v_3,v_5\}, \{v_2,v_4,v_5\}, \{v_3,v_4,v_5\}\},\\ \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(\{v_1,v_3,v_5\})) = \{\{v_1,v_3\}, \{v_1,v_5\}, \{v_3,v_5\}\},\\ \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(\{v_2,v_3,v_4\})) = \{\{v_2,v_3\}, \{v_2,v_4\}, \{v_3,v_4\}\},\end{aligned}$$

and $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X)) = \{X\}$ for the remaining vertex subsets X. A simple computation shows that $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ is exchangeable for each $X \in \wp(V)$.
On the other hand, let us observe that P_5 is not an attractive graph. In fact, just take $X = \{v_2,v_3,v_4\}$, $x = v_3$ and $y = v_1$. Then $\{v_3\} \twoheadleftarrow_{\mathfrak{P}} X \triangle \{v_1,v_3\} = \{v_1,v_2,v_4\}$ since $\{v_1,v_2,v_4\} \approx_{\mathfrak{P}} V$.

Let us consider a locally finite pairing. Then it results that the condition $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X)) = \mathcal{R}_{\mathfrak{P}}(X)$ for any $X \in \wp(\Omega)$ characterizes attractiveness. Thus, in the case of a locally finite pairing, we have that the family of all the minimal partitioners is a closable finitary simplicial complex and, in particular, a matroid when Ω is a finite set [26]. This we now present.

Theorem 3.10. *Let* $\mathfrak{P} \in \mathrm{PR}_{\mathrm{lf}}(\Omega)$. *The following conditions are equivalent:*

(*i*) $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$;
(*ii*) $\forall X \in \wp(\Omega)\ [\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X)) = \mathcal{R}_{\mathfrak{P}}(X)]$.

Furthermore, $\mathcal{N}_{\mathfrak{P}}$ *is a closable finitary simplicial complex when* $\mathfrak{P} \in \mathrm{PR}_{\mathrm{lfa}}(\Omega)$ *and, in particular,* $\mathcal{N}_{\mathfrak{P}}$ *is a matroid such that* $\mathcal{R}_{\mathfrak{P}}(X)$ *is the family of all its* X*-bases, for each* $X \in \wp(\Omega)$*, when* Ω *is a finite set and* $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$*.*

Proof. The statement concerning the fact that $\mathcal{N}_{\mathfrak{P}} \in \mathrm{CFSC}$ follows from Theorem 2.4.

$(i) \implies (ii)$: It has been already shown in part (i) of Theorem 3.8.

$(ii) \implies (i)$: Let us define the non-decreasing map $\rho_{\mathfrak{P}} : \wp_{\mathrm{fin}}(\Omega) \to \mathbb{N}$, where:

$$\rho_{\mathfrak{P}}(F) := \max\{\|F'\| \mid F' \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(F))\}. \tag{15}$$

Fix now $F \in \wp_{\mathrm{fin}}(\Omega)$. We claim that

$$M_{\mathfrak{P}}(F) = \{w \in \Omega \mid \rho_{\mathfrak{P}}(F) = \rho_{\mathfrak{P}}(F \cup \{w\})\}. \tag{16}$$

To this regard, let $z \in M_{\mathfrak{P}}(F)$, so that $F \cup \{z\} \approx_{\mathfrak{P}} F$ by part (i) of Proposition 2.7. Take now $Y \in \mathcal{R}_{\mathfrak{P}}(F \cup \{z\}) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(F \cup \{z\}))$ be such that $\|Y\| = \rho_{\mathfrak{P}}(F \cup \{z\})$. If $z \in \Omega \smallsetminus Y$, then we clearly have $Y \in \mathcal{R}_{\mathfrak{P}}(F)$, whence $\|Y\| \leq \rho_{\mathfrak{P}}(F) \leq \rho_{\mathfrak{P}}(F \cup \{z\}) = \|Y\|$, i.e. $\rho_{\mathfrak{P}}(F) = \rho_{\mathfrak{P}}(F \cup \{z\})$.

On the other hand, if $z \in Y$, we have $Y = C \cup \{z\}$ for some $C \in \wp(F)$. Notice that $C \notin \mathcal{R}_{\mathfrak{P}}(F)$ otherwise, by part (ii) of Remark 2.8 we would have $\pi_{\mathfrak{P}}(Y \smallsetminus \{z\}) = \pi_{\mathfrak{P}}(C) = \pi_{\mathfrak{P}}(F) = \pi_{\mathfrak{P}}(F \cup \{z\})$, contradicting the fact that $Y \in \mathcal{R}_{\mathfrak{P}}(F \cup \{z\})$. Nevertheless, since $Y \in \mathcal{N}_{\mathfrak{P}}$, we must necessarily have $C \in \mathcal{N}_{\mathfrak{P}} \cap \wp(F)$ and, thus, $C \subsetneq D$ for some $D \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(F)) = \mathcal{R}_{\mathfrak{P}}(F)$. This means that $\|C\| < \rho_{\mathfrak{P}}(F)$ and, hence, we deduce that

$$\rho_{\mathfrak{P}}(F \cup \{z\}) = \|Y\| = \|C\| + 1 \leq \rho_{\mathfrak{P}}(F)$$

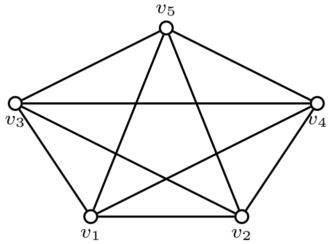
As $\rho_{\mathfrak{P}}$ is non-decreasing, we also have $\rho_{\mathfrak{P}}(F) \leq \rho_{\mathfrak{P}}(F \cup \{z\})$ proving the left inclusion of (16).

Conversely, to prove the reverse inclusion, let $z \in \Omega$ be such that $\rho_{\mathfrak{P}}(F) = \rho_{\mathfrak{P}}(F \cup \{z\})$ and assume by contradiction that $z \in \Omega \smallsetminus M_{\mathfrak{P}}(F)$, i.e. $F \not\approx_{\mathfrak{P}} F \cup \{z\}$ by part (i) of Proposition 2.7. Let moreover $Y \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(F)) = \mathcal{R}_{\mathfrak{P}}(F)$ be such that $\|Y\| = \rho_{\mathfrak{P}}(F) = \rho_{\mathfrak{P}}(F \cup \{z\})$. Clearly, it follows that $Y \in \mathcal{N}_{\mathfrak{P}} \cap \wp(F \cup \{z\})$. So $Y \in \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(F \cup \{z\})) = \mathcal{R}_{\mathfrak{P}}(F \cup \{z\})$, whence $\pi_{\mathfrak{P}}(F) = \pi_{\mathfrak{P}}(Y) = \pi_{\mathfrak{P}}(F \cup \{z\})$ by part (ii) of Remark 2.8, in contrast with the fact that $z \in \Omega \smallsetminus M_{\mathfrak{P}}(F)$. This shows (16).

Let now $X \in \wp(\Omega)$, $y \in \Omega$ and $x \in X$ be such that $\{y\} \not\leftarrowtail_{\mathfrak{P}} X$ and $\{x\} \not\leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\}$. Assume by contradiction that

$$\{x\} \leftarrowtail_{\mathfrak{P}} X \vartriangle \{x, y\}, \tag{17}$$

so that there exists $F_x \in \wp_{\mathrm{fin}}(X \vartriangle \{x, y\})$ such that $\{x\} \leftarrowtail_{\mathfrak{P}} F_x$. We can choose F_x to be minimal with respect to such a property. Then F_x is a minimal element of its same equivalence class $[F_x]_{\approx_{\mathfrak{P}}}$. At this point, in view of our choices of x, y and X, note that:



	v_1	v_2	v_3	v_4	v_5
v_1	0	1	1	1	1
v_2	1	0	1	1	1
v_3	1	1	0	1	1
v_4	1	1	1	0	1
v_5	1	1	1	1	0

Fig. 4. The Graph K_5 and its Adjacence Matrix.

$$\forall F \in \wp_{\mathrm{fin}}(X)\ [\{y\} \not\twoheadleftarrow_{\mathfrak{P}} F] \quad \text{and} \quad \forall G \in \wp_{\mathrm{fin}}(X \smallsetminus \{x\})\ [\{x\} \not\twoheadleftarrow_{\mathfrak{P}} G] \tag{18}$$

In fact, if it were $\{y\} \twoheadleftarrow_{\mathfrak{P}} F$ for some finite subset F of X, then by (D1) we would also have $\{y\} \twoheadleftarrow_{\mathfrak{P}} X$, absurd. In a similar way, if $\{x\} \twoheadleftarrow_{\mathfrak{P}} G$ for some finite subset G of $X \smallsetminus \{x\}$, then by (D1) we would also have $\{x\} \twoheadleftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$, that is absurd.

Let us now check that $y \in F_x$. Suppose by contradiction that $y \notin F_x$. Then we would have $F_x \in \wp_{\mathrm{fin}}(X \smallsetminus \{x\})$ and $\{x\} \twoheadleftarrow_{\mathfrak{P}} F_x$, in contrast with the second condition of (18). Hence $y \in F_x$.

Let now $H := F_x \cup \{x\}$. It is straightforward to check that $F_x \in \mathcal{R}_{\mathfrak{P}}(H)$. Let moreover $\widetilde{F}_x = (F_x \smallsetminus \{y\}) \cup \{x\}$. In view of (18) we easily deduce that

$$\rho_{\mathfrak{P}}(H) = \rho_{\mathfrak{P}}(\widetilde{F}_x \cup \{y\}) = \rho_{\mathfrak{P}}(\widetilde{F}_x) + 1, \qquad \text{and} \qquad \rho_{\mathfrak{P}}(\widetilde{F}_x) = \rho_{\mathfrak{P}}(\widetilde{F}_x \smallsetminus \{x\}) + 1$$

Then we get

$$\begin{aligned} \rho_{\mathfrak{P}}(F_x \smallsetminus \{y\}) &= \|F_x \smallsetminus \{y\}\| = \|F_x\| - 1 = \rho_{\mathfrak{P}}(H) - 1 = \rho_{\mathfrak{P}}(\widetilde{F}_x \cup \{y\}) - 1 \\ &= \rho_{\mathfrak{P}}(\widetilde{F}_x) > \rho_{\mathfrak{P}}(\widetilde{F}_x \smallsetminus \{x\}) = \|F_x\| - 1, \end{aligned}$$

absurd. Therefore, we must have $\{x\} \not\twoheadleftarrow_{\mathfrak{P}} X \vartriangle \{x, y\}$, i.e. $\mathfrak{P} \in \mathrm{PR_a}(\Omega)$. □

In the next results we see an application of the equivalence stated in Theorem 3.10 in the case of the adjacence pairing induced by complete graphs and of the metric pairing induced by n-cycles. In Fig. 4 we represent the adjacence matrix of K_5.

Proposition 3.11. *Let $n \geq 2$ and $\mathfrak{P}[K_n, \mathrm{adj}] \in \mathrm{PR}(V(K_n))$. The following conditions hold:*

(i) for any $X \in \wp(V(K_n))$ we have that

$$\mathcal{R}_{\mathfrak{P}}(X) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X)) = \begin{cases} \{X\} & \text{if } \|X\| \leq n-1 \\ \\ \{A \in \wp(V(K_n)) \mid \|A\| = n-1\} & \text{if } X = V(K_n) \end{cases}$$

(ii) $\mathfrak{P}[K_n, \mathrm{adj}] \in \mathrm{PR}_a(V(K_n))$.

Proof. (i): Straightforward.

(ii): Just use the above part (i) and Theorem 3.10. □

Proposition 3.12. *Let $n \geq 2$ and $\mathfrak{P} := \mathfrak{P}[C_n, d]$. The following conditions hold:*

(i) $\mathcal{M}_{\mathfrak{P}} = \begin{cases} \{\varnothing\} \cup \{\{v_i\} \mid i = 1, \ldots, n\} \cup V(C_n) & \text{if } n = 2k+1 \\ \{\varnothing\} \cup \{\{v_i, v_{i+\frac{n}{2}}\} \mid i = 1, \ldots, \frac{n}{2}\} \cup V(C_n) & \text{if } n = 2k \end{cases}$;

(ii) $\mathfrak{P}[C_n, d] \in \mathrm{PR}_a(V(C_n))$.

Proof. (i): We clearly have $\pi_{\mathfrak{P}}(\varnothing) = v_1 \ldots v_n$ and $\pi_{\mathfrak{P}}(\{v_i\}) \neq \pi_{\mathfrak{P}}(\varnothing)$ for every $i = 1, \ldots, n$. In fact, the vertex v_i constitutes a single block in the set partition with respect to the $\{v_i\}$-symmetry partition, because it is the only vertex whose distance from itself is 0. Thus $M_{\mathfrak{P}}(\varnothing) = \varnothing$.

Let $n = 2k+1$ be an odd integer. Since each $\{v_i\}$-symmetry block distinct from that of v_i (for every $i = 1, \ldots, n$) contains exactly two vertices, we have that $\{v_i\} \not\approx_{\mathfrak{P}} \{v_j\}$ if $i \neq j$. Next, let $X := \{v_i, v_j\}$. Since we cannot find two distinct vertices having the same respective distances from v_i and v_j, we conclude that $\pi_{\mathfrak{P}}(X) = \pi_{\mathfrak{P}}(V(C_n)) = v_1|\ldots|v_n$. This shows our claim when n is odd.

Assume now that $n = 2k$ is an even integer. In the remaining proof, we assume the index sums to be taken modulo n. First of all, notice that both $\{v_i\}$ and $\{v_{i+\frac{n}{2}}\}$ constitute two distinct $\{v_i\}$-symmetry blocks for every $i = 1, \ldots, n$, while the other symmetry blocks are constituted by pairs of vertices. Thus $\{v_i\} \approx_{\mathfrak{P}} \{v_j\}$ if and only if $j = i + \frac{n}{2}$. Thus $\mathcal{M}_{\mathfrak{P}}$ does not contain singletons when n is even. Furthermore, we clearly have $\pi_{\mathfrak{P}}(\{v_i\}) = \pi_{\mathfrak{P}}(\{v_{i+\frac{n}{2}}\}) = \pi_{\mathfrak{P}}(\{v_i, v_{i+\frac{n}{2}}\})$.

