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Geometric and Topological Aspects of Manifolds with Symmetry

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Abstract

In the thesis, we study different geometric and topological aspects of a G-manifold M. These aspects are organized into four different parts, based on four works [1],[2],[3] and [4].

In the first part, we generalize the construction of a complex structure on a product of two odd-dimensional spheres from the classical paper by Calabi and Eckmann [21]. More precisely, let M_i, for $i = 1, 2$, be a Kähler manifold, and let G be a compact Lie group acting on M_i by holomorphic isometries. Suppose that the action admits a *momentum map* μ_i and let $N_i := \mu_i^{-1}(0)$ be a regular level set. We define an almost complex structure on the manifold $N_1 \times N_2$ and provide necessary and sufficient conditions for its integrability. In the integrable case, we find explicit holomorphic charts for $N_1 \times N_2$. As applications, we construct a non integrable almost-complex structure on the product of two complex Stiefel manifolds and we will consider the infinite Calabi-Eckmann manifolds $\mathbb{S}^{2n+1} \times S(\mathcal{H})$, for $n \geq 1$, where $S(\mathcal{H})$ denotes the unit sphere of an infinite dimensional complex Hilbert space $\mathcal{H}$.

The second part of the thesis work is based on [2] and is focused on the reduction properties of a proper action, generalizing the work of Skjelbred and Straume in [105], and of Grove and Searle [43]. More specifically, we generalize the construction of the *core* of a G-manifold M from the case where G is compact to the case in which the G-action is proper. Furthermore, we show that many properties of a proper G-action on M are determined by the action of a group G' on the corresponding core ${}_cM$, and vice versa. We say that such properties admit a *reduction principle*. In particular, we prove that a proper isometric G-action on M is polar (resp. hyperpolar) if and only if the G'-action on ${}_cM$ is polar (resp. hyperpolar). In the case of a proper action by symplectomorphisms on a symplectic manifold, we show that a reduction principle holds for coisotropic and infinitesimally almost homogeneous actions. We further study the coisotropic condition for the case of a proper Hamiltonian action and its relation with the symplectic stratification (see [12]). In particular, we obtain several characterizations for coisotropic actions.

In the third part of the thesis, based on work [3], we study certain topological aspects of a G-manifold M. More precisely, the relation between the geometry of the quotient M/G and the *rational homotopy ellipticity or hyperbolicity* of M. In this part, G is assumed to be compact and M is assumed to be 1-connected. In a paper by Grove, Wilking, and Yeager [44], the authors prove that in the case of a cohomogeneity-2 G-action, M/G can be either a *bad orbifold*, or it is covered by the round sphere, or by the Euclidean plane or by the hyperbolic plane. In all but the last case the manifold M is proved to be *rationally elliptic*. In the same paper it is conjectured that in the last case, i.e. when M/G supports a metric of hyperbolic type, the manifold M is *rationally hyperbolic*. Our first result is a counterexample to this conjecture, that is, we find a rationally elliptic G-manifold such that M/G is a hyperbolic polygon. This counterexample also disproves an analogous conjecture by Grove and Ziller, in the context of polar actions [45]. Subsequently, we look for additional sufficient conditions to prove rational hyperbolicity. Following the classical construction by Bott and Samelson [17], we prove that if the G-action on M is variationally complete and M/G supports a metric of hyperbolic type, then M is rationally hyperbolic if and only if either the isotropy subgroups are Abelian, or a principal isotropy is connected.

In the last part, based on the work in preparation [4], we study a generalization of the *Calabi-Matsushima decomposition* for *real reductive groups*. More precisely, we let Z be a compact connected Kähler manifold and $U \subseteq U(n)$ a compact connected Lie

subgroup of the unitary group, acting in a Hamiltonian fashion and by holomorphic isometries on Z. Under these hypotheses, the action extends to a holomorphic action of the reductive group $U^{\mathbb{C}}$. We consider a *real compatible* subgroup $G \subseteq U^{\mathbb{C}}$, and we prove that the Lie algebra of the stabilizer of a point $z \in Z$, $\mathfrak{g}_z$, relative to the holomorphic sub-action by G admits a decomposition into eigenspaces $\mathfrak{g}_\lambda \subseteq \mathfrak{g}_z$, relative to nonnegative eigenvalues $\lambda \geq 0$, where $\mathfrak{g}_0$ is the reductive part of $\mathfrak{g}_z$. This result was previously known only for the complex case [66], i.e., when $G = U^{\mathbb{C}}$.

Abstract

Nel lavoro di tesi studiamo varii aspetti di natura geometrica e topologica di una G-varietà differenziabile M. La trattazione di tali aspetti è suddivisa in quattro parti, le quali sono basate rispettivamente sui lavori [1], [2], [3] e [4].

Nella prima parte della tesi, forniremo una generalizzazione della costruzione di una struttura complessa sul prodotto di due sfere di dimensione dispari, dovuta al lavoro di Calabi ed Eckmann nell'articolo classico [21]. Più precisamente, sia M_i, con $i = 1, 2$ una varietà di Kähler e sia G un gruppo di Lie compatto che agisce su M_i per isometrie di Kähler. Supponiamo ulteriormente che tale azione ammetta una mappa momento μ_i, sia $N_i := \mu_i^{-1}(0)$ un insieme di livello regolare e assumiamo che la G-azione su N_i sia libera. Tali ipotesi ci permettono di definire una struttura quasi-complessa sul prodotto $N_1 \times N_2$ e ci consentono altresì di determinare condizioni necessarie e sufficienti affinché tale struttura risulti integrabile. Nel caso integrabile, saremo inoltre in grado di trovare esplicitamente delle carte olomorfe per $N_1 \times N_2$. Come applicazioni, mostreremo come la costruzione da noi proposta permetta di definire una struttura quasi complessa sul prodotto di due varietà di Stiefel complesse; nel seguito, considereremo il caso della varietà di Calabi-Eckmann infinita, $\mathbb{S}^{2n+1} \times S(\mathcal{H})$, dove $S(\mathcal{H})$ indica la sfera unitaria di uno spazio di Hilbert complesso infinito-dimensionale.

Nella seconda parte della tesi, basata su [2], ci focalizzeremo sulle proprietà di riduzione di una azione propria, generalizzando i lavori di Skjelbred e Straume [105] e di Grove e Searle [43]. Più specificatamente, generalizziamo la costruzione del *core* di una G-varietà dal caso in cui G è un gruppo di Lie compatto al caso in cui l'azione di G è propria. Inoltre, mostreremo come molte proprietà della G-azione determinino e siano determinate dall'azione di un sottogruppo $G' \subseteq G$ sul core, che denoteremo con ${}_cM$. Diremo che tali proprietà ammettono un principio di riduzione. In particolare, dimostreremo che una G-azione è polare (rispettivamente iperpolare) se e solo se la G'-azione su ${}_cM$ è polare (rispettivamente iperpolare). Nel caso di azioni proprie per simplettomorfismi su varietà simplettiche, proveremo altresì che vale un principio di riduzione per le azioni coisotrope e infinitesimamente quasi omogenee. Nel prosieguo, studieremo ancora le azioni di tipo coisotropo nel caso Hamiltoniano e mostreremo la relazione con la stratificazione simplettica studiata da Bates e Lermann in [12]. In questo modo, otterremo svariate caratterizzazioni per le azioni coisotrope.

