AMPLE BODIES AND TERRACINI LOCI OF PROJECTIVE VARIETIES

ANTONIO LAFACE AND ALEX MASSARENTI

ABSTRACT. We introduce the notion of ample body of a projective variety and use it to prove emptiness results for Terracini loci and specific identifiability results for toric and homogeneous varieties.

CONTENTS

1. INTRODUCTION

Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. The h-Terracini locus of X parametrizes unordered h-uples of distinct points $x_1, \ldots, x_h \in X$ at which the tangent spaces span a linear space of dimension smaller than expected. Terracini loci have been introduced in [BC21] and then studied for several relevant varieties such as Veronese and Segre varieties [BBS20], [Bal22b], [Bal22a], [CG23], [BV23].

These loci are closely related to the concepts of secant defectiveness and identifiability. The h-secant variety $\mathbb{S}ec_h(X) \subset \mathbb{P}^N$ of X is the Zariski closure of the union of the $(h-1)$-planes spanned by collections of h points of X.

The expected dimension of $\mathbb{S}ec_h(X)$ is $\operatorname{expdim}(\mathbb{S}ec_h(X)) := \min\{nh + h - 1, N\}$. The actual dimension of $\mathbb{S}ec_h(X)$ may be smaller than the expected one. The variety X is h-defective if $\dim(\mathbb{S}ec_h(X)) < \operatorname{expdim}(\mathbb{S}ec_h(X))$ and h-identifiable if through a general point of $\mathbb{S}ec_h(X)$ there passes a unique $(h-1)$-plane spanned by h points of X. Furthermore, if this last property holds for a special point $p \in \mathbb{S}ec_h(X)$ we say that p is h-identifiable.

The Terracini's lemma yields that X is h-defective if and only if the h-Terracini locus of X coincides with the symmetric product X^h/S_h. In Section 2 we relate the emptiness of certain Terracini loci to specific identifiability. This property, especially when the ambient projective space parametrized tensors, is relevant also in applied sciences, for instance in psycho-metrics, chemo-metrics, signal processing, numerical linear algebra, computer vision, numerical analysis, neuroscience and graph analysis [BK09].

In Section 3 we associate to a projective variety X a geometric object $\mathscr{A}_X$ that we call the ample body of X. This is the convex hull of ample divisor classes of X. When the Mori cone of X is rational polyhedral $\mathscr{A}_X$ turns out to be a polyhedron which is the Minkowski sum of a rational polytope A_X and of the nef cone of X. In many case we manage to control the geometry of $\mathscr{A}_X$ and to use it to prove emptiness results for Terracini loci. For instance, summing up Proposition 3.4 and Theorem 3.14, for toric varieties we have the following:

Theorem 1.1. *Let $P \subseteq M_{\mathbb{Q}}$ be a full dimensional lattice polytope such that the corresponding projective toric variety $X_P \subseteq \mathbb{P}^{|P \cap M|-1}$, embedded by a complete linear system $|L|$, is smooth, and set*

$$\ell(P) := \min\{|L \cap M| - 1 \,|\, L \text{ is a 1-dimensional face of } P\}.$$

For 2-Terracini loci the following are equivalent:

- $\ell(P) \geq 3$;
- $T_2(X)$ *is empty;*
- X_P *does not contain conics.*

Furthermore, if A_{X_P} is a normal lattice polytope then the following are equivalent:

- $\ell(P) = s$ *with* $s \geq 2h - 1$;
- $L \in s \cdot \mathscr{A}_X$;
- $T_h(X)$ *is empty.*

In Section 4 we apply Theorem 1.1 and the theory of ample bodies to specific toric varieties such as Segre-Veronese varieties, toric varieties of Picard rank two, and to homogeneous varieties. As a sample we summarize our main results for Segre-Veronese varieties in Corollaries 4.3, 4.4.

Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be two r-uples of positive integers, with $n_1 \leq \cdots \leq n_r$ and $N(\boldsymbol{n}, \boldsymbol{d}) = \prod_{i=1}^{r} \binom{n_i + d_i}{n_i} - 1$. The Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is the image in $\mathbb{P}^{N(\boldsymbol{n},\boldsymbol{d})}$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ via the embedding induced by $\mathcal{O}_{\mathbb{P}^{\boldsymbol{n}}}(d_1, \ldots, d_r) = \mathcal{O}_{\mathbb{P}(V_1^*)}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}(V_1^*)}(d_r)$.

Theorem 1.2. *If $h \leq \lceil \frac{d_i}{2} \rceil$ for all $i = 1, \ldots, r$ then $T_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ is empty. Furthermore, if $2h \leq \lceil \frac{d_i}{2} \rceil$ for all $i = 1, \ldots, r$ then any point of $\mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) \setminus \mathbb{S}ec_{h-1}(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ is h-identifiable and $\mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ is smooth outside of $\mathbb{S}ec_{h-1}(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$.*

Finally, in Proposition 4.6 we describe 2-Terracini loci of Grassmannians, and in Section 4.7 we discuss ample bodies of toric Fano varieties and prove the following result:

Theorem 1.3. *Let $P \subseteq M_{\mathbb{Q}}$ be a full dimensional lattice polytope such that the corresponding projective toric variety $X_P \subseteq \mathbb{P}^{|P \cap M|-1}$, embedded by a complete linear system $|L|$, is smooth and Fano of dimension at most four. Then A_{X_P} is a lattice point, and $T_h(X)$ is empty if and only if $\ell(P) = s$ with $s \geq 2h - 1$.*

The proof of Theorem 1.3 is based on our results for toric varieties, the data contained in the Graded Ring Database, and a Magma [BCP97] script we developed to compute A_X.