Next, let $X := \{v_i, v_j\}$. Since we cannot find two distinct vertices having the same respective distances from v_i and v_j when $j \neq i + \frac{n}{2}$, we conclude that $\pi_{\mathfrak{P}}(X) = \pi_{\mathfrak{P}}(V(C_n)) = v_1|\ldots|v_n$ when $j \neq i + \frac{n}{2}$. This shows our claim when n is even.

(ii): Let first $n \geq 2$ be an odd integer. Then, by the above part (i) we have

$$\mathcal{R}_{\mathfrak{P}}(X) = \begin{cases} X & \text{if } \|X\| \leq 2 \\ \wp_2(X) & \text{otherwise} \end{cases}$$

Suppose now that $n \geq 2$ is an even integer and let $\mathcal{G} := \{\{v_i, v_{i+\frac{n}{2}}\} \mid i = 1, \ldots, \frac{n}{2}\}$. Then, by the above part (i) we have

$$\mathcal{R}_{\mathfrak{P}}(X) = \begin{cases} X & \text{if } \|X\| \leq 2 \text{ and } X \notin \mathcal{G} \\ \{\{v_i\}, \{v_{i+\frac{n}{2}}\}\} & \text{if } X \in \mathcal{G} \\ \wp_2(X) \setminus \mathcal{G} & \text{otherwise} \end{cases}$$

Hence, in both cases, we can check that $\mathcal{R}_{\mathfrak{P}}(X) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ for every $X \in \wp(V(C_n))$. Therefore, we conclude that $\mathfrak{P}[C_n, d] \in \mathrm{PR}_a(V(C_n))$ in view of part (ii) of Theorem 3.10. □

Remark 3.13. It may be easily checked that in general the adjacence pairings $\mathfrak{P}[C_n, \mathrm{adj}]$ are not attractive. As a matter of fact, let $G := C_5$ and take $X := \{v_1, v_2\}$, $x := v_2$ and $y := v_3$. Then it results that $\{y\} \not\leftarrowtail_G X$, $\{x\} \not\leftarrowtail_G X \smallsetminus \{x\}$ but $\{x\} \leftarrowtail_G X \vartriangle \{x, y\}$.

4. Quasi-attractive pairings

In this section we introduce the notion of *quasi-attractive pairing*. It originates by assuming a specific property which is indeed weaker than attractiveness, as we will see in Proposition 4.1. Furthermore, we will firstly provide a characterization of quasi-attractiveness in terms of order-theoretical properties of $\mathcal{M}_{\mathfrak{P}}$. Next, we investigate the main properties of quasi-attractive pairings, above all in relation to the behavior of the set system $\mathcal{R}_{\mathfrak{P}}(X)$, for any $X \in \wp(\Omega)$. In particular, we will demonstrate that when Ω is a finite set, quasi-attractiveness becomes equivalent to require the exchangeability of $\mathcal{R}_{\mathfrak{P}}(X)$ for any $X \in \wp(\Omega)$. In this way, we relate quasi-attractiveness to the matroidality of a sub-set system of $\mathcal{N}_{\mathfrak{P}}$.

Let us first provide a necessary condition for attractive pairings, which will be used in the successive part in order to introduce a new subclass of pairings.

Proposition 4.1. *Let* $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$ *and* $X, Y \in \wp(\Omega)$ *be such that* $X \approx_{\mathfrak{P}} Y$. *Then*

$$\forall x \in X \ [\exists y_x \in Y \ (\{x\} \leftarrowtail_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y_x\})]$$

Proof. Let $X, Y \in \wp(\Omega)$ be such that $X \approx_{\mathfrak{P}} Y$ and fix $x \in X$. There is nothing to prove if $x \in Y$. So assume that $x \in \Omega \smallsetminus Y$. First note that if $\{x\} \leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\}$, then the condition $\{x\} \leftarrowtail_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y\}$ holds for each $y \in Y$. Therefore, we may also suppose that $\{x\} \not\leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\}$. Then, using the assumption that $X \approx_{\mathfrak{P}} Y$, we may find an element $y_x \in Y$ such that $\{y_x\} \not\leftarrowtail_{\mathfrak{P}} X \smallsetminus \{x\}$. Hence, by (D1), (D2) and the fact that $Y \approx_{\mathfrak{P}} X$, it follows that $\{y_x\} \leftarrowtail_{\mathfrak{P}} X$. Now, since $\mathfrak{P} \in \mathrm{PR}_{\mathrm{a}}(\Omega)$, by part (*ii*) of Proposition 3.4 we get $\{x\} \leftarrowtail_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y_x\}$. ☐

In view of Proposition 4.1, we may now provide the fundamental notion of quasi-attractive pairing and the corresponding notion of quasi-attractive pairing.

Definition 4.2. We say that a pairing $\mathfrak{P} \in \mathrm{PR}(\Omega)$ is *quasi-attractive* if for each $X, Y \in \wp(\Omega)$ such that $X \approx_{\mathfrak{P}} Y$ and any $x \in X$

$$\exists y_x \in Y \ [\{x\} \leftarrowtail_{\mathfrak{P}} ((X \smallsetminus \{x\}) \cup \{y_x\})]$$

We denote by $\mathrm{PR}_{\mathrm{qa}}(\Omega)$ the collection of all quasi-attractive pairings on Ω and also set $\mathrm{PR}_{\mathrm{lfqa}}(\Omega) := \mathrm{PR}_{\mathrm{lf}}(\Omega) \cap \mathrm{PR}_{\mathrm{qa}}(\Omega)$.

Let us characterize the condition for a pairing of being quasi-attractive in terms of a specific property of the complete lattice $\mathbb{M}(\mathfrak{P})$. More in detail we will demonstrate that a pairing is quasi-attractive if and only if for any $X \in \wp(\Omega)$ the elements of $\mathcal{I}(\mathcal{F})$, where $\mathcal{F}$ is the upset of $M_{\mathfrak{P}}(X)$ in $\mathbb{M}(\mathfrak{P})$, are exactly those of the form $M_{\mathfrak{P}}(X \cup \{z\})$, where $\{z\} \not\twoheadleftarrow_{\mathfrak{P}} X$.

Theorem 4.3. *The following conditions are equivalent:*

- (i) $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$;
- (ii) $\forall Z \in \mathcal{M}_{\mathfrak{P}}, \forall Y \in \wp(\Omega), \forall x \in \Omega$ $[\{x\} \not\twoheadleftarrow_{\mathfrak{P}} Z$ *and* $Y \setminus Z \neq \varnothing$ *and* $Z \cup Y \approx_{\mathfrak{P}} Z \cup \{x\} \implies \exists y \in Y \ (\{x\} \twoheadleftarrow_{\mathfrak{P}} Z \cup \{y\})]$;
- (iii) *for any* $X \in \wp(\Omega)$ *the elements of* $\mathcal{I}((M_{\mathfrak{P}}(X))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}})$ *are exactly those of the form* $M_{\mathfrak{P}}(X \cup \{z\})$*, where* $\{z\} \not\twoheadleftarrow_{\mathfrak{P}} X$.

Proof. $(i) \implies (ii)$: Let $Z \in \mathcal{M}_{\mathfrak{P}}$, $Y \in \wp(\Omega)$ and $x \in \Omega$ be such that $Y \setminus Z \neq \varnothing$ and $Z \cup Y \approx_{\mathfrak{P}} Z \cup \{x\}$. If $\{x\} \twoheadleftarrow_{\mathfrak{P}} Z$, there is nothing to prove. So, assume that $\{x\} \not\twoheadleftarrow_{\mathfrak{P}} Z$. As $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$, there must be some $y \in Y \setminus Z$ such that $\{x\} \twoheadleftarrow_{\mathfrak{P}} Z \cup \{y\}$.
$(ii) \implies (i)$: Let $X, Y \in \wp(\Omega)$ be such that $X \approx_{\mathfrak{P}} Y$ and $x \in X$. We claim that $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$. If $\{x\} \twoheadleftarrow_{\mathfrak{P}} X \setminus \{x\}$, there is nothing to prove. Analogously, if $x \in Y \cap X$, the claim is obvious. So, take $x \in X \setminus Y$ and assume that $\{x\} \not\twoheadleftarrow_{\mathfrak{P}} X \setminus \{x\}$. Set $Z := M_{\mathfrak{P}}(X \setminus \{x\})$. Hence $\{x\} \not\twoheadleftarrow_{\mathfrak{P}} Z$. At this point, it is immediate to check that

$$X \approx_{\mathfrak{P}} Z \cup \{x\}. \tag{19}$$

Furthermore, since $X \approx_{\mathfrak{P}} Y$, there exists $y \in Y$ such that $\{y\} \not\twoheadleftarrow_{\mathfrak{P}} Z$. In particular, $y \in Y \setminus Z$. Notice also that $Y \cup Z \approx_{\mathfrak{P}} Y$. Thus, by (19) and by our choice of Y, it follows that $Z \cup \{x\} \approx_{\mathfrak{P}} Z \cup Y$. So, we can use our assumption to find an element $y_x \in Y$ such that $\{x\} \twoheadleftarrow_{\mathfrak{P}} Z \cup \{y_x\}$. As $Z \approx_{\mathfrak{P}} X \setminus \{x\}$, we conclude that $\{x\} \twoheadleftarrow_{\mathfrak{P}} (X \setminus \{x\}) \cup \{y_x\}$.

$(i) \implies (iii)$: Let $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$. We claim that for any $X \in \wp(\Omega)$ the elements of $\mathcal{I}((M_{\mathfrak{P}}(X))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}})$ are exactly those of the form $M_{\mathfrak{P}}(X \cup \{z\})$, with $\{z\} \not\twoheadleftarrow_{\mathfrak{P}} X$. To this end, fix $X \in \wp(\Omega)$ and an element $z \in \Omega$ such that $\{z\} \not\twoheadleftarrow_{\mathfrak{P}} X$. Consider the subsets

$$Y_z = \{w \in \Omega \mid M_{\mathfrak{P}}(X \cup \{z\}) = M_{\mathfrak{P}}(X \cup \{w\})\} \quad \text{and} \quad X' := M_{\mathfrak{P}}(X \cup \{z\}) \setminus Y_z.$$

It clearly follows that $M_{\mathfrak{P}}(X) \subseteq X' \subsetneq M_{\mathfrak{P}}(X \cup \{z\})$. We now claim that $X' \in \mathcal{M}_{\mathfrak{P}}$. To this regard, assume by contradiction the existence of an element $w \in M_{\mathfrak{P}}(X') \setminus X'$. Thus it results that $w \in Y_z$, whence $X \cup \{w\} \approx_{\mathfrak{P}} X'$. Now, as $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$, there exists $x'_w \in X'$ such that

$$\{w\} \twoheadleftarrow_{\mathfrak{P}} [(X \cup \{w\}) \setminus \{w\}] \cup \{x'_w\} = X \cup \{x'_w\}.$$

Thus, we deduce that $X \cup \{w\} \twoheadleftarrow_{\mathfrak{P}} X \cup \{x'_w\}$. It is also evident that $X \cup \{x'_w\} \twoheadleftarrow_{\mathfrak{P}} X \cup \{w\}$ since $x'_w \in X'$. As $w \in Y_z$, we also get the relation $X \cup \{x'_w\} \approx_{\mathfrak{P}} X \cup \{z\}$, whence $x'_w \in Y_z$,

contradicting our choice of x'_w. This shows that $X' \in \mathcal{M}_{\mathfrak{P}}$.
Let now $X'' \in \mathcal{M}_{\mathfrak{P}}$ be such that $M_{\mathfrak{P}}(X) \subseteq X'' \subsetneqq M_{\mathfrak{P}}(X \cup \{z\})$. We clearly have $X'' \subseteq X'$. Therefore, X' is the only co-cover of $M_{\mathfrak{P}}(X \cup \{z\})$ in the upset of $M_{\mathfrak{P}}(X)$ in the lattice $\mathbb{M}(\mathfrak{P})$.
Conversely, let $W \in \mathcal{I}(\mathcal{F})$, where $\mathcal{F} := (M_{\mathfrak{P}}(X))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}}$. Denote by W' the corresponding co-cover. Then $M_{\mathfrak{P}}(X) \subseteq W' \subsetneqq W$. Suppose by contradiction that no element of the form $M_{\mathfrak{P}}(X \cup \{z\})$, with $\{z\} \not\twoheadleftarrow_{\mathfrak{P}} X$, coincides with W. Since $W' \in \mathcal{I}_{\mathcal{F}}(W)$, it follows that $M_{\mathfrak{P}}(X \cup \{z\}) \subseteq W'$ whenever $\{z\} \not\twoheadleftarrow_{\mathfrak{P}} X$. Thus, taking all the elements $w \in W$ such that $\{w\} \not\twoheadleftarrow_{\mathfrak{P}} X$, we get $W \subseteq W'$, which is an absurd. Therefore, there exists $z \in \Omega$ such that $\{z\} \not\twoheadleftarrow_{\mathfrak{P}} X$ and $W = M_{\mathfrak{P}}(X \cup \{z\})$.
$(iii) \implies (i)$: Take $X \approx_{\mathfrak{P}} Y$ and fix $x \in X$. Set $X' := X \smallsetminus \{x\}$. There is nothing to prove if $\{x\} \twoheadleftarrow_{\mathfrak{P}} X'$. Therefore, assume that $\{x\} \not\twoheadleftarrow_{\mathfrak{P}} X'$. Thus, by our assumption, we have that $M_{\mathfrak{P}}(X' \cup \{x\}) \in \mathcal{I}((M_{\mathfrak{P}}(X'))^{\uparrow}_{\mathcal{M}_{\mathfrak{P}}})$. As a consequence, we can consider the set $B \in (M_{\mathfrak{P}}(X' \cup \{x\}))^{\downarrow,\lessdot}_{\mathcal{M}_{\mathfrak{P}}}$. It is quite easy to check that B cannot contain points of $Z := \{z \in \Omega \mid X' \cup \{x\} \approx_{\mathfrak{P}} X' \cup \{z\}\}$. Thus $M_{\mathfrak{P}}(X') \subsetneqq B \subseteq M_{\mathfrak{P}}(X' \cup \{x\}) \smallsetminus Z \subsetneqq M_{\mathfrak{P}}(X' \cup \{x\})$, so that $B = M_{\mathfrak{P}}(M_{\mathfrak{P}}(X' \cup \{x\}) \smallsetminus Z)$. At this point, notice that $Y \cap Z \neq \varnothing$. Indeed, if it were false, then we would have $Y \subseteq M_{\mathfrak{P}}(X' \cup \{x\}) \smallsetminus Z$, whence $M_{\mathfrak{P}}(Y) = M_{\mathfrak{P}}(X) = M_{\mathfrak{P}}(X' \cup \{x\}) \subseteq B$, absurd.