Nella terza parte della tesi, basata su [3], studieremo la relazione tra la geometria del quoziente M/G di una G-varietà differenziabile M e la razionale ellitticità o iperbolicità di M. In questo contesto la varietà M sarà sempre supposta compatta, connessa e semplicemente connessa. Un lavoro di Grove, Wilking e Yeager [44] dimostra che, se la coomogeneità della G-azione è pari a due, M/G può essere isometrico a un *bad orbifold*, oppure può essere rivestito da una sfera rotonda, dal piano Euclideo o dal piano iperbolico. In tutti i casi precedenti, a parte l'ultimo, in [44] viene mostrato che M è razionalmente ellittica. Nello stesso articolo, è inoltre congetturato che, se M/G supporta una metrica di tipo iperbolico, allora M è razionalmente iperbolica. Il nostro primo risultato è quello di trovare un controesempio per questa congettura, ovverosia una varietà razionalmente ellittica che ammette una G-azione tale che M/G sia un poligono iperbolico. Tale controesempio smentisce una analoga congettura dovuta a Grove e Ziller nell'ambito delle azioni polari [45]. Nell'ottica di fornire risultati positivi per queste congetture, cerchiamo inoltre di trovare ipotesi aggiuntive che siano sufficienti per concludere la razionalità iperbolica di M. Basandoci sul lavoro classico di Bott e Samelson [17], dimostriamo che data una G-azione su M che sia *variazionalmente completa* e tale che M/G supporti una metrica di tipo iperbolico, se le isotropie

singolari sono Abeliane o una isotropia principale è connessa, allora M è razionalmente iperbolica.

Nell'ultima parte della tesi, basata su [4], studiamo una generalizzazione della decomposizione di Calabi-Matsushima nel caso di gruppi riduttivi reali. Più precisamente, sia Z una varietà di Kähler compatta e connessa, sia $U \subseteq U(n)$ un sottogruppo compatto e connesso del gruppo unitario in dimensione complessa n, e supponiamo che U agisca su Z per isometrie di Kähler. Sotto queste ipotesi, la U-azione si può estendere ad una azione olomorfa del gruppo complessificato $U^{\mathbb{C}}$. Consideriamo dunque un sottogruppo reale $G \subseteq U^{\mathbb{C}}$. Mostriamo che l'algebra di Lie dello stabilizzatore di ogni punto $z \in Z$, denotata con $\mathfrak{g}_z \subseteq \mathfrak{g}$, relativa alla sotto-azione olomorfa di G, ammette una decomposizione in autospazi $\mathfrak{g}_\lambda$ relativi ad autovalori non-negativi $\lambda \geq 0$, dove $\mathfrak{g}_0$ è la parte riduttiva di $\mathfrak{g}_z$. Tale risultato era noto in precedenza nel caso riduttivo complesso, ossia quando $G = U^{\mathbb{C}}$, si veda [66].

Introduction

The present thesis focuses on a series of problems of geometric and topological nature, regarding smooth manifolds with symmetry. The concept of symmetry is formalized through the notion of *Lie group action*. This notion can be thought of as a generalization of the notion of Lie group representation, where we do not assume any linear structure on the space acted on by the group. A manifold with symmetry then means the data of a smooth manifold M, of a Lie group G and of a given action of G on M, i.e. a smooth map $\Psi : G \times M \to M$, $(g, x) \mapsto \Psi(g, x)$ such that the map $G \to \mathrm{Diff}(M)$, $g \mapsto \Psi(g, \cdot)$, is a group homomorphism between G and the group of diffeomorphisms of M. When the action is understood, we will speak of M as a G-manifold. The theory of Lie group actions on smooth manifolds becomes much more manageable if one considers only actions by compact Lie groups.

The theory of compact Lie group actions can be considered classical and well established since the 1950's; see e.g. the original articles [81, 83] and the much later comprehensive treatment [19]. Already in the 1960's in the work [91] it was proved that the structural results for compact Lie group actions could be extended to the case of non compact Lie groups under some additional hypothesis. One of the broadest class of actions one can consider in this direction are the so-called *Cartan actions*. However, for simplicity, we will focus on the slightly more restrictive class of *proper Lie group actions*.

The fundamental idea behind the implementation of symmetry in studying geometric and topological problems is the idea of *reduction of variables*. In fact, a group action by G on a smooth manifold M naturally induces an *equivalence relation* in M, namely, two points are related if they are "symmetric", i.e., if there exists $g \in G$ such that one of the points is mapped into the other by the transformation $\Psi(g, \cdot)$. The equivalence classes of this relation are called *G-orbits*. Then we can consider the quotient space, called the *orbit space*, denoted by M/G, which in principle should be "smaller" than M. If we start from any problem in M, e.g. an ODE, a PDE, or any geometric or topological conditions on M, this should translate to a simpler problem in M/G. We refer to this general procedure as to a *reduction process*. Then, we shall call the inverse procedure of retrieving the solution to the original problem a process of *reconstruction*; this nomenclature comes specifically from the Theory of Dynamical Systems (see [73]). Unfortunately, apart from very special conditions (e.g. if the action is proper and *free*), the space M/G is not a manifold anymore and we are forced to work in a different category. Still, in the case of a proper action, M/G is a good topological space. Moreover, M/G is stratified by smooth manifolds (see [13, 29]) and, in principle, the reduction-reconstruction procedures can be applied piece-by-piece.

Thanks to a fundamental theorem of Myers and Steenrod [85], given a Riemannian manifold $(M, \mathbf{g})$ with Riemannian metric $\mathbf{g}$, any closed subgroup (in the compact open topology) of the group of isometries of $(M, \mathbf{g})$, denoted by $\mathrm{Iso}(M, \mathbf{g})$, is a Lie group. In particular, $\mathrm{Iso}(M, \mathbf{g})$ is a Lie group. In view of this result, one can prove that given a proper G-action on M, it is always possible to find a G-invariant Riemannian metric

$\mathbf{g}$ on M. The G-action is then called *isometric with respect to* $\mathbf{g}$. Vice versa, given a Riemannian manifold $(M, \mathbf{g})$, the natural action by any closed subgroup of $\mathrm{Iso}(M, \mathbf{g})$ can be proved to be proper. This series of facts justify the approach we adopt in the thesis, where we will always assume to have a G-action on M which is proper and isometric with respect to some Riemannian metric on M. In this way, our study will be greatly simplified thanks to the use of Riemannian Geometry; for this topic, we refer the reader to the standard treatments [28, 97], or to the very concise and effective [13], and to the Appendix for an extremely brief account of the notions that we will use and the conventions we shall follow.

In general, the geometric objects on a G-manifold M, e.g. Riemannian and metric structures, *symplectic structures*, *Kähler structures*, or more generally functions and tensor fields, that are invariant under the action by G are of special interest. These form the class of so-called *G-invariant objects* on M. We note that the process of reduction in the metric setting is quite natural, as a G-invariant metric structure can be easily transported to M/G (see e.g. [13, 98]). On the contrary, the process of reduction in the symplectic case is highly non trivial and is of great interest in its own right (see [12, 77, 74]). In fact, if M has a G-invariant symplectic structure, even if the action is proper and free, in general, the quotient does not even support a symplectic structure, and hence there is not an immediate way to produce a reduced symplectic space. For an introduction to Symplectic Geometry we refer the reader to the standard treatments [5, 9, 22], and to the Appendix for some elementary notions.

In the rest of this introduction we will summarize the content of each chapter and, apart from Chapter 1, which is introductory by its own nature, we will give specific context to and motivation for each topic treated.

Organization of the Chapters

Chapter 1. The first chapter of the thesis is a summary of basic results of the theory of Lie group actions, with a special focus on proper and isometric actions. We present the local structural results of the theory, namely the *Slice Theorem* and the *Tubular neighborhood Theorem*, and their global implications like the *Principal Orbit Theorem*. We then examine more specific types of actions; e.g. *polar, symplectic and Hamiltonian actions*. All the results of this chapter are well known and standard, maybe with the exception of Theorem 1.2.27, which comes from a sharp remark in the lecture notes [98]. The common references for this chapter are [13, 29, 98] for the general theory of proper, isometric and polar actions and [5, 9, 22] for the symplectic and Hamiltonian actions.