2. Terracini loci, specific identifiability and Bronowski's conjecture

Let $X \subset \mathbb{P}^N$ be an irreducible and non-degenerate variety of dimension n and let $\Gamma_h(X) \subset X \times \cdots \times X \times \mathbb{G}(h-1, N)$, where $h \leq N$, be the closure of the graph of the rational map $\alpha : X \times \cdots \times X \dashrightarrow \mathbb{G}(h-1, N)$ taking h general points to their linear span. Observe that $\Gamma_h(X)$ is irreducible and reduced of dimension hn. Let $\pi_2 : \Gamma_h(X) \to \mathbb{G}(h-1, N)$ be the natural projection, and $\mathcal{S}_h(X) := \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h-1, N)$. Again $\mathcal{S}_h(X)$ is irreducible and reduced of dimension hn. Finally, consider

$$\mathcal{I}_h = \{(x, \Lambda) \,|\, x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N)$$

with the projections π_h^X and ψ_h^X onto the factors. The abstract h-secant variety is the irreducible variety

$$\mathrm{Sec}_h(X) := (\psi_h^X)^{-1}(\mathcal{S}_h(X)) \subset \mathcal{I}_h.$$

The h-secant variety is defined as

$$\mathbb{S}ec_h(X) := \pi_h^X(\mathrm{Sec}_h(X)) \subset \mathbb{P}^N.$$

It immediately follows that $\mathrm{Sec}_h(X)$ is an $(hn+h-1)$-dimensional variety with a $\mathbb{P}^{h-1}$-bundle structure over $\mathcal{S}_h(X)$. We say that X is h-defective if $\dim \mathbb{S}ec_h(X) < \min\{\dim \mathrm{Sec}_h(X), N\}$. We will denote by $\mathbb{S}ec_h(X)^o$ the union of the $(h-1)$-planes spanned by h linearly independent points of X.

Definition 2.1. When $\pi_h^X : \mathrm{Sec}_h(X) \to \mathbb{S}ec_h(X)$ is generically finite we will call its degree the h-secant degree of X, and we will say that X is h-identifiable when its h-secant degree is one.

Remark 2.2. Note that even when $\pi_h^X : \mathrm{Sec}_h(X) \to \mathbb{S}ec_h(X)$ is birational it might have positive dimensional fibers or zero dimensional fibers of degree bigger than one. In this last case $\mathbb{S}ec_h(X)$ will not be normal. We will say that $p \in \mathbb{S}ec_h(X)$ is h-identifiable if $(\pi_h^X)^{-1}(p)$ has degree one.

Definition 2.3. Let $X \subset \mathbb{P}^N$ be a smooth, irreducible and non-degenerate variety of dimension n. The h-th Terracini locus $T_h(X)$ of X is the closure of

$$T_h(X)^o = \{\{x_1,\dots,x_h\} \mid x_i \neq x_j \text{ and } \dim(\langle T_{x_1}X,\dots,T_{x_h}X\rangle) < \min\{hn+h-1,N\}\} \subseteq X^h/S_h$$

in the h-th symmetric product X^h/S_h of X.

Definition 2.4. Given a smooth, irreducible and non-degenerate variety $X \subset \mathbb{P}^N$ of dimension n and a local parametrization

$$\begin{array}{cccc} \phi: & k^n & \longrightarrow & X \subset \mathbb{P}^N \\ & (u_1,\dots,u_n) & \mapsto & [1,\phi_1(u_1,\dots,u_n),\dots,\phi_N(u_1,\dots,u_n)] \end{array}$$

we define the Terracini matrix $T_X(x_1,\dots,x_h)$ of the h points $x_1,\dots,x_h \in \phi(k^n)$ as

$$T_X(x_1,\dots,x_h) = \begin{pmatrix} 1 & \phi_1(p_1) & \dots & \phi_N(p_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_1(p_h) & \dots & \phi_N(p_h) \\ 0 & \phi_{1,u_1}(p_1) & \dots & \phi_{N,u_1}(p_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \phi_{1,u_n}(p_h) & \dots & \phi_{N,u_n}(p_h) \end{pmatrix}$$

where $p_i = \phi^{-1}(x_i)$ and $\phi_{i,u_j} = \frac{\partial \phi_i}{\partial u_j}$.

In the following we relate the emptiness of Terracini loci to specific identifiability.

Proposition 2.5. *If $T_h(X)$ is empty then for any $p \in \mathbb{S}ec_h(X)^o \setminus \mathbb{S}ec_{h-1}(X)^o$ the fiber $(\pi_h^X)^{-1}(p)$ consists of a finite number of points.*

Proof. Let $\langle x_1,\dots,x_h\rangle$ be an h-plane containing p with $x_1,\dots,x_h \in X$. Assume that $(\pi_h^X)^{-1}(p)$ has positive dimension. Then the image of the differential of π_h^X at p has dimension smaller that $\dim(X)h + h - 1$. Since $T_{x_1}X,\dots,T_{x_h}X$ are contained in the image of the differential of π_h^X at p we conclude that $\{x_1\dots,x_h\} \in T_h(X)$. $\qquad\square$

Corollary 2.6. *If $T_h(X)$ is empty, X is h-identifiable and $\mathbb{S}ec_h(X)^o \setminus \mathbb{S}ec_{h-1}(X)^o$ is normal then any point $p \in \mathbb{S}ec_h(X)^o \setminus \mathbb{S}ec_{h-1}(X)^o$ is h-identifiable.*

Proof. Since X is h-identifiable π_h^X is birational, and since $T_h(X)$ is empty Proposition 2.5 yields that π_h^X is finite over $\mathbb{S}ec_h(X)^o \setminus \mathbb{S}ec_{h-1}(X)^o$. To conclude it is enough to note that if the fiber $(\pi_h^X)^{-1}(p)$ over $p \in \mathbb{S}ec_h(X)^o \setminus \mathbb{S}ec_{h-1}(X)^o$ has degree bigger that one then $\mathbb{S}ec_h(X)^o$ is not normal at p. $\qquad\square$