Fix any point $y_x \in Y \cap Z$. Then, by the definition of Z, we get $\{x\} \twoheadleftarrow_{\mathfrak{P}} X' \cup \{y_x\} = (X \smallsetminus \{x\}) \cup \{y_x\}$, from which we conclude that $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$. □

In this section we will demonstrate that when one has a locally finite quasi-attractive pairing, then all the $\mathfrak{P}$-reducts of a maximum partitioner X have the same cardinality. Moreover, we will also deduce that quasi-attractiveness on a finite set Ω may be characterized by the fact that the $\mathfrak{P}$-reducts of any subset $X \in \wp(\Omega)$ form an exchangeable set system, property which characterizes the X-bases of a matroid on Ω, though in general $\mathcal{N}_{\mathfrak{P}}$ is not a matroid.

In the next result we will prove that all the reducts of a maximum partitioner of a locally finite quasi-attractive pairing have the same cardinality.

Theorem 4.4. *Let $\mathfrak{P} \in \mathrm{PR}_{\mathrm{lfqa}}(\Omega)$ and $X \in \mathcal{M}_{\mathfrak{P}}$ be such that $\mathcal{R}_{\mathfrak{P}}(X) \neq \varnothing$. Then $\mathcal{R}_{\mathfrak{P}}(X)$ has uniform cardinality.*

Proof. Let $W = \{w_1, \dots, w_n\}, V = \{v_1, \dots, v_m\} \in \mathcal{R}_{\mathfrak{P}}(X)$ be such that $m < n$. Take $v_1 \in V$. As $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$, there exists an element of W, say w_1, such that $\{v_1\} \twoheadleftarrow_{\mathfrak{P}} (V \smallsetminus \{v_1\}) \cup \{w_1\}$. Set $V^{(1)} := V$ and

$$V^{(2)} := (V \smallsetminus \{v_1\}) \cup \{w_1\} = \{w_1, v_2, \dots, v_m\}.$$

It is immediate to check that $V^{(2)} \approx_{\mathfrak{P}} V^{(1)} \approx_{\mathfrak{P}} V \approx_{\mathfrak{P}} W$. Therefore, we may use again the assumption that $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$ on the subsets $V^{(2)}$ and W and to the element v_2 in order to find $w_2 \in W$ such that $\{v_2\} \twoheadleftarrow_{\mathfrak{P}} (V^{(2)} \smallsetminus \{v_2\}) \cup \{w_2\}$. Set

$$V^{(3)} := (V^{(2)} \setminus \{v_2\}) \cup \{w_2\} = \{w_1, w_2, v_3, \dots, v_m\}.$$

Clearly we get $V^{(3)} \approx_{\mathfrak{P}} W$ and, thus, we may iterate the above procedure until we reach a subset $V^{(m)} := \{w_1, \dots, w_m\} \subsetneqq W$ such that $V^{(m)} \approx_{\mathfrak{P}} W$. Thus, we conclude that $W \notin \mathcal{R}_{\mathfrak{P}}(X)$, contradicting our choice of W. This proves the claim when Ω is finite.
Let now Ω be an infinite set and also assume $W \in \mathcal{R}_{\mathfrak{P}}(X)$ to be infinite. Also the subset V must be infinite. In fact, note that for any $v \in V$ there must correspond some finite subset $F_v \in \wp_{\mathrm{fin}}(W)$ such that $\{v\} \twoheadleftarrow_{\mathfrak{P}} F_v$. Let now $Y := \bigcup\{F_v \mid v \in V\}$. Clearly, it results $V \twoheadleftarrow_{\mathfrak{P}} Y$ and $Y \subseteq W$, whence $Y \approx_{\mathfrak{P}} V \approx_{\mathfrak{P}} W \approx_{\mathfrak{P}} X$. Since $W \in \mathcal{R}_{\mathfrak{P}}(X)$, we must necessarily have $Y = W$. If V were finite, even the subset Y (and so W) would be finite, which is impossible by our assumption. So V is infinite.
At this point, notice that $\|W\| \leq \sum_{v \in V} \|F_v\| \leq \aleph_0 \|V\| = \|V\|$. Repeating the above argument and exchanging the role of W and V, the conclusion of the present theorem holds. □

Corollary 4.5. *If Ω is a finite set, $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$ and $X \in \wp(\Omega)$, then $\mathcal{R}_{\mathfrak{P}}(X)$ has uniform cardinality.*

In the next result we will characterize quasi-attractiveness on finite sets through the fact that the set system of the $\mathfrak{P}$-reducts of any subset X is exchangeable.

Theorem 4.6. *Let Ω be a finite set. Then the following conditions are equivalent:*

(i) *$\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$;*
(ii) *for any $X \in \wp(\Omega)$ the set system $\mathcal{R}_{\mathfrak{P}}(X)$ is exchangeable.*

Proof. $(i) \implies (ii)$: Let $X \in \wp(\Omega)$, $Y, Z \in \mathcal{R}_{\mathfrak{P}}(X)$ and $y \in Y \setminus Z$. As $Y \approx_{\mathfrak{P}} Z$ and $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$, there exists $z \in Z$ such that

$$\{y\} \twoheadleftarrow_{\mathfrak{P}} (Y \setminus \{y\}) \cup \{z\}. \tag{20}$$

Set $W := (Y \setminus \{y\}) \cup \{z\}$. In order to show that $\mathcal{R}_{\mathfrak{P}}(X)$ is exchangeable, we will demonstrate that $z \in Z \setminus Y$ and $W \in \mathcal{R}_{\mathfrak{P}}(X)$. First of all, notice that $z \in Z \setminus Y$, otherwise, by (20), we would get $z \neq y$ and $\{y\} \twoheadleftarrow_{\mathfrak{P}} W = Y \setminus \{y\}$, contradicting the assumption $Y \in \mathcal{R}_{\mathfrak{P}}(X)$. Therefore, since $y \in Y \setminus Z$, it follows that $\|W\| = \|Y\|$. Moreover, again by (20), we get

$$W \approx_{\mathfrak{P}} Y \cup \{z\} \approx_{\mathfrak{P}} Y, \tag{21}$$

where the second equivalence follows by the fact that $Y \approx_{\mathfrak{P}} Z$ and $z \in Z$.

Assume now by contradiction that $W \notin \mathcal{R}_{\mathfrak{P}}(X)$. As $Y \in \mathcal{R}_{\mathfrak{P}}(X)$, we have that $X \twoheadleftarrow_{\mathfrak{P}} Y$, therefore, by (21), we also have that $X \twoheadleftarrow_{\mathfrak{P}} W$. As $W \notin \mathcal{R}_{\mathfrak{P}}(X)$ there exists $w \in W$ such that $X \twoheadleftarrow_{\mathfrak{P}} W \setminus \{w\}$. Take

$$\mathcal{G}_w := \{W' \in \wp(W \smallsetminus \{w\}) \mid X \twoheadleftarrow_{\mathfrak{P}} W'\}.$$

Then $\mathcal{G}_w \neq \varnothing$ because $X \twoheadleftarrow_{\mathfrak{P}} W \smallsetminus \{w\}$ and, since Ω is finite, we have $\mathrm{Min}(\mathcal{G}_w) \neq \varnothing$. Let then $W^* \in \mathrm{Min}(\mathcal{G}_w)$. Then $X \twoheadleftarrow_{\mathfrak{P}} W^*$ and, by the minimality of W^* in $\mathcal{G}_w$, we get

$$W^* \in \mathcal{R}_{\mathfrak{P}}(X) \text{ and } \|W^*\| < \|W\| = \|Y\| \tag{22}$$

As $Y \in \mathcal{R}_{\mathfrak{P}}(X)$, the conditions in (22) contradict Corollary 4.5. So $W \in \mathcal{R}_{\mathfrak{P}}(X)$.
$(ii) \implies (i)$: Let $X, Y \in \wp(\Omega)$ be such that $X \approx_{\mathfrak{P}} Y$ and let $x \in X$. We claim the existence of $y \in Y$ such that $\{x\} \twoheadleftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y\}$. If $\{x\} \twoheadleftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$ we clearly have $\{x\} \twoheadleftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y\}$ for any $y \in Y$, so there is nothing to prove. Similarly, if $x \in X \cap Y$, just choose $y = x$ to get the claim.
Therefore, we may assume that X are incomparable Y with respect to set-theoretical inclusion and take $x \in X \smallsetminus Y$ such that

$$\{x\} \not\twoheadleftarrow_{\mathfrak{P}} X \smallsetminus \{x\} \tag{23}$$

Consider at this point the set system $\mathcal{H}_X := \{A \in [X]_{\approx_{\mathfrak{P}}} \mid x \in A\}$. As $X \in \mathcal{H}_X$ and Ω is finite, we have that $\mathrm{Min}(\mathcal{H}_X) \neq \varnothing$. Let $\hat{X} \in \mathrm{Min}(\mathcal{H}_X)$. We claim that $\hat{X} \in \mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$. As $\hat{X} \in [X]_{\approx_{\mathfrak{P}}}$, we clearly have $M_{\mathfrak{P}}(X) \twoheadleftarrow_{\mathfrak{P}} \hat{X}$.

Let now $X' \subsetneq \hat{X}$ and assume that $X' \in [X]_{\approx_{\mathfrak{P}}}$, i.e. $X' \approx_{\mathfrak{P}} \hat{X}$. Then $x \in \Omega \smallsetminus X'$ and, using (D1)-(D3), we easily deduce that $\{x\} \twoheadleftarrow_{\mathfrak{P}} X \smallsetminus \{x\}$, in contrast with (23). So $\hat{X} \in \mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$.

Denote now by Y' some $\mathfrak{P}$-reduct of Y. In view of our assumptions, we easily deduce that $\hat{X}$ and Y' are incomparable with respect to set-theoretical inclusion. So, as $\hat{X}, Y' \in \mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$, by the exchangeability of $\mathcal{R}_{\mathfrak{P}}(M_{\mathfrak{P}}(X))$ there exists $y_x \in Y' \smallsetminus \hat{X}$ such that $\{x\} \twoheadleftarrow_{\mathfrak{P}} (\hat{X} \smallsetminus \{x\}) \cup \{y_x\}$. As $\hat{X} \in \wp(X)$, it follows that $\{x\} \twoheadleftarrow_{\mathfrak{P}} (\hat{X} \smallsetminus \{x\}) \cup \{y_x\} \twoheadleftarrow_{\mathfrak{P}} (X \smallsetminus \{x\}) \cup \{y_x\}$, i.e. $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$. □

Remark 4.7. When Ω is a finite set and $\mathfrak{P} \in \mathrm{PR}_{\mathrm{qa}}(\Omega)$, the equality $\mathcal{R}_{\mathfrak{P}}(X) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ does not hold, as we will see in the last section. Nevertheless, notice that in general $\mathcal{R}_{\mathfrak{P}}$ coincides with the family of the bases of a matroid on Ω which is a sub-set system of $\mathcal{N}_{\mathfrak{P}}$.

5. The Petersen graph is attractive

In the previous Section 3 we investigated the main general properties of attractive pairings and determined two specific kinds of attractive pairings derived from graphs by means of the theoretical results we gave for a generic attractive pairing. In the present section we prove that the adjacence pairing $\mathfrak{P}[\mathrm{Pet}, \mathrm{adj}]$ induced by the *Petersen graph* is attractive. By the convention stated in Subsection 2.1, here we will replace $\mathfrak{P}[\mathrm{Pet}, \mathrm{adj}]$ by Pet. In Fig. 5 we represent the Petersen graph and its corresponding adjacence matrix

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_1	0	1	0	0	1	1	0	0	0	0
v_2	1	0	1	0	0	0	1	0	0	0
v_3	0	1	0	1	0	0	0	1	0	0
v_4	0	0	1	0	1	0	0	0	1	0
v_5	1	0	0	1	0	0	0	0	0	1
v_6	1	0	0	0	0	0	0	1	1	0
v_7	0	1	0	0	0	0	0	0	1	1
v_8	0	0	1	0	0	1	0	0	0	1
v_9	0	0	0	1	0	1	1	0	0	0
v_{10}	0	0	0	0	1	0	1	1	0	0

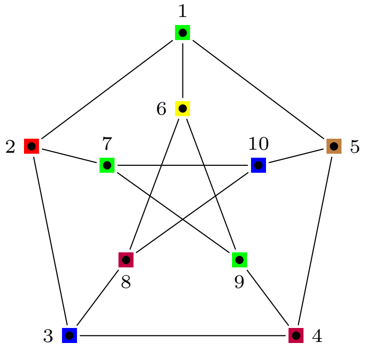


Fig. 5. On the left, the Adjancence Matrix of the Petersen Graph Pet, while on the right with the Symmetry Partition Blocks Induced by $X = \{1, 3, 8\}$.

and, as an example, we exhibit the symmetry partition with respect to the vertex subset $X = \{1, 3, 8\}$, i.e. $\pi_{\mathrm{Pet}}(X) = v_1 v_7 v_9 | v_2 | v_3 v_{10} | v_4 v_8 | v_5 | v_6$.

The vertices of the Petersen graph can be identified with the 2-subsets of $\hat{5} := \{1, 2, 3, 4, 5\}$, in such a way that two vertices X and Y are adjacent if and only if their corresponding 2-subsets are disjoint. In what follows, we write v_{ij} to denote the vertex identified with $\{i, j\}$; moreover, the letters h, i, j, k, l will denote all elements in $\hat{5}$ in an arbitrary order. Recall that Pet is a strongly regular graph, with parameters $(10, 3, 0, 1)$. In other terms, every vertex of Pet admits 3 neighbors and, for any $v, w \in \mathrm{Pet}$,

$$\|N_{\mathrm{Pet}}(v) \cap N_{\mathrm{Pet}}(w)\| = \begin{cases} 0 & \text{if } v \sim w \\ 1 & \text{if } v \nsim w \end{cases}$$

In what follows, we will provide a characterization of the geometric configurations induced by the members of the Moore system $\mathcal{M}_{\mathrm{Pet}}$ and, next, taking into account the previous structural properties of Pet, we demonstrate that it induces an attractive graph.