Chapter 2. In the spirit of the reduction process, we consider an application of the classical *symplectic reduction*, à la Meyer–Marsden–Weinstein (see [77, 74]) and more specifically of one of its consequences, namely, *Kähler reduction* (see [33, 46]). In the classical paper by Calabi and Eckmann [21], the authors show that the compact, simply connected manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ admits a complex structure. When we let $m > 0$, this was the first example of compact, simply connected complex manifold which does not admit a Kähler structure. For $m = 0$, the same authors remark that their construction reduces to the one discovered by Hopf years before [60].

Our work starts with the observation that the construction in [21] can be reinterpreted in terms of Kähler reduction. This allows us to define an almost complex structures on any product of two manifolds $N_1 \times N_2$ that are regular level sets of a momentum map of a given Hamiltonian action (see Definition 2.2.1). The classical construction can be retrieved by considering the natural $U(1)$-actions on $\mathbb{C}^{n+1}$ and $\mathbb{C}^{m+1}$, which are symplectic and Hamiltonian and whose momentum maps are proportional to the standard Hermitian product of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{m+1}$, respectively. A classical result by Newlander and Niremberg [87], allows us to find necessary and sufficient conditions to determine the integrability of this almost complex structure (see Theorem 2.2.2). It turns out that integrability is equivalent to asking for the Lie group acting being Abelian, which is in accordance with the classical result. Next, we show that the construction in [21] can easily be generalized to the action of a complex torus (see Theorem 2.2.4).

As an application, we show that thanks to our construction the product of two complex Stiefel Manifolds $V_k(\mathbb{C}^n) \times V_k(\mathbb{C}^n)$ supports a non-integrable almost complex structure coming from a natural $U(k)$-action on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$. Moreover, the subaction by any torus of $U(k)$ provides a foliation of $V_k(\mathbb{C}^n) \times V_k(\mathbb{C}^n)$ into J-holomorphic tori. As a second application, we observe that our Theorem 2.2.4 provides a complex structure J on the infinite dimensional Calabi-Eckmann manifold $\mathbb{S}^{2n+1} \times S(\mathcal{H})$, where $S(\mathcal{H})$ is the unit sphere of an infinite dimensional complex Hilbert space $\mathcal{H}$. This allows us to conclude that there are no symplectic structures on $\mathbb{S}^{2n+1} \times S(\mathcal{H})$ compatible with J, making $\mathbb{S}^{2n+1} \times S(\mathcal{H})$ a Kähler manifold.

Chapter 2 is based on the published paper [1], which is a joint work with Prof. L. Biliotti.

Chapter 3. In this chapter, we investigate a particular aspect of the reduction process for a G-manifold M. We generalize the construction of the *core* of a group action from the compact case treated in [43, 105] to the case of proper actions (see also [11, 106]). This amounts to considering a special submanifold ${}_cM$ of M, called the *core of the action* which supports the action of a subgroup $N \subseteq G$ and is such that the inclusion ${}_cM \hookrightarrow M$ induces an isomomorphism of stratified spaces in the quotients $M/G \simeq {}_cM/N$. We can regard the core of a proper isometric G-action as an example of a k-section (see [37] and Section 3.2), which is in turn a generalization of the concept of a *section*. Recall that an isometric G-action on M is called polar if there exists a totally geodesic submanifold Σ (either embedded or injectively immersed, depending on the convention[1]), that meets all the G-orbits orthogonally. This submanifold is called a *section* for the action. Polar actions are of prime interest especially from the reduction point of view. In fact, the section provides a smaller manifold having the same dimension of the quotient, which is acted on by a finite group, W, called the generalized Weyl group of the section. In other words, we can avoid the singularities in M/G, by looking at the section (which covers it, in the sense of orbifolds, see [36]). By the work of Palais and Terng [92], Theorem 4.15, there is an isomorphism between the G-invariant functions on M and the W-invariant functions on Σ. This type of result also holds, in a weaker sense, for Riemannian metrics and tensor fields (see [76]). The core construction allows us to obtain a generalization of Theorem 4.15 in [92]. That is, we find that there exists an isomorphism between the G-invariant functions on M and the N-invariant functions on ${}_cM$ (see Theorem 3.1.13). It would be interesting to extend this theorem to other invariant objects.

From a different perspective, ${}_cM$ seems to be the appropriate space where the information on the G-action on M can be encoded. More precisely, we would like to investigate when given a property $\mathcal{P}$ of the G-action on M, $\mathcal{P}$ is satisfied if and only if $\mathcal{P}$ is satisfied by the N-action on ${}_cM$. We find that many properties satisfy this

[1]For instance, [98] uses the embedded version and [13] the injectively immersed version; we shall follow the second one for having more flexibility.

condition, and for such properties, we will say that they admit a *reduction principle*. The first examples we provide are polar actions (see Theorem 3.2.1) and hyperpolar polar actions (see Theorem 3.2.3). That is, we prove that a proper isometric G-action on M is polar (hyperpolar) if and only if the N-action on ${}_cM$ is polar (hyperpolar).

Subsequently, we investigate *coisotropic actions* and *multiplicity free actions*. These actions were studied in [48], in the context of the Hamiltonian description of Quantum Mechanical Systems. The name *multiplicity free actions* is justified by their Theorem 5.4, where they prove that the *lifted action* on T^*Q of any action by a compact group G on a manifold Q is multiplicity free if and only if the natural unitary representation of G on the Hilbert space $L^2(Q)$ of square-integrable functions on Q is multiplicity free in the usual sense of representations[2]. The same authors showed in [47] that a *Hamiltonian* action by a compact group G on a symplectic manifold M is coisotropic if and only if it is multiplicity free (see also [62]). Our Proposition 3.3.3 generalizes the same result for proper coisotropic actions on symplectic manifolds. Then we prove that the coisotropic condition admits a reduction principle (see Theorem 3.3.8). Moreover, by combining our Proposition 3.3.3 with this last result, we obtain a wide characterization of coisotropic actions in Theorem 3.3.10.

Justified by certain results in the context of Kähler actions (see [61]), we examine a real analog of the notion of *almost homogeneous actions*, namely *infinitesimally almost homogeneous actions*. These actions are more general than coisotropic actions (see Remark 3.3.12), and still admit a reduction principle (see Theorem 3.3.15).

In the last section of Chapter 3, we study further characterizations for coisotropic actions under the additional hypothesis that the G-action is Hamiltonian. This is equivalent to asking that the action admits an *equivariant momentum map*. If we also assume that the coadjoint orbits are embedded, we can rely on the stratification results of Bates and Lerman in [12] and on the expression of the momentum map in the so called *normal form* (see again [12] or [49] for the compact case). Basing on these results, in Theorem 3.4.8, we obtain a characterization of coisotropic Hamiltonian actions which can be seen as a refinement of Theorem 3.3.10.

As an application, we prove how a Hamiltonian action by a product group $G = G_0 \times G_1$ is coisotropic if and only if the (sub)action by one of the G_i is coisotropic on the symplectic pieces of $M_{\alpha_{i+1}}$ for all $\alpha_{i+1} \in \mathfrak{g}_{i+1} = \mathrm{Lie}(G_{i+1})$, for $i \in \mathbb{Z}_2$ (see Theorem 3.4.16).

Following the results of [62], we point out that in the Hamiltonian case, the coisotropic condition can be read off from the *homogeneity rank* of the *symplectic slice representation* (see Corollary 3.4.5).