Corollary 2.7. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n embedded by a line bundle $L = \omega_X \otimes A^{2(n+1)} \otimes B$, where A is very ample and B is nef. If $T_2(X)$ is empty and X is 2-identifiable then any point $p \in \mathbb{S}ec_2(X) \setminus X$ is 2-identifiable.*

Proof. By [Ull16, Corollary C] $\mathbb{S}ec_2(X)$ is normal. Hence, Corollary 2.6 yields that any point $p \in \mathbb{S}ec_2(X)^o \setminus X^o$ is 2-identifiable. Now, let $p \in \mathbb{S}ec_2(X) \setminus X$ be a point lying on a line L tangent to X at $x \in X$. Assume that the is another line L' through p that is secant to X. If L' is tangent to X at a point x' then $T_x X \cap T_{x'} X \neq \emptyset$ and hence $T_2(X) \neq \emptyset$. If L' is a proper secant then arguing as in the proof of Proposition 2.5 we get that the fiber $(\pi_2^X)^{-1}(p)$ is finite and hence $\mathbb{S}ec_2(X)$ is not normal at p contradicting [Ull16, Corollary C]. □

Proposition 2.8. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n. If $T_{2h}(X)$ is empty then any point of $\mathbb{S}ec_h(X) \setminus \mathbb{S}ec_{h-1}(X)$ is h-identifiable. Furthermore, $\mathrm{Sing}(\mathbb{S}ec_h(X)) \subset \mathbb{S}ec_{h-1}(X)$.*

Proof. Let $p \in \mathbb{S}ec_h(X) \setminus \mathbb{S}ec_{h-1}(X)$ and assume that there are two $(h-1)$-planes H, H' through p intersecting X in schemes of dimension zero and degree h supported respectively on $\{x_1, \ldots, x_a\}$ and $\{x'_1, \ldots, x'_b\}$.

Then $H \subset \langle T_{x_1} X, \ldots, T_{x_a} X \rangle$ and $H' \subset \left\langle T_{x'_1} X, \ldots, T_{x'_a} X \right\rangle$ yield that $p \in \langle T_{x_1} X, \ldots, T_{x_a} X \rangle \cap \left\langle T_{x'_1} X, \ldots, T_{x'_a} X \right\rangle$ and hence $T_{a+b}(X) \neq \emptyset$. To conclude it is enough to note that $a + b \leq 2h$ and that $T_{2h}(X) = \emptyset$ implies $T_{h'}(X) = \emptyset$ for all $h' \leq 2h$.

Therefore, π_h^X is 1-to-1 over $\mathbb{S}ec_h(X) \setminus \mathbb{S}ec_{h-1}(X)$ and since $T_h(X) = \emptyset$, arguing as in Proposition 2.5, we have that π_h^X is a submersion over $\mathbb{S}ec_h(X) \setminus \mathbb{S}ec_{h-1}(X)$ and hence $\mathbb{S}ec_h(X) \setminus \mathbb{S}ec_{h-1}(X)$ is smooth. □

Thanks to the theory developed in [MM22, Section 2] it is possible to relate Terracini loci to the Bronowski's conjecture [CR06, Remark 4.6] which has been proved false and amended, by requiring the non degeneracy of the Gauss map of X, in [MM22, Theorem 1.3, Conjecture 1.4].

Notation 2.9. Let $X \subset \mathbb{P}^N$ be an irreducible and non-degenerate variety. A general h-tangential projection of X is a linear projection $\tau_{x_1, \ldots, x_h}^X : X \dashrightarrow X_h \subset \mathbb{P}^{N_h}$ from the linear span of h tangent spaces $\langle T_{x_1} X, \ldots, T_{x_h} X \rangle$ where $x_1, \ldots, x_h \in X$ are general points. When there will be no danger of confusion we will denote a general h-tangential projection $\tau_{x_1, \ldots, x_h}^X$ simply by τ_h^X.

Notation 2.10. Consider the map $\pi_{h+1}^X : \mathrm{Sec}_{h+1}(X) \to \mathbb{S}ec_{h+1}(X) \subset \mathbb{P}^N$. For a general point $p \in \mathbb{S}ec_h(X) \subset \mathbb{S}ec_{h+1}(X)$ we split the fiber $\pi_{h+1}^X(p)$ in two parts T_p^h, R_p^h defined as follows:

- the general point of the trivial part T_p^h is a pair (p, Λ) where Λ is an h-plane $(h+1)$-secant to X of the form $\Lambda = \langle x, \Lambda' \rangle$ with $x \in X$ and Λ' an $(h-1)$-plane h-secant to X and passing through p;
- the residual part R_p^h is the closure of the complement of T_p in $(\pi_{h+1}^X)^{-1}(p)$.

The following is the revised version of Bronowski's conjecture in [MM22, Conjecture 1.4].

Conjecture 2.11. *Let $X \subset \mathbb{P}^N$ be an irreducible and non-degenerate variety with non degenerate Gauss map. The variety X is h-identifiable if and only if a general $(h-1)$-tangential projection $\tau_{h-1}^X : X \dashrightarrow X_{h-1} \subset \mathbb{P}^{N_{h-1}}$ is birational.*

Proposition 2.12. *Let $X \subset \mathbb{P}^N$ be an irreducible and non-degenerate variety such that $T_{2h-1}(X)$ is empty. Then X has non degenerate Gauss map and Conjecture 2.11 holds true for X and the integer h, that is X is h-identifiable if and only if a general $(h-1)$-tangential projection $\tau_{h-1}^X : X \dashrightarrow X_{h-1} \subset \mathbb{P}^{N_{h-1}}$ is birational.*

Proof. First, note that the Gauss map of X is degenerate if and only if a general tangent space of X is tangent to X along a positive dimensional subvariety and in this case $T_2(X)$ is non empty.