We divide the proof in some propositions, where we analyze the various cases occurring, and next we recollect them in a single statement. First of all notice that if $\|X\| \le 1$, then $X \in \mathcal{M}_{\mathrm{Pet}}$ since the Petersen graph is connected and twin-free (i.e. any two vertices v and w are such that $N_G(v) \ne N_G(w)$). Furthermore, 2-subsets of $V(\mathrm{Pet})$ are maximum partitioners. This we now present.

Proposition 5.1. *We have that* $\wp_2(V(\mathrm{Pet})) \subseteq \mathcal{M}_{\mathrm{Pet}}$.

Proof. Set $G := \mathrm{Pet}$. We claim that any 2-subset X belongs to $\mathcal{M}_G$. To this regard, by part (i) of Proposition 2.7 it suffices to show that whenever we take $u \in V(G) \setminus X$, then $\pi_G(X) \ne \pi_G(X \cup \{u\})$, i.e. there exists $z, z' \in V(G)$ such that $z \equiv_X z'$ but $z \not\equiv_{X \cup \{u\}} z'$.

To this end, let $X = \{v, w\}$ and $u \in V(G) \setminus X$. Assume first that $v \sim w$. Two cases may occur: either $u \nsim v$ and $u \nsim w$ or u is adjacent to only one vertex of X. In the first case, by our choice of v and w and since $u \nsim v$, by the strong regularity we get $N_G(u) \cap N_G(v) = \{z\}$ and $N_G(v) \cap N_G(w) = \varnothing$. Let moreover $z' \in N_G(v) \setminus \{z\}$. It may be easily checked that $z \equiv_X z'$ and $z \not\equiv_{X \cup \{u\}} z'$.

In the second case, suppose that $u \sim v$ and $u \nsim w$. Then there exist $h, i, j, k \in \hat{5}$ such that $w = v_{ij}$, $v = v_{hk}$ and $u = v_{il}$. Using the strong regularity of G, it may be easily checked that $N_G(X) = \{v_{ij}, v_{ih}, v_{jh}, v_{hk}, v_{kl}, v_{hl}\} \neq V(G)$. Therefore there exists $z \in V(G) \setminus [N_G(w) \cup N_G(v) \cup N_G(u)]$. Moreover, let us note that there exists $z' \in V(G)$ such that $z' \nsim v$, $z' \nsim w$ and $z' \sim u$. In fact, as $N_G(u) = \{v, u', u''\}$, using the strong regularity of G it is straightforward to see that $u', u'' \notin N_G(v) \cup N_G(w)$. Let $z' := u'$ (the case $z' := u''$ is the same). Then $z \equiv_X z'$ but $z \not\equiv_{X \cup \{u\}} z'$.
Take now $X = \{v, w\}$, with $v \nsim w$. Three cases may occur: or v, w and u form a 3-path, or $u \sim v$ and $u \nsim w$ or u, v and w are three non-adjacent vertices. In the first situation, the vertex u is the only element of $N_G(v) \cap N_G(w)$. Let $v = v_{ij}$ and $w = v_{ih}$ for some $h, i, j \in \hat{5}$. So $u = v_{kl}$. Take then $z = v_{hj}$ and $z' = v_{ik}$. It is straightforward to see that $z \equiv_X z'$ and $z \not\equiv_{X \cup \{u\}} z'$.
In the second situation, let $v = v_{ij}$ and $w = v_{ih}$ for some $h, i, j \in \hat{5}$. We may take $u = v_{hk}$. Notice that there exist $z, t \in V(G)$ such that $\{z\} = N_G(u) \cap N_G(w)$ and $\{t\} = N_G(w) \cap N_G(v)$. Denote by z' the third vertex which is adjacent to w. It may be easily seen that $z \equiv_X z'$ and $z \not\equiv_{X \cup \{u\}} z'$.
In the third case, we have that u, v and w are three non-adjacent vertices. If $v = v_{ij}$ and $w = v_{ih}$ for some $h, i, j \in \hat{5}$, then there are exactly three vertices non-adjacent to both v and w, namely $u = v_{ik}$, $z = v_{jh}$ and $z' = v_{il}$. Hence, we get $z \sim u$ and $z' \nsim u$. Therefore $z \equiv_X z'$ and $z \not\equiv_{X \cup \{u\}} z'$. Thus $X \in \mathcal{M}_G$ when X is a 2-subset. □

Let us now characterize the maximum partitioners with three elements.

Proposition 5.2. *Let $X \in \wp_3(V(\mathrm{Pet}))$. Then $X \in \mathcal{M}_{\mathrm{Pet}}$ if and only if $\mathrm{Pet}[X] \cong P_{2,1}$.*

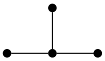
Proof. Set $G := \mathrm{Pet}$ and let $X = \{u, v, w\}$ be 3-subset. Let first $G[X] \cong P_{2,1}$. In view of the strong regularity of G, it may be easily shown that $\|N_G(X)\| = 7$. Thus, the three vertices in $V(G) \setminus N_G(X)$ form a single X-symmetry block. It is a simple exercise to check that $\varnothing \neq V(G) \setminus N_G(X \cup \{t\}) \subsetneq V(G) \setminus N_G(X)$ whenever $t \notin X$. So $X \in \mathcal{M}_G$.

Assume now that $G[X] \cong P_{1,1,1}$. Then there exists $z \in V(G) \setminus X$ such that either $X = N_G(z)$ or z is not adjacent to any of the three vertices of X. In the first case, the addition of the vertex z to X does not affect the induced symmetry partition, i.e. $\pi_G(X) = \pi_G(X \cup \{z\})$. In the second case, let us note that $\pi_G(X) = X \cup \{z\} | \{t\}_{t \in V(G) \setminus (X \cup \{z\})}$. In fact, there is no pair of adjacent vertices in $X \cup \{z\}$ and, because of the strong regularity of G, the vertices of $V(G) \setminus (X \cup \{z\})$ are the common neighbors of pairs of vertices of $X \cup \{z\}$. Nevertheless, by the same reason, it results that $\pi_G(X \cup \{z\}) = \pi_G(X)$. Therefore, when $G[X] \cong P_{1,1,1}$, we conclude that $X \notin \mathcal{M}_G$.

Let $G[X] \cong P_3$. Without loss of generality, suppose that $u \sim v$ and $v \sim w$. Let $z \in V(G)$ be such that $v \sim z$, $z \nsim u$ and $z \nsim w$. Then we have that $\pi_G(X) = \pi_G(X \cup \{z\})$. In fact, the other two vertices $t, t' \in N_G(z) \setminus \{v\}$ are not adjacent to the vertices of X, so $t \equiv_X t'$ and $t \equiv_{X \cup \{z\}} t'$. The other X-symmetry blocks remain the same since the involved vertices are not adjacent to z. Thus $\pi_G(X) = \pi_G(X \cup \{z\})$, i.e. $X \notin \mathcal{M}_G$. □

In the following result we will exhibit all the 4-subsets of Pet which are also maximum partitioners.

Proposition 5.3. *Let $X \in \wp_4(V(\mathrm{Pet}))$. Then $X \in \mathcal{M}_{\mathrm{Pet}}$ if and only if $G[X] \cong P_{2,2}$ or $G[X] \cong P_{1,1,1,1}$ or $G[X] \cong H_1$, where H_1 has the following form:*



Proof. Set $G := \mathrm{Pet}$ and let $X = \{u, v, w, t\}$. Assume that $G[X] \cong P_4$, with $u \sim v$, $v \sim w$ and $w \sim t$. Without loss of generality, we may suppose that $v = v_{ij}$, $u = v_{hl}$, $w = v_{hk}$ and $t = v_{il}$. Therefore, it may be easily checked through the strong regularity of G that $N_G(X) = \{v_{ij}, v_{hl}, v_{hk}, v_{il}, v_{hj}, v_{jk}, v_{jl}, v_{kl}, v_{ik}\}$. Thus $\|N_G(X) \setminus N_G(X)\| = 1$. Denote by c the third vertex in $N_G(v)$. It does not belong to X. Set $B := X \cup \{c\}$. We claim that $\pi_G(X) = \pi_G(X \cup \{c\})$. To this end, take two distinct vertices $z, z' \in V(G)$ such that $z \equiv_X z'$. Using the strong regularity, it may be easily proved the existence of a unique element $x \in X$ such that $z \sim x$ and $z' \sim x$. As a consequence, we also have $z \nsim z'$.

Notice now that neither $x = u$ nor $x = t$. In fact, if $z, z' \in N_G(u)$ (the case of t is similar), since $u \nsim t$, only one among z and z' must be adjacent to t and this implies $z \not\equiv_X z'$, in contrast with our assumption. So, either $x = v$ or $x = w$. If $z, z' \in N_G(v)$, it may be easily checked that $z = u$ and $z' = c$, so that $z \equiv_B z'$.

On the other hand, if $z, z' \in N_G(w)$, it is straightforward to check that $z = t$ and $z' = y$, where y denotes the third vertex of $N_G(w)$. Clearly, we cannot have $c \sim y$, otherwise we would find a 4-cycle in G. This implies that $z \equiv_B z'$, showing that $\pi_G(X) = \pi_G(X \cup \{c\})$, whence $X \notin \mathcal{M}_G$.

Assume now that $G[X] \cong P_{3,1}$, with $u \sim v$ and $v \sim w$. Without loss of generality, we may suppose that $v = v_{ij}$, $u = v_{hl}$, $w = v_{hk}$ and $t = v_{hi}$. Then, using the strong regularity of G, we have $N_G(X) = \{v_{ih}, v_{ij}, v_{hk}, v_{ik}, v_{il}, v_{kl}, v_{jk}, v_{jl}\}$, so that $\|N_G(X) \setminus N_G(X)\| = 2$ and $t \in V(G) \setminus N_G(X)$. Denote by c the vertex in $V(G) \setminus (N_G(X) \cup \{t\})$ and set $B := X \cup \{c\}$. We claim that $\pi_G(X) = \pi_G(\mathrm{B})$. To this end, take two distinct vertices $z, z' \in V(G)$ such that $z \equiv_X z'$.

It may happen that either $z, z' \in V(G) \setminus N_G(X)$ or $z = u$ and $z' = w$. In fact, let $x \in X$ be the only vertex which is both adjacent to z and to z'. It cannot be $x = t$ since the strong regularity of G implies that every neighbor of t is also a neighbor of one among u, v and w; moreover it cannot be $x = u$ (the case $x = w$ is similar) since the vertices u and t share only a common neighbor by the strong regularity of G. Thus $x = v$ and then $z = u$ and $z' = w$.

At this point, it is straightforward to check that $\pi_G(X) = \pi_G(\mathrm{B})$, whence $X \notin \mathcal{M}_G$.

Assume now that $G[X] \cong P_{2,1,1}$, with $v \sim w$. Without loss of generality, we may suppose that $v = v_{ij}$, $w = v_{hl}$, $u = v_{jl}$ and $t = v_{il}$. Using the strong regularity of G, we have $N_G(X) = \{v_{jl}, v_{kl}, v_{ij}, v_{hk}, v_{hj}, v_{hl}, v_{ik}, v_{jk}\}$. Thus $V(G) \setminus N_G(X) = \{v_{ih}, v_{il}\} = \{u, t\}$. Furthermore, by the previous computation, we observe that $N_G(w) \cap N_G(u) \cap N_G(t) = \varnothing$ and $N_G(v) \cap N_G(u) \cap N_G(t)$ has only one element, that we denote by $\{c\}$.

Set $B := X \cup \{c\}$. We claim that $\pi_G(X) = \pi_G(\mathrm{B})$. To this end, take two distinct vertices $z, z' \in V(G)$ such that $z \equiv_X z'$. If $z \sim z'$, then by the strong regularity it follows that $z, z' \in V(G) \setminus N_G(X)$, while if $z \nsim z'$ then there exists a unique vertex $x \in \Omega$ such that $z \sim x$ and $z' \sim x$ so that, if $x \notin X$, then we again have $z, z' \in V(G) \setminus N_G(X)$. The previous computations force the vertex x to coincide with v. Therefore we get $z = y$ and $z' = w$, where y denotes the third vertex in $N_G(v)$. As $z \equiv_X z'$, we may have either $z = u$ and $z' = t$ or $z = y$ and $z' = w$. It is now straightforward to check that $\pi_G(X) = \pi_G(\mathrm{B})$.

Let $G[X] \cong P_{1,1,1,1}$. The vertices of X are all isolated and, by the strong regularity of G, any pair of vertices of X admits one common neighbor. This suffices to show that $\pi_G(X) = X|\{v\}_{v \in N_G(X)}$. Take now $c \in V(G) \setminus X$. Hence there are two vertices of X, say v and w, such that $c \in N_G(v) \cap N_G(w)$. Then we get $v \equiv_X u$ and $v \not\equiv_{X \cup \{c\}} u$, proving that $X \in \mathcal{M}_G$.

Let now assume that $G[X] \cong H_1$, with $u \sim v$, $v \sim w$ and $v \sim t$. Then there exists a vertex a such that $X = \{a\} \cup N_G(a)$. As v is the common neighbor of each pair of non-adjacent vertices in X, we deduce that any vertex of $X \setminus \{a\}$ admits two other distinct neighbors, so $N_G(X) = V(G)$. Now, let $z, z' \in V(G)$ be such that $z \equiv_X z'$. Then there exists only a vertex $x \in X$ such that $z \sim x$ and $z' \sim x$. So, it may be easily checked that $\pi_G(X) = v|N_G(v)\,|\,(N_G(y) \setminus \{v\})_{y \in N_G(v)}$. Take now $c \in V(G) \setminus X$. Denote by z, z' the elements of $N_G(t) \setminus \{v\}$. If $c \in N_G(z)$ (the case $c \in N_G(z')$ is similar), then $z \not\equiv_{X \cup \{c\}} z'$. Moreover, if $c = z$ (the case $c = z'$ is similar), then $t \not\equiv_{X \cup \{z\}} u$. Thus $X \in \mathcal{M}_G$.