At the end of the chapter, we apply the core construction to the study of the action by a compact Lie group G by holomorphic isometries, on a Kähler manifold M. We assume that this action admits a momentum map μ and that M also supports a holomorphic action by the complexified group $G^{\mathbb{C}}$ (e.g., this happens if M is compact). A fundamental role in this setting is played by so called *stable, semistable* and *polystable points*, $M^s, M^{ss}, M^{ps} \subseteq M$, since there are isomorhisms $M/\!\!/G \simeq M^s/\!\!/G \simeq M^s/G^{\mathbb{C}} \simeq M^{ss}/G^{\mathbb{C}} \simeq M^{ps}/G^{\mathbb{C}}$, where for any subset of $X \subseteq M$ we write $X/\!\!/G := (\mu^{-1}(0) \cap X)/G$ for the Meyer–Marsden–Weinstein quotient (for a treatment of these matters in the context of a differentiable version of *Geometric Invariant Theory*, see [34, 63]). Let $F \subseteq {}_cM$ be a connected component of the core of the G-action, and let G_F be the largest subgroup of G such that $G_F(F) \subseteq F$. Then the G_F-action has an associated momentum map (given by $\mu|_F$), and F also supports a $G_F^{\mathbb{C}}$-action. By Theorem 3.4.25,

[2] A unitary representation of G on a complex Hilbert space $\mathcal{H}$ is called *multiplicty-free* if every irreducible representation of G occurs in $\mathcal{H}$ only with multiplicity zero or one.

if the G-action is *Energy complete* (see Definition 3.4.23), we have $F^{ss} = M^{ss} \cap F$, which we can regard as another example of the flexibility of the core construction. Moreover, using the *squared momentum map* of the G_F-action, we get a stratification-like result for M in Theorem 3.4.31; compare with Corollary 6.6 in [34].

Chapter 3 is based on the submitted preprint [2], which is a joint work with Prof. L. Biliotti and G. May.

Chapter 4. The presence of a symmetry can also simplify the study of the relations between the topology and the geometry of a manifold M. In order to give a context, let us recall some classical results on these relations.

Let M be a simply connected manifold. When M supports a metric of nonpositive sectional curvature, by the classical Hadamard Theorem (see, e.g. [28, 97]), M is forced to be diffeomorphic to $\mathbb{R}^{\dim(M)}$. On the other hand, if M is non-compact admitting a metric of nonnegative curvature then M is diffeomorphic to the normal bundle of a closed, convex, totally geodesic submanifold of M, called the *soul* of M (see the *Soul Theorem*, Theorem 12.4.1 in [97]). In particular, if the sectional curvature is positive, the soul is a point and M is again diffeomorphic to $\mathbb{R}^{\dim(M)}$. It is worth mentioning that in the regime of sectional curvature bounded away from zero, where M is forced to be compact (by the *Bonnet–Meyers Theorem*, see [28, 97]), by the celebrated *Sphere Theorem* (see again [28, 97]), M is homeomorphic to a sphere (to be precise, we need that $0 < \frac{1}{4}K_{\max} < K < K_{\max}$, where K is the sectional curvature and $K_{\max}$ is its maximum over $\mathrm{Gr}_2(TM)$). More refined results tell us that in the regime $0 < \frac{1}{4}K_{\max} \leq K \leq K_{\max} + \varepsilon$, for some $\varepsilon > 0$, M is forced to be homeomorphic to a *compact rank-1 symmetric space (CROSS)* (see Theorem 12.3.3 in [97]).

It follows from the above discussion that the case where M is a closed, simply connected manifold of nonnegative curvature is of special interest. In this case, the question of finding sharp upper bounds for the total Betti numbers of M over any coefficient field F, in the form[3]

$$\sum_{i=0}^{n} b_i(M, F) \leq 2^n \ , \tag{1}$$

and the positivity of the Euler characteristic of M (when $\dim(M)$ is even),

$$\chi(M, F) \geq 0 \ , \tag{2}$$

are two very important conjectures.

We remark that conditions (1) and (2) are satisfied if M is a closed, simply connected *rationally elliptic* manifold (see e.g. [32] and [31]). For this reason, the so-called *Bott Conjecture*, stating that every non-negatively curved, closed, simply connected manifold is rationally elliptic, is of prime importance. More generally, any sufficient geometric condition implying rational ellipticity is most desirable. In this direction, several results were obtained thanks to the additional hypothesis that M supports an isometric action by a compact group G (see [42, 44, 45, 101]).

We are particularly interested in [45] (in the case of polar actions) and in [44] (in the case of *cohomogeneity-2* actions), where the rational ellipticity of a simply connected G-manifold M is obtained under the hypothesis that the quotient M/G is either flat or positively curved. Then, in both settings it is conjectured that if the quotient supports

[3]Note that (1) is justified by $2^n = \sum_{i=0}^{n} b_i(\mathbb{T}^n, F)$, i.e. the torus is believed to be the topologically largest manifold of non negative sectional curvature (see [41]).

a hyperbolic metric, M is not rationally elliptic and is hence *rationally hyperbolic* because of the *rational dichotomy* (Section 33, Chapter VI of [31]).

Our first main result in this chapter is Theorem 4.2.6, where we provide a counterexample to the conjecture above in both settings. This amounts to performing equivariant connected sums of rationally elliptic cohomogeneity-2 polar G-manifolds, obtaining quotients with an increasing number of sides, but leaving fixed the rational homotopy behavior. Moreover, a comparison between our counterexamples and the $SU(3)$-manifold $SU(3)^{\#k}$, which is constructed from the conjugation action of $SU(3)$ on itself, will provide a negative answer to a question by Radeschi, Samani and Wilhelm (see Conjecture 4.2.8) asking if given M, M' manifolds admitting isometric actions by G, G', respectively, such that M/G is isometric to M'/G', then M is rationally elliptic if and only if M' is (see our Theorem 4.2.9).

Rational ellipticity is also related to the concept of *topological entropy* (see [94, 50]). In the work [96], it is proved that if a closed *nilpotent manifold* M admits a metric of zero topological entropy, then M is rationally elliptic (see also [101] for a relation with the results obtained in the case of a G-manifolds mentioned above). The converse of this statement can be formulated as the following question.

> If every Riemannian metric on a closed nilpotent manifold M has positive topological entropy, is M rationally hyperbolic?

Each G-manifold M in Theorem 4.2.6 provides a partial negative answer to this question. In fact, we prove that all G-invariant metrics on M have positive topological entropy.

We observe that the actions considered in the conjectures formulated in [45] and [44] (see conjectures 4.1.4 and 4.1.5) are *variationally complete* actions. These actions were introduced in [17] and later generalized to the case of *Singular Riemannian Foliations* in [71]. Using Morse Theory, we are able to prove that if G is a compact Lie group acting isometrically on a closed, simply connected manifold M, the G-action is variationally complete and M/G supports a metric of hyperbolic type, then if either the *singular isotropies* are Abelian or a *principal isotropy* is connected, M is rationally hyperbolic (see our Theorem 4.3.1). In order to prove this result, we will rely on the classical construction of Bott-Samelson cycles (see [17, 40, 86]). This construction seems to be very flexible, and it was shown to be generalizable to Singular Riemannian Foliations (see [89, 111]).

The content of Chapter 4 is based on the joint work in preparation [3], with Prof. M. Radeschi and Prof. R. Mendes.