Let $p \in \mathbb{S}ec_{h-1}(X)$ be a general point, and consider an $(h-2)$-plane $\langle x_1, \ldots, x_{h-1} \rangle$ through p with $x_1, \ldots, x_{h-1} \in X$. Assume that the residual part R_p^{h-1} is non empty. Then there is an $(h-1)$-plane

$\langle x'_1, \ldots, x'_h \rangle$ through p with $x'_1, \ldots, x'_h \in X$ and at least two of the x'_i do not belong to $\{x_1, \ldots, x_{h-1}\}$. Therefore, $T_{2h-1}(X) \neq \emptyset$ contradicting the hypotheses. Therefore, R_p^{h-1} is empty and to conclude it is enough to apply [MM22, Corollary 2.18]. □

3. Ample bodies and toric varieties

Let N be a rank n free abelian group, $M := \mathrm{Hom}(N, \mathbb{Z})$ its dual and $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ the corresponding rational vector space. Let $P \subseteq M_{\mathbb{Q}}$ be a full-dimensional lattice polytope, that is the convex hull of finitely many points in M which do not lie on a hyperplane. The polytope P defines a polarized pair (X_P, H) consisting of the toric variety X_P together with a very ample Cartier divisor H of X_P. More precisely X_P is the Zariski closure of the image of the monomial map

$$(3.1) \qquad \begin{array}{rccc} \phi_P : & (k^*)^n & \longrightarrow & \mathbb{P}^N \\ & u & \mapsto & [\chi^m(u) : m \in P \cap M] \end{array}$$

where $P \cap M = \{m_0, \ldots, m_N\}$, $\chi^m(u)$ denotes the Laurent monomial in the variables $(u_1, \ldots, u_n)$ defined by the point m, and H is a hyperplane section of X_P.

Lemma 3.2. *Let X be a smooth projective toric variety with one-parameter subgroup lattice N and let $p_1, p_2 \in X$ be two distinct points. Then there exists $v \in N$ such that the two limits*

$$q_1 := \lim_{t \to 0} t^v \cdot p_1, \qquad q_2 := \lim_{t \to 0} t^v \cdot p_2$$

are two distinct points which are either torus invariant or lie on a torus invariant curve.

Proof. Let T be the big torus of X and let X_i be the Zariski closure of the orbit $T \cdot p_i$. Recall that both q_1 and q_2 are invariant fixed points for a general $v \in N$. If $X_1 \cap X_2$ is empty, then for such a general $v \in N$ we get two distinct invariant points. Assume now that $X_1 \cap X_2$ is non empty. Then there exists an invariant point $q \in X_1 \cap X_2$. Let $U \subseteq X$ be an open invariant affine subset which contains q. Since U has non-empty intersection with $T \cdot p_i$ it follows that $p_i \in U$.

The above analysis allows one to reduce to the case where both p_1 and p_2 are contained in an open affine subset $U \simeq k^n$ with the standard action $(k^*)^n \times k^n \to k^n$ given by $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1 x_1, \ldots, t_n x_n)$. Since p_1 and p_2 are distinct they must differ for at least one coordinate, say the first one. Choosing $v = (0, 1, \ldots, 1)$ one has $t^v \cdot (x_1, x_2, \ldots, x_n) = (x_1, t x_2, \ldots, t x_n)$ so that the two limits q_1 and q_2 remain distinct and both lie on the invariant curve $\{x_2 = \cdots = x_n = 0\}$. □

Definition 3.3. Let $P \subseteq M_{\mathbb{Q}}$ be a lattice polytope. The *length* of P is

$$\ell(P) := \min\{|L \cap M| - 1 \,|\, L \text{ is a 1-dimensional face of } P\}.$$

Proposition 3.4. *Let $P \subseteq M_{\mathbb{Q}}$ be a full dimensional lattice polytope such that the corresponding projective toric variety $X_P \subseteq \mathbb{P}^{|P \cap M|-1}$ is smooth and linearly normal. Then the following are equivalent:*

(a) $\ell(P) \geq 3$;
(b) the 2-Terracini locus of X_P is empty;
(c) X_P does not contain conics.

Proof. The implication $(b) \Rightarrow (c)$ is trivial since tangent lines to a conic intersect. The implication $(c) \Rightarrow (a)$ follows from the fact that if X_P does not contain conics in particular it does not contain invariant conics. Thus any edge of P has length at least three.

We now prove $(a) \Rightarrow (b)$. Thanks to Lemma 3.2 it is enough to prove that $T_{x_1} X_P \cap T_{x_2} X_P = \emptyset$ for two distinct points $x_1, x_2 \in X_P$ lying on an invariant curve. If the two points are invariant ones, we claim that the corresponding double points impose independent conditions on the linear system of hyperplane sections since imposing each invariant double point means removing from P a vertex together with all the points at distance one from it.

Since any two vertexes have distance at least three, it follows that two vertexes can not share the same point at distance one, which proves the claim.

If at least one point is not invariant then both points are contained in a common invariant affine chart. Consider the local parametrization (3.1) of X_P. We may prove the claim for the invariant

curves in this chart since for the other ones it is enough to consider a change of variables in the torus. Up to a change of variables we may write (3.1) as follows:

$$\phi_P : \quad (k^*)^n \quad \longrightarrow \quad \mathbb{P}^N$$
$$(u_1, \ldots, u_n) \quad \mapsto \quad [1 : u_1 : \cdots : u_1^{d_1} : \cdots : u_n : \cdots : u_n^{d_n}, \phi_1 : \cdots : \phi_{N-d_1-\cdots-d_n}]$$

where $\phi_i(u_1, \ldots, u_n)$ is a monomial depending on at least two of the u_j. Note that

(i) since X_P is smooth the monomials $u_1, \ldots, u_n$ must appear in the expression of ϕ_P;

(ii) the monomials $u_1u_2, \ldots, u_1u_n$ must also appear in the expression of ϕ_P since $X_P \subseteq \mathbb{P}^N$ is linearly normal and u_1u_i is in the convex hull of u_1^2 and u_i^2.