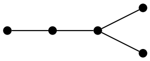
Finally, assume that $G[X] \cong P_{2,2}$, with adjacencies $u \sim v$ and $w \sim t$. We first find the symmetry partition induced by X. Since $u \sim v$, there exist four distinct indices h, j, k, $l \in \hat{5}$ such that $u = v_{hj}$ and $v = v_{kl}$. Furthermore, as $w \sim t$ without being adjacent to u and v, we may have $\{w, t\} = \{v_{ij}, v_{jl}\}$ or $\{w, t\} = \{v_{il}, v_{jk}\}$. Assume that $\{w, t\} = \{v_{ik}, v_{jl}\}$ (the proof in the other case is similar). Since $v_{ij} \nsim v_{jl}$, there exists only a vertex, namely v_{hk}, which is adjacent to both v_{ij} and v_{jl}. Similarly, let v_{hl} be the only vertex adjacent to both v_{ij} and v_{ik}, v_{hi} be the only vertex adjacent to both v_{kl} and v_{jl}, and v_{hj} be the only vertex adjacent to both v_{kl} and v_{ik}. In this way, we get $N_G(X) = \{v_{ij}, v_{kl}, v_{ik}, v_{jl}, v_{hk}, v_{hl}, v_{hi}, v_{hj}\} = V(G) \setminus \{v_{il}, v_{jk}\}$. Thus, it results that $v_{il} \equiv_X v_{jk}$ since they are not adjacent to the vertices in X.

At this point, take $z, z' \in N_G(X)$ be such that $z \equiv_X z'$. Using the strong regularity of G, we find a unique vertex $x \in X$ such that $z \sim x$ and $z' \sim x$. Without loss of generality, let $x = v_{ij}$. Thus $\{z, z'\} \subseteq \{v_{kl}, v_{hl}, v_{hk}\}$. Nevertheless, $v_{hl} \sim v_{ik}$ but $v_{kl} \nsim v_{ik}$; similarly, $v_{hk} \sim v_{jl}$ but $v_{kl} \nsim v_{jl}$ and, finally, $v_{hk} \nsim v_{ik}$. This forces $z = z'$ and, therefore, we get $\pi_G(X) = \{v\}_{v \in N_G(X)}|V(G) \setminus N_G(X)$.

To conclude the proof, we claim that $X \in \mathcal{M}_G$. It suffices to demonstrate that any vertex $c \notin X$ is always adjacent to only one among the vertices of $V(G) \setminus N_G(X)$. To this end, notice that $c \in \{v_{hl}, v_{hj}, v_{hk}, v_{hi}, v_{jk}, v_{il}\}$, that $N_G(v_{il}) = \{v_{hj}, v_{jk}, v_{hk}\}$ and $N_G(v_{jk}) = \{v_{hl}, v_{hi}, v_{il}\}$. Hence, there always exists an edge between the vertices in $V(G) \setminus X$ and those of $V(G) \setminus N_G(X)$. As no pair of vertices may have two common neighbors, we get $\pi_G(X) \neq \pi_G(X \cup \{c\})$, i.e. $X \in \mathcal{M}_G$. □

In the next technical result, whose proof is straightforward, we exhibit the following properties of specific 4-subsets to be used when showing the attractiveness of the relation $\twoheadleftarrow_{\mathrm{Pet}}$.

Lemma 5.4. *Let* $X = \{u, v, w, t\} \in \wp_4(V(\mathrm{Pet}))$ *be such that* $\mathrm{Pet}[X]$ *is isomorphic to one among* $P_{3,1}$, $P_{2,1,1}$, P_4. *Then there exists* $Y \in \wp(V(\mathrm{Pet}))$ *containing* X, *with* $\pi_{\mathrm{Pet}}(X) = \pi_{\mathrm{Pet}}(Y)$ *and* $\mathrm{Pet}[Y] \cong H_3$, *where* H_3 *has the following form:*



H_3

In the next two propositions we respectively investigate the behavior of the 5-subsets and of the 6-subsets of Pet.

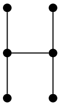
Proposition 5.5. *Let* $X \in \wp_5(V(\mathrm{Pet}))$. *Then one of the following cases occurs:*

(*i*) $X \in \mathcal{R}_{\mathrm{Pet}}$;
(*ii*) *there exists a* 6*-subset* $Y \in \mathcal{M}_{\mathrm{Pet}}$ *such that* $X \subseteq Y$ *and* $\pi_{\mathrm{Pet}}(X) = \pi_{\mathrm{Pet}}(\mathrm{Y})$.

Proof. Let $X \in \wp(V(\mathrm{Pet}))$ be a 5-subset. By the proof of Theorem 7.10 of [10], it results that $\mathrm{Pet}[X]$ is isomorphic to one among the geometric configurations C_5, P_5, $P_{4,1}$, $P_{2,2,1}$, $P_{3,1,1}$ or H_3, where H_3 is as in the statement of Lemma 5.4. In the first four cases, we have that $X \in \mathcal{R}_{\mathrm{Pet}}$, while in the latter two cases, just following the proof of Theorem 7.10 of [10], we can check the existence of a 6-subset $Y \in \mathcal{M}_{\mathrm{Pet}}$ such that $X \subseteq Y$ and $\pi_{\mathrm{Pet}}(X) = \pi_{\mathrm{Pet}}(\mathrm{Y})$. □

Lemma 5.6. *Let* $X \in \wp_6(V(\mathrm{Pet}))$. *Then the following conditions are equivalent:*

(*i*) $X \not\twoheadleftarrow_{\mathrm{Pet}} V(\mathrm{Pet})$;
(*ii*) $X = N_{\mathrm{Pet}}(v) \vartriangle N_{\mathrm{Pet}}(w)$ *where* $v \sim w$;
(*iii*) $\mathrm{Pet}[X] \cong H_2$, *where* H_2 *has the form:*



H_2

Proof. In our proof we set $G := \mathrm{Pet}$.

$(ii) \iff (iii)$: It has been already proved in Proposition 7.2 of [10].

$(ii) \implies (i)$: Assume that $X = N_G(v) \bigtriangleup N_G(w)$ for some two adjacent vertices v and w. Without loss of generality, we can suppose that $N_G(v) := \{w, v', v''\}$ and $N_G(w) := \{v, w', w''\}$, with v', v'', w', w'' distinct vertices. Let us check that $v' \equiv_X v''$ (and similarly $w' \equiv_X w''$). We have $v' \sim v$ and $v'' \sim v$ by our choice of $N_G(v)$. Moreover, by the strong regularity, we have $v' \nsim w$, $v'' \nsim w$ and $v' \nsim v''$. Next, we also have $v' \nsim w'$. Indeed, if had $v' \sim w'$, then we would obtain $N_G(v') \cap N_G(w) = \{v, w'\}$, in contrast with the fact that non-adjacent vertices in G must have only one common neighbor. A similar argument shows that $v' \nsim w''$, $v'' \nsim w'$ and $v'' \nsim w''$. This proves that $v' \equiv_X v''$. Hence, since $\pi_G(V(G)) = v_1|\dots|v_{10}$ because G is a twin-free graph, we conclude that $X \not\approx_G V(G)$.

$(i) \implies (ii)$: Assume $X \not\approx_G V(G)$. We claim that $X = N_G(v) \bigtriangleup N_G(w)$, for some two adjacent vertices v and w. As $X \not\approx_G V(G)$, there exist two distinct vertices $v', v'' \in V(G)$ such that $v' \equiv_X v''$. Let us check that $v' \nsim v''$. Suppose by contradiction that $v' \sim v''$. Then $v', v'' \notin X$ because otherwise $F_G(v', v') = 0 \neq 1 = F_G(v', v'')$, whence $v' \not\equiv_X v''$, in contrast with the assumption. Now, in view of the strong regularity, v' and v'' cannot have a common neighbor. Thus, as $v' \equiv_X v''$, we have that $v' \nsim x$ and $v'' \nsim x$ for every $x \in X$. Being $N_G(v') = \{v'', v'_1, v'_2\}$ and $N_G(v'') = \{v', v''_1, v''_2\}$, we deduce that $N_G(v') \cup N_G(v'') \subseteq V(G) \smallsetminus X$, but $\|N_G(v') \cup N_G(v'')\| = 6$ by the strong regularity of G, while $\|V(G) \smallsetminus X\| = 4$, absurd. Thus, we proved that $v' \nsim v''$.

Again by the strong regularity, there exists a unique vertex $v \in V(G)$ such that $v \sim v'$ and $v \sim v''$. Set $N_G(v') := \{v, v'_1, v'_2\}$ and $N_G(v'') := \{v, v''_1, v''_2\}$, with v'_1, v'_2, v''_1, v''_2 be distinct vertices. Observe that $v'_1 \notin X$ because otherwise $F(v', v'_1) = 1 \neq 0 = F(v'', v'_1)$, in contrast with the fact that $v' \equiv_X v''$. Similarly, we have $v'_2 \notin X$, $v''_1 \notin X$ and $v''_2 \notin X$. As X is a 6-subset, we deduce that $v \in X$. Analogously, being $\{v', v''\} \cap \{v'_1, v'_2, v''_1, v''_2\} = \varnothing$, we get $\{v', v''\} \subseteq X$.

At this point, consider $N_G(v) = \{v', v'', w\}$. Note that $w \neq v'_1$, otherwise $\{w\} \in N_G(v') \cap N_G(v)$, in contrast with the strong regularity of G. Similarly, we have that $w \neq v''_1$, $w \neq v'_2$ and $w \neq v''_2$, i.e. $w \in X$.

Finally, let $N_G(w) = \{v, w', w''\}$. Hence $\{w', w''\} \cap \{v', v''\} = \varnothing$ because $v \sim w$. Let us see that $w' \in X$ and $w'' \in X$. Suppose by contradiction that $w' \notin X$ (the proof for w'' is the same). Without loss of generality, we may suppose that $w' = v'_1$. Then, by what we proved above, it results that $w \nsim v'$. Furthermore, we have $N_G(w) \cap N_G(v') = \{v, w'\}$, contradicting the strong regularity of G. So, we must have $w' \in X$. This proves that $X = N_G(v) \cup N_G(w) = N_G(v) \bigtriangleup N_G(w)$, with $v \sim w$. □

Proposition 5.7. *Let $X \in \wp_6(V(\mathrm{Pet}))$ be such that* $\mathrm{Pet}[X] \cong H_2$. *Then $X \in \mathcal{M}_{\mathrm{Pet}}$.*

Proof. Let $u \in V(G) \smallsetminus X$. In view of part (i) of Proposition 2.7, we need to find two vertices $z, z' \in V(G)$ such that $z \equiv_X z'$ and $z \not\equiv_{X \cup \{u\}} z'$. Using the same notations as those of the proof of Lemma 5.6, without loss of generality we may assume $u = v'_1$. Set

$z := v'$ and $z' := v''$. Then $z \equiv_X z'$ and, moreover, $z \sim u$ and $z' \nsim u$, whence $z \not\equiv_{X \cup \{u\}} z'$. This proves that $X \in \mathcal{M}_G$. □

Corollary 5.8. *We have that* $\wp_7(V(\mathrm{Pet})) \subseteq [V(\mathrm{Pet})]_{\approx_{\mathrm{Pet}}}$.

Let us recollect the above results in the following theorem, where we provide all the members of $\mathcal{M}_{\mathrm{Pet}}$.

Theorem 5.9. *Let* $X \in \wp(V(\mathrm{Pet}))$. *Then* $X \in \mathcal{M}_{\mathrm{Pet}}$ *if and only if one of the following conditions holds:*

- $\|X\| \leq 2$;
- $X = V(\mathrm{Pet})$;
- $\|X\| = 3$ *and* $G[X] \cong P_{2,1}$;
- $\|X\| = 4$ *and* $G[X] \cong P_{2,2}$, *or* $G[X] \cong P_{1,1,1,1}$ *or* $G[X] \cong H_1$;
- $\|X\| = 6$ *and* $X = N_{\mathrm{Pet}}(v) \triangle N_{\mathrm{Pet}}(w)$, *for some* $v \sim w$.

In the next result we will demonstrate that the Petersen graph is attractive.

Theorem 5.10. *Let* $V := V(\mathrm{Pet})$. *We have that* $\mathrm{Pet} \in \mathrm{PR_a}(V)$.

Proof. Set $G := \mathrm{Pet}$. In view of Theorem 5.9, we may assume that $1 \leq \|A\| \leq 6$. Furthermore, by Proposition 5.5, Lemma 5.6 and Proposition 5.7, notice that when A is a 6-subset it is never possible to choose at the same time two vertices $b \in \Omega$ and $a \in A$ such that $\{b\} \not\prec_G A$ and $\{a\} \not\prec_G A \setminus \{a\}$.
Let now A be a singleton or a 2-subset. Hence, in view of Theorem 5.9 for any choice of $b \in \Omega$ and $a \in A$ such that $\{b\} \not\prec_G A$ and $\{a\} \not\prec_G A \setminus \{a\}$ it easily follows that $\{a\} \not\prec_G A \triangle \{a, b\}$.
Let A be an arbitrary 3-subset. Clearly, $\{a\} \not\prec_G A \setminus \{a\}$ for any $a \in A$. Let $\{b\} \not\prec_G A$ and set $X := (A \setminus \{a\}) \cup \{b\}$. Three possible cases may occur:

- $G[A] \cong P_{1,1,1}$.

As we argued in Proposition 5.2, there exists $z \in V(G)$ such that either $A = N_G(z)$ or z is not adjacent to any of the three vertices of A. Suppose first that $A = N_G(z)$ for some $z \in V(G)$. Then we get $b \in V(G) \setminus (A \cup \{z\})$. Hence either $G[X] \cong P_{2,1}$ or $G[X] \cong P_{1,1,1}$. If $G[X] \cong P_{2,1}$, in view of Proposition 5.2 we clearly have $\{a\} \not\prec_G X$. Otherwise, if $G[X] \cong P_{1,1,1}$, there exists $w \in V(G)$ which is not adjacent to the vertices of X. Again by the proof of Proposition 5.2, it results that $X \approx_G X \cup \{w\}$ and, by Proposition 5.3, that $X \cup \{w\} \in \mathcal{M}_G$. So, $\{a\} \not\prec_G X$.