Chapter 5. We study the action of a compact subgroup $U \subseteq U(n)$ on a compact Kähler manifold Z and the induced holomorphic action by the complexified group $U^{\mathbb{C}}$. We further suppose that the U-action is Hamiltonian, with momentum map μ. As we mentioned in the introduction to Chapter 3, in the spirit of a differentiable version of the classical Geometric Invariant Theory, one is interested in the study of the interplay between the orbits of the complexified group $U^{\mathbb{C}}(z)$ for $z \in Z$ and the momentum map. More specifically, one looks for numerical criteria for *geometric stability conditions* (see Hilbert-Mumford Numerical Criteria in Chapter 12 of [34] and [14]). In this context, the so-called *Calabi-Matsushima decomposition* is particularly useful (see, e.g., Theorems 13.4 and 13.5 in [34]). In our work, we consider a *real compatible* subgroup $G \subseteq U^{\mathbb{C}}$, and we prove that the Lie algebra of the stabilizer of a point $z \in Z$, $\mathfrak{g}_z \subseteq \mathfrak{g}$, relative to the holomorphic sub-action of G, admits a decomposition into non-negative eigenspaces, $\mathfrak{g}_\lambda$, $\lambda \geq 0$, where $\mathfrak{g}_0$ is the reductive part of $\mathfrak{g}_z$ (see Theorem 5.2.2). This result was previously known in the complex case (see [66]), i.e., when $G = U^{\mathbb{C}}$, which is the case considered in [34].

More specifically, we will use the so-called *gradient map*, which is an adaptation of the momentum map construction for the case of an action by a real compatible subgroup (see [15, 56]). In fact, given a real compatible subgroup $G \subseteq U^{\mathbb{C}}$, we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus p$, where $\mathfrak{k}$ is a compact Lie subalgebra of $\mathfrak{g}$ and $p \subseteq \mathfrak{g}$ is a vector subspace. This leads to a splitting $G = K \exp(p)$, where K is a maximal compact subgroup of G. Then the gradient map will be an Ad_K-equivariant map $\mu_p : Z \to p$ and $\mathfrak{g}_\lambda = \{\xi \in \mathfrak{g}_z \ : \ \mathrm{ad}_{\mu_p}\xi = \lambda\xi\}$.

A possible future application of our generalization is the extension of results that are established in the complex reductive case to the real compatible case. For instance, one could ask whether Theorem 13.5 in [34] can be generalized as follows. Let $z \in Z$ be an extremal point of the squared momentum map $|\mu|^2$. Then $U_z^0 = G_z^0 \cap U$ is a maximal compact subgroup of G_z^0. Unfortunately, the proof of this and other similar results in this more general setting still present some difficulties.

The content of Chapter 5 is based on the joint work in preparation [4], with Prof. L. Biliotti and Prof. F. Podestà.

ORIGINAL RESULTS. The main original and new results presented in this thesis are marked with the symbol $(\star)$.

NOTATION. Throughout the thesis, we will denote Lie group actions either by the letter $\Psi : G \times M \to M$ or $A : G \times M \to M$. Moreover, we shall often use the simpler notations $\Psi(g, x) = g.x = gx$. If the action is understood, we will also suppress the symbol Ψ. As will become apparent, we remark that the notation $\Psi(g, x)$ is more useful when we need to differentiate the action with respect to g or x.

Chapter 1

Preliminaries on Group Actions

We recall some basic definitions and foundational results on Lie group Actions. We mainly follow the treatment given in [13] and in [98]. We will rely on many results from the theory of Lie groups, for which we generally refer to the classical references [29, 58]. For the notions of Riemannian Geometry, we refer to the equally standard treatments found in [28, 97], and for the notions of Symplectic Geometry, we refer to [5, 9]. For an extremely concise summary of the fundamental concepts that we will use and the conventions we will follow, we also refer the reader to the Appendix, specifically Sections 5.3 and 5.4.

1.1 Lie Group actions

Let G be a Lie group and let M be a smooth manifold. In the following, we will denote by $e \in G$ the identity element of G.

Definition 1.1.1. We say that G *acts on the left* on M or that G defines a *left action* on M if there exists a smooth map $\Psi : G \times M \to M$ such that

$$i)\ \Psi(e, x) = x\ ;$$

$$ii)\ \Psi(g_1 g_2, x) = \Psi(g_1, \Psi(g_2, x))\ ,$$

for all $x \in M$ and $g_1, g_2 \in G$.

When the action is understood, we will also use the notation $G \circlearrowleft M$. For later convenience, we define the maps $\Psi_g : M \to M$, $\Psi_g(x) := \Psi(g, x)$, and $\Psi^x : G \to M$, $\Psi^x(g) := \Psi(g, x)$. Notice that Ψ_g is a diffeomorphism for all $g \in G$ and we see that the action Ψ defines a group homomorphism $G \to \mathrm{Diff}(M)$, $g \mapsto \Psi_g$. In fact, for all $g_1, g_2 \in G$, $g_1 g_2 \mapsto \Psi_{g_1 g_2} = \Psi_{g_1} \circ \Psi_{g_2}$. If the map $G \to \mathrm{Diff}(M)$ is injective, we say that the action is *effective*.

Remark 1.1.2. Definition 1.1.1 is a generalization of the notion of a *group representation*. In fact, if we ask that $M = V$ is a vector space and that all the Ψ_g are linear isomorphisms of V, what we shall call a *linear action*, the action defines a group homomorphism $G \to \mathrm{GL}(V)$, i.e. a representation of G on V. Notice also that an effective linear action corresponds to a *faithful* representation.

We will also use the notion of a *right action*. In this case, the action will be a smooth map $\tilde{\Psi} : M \times G \to M$, such that

$$i)\ \tilde{\Psi}(x, e) = x\ ;$$

$$ii)\ \tilde{\Psi}(x, g_1 g_2) = \tilde{\Psi}(\tilde{\Psi}(x, g_1), g_2)\ ,$$

for all $x \in M$ and $g_1, g_2 \in G$. In this case, we use the notation $G \circlearrowright M$. As we did for the left actions, we define maps $\tilde{\Psi}^g : M \to M$, $\tilde{\Psi}_x : G \to M$. A right action $\tilde{\Psi}$ defines an anti-homomorphism of groups $G \to \mathrm{Diff}(M)$, because $g_1 g_2 \mapsto \tilde{\Psi}^{g_1 g_2} = \tilde{\Psi}^{g_2} \circ \tilde{\Psi}^{g_1}$.

Remark 1.1.3. It is easy to see that if we define $\mathcal{I} : G \times M \to M \times G$ as $(g, x) \mapsto (x, g^{-1})$, every right action $\tilde{\Psi}$ gives rise to a left action $\tilde{\Psi} \circ \mathcal{I}$. Vice versa, every left action Ψ gives rise to a right action $\Psi \circ \mathcal{I}^{-1}$. In the following, if not otherwise stated, we refer to left actions, since completely analogous definitions and statements can be expressed in terms of right actions.

Let Ψ be a left action of G on M. In the following, we use a shorthand notation for the action, $\Psi(g, x) := g.x$. There are notable subsets of G and M that we are going to consider extensively.

Definition 1.1.4. We define the following subsets

i) *stabilizer of* or *isotropy subgroup of* $x \in M$, $G_x := \{g \in G\ :\ g.x = x\} \subseteq G$;

ii) *orbit of* $x \in M$, $G(x) := \{y \in M\ :\ y = g.x, \text{ for some } g \in G\} \subseteq M$;

iii) *ineffective kernel* of Ψ, $\bigcap_{x \in M} G_x \subseteq G$.

Moreover, we call $x \in M$ a *fixed point* for the action if $G_x = G$ or $G(x) = \{x\}$.

More generally, we define the orbit of a subset Y of M as the union of the orbits of its points and we will denote it as $G(Y)$. We notice that the ineffective kernel coincides with the kernel of the homomorphism $G \to \mathrm{Diff}(M)$, $g \mapsto \Psi_g$. Therefore, the action is effective if and only if $\bigcap_{x \in M} G_x = \{e\}$.