Let $C \subset X_P$ be a torus invariant curve say $C = \phi_P(\{u_2 = \cdots = u_n = 0\})$, and consider two distinct points $x_1 = (1, a, \ldots, a^{d_1}, 0, \ldots, 0)$, $x_2 = (1, b, \ldots, b^{d_1}, 0, \ldots, 0)$ of C. Now, since all the 1-dimensional faces of P have length at least three (i) and (ii) yield that the Terracini matrix $T_{X_P}(x_1, x_2)$ has a minor $M_{a,b}$ of size $(2n + 2)$ of the following form

$$M_{a,b} = \begin{pmatrix} A_{a,b} & 0_{4,2n-2} \\ 0_{2n-2,4} & B_{a,b} \end{pmatrix}$$

where $0_{i,j}$ denotes the $i \times j$ zero matrix

$$A_{a,b} = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 \end{pmatrix} \quad \text{and} \quad B_{a,b} = \text{Diag}_{2n-2}(C_{a,b})$$

is the size $2n - 2$ matrix having $n - 1$ copies of

$$C_{a,b} = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}$$

on the main diagonal and whose other entries are zero. Note that $\det(A_{a,b}) = (b-a)^4$ and $\det(B_{a,b}) = (b-a)^{n-1}$. Therefore, $\det(M_{a,b}) = \det(A_{a,b})\det(B_{a,b}) = (b-a)^{n+3}$ proving the claim. $\square$

Remark 3.5. The closure of the image of the map

$$\phi : \quad (k^*)^2 \quad \longrightarrow \quad \mathbb{P}^9$$
$$(u_1, u_2) \quad \mapsto \quad [1 : u_1 : u_1^2 : u_1^3 : u_2 : u_2^2 : u_2^3 : u_1u_2 : u_1^2u_2 : u_1u_2^2]$$

is the degree three Veronese embedding $V_3^2 \subset \mathbb{P}^9$. Proposition 3.4 yields that $T_2(V_3^2)$ is empty. Now, let $X_P \subset \mathbb{P}^8$ be the projection of V_3^2 from $[0 : \cdots : 0 : 1 : 0 : 0]$, that is the closure of the image of the map ϕ_P obtained by removing u_1u_2 from the expression of ϕ. The variety X_P is smooth.

For any pair of points of the form $x_1 = \phi_P(a, 0)$, $x_2 = \phi_P(-a, 0)$ we have $T_{x_1}X_P \cap T_{x_2}X_P \neq \emptyset$ and so $T_2(X_P)$ is non empty. Therefore, the assumption on the linear normality of X_P in Proposition 3.4 can not be dropped.

Example 3.6. We show that 2-Terracini loci can originate from curves of degree greater than two. The closure of the image of the map

$$\phi : \quad k^2 \quad \longrightarrow \quad \mathbb{P}^6$$
$$(u_1, u_2) \quad \mapsto \quad [u_1^3 : u_1^2u_2^3 : u_1^2u_2^2 : u_1^2u_2 : u_1u_2^2 : u_1u_2 : u_2]$$

defined by the following lattice polygon
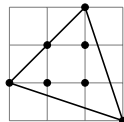
is a surface $X \subset \mathbb{P}^6$ of degree eight, and $C = \overline{\phi(\{u_1 - u_2 = 0\})} \subset X$ is a rational normal quartic. The Terracini matrix of ϕ at any two general points of the curve $\{u_1 - u_2 = 0\}$ is the following

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 2 & 2 & 2 & 1 & 1 & 0 \\
0 & 3 & 2 & 1 & 2 & 1 & 1 \\
t^3 & t^5 & t^4 & t^3 & t^3 & t^2 & t \\
3t^2 & 2t^4 & 2t^3 & 2t^2 & t^2 & t & 0 \\
0 & 3t^4 & 2t^3 & t^2 & 2t^2 & t & 1
\end{pmatrix}
$$

where we assume one of the two points to be the image of $(1,1)$. This matrix has always rank five since the vector

$$(t^5 - t^4 - 2t^3, -t^4 + t^3, -t^5 + t^4, 2t^2 + t - 1, -t^3 + t^2, -t^2 + t)$$

is in the kernel of its transpose.

In what follows we make use of [BC21, Theorem 3.5] and [BC21, Lemma 3.6] to prove emptiness of Terracini loci for a class of embedded projective toric variety. In order to this we introduce an unbounded convex set attached to a projective variety.

Definition 3.7. Let X be a smooth projective variety. The *ample body* of X is

$$\mathscr{A}_X := \{D \in \mathrm{N}^1(X)_{\mathbb{R}} \;:\; D \cdot C \geq 1 \text{ for any curve } C\}.$$

Observe that $\mathscr{A}_X$ is an unbounded convex set such that $\mathscr{A}_X + \mathrm{Nef}(X) = \mathscr{A}_X$.

Proposition 3.8. *Let X be a smooth projective variety whose monoid of classes of curves is finitely generated. Then $\mathscr{A}_X$ is a polyhedron which is the Minkowski sum*

$$\mathscr{A}_X = A_X + \mathrm{Nef}(X)$$

of a rational polytope $A_X \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$ together with its recession cone $\mathrm{Nef}(X) \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$.