Suppose now that there exists $z \in V(G)$ which is not adjacent to any of the three vertices of A. By the proof Proposition 5.2 it results that $X \approx_G X \cup \{z\}$ and we also have $X \cup \{z\} \in \mathcal{M}_G$ by Proposition 5.3. So, we get $b \neq z$. On the one hand, if b is adjacent to two vertices of A, when a is one of them and we replace it with b, we obtain $G[X] \cong P_{2,1}$ and, thus $\{a\} \nprec_G X$; while when a is the remaining vertex and we replace it with b, we obtain $G[X] \cong P_3$. By Proposition 5.2, there exists $y \in V(G) \setminus \{a\}$ such that $M_G(X) = X \cup \{y\}$ and $X \cup \{y\} \cong H_1$. Thus, again, $\{a\} \nprec_G X$.

On the other hand, if b is adjacent to only one vertex of A, when a is one of the two remaining vertices and we replace it with b, we obtain $G[X] \cong P_{2,1}$ and, thus $\{a\} \nprec_G X$; while when $a \sim b$, we get $G[X] \cong P_{1,1,1}$. By our choice of b, there must necessarily exists $t \in V(G)$ such that $N_G(t) = X$. Therefore $M_G(X) = X \cup \{t\}$ and, thus, $\{a\} \nprec_G X$.

- $G[A] \cong P_3$.

Let z be the third vertex adjacent to the central vertex of the 3-path A. Then $b \neq z$ in view of Proposition 5.3. Thus, either b is adjacent to only one of the remaining vertices of the 3-path A or $b \in V(G) \setminus N_G(\mathrm{A})$. In the first case, when we choose a to be the second extreme of the 3-path A, we obtain $G[X] \cong P_3$ so, taking the vertex y which is adjacent to the central vertex of the 3-path X, we conclude that $y \neq a$ and $M_G(X) = X \cup \{y\}$, whence $\{a\} \nprec_G X$; otherwise when we choose a to be one of the other vertices, it results that $G[X] \cong P_{2,1}$ and, hence $\{a\} \nprec_G X$. In the second case, whenever we replace a vertex a with b we get $G[X] \cong P_{2,1}$ and, hence $\{a\} \nprec_G X$.

- $G[A] \cong P_{2,1}$

If a is the isolated vertex, b can be adjacent to at least one of the remaining vertices, so that $G[X] \cong P_3$. In such a situation, denote by v the central vertex of the resulting 3-path $G[X]$. As $M_G(X) = X \cup \{y\}$ where y is the third vertex of $N_G(v)$ and $y \neq a$, we conclude that $\{a\} \nprec_G X$. On the other hand, b can be an isolated vertex, i.e. $G[X] \cong P_{2,1}$, and then $\{a\} \nprec_G X$ as $X \in \mathcal{M}_G$.

Let now $A = \{u, v, w, t\}$ be a 4-subset. In view of Propositions 5.2 and 5.3 it may be easily checked that $A \setminus \{a\} \approx_G A$ for any $a \in A$ if and only if either $G[A] \cong P_{1,1,1,1}$ or $G[A] \cong H_1$. So, we need to examine the remaining four possible main cases.

- $G[A] \cong P_{2,2}$.

Without loss of generality, assume that $u \sim v$ and $w \sim t$. Take $a \in A$ and $b \in V(G) \setminus A$ and set $X := A \triangle \{a, b\}$. We may have either $b \in V(G) \setminus N_G(\mathrm{A})$ or b is adjacent to only one vertex of any of the two 2-paths, for instance $b \in N_G(u) \cap N_G(w)$. On the one hand, if $b \in V(G) \setminus N_G(\mathrm{A})$, then $G[X] \cong P_{2,1,1}$. Moreover, in view of Proposition 5.3 and Lemma 5.4, we get $M_G(X) = X \cup \{x, y\}$ where x is adjacent to b, to the other isolated vertex of X and to one of the vertices of the 2-path of X, while y is the third neighbor of the previous vertex of the 2-path of X. As $a \notin \{x, y\}$, we conclude that $\{a\} \nprec_G X$. On the other hand, if $b \in N_G(u) \cap N_G(w)$ we have either $G[X] \cong P_{3,1}$ or $G[X] \cong P_4$.

When $G[X] \cong P_{3,1}$, by the proof of Proposition 5.3 we get $M_G(X) = X \cup \{x,y\}$, where $x \in V(G) \smallsetminus (N_G(\mathrm{A}) \cup \{v\})$ and $y \in N_G(x) \cap N_G(v)$; while, when $G[X] \cong P_4$, we get $M_G(X) = X \cup \{x,y\}$, where x is the third vertex of $N_G(b)$ and y is the third vertex of $N_G(u)$. In both the previous situations, we get $a \notin \{x,y\}$, whence $\{a\} \not\leftarrowtail_G X$.

- $G[A] \cong P_4$.

Without loss of generality, assume that $u \sim v$, $v \sim w$ and $w \sim t$. Let $\{b\} \not\leftarrowtail_G A$. In view of Proposition 5.3, it follows that $b \notin N_G(v) \cup N_G(w)$. Set $X := A \vartriangle \{a,b\}$. Some possible subcases may occur.

We may remove an extreme a of the 4-path, say u, or an inner vertex, say v. Let first $a = u$. On the one hand, we may have $b \sim t$, whence $G[X] \cong P_4$. Then $M_G(X) = X \cup \{x,y\}$ where $x \in N_G(w)$ and $y \in N_G(t)$. Clearly, $a \notin \{x,y\}$ and hence $\{a\} \not\leftarrowtail_G X$. On the other hand, we may have $b \sim u$ or $b \notin N_G(\mathrm{A})$. We get $G[X] \cong P_{3,1}$ and reasoning as in the proof of Proposition 5.3, it follows that $M_G(X) = X \cup \{x,y\}$ where $x \in N_G(w) \cap N_G(b)$ and y is the third vertex of $N_G(x)$. Thus again $a \notin \{x,y\}$ and $\{a\} \not\leftarrowtail_G X$.

Let now $a = v$. Some possible cases may occur. We may have $b \sim u$ and $b \nsim t$, whence $G[X] \cong P_{2,2}$. By Proposition 5.3 it easily follows that $\{a\} \not\leftarrowtail_G X$.

We may also have $b \in N_G(u) \cap N_G(t)$. Then $G[X] \cong P_4$ and, by Lemma 5.4, $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(t)$ and $y \in N_G(b)$. Therefore, $a \notin \{x,y\}$ and again $\{a\} \not\leftarrowtail_G X$.

We may have $b \sim t$ and $b \nsim u$. In such a case, we get $G[X] \cong P_{3,1}$. By Lemma 5.4, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(t) \cap N_G(u)$, $y \in N_G(x)$ and $y \neq u$. Clearly $a \notin \{x,y\}$ and therefore $\{a\} \not\leftarrowtail_G X$.

Finally, we may have $b \notin N_G(\mathrm{A})$, whence $G[X] \cong P_{2,1,1}$. In this situation, by Lemma 5.4 notice that $M_G(X) = X \cup \{x,y\}$ where $x \in N_G(t) \cap N_G(u)$ and $y \in N_G(t)$. Thus, once again $\{a\} \notin \{x,y\}$, so $\{a\} \not\leftarrowtail_G X$.

- $G[A] \cong P_{3,1}$.

Without loss of generality, assume that $u \sim v$ and $v \sim w$. In view of Lemma 5.4, it follows that b cannot be the common neighbor of v and t and, moreover, called such a common neighbor z, we cannot have $b \sim z$. Therefore, we may choose b to be adjacent to only one extreme of the 3-path or to be adjacent to one extreme of the 3-path and to the isolated vertex. In any case, we are assuming $b \sim u$. Let then $a \in A$ and set $X := A \vartriangle \{a,b\}$. Suppose that b is only adjacent to u as in the first situation listed above. If $a = v$ we get $G[X] \cong P_{2,1,1}$ and thus, by Lemma 5.4, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(w) \cap N_G(b)$ and $y \sim b$. Clearly $v \notin \{x,y\}$, so that $\{a\} \not\leftarrowtail_G X$. If $a = w$ we get $G[X] \cong P_{3,1}$ and thus, by Lemma 5.4, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(u) \cap N_G(t)$ and $y \sim x$. Clearly $w \notin \{x,y\}$, whence $\{a\} \not\leftarrowtail_G X$. If $a = t$ we get $G[X] \cong P_4$ and thus, by Lemma 5.4, we get $M_G(X) = X \cup \{x,y\}$, where $x \in N_G(u)$ and $y \in N_G(v)$. Clearly, $t \notin \{x,y\}$ and hence $\{a\} \not\leftarrowtail_G X$.

In the second case, if $a = w$ we get $G[X] \cong P_4$ and thus, by Lemma 5.4, we get $M_G(X) =$

$X \cup \{x, y\}$, where $x \in N_G(b)$ and $y \in N_G(u)$. Clearly $w \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$. If $a = v$ we get $G[X] \cong P_{3,1}$ and thus, by Lemma 5.4, we get $M_G(X) = X \cup \{x, y\}$, where $x \in N_G(b) \cap N_G(w)$ and $y \in N_G(x)$. Clearly $v \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$. If $a = t$ we get $G[X] \cong P_4$ and thus, by Lemma 5.4, we get $M_G(X) = X \cup \{x, y\}$, where $x \sim v$ and $y \sim u$. Clearly $t \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$. Finally, if $a = u$, we get $G[X] \cong P_{2,2}$ which belongs to $\mathcal{M}_G$ by Theorem 5.9. So again $\{a\} \not\leftarrowtail_G X$.

- $G[A] \cong P_{2,1,1}$.

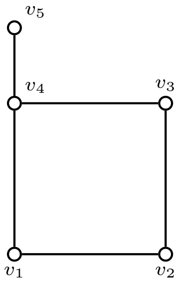
Without loss of generality, assume that $u \sim v$. Let $z \in N_G(w) \cap N_G(t)$. In view of the proof of Proposition 5.3, it follows that only one of the vertices u and v is adjacent to z. Suppose that u is such a vertex (the case of the vertex v is similar). Then the condition $\{b\} \not\leftarrowtail_G A$ becomes $b \notin N_G(u) \cup A$ by Lemma 5.4. Therefore, it may happen that either b is adjacent to only one isolated vertex of A or $b \sim v$ and it is also adjacent to one isolated vertex of A.

Suppose first that b is adjacent to only one isolated vertex of A, say $b \sim w$. Let now $a \in A$ and set $X := A \triangle \{a, b\}$. If $a = t$, we get $G[X] \cong P_{2,2}$ and, using Theorem 5.9, we easily deduce that $\{a\} \not\leftarrowtail_G X$. If $a = w$, we get $G[X] \cong P_{2,1,1}$. Our choice of b and Lemma 5.4 imply that $M_G(X) = X \cup \{x, y\}$, where $x \in N_G(t) \cap N_G(v)$ and $y \in N_G(v)$. Clearly, $w \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$. If $a = u$, we again get $G[X] \cong P_{2,1,1}$. Now, as $N_G(w) \cap N_G(v) \cap N_G(t) = \varnothing$, by Lemma 5.4 we deduce that $M_G(X) = X \cup \{x, y\}$, where $x \in N_G(v) \cap N_G(b)$ and $y \in N_G(b)$. Clearly, $u \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$. If $a = v$, once again we get $G[X] \cong P_{2,1,1}$. Now, as $N_G(w) \cap N_G(u) \cap N_G(t) \neq \varnothing$, by Lemma 5.4 we deduce that $M_G(X) = X \cup \{x, y\}$, where $x \in N_G(u) \cap N_G(w)$ and $y \in N_G(w)$. Clearly, $v \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$.

Suppose now that $b \sim v$ and it is also adjacent to one isolated vertex of A, say $b \sim w$. Fix $a \in A$ and set $X := A \triangle \{a, b\}$. If $a = t$, we get $G[X] \cong P_4$. By Lemma 5.4 we deduce that $M_G(X) = X \cup \{x, y\}$, where $x \in N_G(b)$ and $y \in N_G(v)$ and, as $t \notin \{x, y\}$, we conclude that $\{a\} \not\leftarrowtail_G X$. If $a = w$, we get $G[X] \cong P_{3,1}$. By Lemma 5.4 we deduce that $M_G(X) = X \cup \{x, y\}$, where $x \in N_G(v) \cap N_G(t)$ and $y \in N_G(x)$. Clearly $w \notin \{x, y\}$ and thus $\{a\} \not\leftarrowtail_G X$. If $a = u$, we again get $G[X] \cong P_{3,1}$. By Lemma 5.4 we deduce that $M_G(X) = X \cup \{x, y\}$, where $x \in N_G(b) \cap N_G(t)$ and $y \in N_G(x)$. Clearly, $u \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$. If $a = v$, we get $G[X] \cong P_{2,1,1}$. In view of Lemma 5.4, there exists a 6-subset B containing X and inducing the same symmetry partition as X. Let $x \in N_G(u) \cap N_G(t)$. If $y \in N_G(b)$, we would have $v \in B$ and, in particular, $\{b\} \leftarrowtail_G A$, contradicting our choice of b. Therefore, $B = M_G(X) = X \cup \{x, y\}$, where $x \in N_G(w) \cap N_G(t)$ and $y \in N_G(w)$. Clearly $u \notin \{x, y\}$ and hence $\{a\} \not\leftarrowtail_G X$.

Finally, assume that $A = \{u, v, w, t, z\}$ is a 5-subset. By Proposition 5.5 and Theorem 5.9, we may only suppose either $G[A] \cong P_{3,1,1}$ or $G[A] \cong H_3$, where H_3 is as in the statement of Lemma 5.4.