It is important to observe that G_x is a subgroup of G and that the ineffective kernel is a normal subgroup of G. To see this second fact, let $h \in \bigcap_{x \in M} G_x$. Then for any $x \in M$ and $g \in G$, we have $ghg^{-1}.x = gg^{-1}.x = x$. It follows that the quotient $\tilde{G} := G/(\bigcap_{x \in M} G_x)$ is a Lie group and the natural action of $\tilde{G}$ on M given by $[g].x := g.x$ is well defined and effective, where $[g]$ denotes the equivalence class of g, i.e. the coset $g(\bigcap_{x \in M} G_x)$.

Remark 1.1.5. If $y \in G(x)$, then, in general, $G_x \neq G_y$. However, suppose $y = g.x$ for some $g \in G$. Then if $h \in G_x$, we have $ghg^{-1}.y = ghg^{-1}g.x = g.x = y$, hence $ghg^{-1} \in G_y$. Therefore, $G_y = gG_x g^{-1}$.

We also define two important types of actions, that we will discuss in the next sections.

Definition 1.1.6. If $G_x = \{e\}$ for all $x \in M$ we say that the action if *free*. If there is only one orbit, i.e. $G(x) = M$ for $x \in M$, we say that the action is *transitive*.

Naturally induced actions

We now briefly discuss common, natural actions that can be constructed from given ones.

Restriction. Let $\Psi : G \times M \to M$ be an action and let $H \subseteq G$ be a subgroup. Then the restriction $\Psi|_{H \times M} : H \times M \to M$, is an action of H on M, called the *subaction* by H. We call a subset $Y \subseteq M$ *G-invariant* if $G(Y) \subseteq Y$. If such Y is a submanifold of M we can define a G-action on Y by the restriction $\Psi|_{G \times Y}$. Putting the previous two scenarios together, if Y is invariant under a given subaction of $H \subseteq G$, we can define an H-action on Y simply by $\Psi|_{H \times Y}$.

Product. Let us consider two actions $\Psi_i : G_i \times M_i \to M_i$, $i = 1, 2$. We can define the *product action*, $\Psi : (G_1 \times G_2) \times (M_1 \times M_2)$, in the obvious way: $\Psi((g_1, g_2), (x_1, x_2)) \mapsto (\Psi_1(g_1, x_1), \Psi_2(g_2, x_2))$. In particular, if $G_1 = G = G_2$, we can also consider the *diagonal action* $\Delta_{G\times G} \times (M_1 \times M_2) \to M_1 \times M_2$, where $\Delta_{G\times G} := \{(g, g) \ : \ g \in G\} \simeq G$ is the diagonal subgroup.

Quotients. Let us consider two actions $\Psi_i : G_i \times M \to M$, on the same manifold M by G_i, $i = 1, 2$. Suppose that for all $g_i \in G_i$ and $x \in M$, we have $\Psi_2(g_2, \Psi_1(g_1, x)) = \Psi_1(g_1, \Psi_2(g_2, x))$. Then there are natural (in principle only continuous) actions $\Psi_{1,2} : G_1 \times M/G_2 \to M/G_2$ and $\Psi_{2,1} : G_2 \times M/G_1 \to M/G_1$.

Let $G \times M \to M$ be an action. Then if H is any closed normal subgroup of the ineffective kernel of Ψ, there is a natural action of G/H on M.

Example 1.1.7. We list some easy examples of these phenomena.

i) Let $O(n)$, denote the orthogonal group. Then the natural representation $O(n) \circlearrowleft \mathbb{R}^n$ can be restricted to each sphere centered at the origin with radius r, $O(n) \circlearrowleft \mathbb{S}^{n-1}_r$. Let $v \in \mathbb{R}^n$ be a nonzero vector. Then if $v^\perp$ is the hyperplane orthogonal to v, there is also an action $O(n-1) \circlearrowleft (v^\perp \cap \mathbb{S}^{n-1}_r)$.

ii) Let $G = SO(2) \times \mathbb{R}^*$ and let $G \circlearrowleft \mathbb{R}^3$ as $((A, b), (x, y, z)) \mapsto (A(x, y), zb)$. The orbits are cylinders around the positive (negative) half of the z-axis if $(x, y) \neq 0$ and $z \geq 0$ ($z \leq 0$) and the positive (negative) half of the z-axis if $(x, y) = 0$ and $z \geq 0$ ($z \leq 0$). $(0, 0, 0)$ is a fixed point and the orbits of points $(x, y, 0)$ are circles. The xy-plane and the union of the upper and lower open half space are invariant subset of the action. This is obvious since, as we have just seen, these submanifolds are union of G-orbits.

iii) As a consequence of Remark 1.1.5, we can see that given a subgroup $K \subseteq G$, the fixed point set of the K-subaction, that we denote as $M^K = \{x \in M \ : \ K \subseteq G_x\}$ is invariant under the action of $N(K) := \{g \in G \ : \ gKg^{-1} = K\}$, the normalizer of K in G. We will see that such a subset is in fact a submanifold of M. Then there is a natural action $N(K) \circlearrowleft M^K$.

Equivariant maps

Suppose that we have two G-actions $\Psi : G \times M \to M$ and $\tilde{\Psi} : G \times N \to N$. Given a smooth map $f : M \to N$, we call f *G-equivariant* if $\tilde{\Psi}_g \circ f = f \circ \Psi_g$, for all $g \in G$. If $\tilde{\Psi}$ is the trivial action, i.e., $\tilde{\Psi}_g = \mathrm{id}_N$ for all $g \in G$, a G-equivariant map f is called *G-invariant*.

1.1.1 Infinitesimal aspects.

Given an action $\Psi : G \circlearrowleft M$ on a smooth manifold M, we can define the so-called *lifted action* or *tangent lift* of the action on TM by

$$(g, (x, v)) \mapsto g.(x, v) := (\Psi_g(x), (\mathrm{d}\Psi_g)_x(v)) \in T_{\Psi_g(x)}M \ .$$

The natural projection $\pi : TM \to M$, is clearly a G-equivariant map, since $\pi(g.(x, v)) = g.x = g.\pi(x, v)$. There is also a notion of *co-tangent lift* that allows us to define an action on T^*M, namely

$$(g, (x, \alpha)) \mapsto g.(x, \alpha) := (\Psi_g(x), (\mathrm{d}\Psi_{g^{-1}})^*(\alpha)) \in T^*_{\Psi_g(x)}M \ .$$

Then we can define an action on any bundle constructed from TM. Given $(x,\xi) := (x, v_1 \otimes \cdots \otimes v_k \otimes \alpha_1 \otimes \cdots \otimes \alpha_h) \in (TM^{\otimes_k} \otimes T^*M^{\otimes_h})$, we set

$$g.(x,\xi) := (\Psi_g(x), (\mathrm{d}\Psi_g)_x(v_1) \otimes \cdots \otimes (\mathrm{d}\Psi_g)_x(v_k) \otimes (\mathrm{d}\Psi_{g^{-1}})^*(\alpha_1) \otimes \cdots \otimes (\mathrm{d}\Psi_{g^{-1}})^*(\alpha_h)) \ .$$

In particular, if $x \in M$ is a fixed point of the action, there is also a linear action on T_xM, called the *isotropy representation*, $G \circlearrowleft T_xM$, defined by $g.v := (\mathrm{d}\Psi_g)_x(v)$.