Proof. By hypothesis there exist finitely many irreducible curves $C_1, \ldots, C_r$ whose classes in $\mathrm{N}_1(X)_{\mathbb{R}}$ form a Hilbert basis of the Mori cone of X. Any curve of X is rationally equivalent to a non-negative sum of these curves.

It follows that $\mathscr{A}_X$ is intersection of the finitely many half-spaces $\{D \in \mathrm{N}^1(X)_{\mathbb{R}} \;:\; D \cdot C_i \geq 1\}$, which proves the statement. Being $\mathscr{A}_X$ a polyhedron it is Minkowski sum of a polytope A_X together with its recession cone σ_X. From the definition of $\mathscr{A}_X$ it immediately follows that $\mathrm{Nef}(X) = \sigma_X$. $\square$

Proposition 3.9. *Let X be a smooth projective variety whose Mori cone is rational polyhedral. Then A_X is a normal lattice polytope if and only if for any h the equality*

$$(h \cdot \mathscr{A}_X) \cap \mathrm{N}^1(X) = \sum_{i=1}^{h} (\mathscr{A}_X \cap \mathrm{N}^1(X))$$

holds.

Proof. By the definition of normal lattice polytope we have that

$$(h \cdot A_X) \cap \mathrm{N}^1(X) = \sum_{i=1}^{h} (A_X \cap \mathrm{N}^1(X))$$

and this is exactly what we need for the second condition to be satisfied. $\square$

Remark 3.10. Observe that if A_X is a point then is must necessarily be a lattice point and of course a polytope consisting of just one lattice point is normal.

Lemma 3.11. *Let X be a projective variety with smooth locus X^o and let A, B be Weil divisors on X. Assume that for any subsets $S_A, S_B \subseteq X^o$ of cardinality $n := |S_A| > 1$ and $|S_B| = 2$ one has*

$$H^1(X, \mathcal{O}_X(A) \otimes \mathcal{I}_{S_A}) = H^1(X, \mathcal{O}_X(B) \otimes \mathcal{I}_{S_B}) = 0.$$

Then $H^1(X, \mathcal{O}_X(A + B) \otimes \mathcal{I}_S) = 0$ for any subset $S \subseteq X^o$ of cardinality $n + 1$.

Proof. Let $S \subseteq X^o$ be a subset of cardinality $n + 1$. Given any $p \in S$ write $S = U \cup \{p, q\}$ with $|U| = n - 1$ non empty and $p \neq q$. By hypothesis there exist

$$f \in H^0(X, \mathcal{O}_X(A) \otimes \mathcal{I}_U) \text{ and } g \in H^0(X, \mathcal{O}_X(B) \otimes \mathcal{I}_p)$$

which do not vanish at q. Thus $fg \in H^0(X, \mathcal{O}_X(A + B) \otimes \mathcal{I}_{U \cup \{p\}})$ does not vanish at q. $\qquad\square$

Corollary 3.12. *Let X be a projective variety with smooth locus X^o and let D be a Weil divisor which is sum of n very ample divisors. Then any subset of X^o of cardinality $n + 1$ imposes independent conditions on D.*

Proof. Observe that if B is very ample then it satisfies the hypothesis on the divisor B in Lemma 3.11. The statement follows by induction on n. $\qquad\square$

Proposition 3.13. *Let $X \subseteq \mathbb{P}^N$ be a projective variety embedded by a complete linear system $|L|$. Assume that L is sum of $2h - 1 \geq 3$ very ample divisors. Then $T_h(X)$ is empty.*

Proof. Let $S \subseteq X^o$ be a subset of h distinct points. Write $L = A + B + C$ where both A and B are sum of $h - 1$ very ample divisors and C is very ample. Since C is very ample for any $p \in S$, there is a smooth element in the linear system of C through p which does not contain any point of $S \setminus \{p\}$.

By Corollary 3.12 S imposes independent conditions on A. Since also $B + C$ is very ample the hypotheses of [BC21, Lemma 3.6] are satisfied so that for any $p \in S$ the scheme $(S \setminus \{p\}) \cup \{2p\}$ imposes independent conditions on $B + C$. Finally, to conclude it is enough to apply [BC21, Theorem 3.5] with $L_1 = B + C$ and $L_2 = A$. $\qquad\square$

Now, we are ready to prove the main result of this section.

Theorem 3.14. *Let $P \subseteq M_{\mathbb{Q}}$ be a full dimensional lattice polytope such that the corresponding projective toric variety $X_P \subseteq \mathbb{P}^{|P \cap M| - 1}$, embedded by a complete linear system $|L|$, is smooth. If A_{X_P} is a normal lattice polytope then the following are equivalent:*

    (a) $\ell(P) = s$ with $s \geq 2h - 1$;
    (b) $L \in s \cdot \mathscr{A}_X$;
    (c) $T_h(X)$ is empty.

Proof. Note that $(b) \Rightarrow (c)$ follows from Propositions 3.9 and 3.13. Now, assume that $s < 2h - 1$. Then X_P contains an invariant curve embedded as a curve $C \subset \mathbb{P}^{|P \cap M| - 1}$ of degree s. Since X_P is smooth, arguing as in the proof of Proposition 3.4, we get that $C \subset \mathbb{P}^{|P \cap M| - 1}$ is a rational normal curve of degree s. Since $s < 2h - 1$ the span of any h tangent lines of C has dimension smaller than expected and hence the span of h tangent spaces of X_P at points of C has dimension smaller than expected as well. Therefore, $T_h(X)$ is non empty proving that $(c) \Rightarrow (a)$.