In the first case, let $u \sim v$ and $v \sim w$. Let $\{b\} \not\leftarrowtail_G A$. In view of Proposition 5.5, we can choose b to be adjacent to one extreme of the 3-path and to one isolated vertex. Without



	v_1	v_2	v_3	v_4	v_5
v_1	0	1	0	1	0
v_2	1	0	1	0	0
v_3	0	1	0	1	0
v_4	1	0	1	0	1
v_5	0	0	0	1	0

Fig. 6. The Graph of Example 5.11 and its Adjacence Matrix.

loss of generality, take $b \in N_G(u) \cap N_G(t)$. Let $a \in A$ be such that $\{a\} \not\prec_G A \setminus \{a\}$. In view of Lemma 5.4, the only choice of a is v. Set $X := A \vartriangle \{a, b\}$. Then $G[X] \cong P_{3,1,1}$. By Propositions 5.5 and 5.7 we get $X \subseteq B$, where $B \in \mathcal{M}_G$ is a 6-subset such that $G[B] \cong H_2$. The unique way to obtain such a B consists of adding to X the vertex $y \in N_G(b) \cap N_G(z) \cap N_G(w)$. Clearly, $y \neq a$ and thus $\{a\} \not\prec_G X$.
In the second case, let $v \sim w$, $w \sim u$, $u \sim t$ and $u \sim z$. Let $\{b\} \not\prec_G A$. Then $b \notin N_G(w)$. Without loss of generality, we may have either $b \in N_G(v) \cap N_G(t)$ or $b \in N_G(t) \setminus N_G(v)$. In both situations, it follows that $X \cong H_3$ and adding the remaining vertex y of $N_G(t)$, we get $X \subseteq X \cup \{y\}$ and $X \approx_G X \cup \{y\}$. As $y \neq a$, we conclude that $\{a\} \not\prec_G X$. This completely proves our result. □

The example of the Petersen graph may induce to think the existence of a relation between strongly regularity and attractiveness. Nevertheless, there is no link between the above properties. In fact, in reference to the discussion of the 5-cycle at the beginning of Section 5, notice that when a graph is strongly regular, it does not necessarily turn out to be an attractive graph. The converse does not even hold as we will see in the following example.

Example 5.11. Consider the non-regular graph G in Fig. 6 and its corresponding $\mathfrak{P}[G, \mathrm{adj}]$, denoted again by G due to the convention stated in Subsection 2.1.

It may be easily shown that

$$\mathcal{M}_G = \{\varnothing, \{v_1, v_3, v_4\}, \{v_2\}, \{v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_3, v_4, v_5\}, V(G)\}$$

and that

$$\begin{gathered}
[\varnothing]_{\approx_G} = \{\varnothing\}, \ [\{v_1, v_3, v_4\}]_{\approx_G} = \{\{v_1\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_1, v_3, v_4\}\}, \\
[\{v_2\}]_{\approx_G} = \{\{v_2\}\}, \\
[\{v_5\}]_{\approx_G} = \{\{v_5\}\}, \quad [\{v_2, v_5\}]_{\approx_G} = \{\{v_2, v_5\}\}, \\
[\{v_1, v_2, v_3, v_4\}]_{\approx_G} = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_2, v_3, v_4\}, \\
\{v_1, v_2, v_3, v_4\}\},
\end{gathered}$$

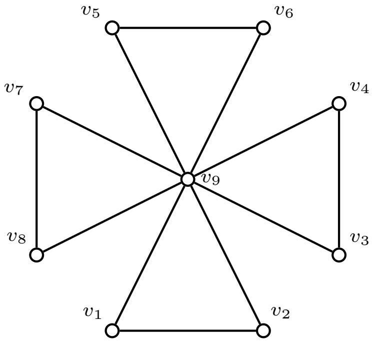


Fig. 7. The Graph F_4.

$$\begin{aligned}[\{v_1,v_3,v_4,v_5\}]_{\approx_G} &= \{\{v_1,v_5\},\{v_3,v_5\},\{v_4,v_5\},\{v_1,v_3,v_5\},\{v_1,v_4,v_5\},\{v_3,v_4,v_5\},\\ &\quad \{v_1,v_3,v_4,v_5\}\},\\ [V(G)]_{\approx_G} &= \{\{v_1,v_2,v_5\},\{v_2,v_3,v_5\},\{v_2,v_4,v_5\},\{v_1,v_2,v_3,v_5\},\{v_1,v_2,v_4,v_5\},\\ &\quad \{v_2,v_3,v_4,v_5\},V(G)\}.\end{aligned}$$

The reader can now easily check that G is an attractive graph by using Theorem 3.10.

6. Friendship graphs are quasi-attractive but not attractive

We now find a model of quasi-attractive pairing induced by the adjacence matrix of a specific graph family. To be more detailed, in this section we deal with a family of graphs inducing a quasi-attractive relation. This is the so-called *Erdös' friendship graph* F_n. We will characterize the members of the Moore system $\mathcal{M}_{F_n}$ and, next, we will prove that adjacence pairing of F_n is quasi-attractive but not attractive by using the structural properties of F_n and the results of the previous sections.

Let us consider the set $V := \{v_1, v_2, \ldots, v_{2n+1}\}$ and take the partition $\mathcal{T} := \{S_1, \ldots, S_n, S_{n+1}\}$ of V, where $S_i := \{v_{2i-1}, v_{2i}\}$ for each $i = 1, \ldots, n$, and $S_{n+1} := \{v_{2n+1}\}$. Let F_n be the graph with vertex set $V(F_n) := V$ and edge set $E(G) := \{S_1, \ldots, S_n, \{v_j, v_{2n+1}\}_{j=1}^{2n}\}$. As an example, in Fig. 7 we represent F_4.

Fix a non-empty vertex subset $X \in \wp(V)$. We set

$$\mathcal{G}_X := \{S \in \mathcal{T} \setminus \{S_{n+1}\} \mid S \subseteq X\}, \quad W_X := \bigcup \mathcal{G}_X$$

$$\mathcal{F}_X := \{S \in \mathcal{T} \mid \|S \cap X\| = 1\}, \quad Z_X := \bigcup \{S \cap X \mid S \in \mathcal{F}_X\} \text{ and}$$

$$Y_X := \bigcup \{S \setminus X \mid S \in \mathcal{F}_X\}.$$

We now exhibit the X-symmetry partition of $\mathfrak{P}[F_n, \mathrm{adj}]$ (from now on F_n, in agreement with the convention of Subsection 2.1) for each non-empty vertex subset $X \in \wp(V(F_n)) \setminus \{\varnothing\}$. We leave the details to readers.

Proposition 6.1. *For each non-empty vertex subset $X \in \wp(V(F_n)) \setminus \{\varnothing\}$, we have that*

$$\pi_{F_n}(X) = \begin{cases} Y_X \cup \{v_{2n+1}\} | V(G) \setminus (Y_X \cup \{v_{2n+1}\}) \\ \quad \text{if } X = \{v_i\} \text{ for some } i = 1, \ldots, 2n \\ \{v\}_{v \in Y_X} | \{w\}_{w \in W_X} | v_{2n+1} | V(G) \setminus (Y_X \cup W_X \cup \{v_{2n+1}\}) \quad \text{otherwise} \end{cases} \tag{24}$$

In the next result we will characterize the maximum partitioners of the graph F_n.

Proposition 6.2. *Let $X \in \wp(V(F_n))$. Then $X \in \mathcal{M}_{F_n}$ if and only if it results that*

(1) $\|X\| \leq 1$ *or*
(2) $X = V(F_n)$ *or*
(3) $X = Z_X \cup W_X \cup \{v_{2n+1}\}$, *where Z_X and W_X satisfy one of the following conditions:*
  (A) $\|Z_X \setminus \{v_{2n+1}\}\| \leq 1$ *and* $\|Z_X \setminus \{v_{2n+1}\}\| + \frac{\|W_X\|}{2} \leq n - 1$;
  (B) $\|Z_X \setminus \{v_{2n+1}\}\| \geq 2$.

Proof. Set $G := F_n$. There is nothing to prove when $X = V(G)$. Moreover, in view of (24), it easily follows that $\varnothing \in \mathcal{M}_G$. It may be also easily checked that all singletons are maximum partitioners of G. In fact, let $X = \{v, w\}$ and fix one of its vertices, say v. Assume that $v \in S_i$ for some $i = 1, \ldots, n$ and denote by v' the remaning vertex of S_i. Then either $v \in W_X$ or $v' \in Y_X$. In both cases, by (24) we see that $\pi_G(X) \neq \pi_G(\{v\})$ and $\pi_G(X) \neq \pi_G(\{w\})$.

Consider now a vertex subset $X \neq V(G)$ such that $\|X\| \geq 2$ and $X = Z_X \cup W_X \cup \{v_{2n+1}\}$. We will show that $X \in \mathcal{M}_G$ if and only if one among the conditions (A) and (B) of the statement holds. By Proposition 2.7, it suffices to show that for each $w \in V(G) \setminus X$ there exist two vertices u, u' such that $u \equiv_X u'$ and $u \not\equiv_{X \cup \{w\}} u'$ if and only if X satisfies one among the conditions (A) and (B) of the statement.

Take then $w \notin X$, so that $w \in S_i$ for some $i = 1, \ldots, n$, and denote by w' the remaining vertex of S_i. Notice first that if $w' \notin X$, then $w, w' \in V(G) \setminus (Y_X \cup W_X \cup \{v_{2n+1}\})$, so that $w \equiv_X w'$. Furthermore, as $w' \in Y_{X \cup \{w\}}$ and $w \in Z_{X \cup \{w\}} \setminus \{v_{2n+1}\}$, we get $w \not\equiv_{X \cup \{w\}} w'$. So, if $Z_X \setminus \{v_{2n+1}\} = \varnothing$, then $X \in \mathcal{M}_G$.

Suppose therefore that $w' \in X$, whence $w' \in Z_X \setminus \{v_{2n+1}\}$. Two possible cases may occur:

- $\|Z_X \setminus \{v_{2n+1}\}\| = 1$ and $\|Z_X \setminus \{v_{2n+1}\}\| + \frac{\|W_X\|}{2} \leq n - 1$;

so that we get $\frac{\|W_X\|}{2} \leq n - 2$, i.e. there exists S_j, where $j \neq i$, such that $S_j \cap X = \varnothing$. Let $u \in S_j$ and set $u' := w'$. As $u, u' \in V(G) \setminus (Y_X \cup W_X \cup \{v_{2n+1}\})$, by (24) we easily get $u \equiv_X u'$. Nevertheless, as $w' \in W_{X \cup \{w\}}$ and $S_j \cap (X \cup \{w\}) = \varnothing$, again by (24) we conclude that $u \not\equiv_{X \cup \{w\}} u'$.

- $\|Z_X \setminus \{v_{2n+1}\}\| \geq 2$;

so there must be a vertex $u \in Z_X \setminus \{w', v_{2n+1}\}$ and which clearly belongs to S_j, for some $j \neq i$. Set $u' := w'$. We get $u, u' \in V(G) \setminus (Y_X \cup W_X \cup \{v_{2n+1}\})$ whence, by (24), it follows that $u \equiv_X u'$. Nevertheless, as $u \in W_{X\cup\{w\}}$ and $u' \in Z_{X\cup\{w\}}$, we conclude that $u \not\equiv_{X\cup\{w\}} u'$.

In this way we showed that if X satisfies one among the conditions (A) and (B), then $X \in \mathcal{M}_G$. Conversely, let $X \in \mathcal{M}_G \setminus \{V(G)\}$ be such that $\|X\| \geq 2$. In view of (24), if $v_{2n+1} \in V(G) \setminus X$, then $\pi_G(X) = \pi_G(X \cup \{v_{2n+1}\})$. Thus, also by the definition of Z_X and W_X, we get $X = Z_X \cup W_X \cup \{v_{2n+1}\}$. Clearly, if $\|Z_X \setminus \{v_{2n+1}\}\| \geq 2$, we obtain Condition (B) of the statement. Therefore, in our successive argument, we can suppose that $\|Z_X \setminus \{v_{2n+1}\}\| \leq 1$.

We will demonstrate that Condition (A) of the statement holds. To this regard, let $w \notin X$ and suppose that $w \in S_i$ for some $i = 1, \dots, n$. As $X \in \mathcal{M}_G$, we have that $\pi_G(X) \neq \pi_G(X \cup \{w\})$. Moreover, denote by w' the remaining vertex of S_i. Suppose first that $w' \notin X$. Then $S_i \cap X = \varnothing$ so that, in view of the definition of $\mathcal{G}_X$, we have $S_i \notin \mathcal{G}_X$. By our assumption on the number of elements of $Z_X \setminus \{v_{2n+1}\}$, we can find another S_l (with $i \neq l$) for which $S_l \notin \mathcal{G}_X$. Thus $\frac{\|W_X\|}{2} \leq n-2$ and $\|Z_X \setminus \{v_{2n+1}\}\| + \frac{\|W_X\|}{2} \leq n-1$, i.e. Condition (A) holds in this case.

Finally, suppose that $w' \in X$. Hence $w' \in Z_X \setminus \{v_{2n+1}\}$ and $S_i \notin \mathcal{G}_X$. Since $X \in \mathcal{M}_G$, by (6.1) we cannot have $X = \bigcup_{\substack{1\leq j\leq 2n+1 \\ i\neq j}} S_j$. Hence there exists another S_l (with $i \neq l$) for which $S_l \notin \mathcal{G}_X$. Thus $\frac{\|W_X\|}{2} \leq n-2$ and $\|Z_X \setminus \{v_{2n+1}\}\| + \frac{\|W_X\|}{2} \leq n-1$, i.e. Condition (A) holds also in this case. □

We are now able to characterize the minimal partitioners of F_n.

Proposition 6.3. *Let $X \in \wp(V(F_n)) \setminus \{\varnothing\}$. Then $X \in \mathcal{N}_{F_n}$ if and only if one of the following conditions holds:*

- *(i)* $\|X\| \leq n-2$ *and* $v_{2n+1} \in V(F_n) \setminus X$;
- *(ii)* $X = \{v_{2n+1}\}$;
- *(iii)* $\|X\| = 2$ *and* $v_{2n+1} \in X$.

Proof. Set $G := F_n$. There is nothing to prove if $\|X\| \leq 1$. Assume therefore that $\|X\| \geq 2$. We firstly claim that $X \approx_G V(G)$ if and only if either $\|X\| \geq 2n$ or $\|X\| = 2n-1$ and $v_{2n+1} \notin X$. To this end, by Proposition 6.2 it results that $X \approx_G V(G)$ if and only if the two following cases occur: either $\|Z_X \setminus \{v_{2n+1}\}\| = 1$ and $\frac{\|W_X\|}{2} = n-1$ or $Z_X \setminus \{v_{2n+1}\} = \varnothing$ and $\frac{\|W_X\|}{2} = n$. The latter case implies that $\|X\| = 2n$, while the former implies that either $\|X\| = 2n$ or $\|X\| = 2n-1$, depending on whether v_{2n+1} belongs or not to X.