Remark 1.1.8. Let $G \circlearrowleft M$ be a left action. In the following, we will often refer to G-equivariant or G-invariant tensor fields, e.g., Riemannian metrics, symplectic forms, and endomorphism of TM. In particular, we will say that a Riemannian metric $\mathbf{g}$ on M is G-invariant if $\Psi_g^*\mathbf{g} = \mathbf{g}$ for all $g \in G$. This is consistent with the previous definition of a G-invariant map if we consider a Riemannian metric as a fibre-wise linear map $\mathbf{g} : TM \odot TM \to \mathbb{R}$, where $TM \odot TM$ is the symmetric tensor product, $G \circlearrowleft TM$ is the lifted action, and $G \circlearrowleft \mathbb{R}$ trivially. A similar discussion holds for a symplectic form ω, seen as a fibre-wise linear map $\omega : TM \wedge TM \to \mathbb{R}$. Finally, an endomorphism of TM, $J : TM \to TM$, is G-equivariant if $J \circ (\mathrm{d}\Psi_g) = (\mathrm{d}\Psi_g) \circ J$, for all $g \in G$. In particular, if J is an almost complex structure, this is equivalent to say that Ψ_g is J-holomorphic for all $g \in G$.

Furthermore, by the discussion above, any action $\Psi : G \circlearrowleft M$ on a smooth manifold induces an action on the space of sections of every tensor bundle over M. Then, G-equivariant tensor fields can also be seen as fixed points of such induced actions. For example, covariant tensor fields are G-equivariant if they are left fixed by the pull-back. Following the example above, the meaning of $\Psi_g^*\mathbf{g} = \mathbf{g}$ for all $g \in G$ is that $\mathbf{g} \in \mathcal{C}^\infty(M, T^*M \odot T^*M)$, is a fixed point of the action $\Psi^* : G \circlearrowleft \mathcal{C}^\infty(M, T^*M \odot T^*M)$, where $\Psi^*(g, \mathbf{g}) := \Psi_g^*\mathbf{g}$.

Let now $\mathfrak{g}$ denote the Lie algebra of G and let $\{\exp(t\xi)\}_{t\in\mathbb{R}} \subseteq G$ be the 1-parameter subgroup relative to a non-zero element $\xi \in \mathfrak{g}$.

Definition 1.1.9. We define the *infinitesimal generator of the action* $\Psi : G \circlearrowleft M$ as the vector field $\xi_M \in \mathcal{C}^\infty(M, TM)$ such that

$$\xi_M(x) := \frac{d}{dt}\Psi_{\exp(t\xi)}(x)\Big|_{t=0} = (\mathrm{d}\Psi^x)_e(\xi) \in T_xM \ ,$$

for all $x \in M$.

One should check that this is actually a smooth vector field on M, but we think this can be seen as obvious since ξ_M is the vector field associated to the smooth flow $\Phi^{\xi_M} : \mathbb{R} \times M \to M$, $(t,x) \mapsto \Phi_t^{\xi_M}(x) := \Psi_{\exp(t\xi)}(x)$.

We immediately notice that if ξ is in the Lie algebra $\mathfrak{g}_x$ of G_x, the stabilizer of a point $x \in M$, then $\xi_M(x) = 0$. In particular, if x is a fixed point of the action, $\xi_M(x) = 0$ for all $\xi \in \mathfrak{g}$.

Proposition 1.1.10. *The map $\mathfrak{g} \to \mathcal{C}^\infty(M, TM)$, $\xi \mapsto \xi_M$ is a Lie algebra anti-homomorphism, with respect to the Lie bracket of $\mathfrak{g}$ and the Lie bracket of vector fields of $\mathcal{C}^\infty(M, TM)$. That is, for all $\xi, \eta \in \mathfrak{g}$, we have*

$$[\xi, \eta]_M = -[\xi_M, \eta_M] \ .$$

Proof. In fact, for all $x \in M$, $g \in G$ and $\xi \in \mathfrak{g}$, we have

$$\xi_M(\Psi_g(x)) = \frac{d}{dt}\Psi_{\exp(t\xi)}(\Psi_g(x))\Big|_{t=0} = \frac{d}{dt}\Psi_{\exp(t\xi)g}(x)\Big|_{t=0} = \frac{d}{dt}\Psi_{gg^{-1}\exp(t\xi)g}(x)\Big|_{t=0} =$$

$$= \frac{d}{dt}\Psi_g(\Psi_{g^{-1}\exp(t\xi)g}(x))\Big|_{t=0} = (\mathrm{d}\Psi_g)_x\left(\frac{d}{dt}\Psi_{g^{-1}\exp(t\xi)g}(x)\right)\Bigg|_{t=0} = (\mathrm{d}\Psi_g)_x\left((\mathrm{Ad}_g^{-1}\xi)_M(x)\right) .$$

Then, using the standard definition of Lie brackets, we have

$$[\eta_M,\xi_M](x) = \frac{d}{dt}(\mathrm{d}\Psi_{\exp(-t\eta)})_{\Psi_{\exp(t\eta)}(x)}(\xi_M(\Psi_{\exp(t\eta)}(x))\Big|_{t=0} =$$

$$= \frac{d}{dt}(\mathrm{d}\Psi_{\exp(-t\eta)})_{\Psi_{\exp(t\eta)}(x)}\left((\mathrm{d}\Psi_{\exp(t\eta)})_x\left((\mathrm{Ad}_{\exp(-t\eta)}\xi)_M(x)\right)\right)\Big|_{t=0} =$$

$$= \frac{d}{dt}(\mathrm{Ad}_{\exp(-t\eta)}\xi)_M(x)\Big|_{t=0} = (\mathrm{d}\Psi^x)_e\left(\frac{d}{dt}\mathrm{Ad}_{\exp(-t\eta)}\xi\right) = (\mathrm{d}\Psi^x)_e([\xi,\eta]) = [\xi,\eta]_M(x) .$$

□

A clarification about the anti-homomorphism can be given by the following argument. We have just said that the action Ψ can be seen as a map $\psi : G \to \mathrm{Diff}(M)$, $g \mapsto \Psi_g$, and let us think of $\mathrm{Diff}(M)$ as an infinite dimensional Lie group whose Lie algebra is given by $\mathcal{C}^\infty(M,TM)$. Then the differential of this map at $e \in G$ is $(\mathrm{d}\psi)_e : \mathfrak{g} \to \mathcal{C}^\infty(M,TM)$ and we have $(\mathrm{d}\psi)_e(\xi) = \xi_M$. By the calculations just done, such a map is G-equivariant, with respect to the adjoint action of G on $\mathfrak{g}$ and the G-action on $\mathcal{C}^\infty(M,TM)$ by push-forward of vector fields, $(g,X) \mapsto (\Psi_g)_*X$. Therefore, because of G-equivariance,

$$(\mathrm{Ad}_g\xi)_M = (\mathrm{d}\psi)_e(\mathrm{Ad}_g\xi) = (\Psi_g)_*((\mathrm{d}\psi)_e\xi) = (\Psi_g)_*\xi_M ,$$

and since $(\mathrm{d}\psi)_e$ is linear, taking $g(t) = \exp(t\eta)$ we get, deriving both sides of the previous equation,

$$\frac{d}{dt}(\mathrm{d}\psi)_e(\mathrm{Ad}_{\exp(t\eta)}\xi)\Big|_{t=0} = (\mathrm{d}\psi)_e([\eta,\xi]) = [\eta,\xi]_M ,$$

on the other hand,

$$\frac{d}{dt}(\Psi_{\exp(t\eta)})_*(\xi_M)\Big|_{t=0} = -[\eta_M,\xi_M] .$$

We finish this subsection with the following remark.