Finally, to prove that $(a) \Rightarrow (c)$ note that since $\ell(P) = s$ all the toric invariant curves in X_P have degree at least s, and since these curves generate the Mori cone of X_P we get that $L \in s \cdot \mathscr{A}_X$. $\qquad\square$

There exist smooth toric varieties for which A_X is not a normal lattice polytope as the following example shows.

Example 3.15. Let X be the smooth toric surface whose fan is the following:
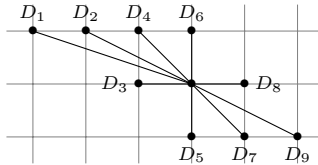
Then A_X is not a lattice polytope. To prove this observe that the $\mathbb{Q}$-divisor

$$D = D_3 + D_4 + 4D_5 + 3D_6 + 4D_7 + \frac{7}{2}D_8 + 5D_9$$

has intersection product at least one with any D_i and the equality holds for $i \in \{1, 2, 3, 4, 5, 7, 8\}$. Since the classes of these seven divisors form a basis of the rational Picard lattice, it follows that the

class of D is a vertex of A_X. On the other hand $D \cdot D_6 = \frac{3}{2}$, so that the class of D can not be in the integral Picard group and so A_X con not be a lattice polytope.

4. APPLICATIONS

In this section we apply our main results in Sections 2 and 3 to several classes of projective varieties.

Proposition 4.1. *Let $X \subset \mathbb{P}^N$ be a smooth toric variety of Picard rank two embedded by the complete linear system of a divisor $L \in (2h-1) \cdot \mathscr{A}_X$. Then $T_h(X)$ is empty.*

Proof. If X is a smooth projective toric variety with Picard rank two, then its nef cone is generated by two primitive rays, so that it is simplicial.

To prove that it is also smooth we proceed as follows. Let $D_1, \ldots, D_r$ be the prime torus invariant divisors of X and say that D_1, D_2 are the two whose classes generate $\mathrm{Nef}(X)$. The intersection $\bigcap_{i=3}^r D_i$ is a torus invariant point $p \in X$ and the local divisor class group $\mathrm{Cl}(X, p)$ is generated by the class of the prime invariant divisors which contain p, that is by the D_i with $i \geq 3$. Thus $\mathrm{Cl}(X, p) = \mathrm{Cl}(X \setminus D_1 \cup D_2) = \mathrm{Cl}(X)/\langle D_1, D_2 \rangle$.

On the other hand $\mathrm{Cl}(X, p)$ is trivial, being X smooth, so that we can conclude that $\mathrm{Cl}(X) = \langle D_1, D_2 \rangle$, which proves that the classes of D_1 and D_2 form a basis of $\mathrm{Cl}(X)$. This shows that $\mathrm{Nef}(X)$ is smooth. Being this cone smooth and simplicial we get that A_X is a point. Finally, to conclude it is enough to apply Theorem 3.14. $\square$

Choose positive integers $a_1 \leq a_2 \leq \cdots \leq a_n$ such that $\sum_{i=1}^n a_i = N - n + 1$, $\Lambda_i \cong \mathbb{P}^{a_i} \subset \mathbb{P}^N$ complementary linear subspaces, $C_i \subset \mathbb{P}^{a_i}$ rational normal curves, isomorphisms $\phi_i : C_1 \to C_i$, and consider the rational normal scroll $S_{a_1, \ldots, a_n} = \bigcup_{p \in C_1} \langle p, \phi_2(p), \ldots, \phi_n(p) \rangle$. Let H be the restriction to $S_{a_1, \ldots, a_n}$ of the hyperplane section of $\mathbb{P}^N$, $D = dH$ and $S_{a_1, \ldots, a_n, d} \subset \mathbb{P}^{N_d}$ the image of $S_{a_1, \ldots, a_n}$ via the embedding induced by D.

Corollary 4.2. *If $h \leq \lceil \frac{d}{2} \rceil$ then $T_h(S_{a_1, \ldots, a_n, d})$ is empty.*

Proof. Since $S_{a_1, \ldots, a_n, d}$ is a smooth toric variety of Picard rank two the claim follows from Proposition 4.1. $\square$

Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ and $\boldsymbol{d} = (d_1, \ldots, d_r)$ be two r-uples of positive integers, with $n_1 \leq \cdots \leq n_r$ and $N(\boldsymbol{n}, \boldsymbol{d}) = \prod_{i=1}^r \binom{n_i + d_i}{n_i} - 1$. The Segre-Veronese variety $SV_{\boldsymbol{d}}^{\boldsymbol{n}}$ is the image in $\mathbb{P}^{N(\boldsymbol{n}, \boldsymbol{d})}$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ via the embedding induced by $\mathcal{O}_{\mathbb{P}^{\boldsymbol{n}}}(d_1, \ldots, d_r) = \mathcal{O}_{\mathbb{P}(V_1^*)}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}(V_1^*)}(d_r)$.

As a consequence of Theorem 3.13 we recover the results on Terracini loci of Segre-Veronese varieties in [Bal22a, Theorem 1.3].

Corollary 4.3. *If $h \leq \lceil \frac{d_i}{2} \rceil$ for all $i = 1, \ldots, r$ then $T_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ is empty.*

Proof. Since $h \leq \lceil \frac{d_i}{2} \rceil$ for all $i = 1, \ldots, r$ we may write $L = 2A + B$ where $L = \mathcal{O}_{\mathbb{P}^{\boldsymbol{n}}}(d_1, \ldots, d_r)$, $A = \mathcal{O}_{\mathbb{P}^{\boldsymbol{n}}}(h-1, \ldots, h-1)$ and $B = \mathcal{O}_{\mathbb{P}^{\boldsymbol{n}}}(b_1, \ldots, b_r)$ with $b_i \geq 1$ for all $i = 1, \ldots, r$.