Suppose now $X \not\approx_G V(G)$. Let us prove that $[X]_{\approx_G}$ contains two elements if and only if $X \neq \{v_i, v_{2n+1}\}$. To this end, assume that $[X]_{\approx_G}$ contains two elements. Assume by contradiction that $X = \{v_i, v_{2n+1}\}$ for some $i = 1, \dots, 2n$. By Proposition 6.2 it results that $X \in \mathcal{M}_G$ and, since each singleton belongs to $\mathcal{M}_G$, we conclude that $[X]_{\approx_G} = \{\{v_i, v_{2n+1}\}\}$, in contrast with our choice of X.

Conversely, let us assume that $X \neq \{v_i, v_{2n+1}\}$ and suppose moreover that $v_{2n+1} \in V(G) \setminus X$. By (24) we clearly have $X \approx_G X \cup \{v_{2n+1}\}$ and, using Proposition 6.2, it may be easily checked that $[X]_{\approx_G}$ contains only X and $X \cup \{v_{2n+1}\}$. The conclusion of our Proposition is now immediate. □

Remark 6.4. Proposition 6.3 and part (iv) of Proposition 2.7 ensure that $\mathcal{R}_{\mathfrak{P}} \subsetneq \mathrm{Max}(\mathcal{N}_{\mathfrak{P}})$ since the last family contains the vertex subsets of the form $\{v_i, v_{2n+1}\}$, with $i = 1, \dots, 2n$, while the former does not.

At this point, notice that $F_1 = K_3$, i.e. it is complete graph on 3 vertices. By what we said at the beginning of Section 5, K_3 is attractive and, hence, it is also quasi-attractive. In the next theorem we will demonstrate that friendship graphs are quasi-attractive but not attractive. To this regard, we will prove that $\mathcal{R}_{F_n}(X)$ forms an exchangeable set system for any $X \in \wp(\Omega)$, so that Theorem 4.6 ensures the quasi-attractiveness of F_n, while using Theorem 3.10 we will check that F_n cannot be attractive.

Theorem 6.5. *Let* $V := V(F_n)$*. Then* $F_n \in \mathrm{PR}_{\mathrm{qa}}(V) \setminus \mathrm{PR}_a(V)$ *for each* $n \geq 2$*.*

Proof. In view of Proposition 6.3, it is immediate to check that $\mathcal{R}_{F_n}(X)$ forms an exchangeable set system for any $X \in \wp(\Omega)$. Thus, by Theorem 4.6 we conclude that F_n is quasi-attractive for each $n \geq 2$.
To show that F_n is not attractive, by the last statement of Theorem 3.10 it suffices to prove that $\mathcal{N}_{F_n}$ is not a matroid. To this regard, take the vertex subsets $X = \{v_{2n}, v_{2n+1}\}$ and $Y = \{v_1, v_2, v_{2n}\}$. In view of Corollary 6.3, it results that $X, Y \in \mathcal{N}_{F_n}$. Nevertheless, notice that neither $X \cup \{v_1\}$ nor $X \cup \{v_2\}$ belong to $\mathcal{N}_{F_n}$. Thus $\mathcal{N}_{F_n}$ is not a matroid and the graph F_n cannot be attractive. □

Remark 6.6. Theorem 6.5 gives an example of quasi-attractive graph which is not regular and, a fortiori, which is not strongly regular. The converse does not even hold, as one may see taking again the 5-cycle C_5. Let $X := \{v_1, v_2, v_4\}$, $Y := \{v_1, v_3, v_5\}$ and $x := v_4$. It may be easily checked that $X \approx_{C_5} Y$ and that neither $\{x\} \twoheadleftarrow_{C_5} (X \setminus \{x\}) \cup \{v_3\}$ nor $\{x\} \twoheadleftarrow_{C_5} (X \setminus \{x\}) \cup \{v_5\}$ hold.

7. Conclusions

In the present work we analyzed the representability of abstract simplicial complexes on Ω as the minimal partitioners of some pairing on the same ground set and, next, we dealt with the matroidal-like properties of the set systems $\mathcal{N}_{\mathfrak{P}}$ and $\mathcal{R}_{\mathfrak{P}}(X)$, for any subset X of a given ground set Ω. Indeed, when Ω is finite and $\mathfrak{P}$ is attractive, the minimum partitioner family forms a matroid on Ω. In addition, we characterized attractiveness on finite ground sets in terms of the equality $\mathcal{R}_{\mathfrak{P}}(X) = \mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$ (for each $X \in \wp(\Omega)$) and checked that in general it is only necessary for attractiveness, as well as the *exchangeability* of $\mathrm{Max}(\mathcal{N}_{\mathfrak{P}} \cap \wp(X))$. However, the most important contribution of this paper is the introduction of the larger class of *quasi-attractive pairings*. We characterized them from a lattice-theoretic point of view in Theorem 4.3 and, successively, we proved that quasi-attractiveness, together with a local finiteness condition, ensures the uniformity of the cardinality of the members of $\mathcal{R}_{\mathfrak{P}}(X)$ (for each $X \in \wp(\Omega)$) and that, on finite ground sets, quasi-attractiveness is equivalent to the exchangeability of $\mathcal{R}_{\mathfrak{P}}(X)$. Thus, as for attractiveness, we related quasi-attractiveness to suitable properties of $\mathcal{R}_{\mathfrak{P}}(X)$.

Different research lines may be outlined starting from our manuscript. Firstly, it seems appropriate to enhance the formal and mathematical framework, possibly analyzing the abstract theoretical properties occurring in the definition of specific subcollections of pairings, even on infinite ground sets. So doing, on the one hand, it is possible to carry out some interpretations of the theoretical results in concrete situations occurring in and deriving from applications and, on the other hand, it is possible to improve the formal techniques of rough set theory and data mining.

Secondly, Theorem 2.4 gives information about the possibility of exhibiting cryptomorphic axiomatizations of attractiveness in terms of suitable kinds of simplicial complexes, closure operators or simplicial operators. However, nothing is known about such a possibility for quasi-attractiveness. Consequently, a problem that our paper still leaves open concern the attempt of axiomatizing attractiveness through suitable set systems or set operators.

Thirdly, the relationship between matroidality and minimal partitioners may be investigated more deeply, trying to exhibit a complete characterization of all those situations where $\mathcal{N}_{\mathfrak{P}}$ turns out to be a matroid. In such a context, a valid help may come from the attempt of characterizing all the abstract simplicial complexes representable as the minimal partitioner family of some pairing, so extending the analysis undertaken at the end of Section 2.

In addition, in [30] linking maps and sub-bijections have been extended at a functorial level to get a categorical setting for the analysis of mathematical structures. Therefore, the categorical counterpart of our analysis may be reached after defining a suitable category **PR** of pairings and pairing homomorphisms and taking some of its specific subcategories and suitable functors from them to the category of matroids [28] or of simplicial complexes [15]. So, the questions we may pose are: which subcategories do we have to consider to enrich our investigation? After choosing them suitably, which are the properties they satisfy and, in particular, which are those inherited from **PR**? How to relate the categorical analysis of these subcategories of pairings with those of matroids and simplicial complexes?

Finally, following the idea of the last two sections of the paper, the models exhibited enable us to leave open the problems of characterizing all those graphs whose adjacence pairing is attractive or quasi-attractive. Nevertheless, such a problem arises in all those cases where we interpret a graph as a pairing, so that the analysis may lead to different characterizations depending on whether we deal with metric pairings, or laplacian pairings or incidence pairings or any other pairing that may be induced by means of a graph.

CRediT authorship contribution statement

C. Bisi and F.G. Infusino wrote and revised the whole manuscript.

Declaration of competing interest

The authors declare none.

Data availability

The manuscript has no associated data.

Acknowledgments

Federico G. Infusino is affiliated to INDAM - GNSAGA. We are extremely grateful to the anonymous reviewer who has devoted the time and effort to the review process, helping us to improve the quality of our manuscript.

Funding declarations. No funding is available.

References

[1] J.A. Aledo, S. Martínez, J.C. Valverde, Parallel dynamical systems over directed dependency graphs, Appl. Math. Comput. 219 (3) (2012) 1114–1119.

[2] J.A. Aledo, L.G. Diaz, S. Martinez, J.C. Valverde, Solution to the predecessors and Gardens-of-Eden problems for synchronous systems over directed graphs, Appl. Math. Comput. 347 (2019) 22–28.

[3] B. Alspach, L. Goddyn, C.-Q. Zhang, Graphs with the circuit cover property, Trans. Am. Math. Soc. 344 (1994) 131–154.

[4] G. Birkhoff, Lattice Theory, third edition, American Mathematical Society, Providence, Rhode Island, 1967.

[5] C. Bisi, G. Chiaselotti, T. Gentile, Real subset sums and posets with an involution, Int. J. Algebra Comput. 32 (1) (2022) 127–157.

[6] P. Bonacini, M. Gionfriddo, L. Marino, Nesting house-designs, Discrete Math. 339 (4) (2016) 1291–1299.

[7] J.E. Bonin, Basis-exchange properties of sparse paving matroids, Adv. Appl. Math. 50 (1) (2013) 6–15.

[8] E. Boros, V.A. Gurvich, I.E. Zverovich, Friendship two-graphs, Graphs Comb. 26 (2010) 617–628.

[9] G. Chiaselotti, T. Gentile, F. Infusino, Simplicial complexes and closure systems induced by indistinguishability relations, C. R. Acad. Sci. Paris, Ser. I 355 (2017) 991–1021.

[10] G. Chiaselotti, T. Gentile, F. Infusino, P.A. Oliverio, The adjacency matrix of a graph as a data table. A geometric perspective, Ann. Mat. Pura Appl. 196 (3) (2017) 1073–1112.

[11] G. Chiaselotti, T. Gentile, F. Infusino, New perspectives of granular computing in relation geometry induced by pairings, Ann. Univ. Ferrara 65 (1) (May 2019) 57–94.

[12] G. Chiaselotti, T. Gentile, F. Infusino, Lattice representation with set partitions induced by pairings, Electron. J. Comb. 27 (1) (2020) P1.19.

[13] G. Chiaselotti, F. Infusino, P.A. Oliverio, Set relations and set systems induced by some families of integral domains, Adv. Math. 363 (25) (March 2020) 106999.

[14] G. Chiaselotti, F. Infusino, Alexandroff topologies and monoid actions, Forum Math. 32 (3) (2020) 795–826.

[15] G. Chiaselotti, F. Infusino, Some classes of abstract simplicial complexes motivated by module theory, J. Pure Appl. Algebra 225 (1) (January 2021) 106471.

[16] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, second edition, Cambridge University Press, New York, 2002.

[17] A. Day, Filter monads, continuous lattices and closure systems, Can. J. Math. XXVII (1) (1975) 50–59.

[18] A. Day, The lattice theory of functional dependencies and normal decompositions, Int. J. Algebra Comput. 2 (4) (1992) 409–431.

[19] J. Demetrovics, G.O.H. Katona, D. Miklos, Functional dependencies distorted by errors, Discrete Appl. Math. 156 (2008) 862–869.

[20] R. Diestel, Graph Theory, 4th edition, Graduate Text in Mathematics, Springer, 2010.

[21] K. Engel, T.D. Thu, Boundary optimization for rough sets, Discrete Math. 341 (9) (2018) 2465–2477.

[22] P. Erdös, A. Rényi, Asymmetric graphs, Acta Math. Hung. 14 (3–4) (1963) 295–315.

[23] P. Erdös, A. Rényi, V.T. Sos, On a problem of graph theory, Studia Sci. Math. Hung. 1 (1966) 215–235.

[24] M. Gionfriddo, E. Guardo, L. Milazzo, Extending bicolorings for Steiner triple systems, Appl. Anal. Discrete Math. (2013) 225–234.

[25] C. Godsil, G.F. Royle, Algebraic Graph Theory, Springer Science & Business Media, vol. 207, 2001.

[26] G. Gordon, J. McNulty, Matroids: A Geometric Introduction, Cambridge University Press, 2012.

[27] F. Harary, R.A. Melter, On the metric dimension of a graph, Ars Comb. 2 (1976) 191–195.

[28] C. Heunen, V. Patta, The category of matroids, Appl. Categ. Struct. 26 (2) (2018) 205–237.

[29] A. Huang, H. Zhao, W. Zhu, Nullity-based matroid of rough sets and its application to attribute reduction, Inf. Sci. 263 (2014) 153–165.

[30] F. Infusino, Prekernels of topologically axiomatized subcategories of concrete categories, J. Algebra 622 (2023) 469–505.

[31] J. Jonsson, Simplicial Complexes of Graphs, vol. 3, Springer, Berlin, 2008.

[32] G.O.H. Katona, A. Sali, New type of coding problem motivated by database theory, Discrete Appl. Math. 144 (2004) 140–148.

[33] D. Kozlov, Combinatorial Algebraic Topology, vol. 21, Springer Science & Business Media, 2008.

[34] M. Lemos, A characterization of graphic matroids using non-separating cocircuits, Adv. Appl. Math. 42 (1) (2009) 75–81.

[35] X. Li, S. Liu, Matroidal approaches to rough sets via closure operators, Int. J. Approx. Reason. 53 (2012) 513–527.

[36] X. Li, H. Yi, S. Liu, Rough sets and matroids from a lattice-theoretic viewpoint, Inf. Sci. 342 (2016) 37–52.

[37] J. Martí-Farré, From clutters to matroids, Electron. J. Comb. 21 (1) (2014) P1.11.

[38] J. Martí-Farré, A. de Mier, Transformation and decomposition of clutters into matroids, Adv. Math. 312 (2017) 286–314.

[39] Z. Pawlak, Rough Sets. Theoretical Aspects of Reasoning about Data, Kluwer Academic Publisher, 1991.

[40] J.A. Samper, Quasi-matroidal classes of ordered simplicial complexes, J. Comb. Theory, Ser. A 175 (2020) 105274.

[41] D.A. Simovici, C. Djeraba, Mathematical Tools for Data Mining, Springer-Verlag, London, 2014.

[42] L. Traldi, Clutters and circuits, Adv. Appl. Math. 18 (2) (1997) 220–236.

[43] L. Traldi, R. Brijder, H.J. Hoogeboom, The adjacency matroid of a graph, Electron. J. Comb. (2013) P27.

[44] W.T. Tutte, An algorithm for determining whether a given binary matroid is graphic, Proc. Am. Math. Soc. 11 (1960) 905–917.