Since the Mori cone of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is generated by the classes $[l_i]$, where $l_i \subset \mathbb{P}^{n_i}$ is a line to conclude it is enough to apply Theorem 3.13. $\square$

Corollary 4.4. *If $2h \leq \lceil \frac{d_i}{2} \rceil$ for all $i = 1, \ldots, r$ then any point of $\mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}}) \setminus \mathbb{S}ec_{h-1}(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ is h-identifiable and $\mathbb{S}ec_h(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$ is smooth outside of $\mathbb{S}ec_{h-1}(SV_{\boldsymbol{d}}^{\boldsymbol{n}})$.*

Proof. The claim follows immediately from Corollary 4.3 and Proposition 2.8. $\square$

Proposition 4.5. *Let $X \subset \mathbb{P}^N$ be a homogeneous variety embedded by the complete linear system of a divisor $L \in (2h-1) \cdot \mathscr{A}_X$. Then $T_h(X)$ is empty.*

Proof. If X is a homogeneous variety then $\mathrm{Nef}(X) = \mathrm{Eff}(X)$ and both cones are smooth and simplicial [Bri05, Proposition 1.4.1]. In particular A_X is the lattice point given by the sum of the primitive generators of the rays of the nef cone. Hence, to conclude it is enough to apply Propositions 3.9 and 3.13. $\square$

If $L \in r \cdot \mathscr{A}_X$ with $r \leq 2h - 2$ the Terracini locus is in general non empty as the following result shows.

Proposition 4.6. *Let* $\mathbb{G}(r, n) \subset \mathbb{P}^N$ *be the Grassmannian of r-linear spaces in $\mathbb{P}^n$ embedded with the Plücker embedding. Then* $T_2(\mathbb{G}(r, n))$ *is the closure of*

$$\{([U], [U']) \in \mathbb{G}(r, n) \times \mathbb{G}(r, n) \mid U \neq U' \text{ and } \dim(U \cap U') \geq r - 2\}/S_2$$

in $(\mathbb{G}(r, n) \times \mathbb{G}(r, n))/S_2$.

Proof. By [MR18, Lemma 6.5] we have the following characterization of the tangent space of the Grassmannian at a point $[U] \in \mathbb{G}(r, n)$:

$$T_{[U]}(\mathbb{G}(r, n)) = \langle e_I \mid d(I, \{0, \ldots, r\}) \leq 1 \rangle$$

where $(e_0, \ldots, e_n)$ is a basis of k^{n+1}, U is generated by $e_0, \ldots, e_r \in \mathbb{P}^n$, $e_I = e_{i_0} \wedge \cdots \wedge e_{i_r}$ and $d(I, J)$ is the Hamming distance between the two lists I and J.

Now, given $[U], [V] \in \mathbb{G}(r, n)$ let $s := \dim(U \cap V)$, after a base change, we can write $U = \langle e_0, \ldots, e_r \rangle$ and $U' = \langle e_0, \ldots, e_s, e_{r+1}, \ldots, e_{r+(r-s)} \rangle$. Note that $s = \dim(U \cap U')$. Therefore

$$\begin{aligned} T_{[U]}(\mathbb{G}(r, n)) \cap T_{[U']}(\mathbb{G}(r, n)) &= \langle e_I \mid d(I, \{0, \ldots, r\}) \leq 1 \rangle \cap \langle e_I \mid d(I, \{0, \ldots, s, r+1, \ldots, r+(r-s)\}) \leq 1 \rangle \\ &= \langle e_I \mid d(I, \{0, \ldots, r\}) \leq 1 \text{ and } d(I, \{0, \ldots, s, r+1, \ldots, r+(r-s)\}) \leq 1 \rangle. \end{aligned}$$

Hence, $T_{[U]}(\mathbb{G}(r, n)) \cap T_{[U']}(\mathbb{G}(r, n))$ is empty if and only if $d(\{0, \ldots, r\}, \{0, \ldots, s, r+1, \ldots, r+(r-s)\}) \geq 3$ which is equivalent to $s \leq r - 3$. $\square$

4.7. **Fano varieties.** We show that Theorem 3.14 applies to smooth toric Fano varieties of dimension at most four.

Proposition 4.8. *Let X be a smooth toric Fano variety of dimension at most four. Then A_X is a lattice point.*

Proof. We load the n-th entry from the smooth toric Fano varieties database using the function `FanoX` from the following library: `https://github.com/alaface/Terracini_Loci`. The following function computes the ample polytope of the toric variety X.

```
AX := function(X)
 forms := IntersectionForms(X);
 pol := &meet[HalfspaceToPolyhedron(v,1) : v in forms];
 return CompactPart(pol);
end function;
```

Smooth toric varieties of dimension at most four are the first 147 entries of the database. For each such variety we check with the following script

```
> time {#Points(AX(FanoX(n))) : n in [1..147]};
{ 1 }
Time: 153.710
```

that the ample polytope consists of a point. $\square$

However, there are smooth toric Fano 5-fold for which A_X is not a point.

Example 4.9. Let $X \subseteq \mathbb{P}^{192}$ be the smooth projective toric Fano 5-fold, ID. 556 in the Graded Ring Database, whose Cox ring $\mathbb{C}[x_1, \ldots, x_{10}]$ has the following grading matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Let D_i be the i-th prime invariant torus divisor of X. Then A_X is a lattice segment of length one whose lattice points are the classes of $-K_X$ and $-K_X + D_3$.

Proof of Theorem 1.3. Since by Proposition 4.8 A_X is a lattice point Theorem 3.14 applies. $\square$

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Antonio Laface, Departamento de Matematica, Universidad de Concepción, Casilla 160-C, Concepción, Chile
    Email address: `alaface@udec.cl`

Alex Massarenti, Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli 30, 44121 Ferrara, Italy
    Email address: `alex.massarenti@unife.it`