Remark 1.1.11. Let $x \in M$. For all $g \in G$ we have $\ker((\mathrm{d}\Psi^x)_g) = T_g(gG_x)$. In the case of a right action $\tilde{\Psi} : G \circlearrowright M$, for the map $\tilde{\Psi}_x$, instead we get $\ker((\mathrm{d}\tilde{\Psi}_x)_g) = T_g(G_xg)$. We verify this for a left action. Let $\xi \in \mathfrak{g}$ and $g \in G$, then $(\mathrm{d}L_g)_e\xi \in T_gG$, where $L_g : G \to G$ is the left translation by g. We have

$$\frac{d}{dt}\Psi(g\exp(t\xi),x)\Big|_{t=0} = (\mathrm{d}\Psi^x)_g((\mathrm{d}L_g)_e\xi) ,$$

and on the other hand,

$$\frac{d}{dt}\Psi(g\exp(t\xi),x)\Big|_{t=0} = (\mathrm{d}\Psi_g)_x\left(\frac{d}{dt}\Psi^x(\exp(t\xi))\right)\Bigg|_{t=0} = (\mathrm{d}\Psi_g)_x(\xi_M(x)) .$$

So, $(\mathrm{d}\Psi^x)_g((\mathrm{d}L_g)_e\xi) = 0$ if and only if $\xi_M(x) = 0$, i.e. if and only if $\xi \in \mathfrak{g}_x$, the Lie algebra of G_x, and therefore, if and only if $(\mathrm{d}L_g)_e\xi \in T_g(gG_x)$.

1.1.2 Orbits and Orbit space

In general, given $x \in M$, the orbit of x is an immersed submanifold $G(x) \subseteq M$, where the immersion is given by the map $\varphi : G/G_x \to M$, $\varphi([g]) := \Psi^x(g)$, which is well defined and smooth. However, in general, φ is not an embedding. The tangent space of $G(x)$ at x is clearly given by $(\mathrm{d}\Psi^x)_e(\mathfrak{g})$. Moreover, $(\mathrm{d}\Psi_g)_x$ leaves $T_x(G(x))$ invariant for all $g \in G_x$. In fact, if $v \in T_x(G(x))$, we have seen that $v = (\mathrm{d}\Psi^x)_e(\xi) = \xi_M(x)$, for some $\xi \in \mathfrak{g}$. In general, we have computed that for $g \in G$

$$\xi_M(\Psi_g(x)) = (\mathrm{d}\Psi_g)_x \left((\mathrm{Ad}_g^{-1}\xi)_M(x)\right) \ .$$

Then, if $g \in G_x$, we get that

$$\xi(\Psi_g(x)) = \xi_M(x) = (\mathrm{d}\Psi_g)_x \left((\mathrm{Ad}_g^{-1}\xi)_M(x)\right) \ ,$$

therefore, $(\mathrm{d}\Psi_g)_x^{-1}(\xi_M(x)) = (\mathrm{Ad}_g^{-1}\xi)_M(x)$. By choosing g^{-1} instead of g, we get

$$(\mathrm{d}\Psi_g)_x(\xi_M(x)) = (\mathrm{Ad}_g\xi)_M(x) = (\mathrm{d}\Psi^x)_e(\mathrm{Ad}_g\xi) \ .$$

We conclude that $(\mathrm{d}\Psi_g)_x(v) \in (\mathrm{d}\Psi^x)_e(\mathfrak{g}) = T_x(G(x))$.

Since two orbits intersecting each other must be the same orbit, the binary relation on M of "belonging to the same orbit" is an equivalence relation. We can therefore consider the quotient space M/G, called the *orbit space* of the action. There is a natural projection $\pi : M \to M/G$, sending each point to the corresponding orbit through it.

We endow M/G with the quotient topology, the finest rendering π a continuous map. The map π is also open. In fact, let $U \subseteq M$ be an open subset. Then $\pi(U) \subseteq M/G$ can be written as $\pi(U) = \pi(\pi^{-1}(\pi(U)))$, but $\pi^{-1}(\pi(U)) = \bigcup_{g\in G} g.U$, which is a union of open sets. Therefore, $\pi(U)$ is open in the quotient topology. This topology is in general highly pathological, e.g., not Hausdorff or not even T0.

Example 1.1.12. We give some examples of ugly topologies arising from very simple actions.

i) Consider the action $\mathbb{C}^* \circlearrowleft \mathbb{C}^n$, $(w, z) \mapsto wz$. The quotient consists of two points, but there is no way of separating them in the quotient topology, which is not T1 since there is no open subset containing the orbit of $0 \in \mathbb{C}^n$ but not the other. Or, we observe that one point is closed but the other is not.

ii) Let $\mathbb{Z} \circlearrowleft \mathbb{S}^1$ be the rotation by an irrational angle, then the action is free and the only open sets of the orbit space are the empty set and the whole orbit space. Since $\mathbb{S}^1/\mathbb{Z}$ consists of more than one point, this space is not even T0.

1.2 Proper actions

To avoid pathological topologies for the quotients, one way is to consider *proper actions*.

Definition 1.2.1. We call an action $\Psi \circlearrowleft M$ proper, if the map $\phi : G \times M \to M \times M$, defined by $(g, x) \mapsto (\Psi(g, x), x)$ is proper in the topological sense, i.e. preimages of compact subsets are compact.

Notice that if $G \circlearrowleft M$ is proper, the stabilizers G_x, for all $x \in M$, are compact subgroups of G. Consequently, the ineffective kernel of the action is a compact normal subgroup.

Since every manifold admits a Riemannian metric and is a metric space with distance induced by such a metric, we can think of G and M as metric spaces. Then we have a useful characterization of proper actions in terms of sequences.

Proposition 1.2.2. *An action $\Psi \circlearrowleft M$ is proper if and only if for all sequences $\{g_n\}_{n\in\mathbb{N}} \subset G$, $\{x_n\}_{n\in N} \subset M$, if the sequences $\{x_n\}_{n\in N}$ and $\{\Psi(g_n, x_n)\}_{n\in\mathbb{N}} \subset M$ converge, then there exists a converging subsequence $\{g_{n_k}\}_{k\in\mathbb{N}} \subset G$.*

Proof. The proof follows from the very definition, since a metric space is sequentially compact if and only if it is compact. □

Remark 1.2.3. We note that if $H \subseteq G$ is a closed subgroup, then if the action $G \circlearrowleft M$ is proper, also the subaction $H \circlearrowleft M$ is proper. In fact, given $h_n \in H$, and $x_n \longrightarrow x \in M$ and $h_n.x_n \longrightarrow y \in M$, by properness there exists $h_{k_n} \longrightarrow h \in G$, but then $h \in H$, because H is closed.

We add another characterization of proper actions that can be of use.

Proposition 1.2.4. *An action is proper if and only if for all $x, y \in M$, there exists $U_x, U_y \subseteq M$ open neighborhoods of x and y respectively, such that the subset*

$$\Xi(U_x, U_y) := \{g \in G \ : \ g(U_x) \cap U_y \neq \emptyset\} \ ,$$

is relatively compact in G.

Proof. Suppose that the action is proper, and consider two cases. If $y \notin G(x)$, we see that $(y, x) \notin \phi(G \times M)$. Then, because $\phi(G \times M)$ is closed in $M \times M$, there is a neighborhood $(U_y \times U_x) \subseteq (M \times M)$ of (y, x) such that $(U_y \times U_x) \cap \phi(G \times M) = \emptyset$. Therefore, $g(U_x) \cap U_y = \emptyset$ for all $g \in G$, and hence $\Xi(U_x, U_y) = \emptyset$, which is compact. In the other case, if $y \in G(x)$, we first notice that $\phi^{-1}(\{y\} \times \{x\}) = \Xi(\{x\}, \{y\}) \times \{x\}$. By the properness of Ξ, we have that $\Xi(\{x\}, \{y\})$ is compact. Take a compact neighborhood K of $\Xi(\{x\}, \{y\})$, then K is a neighborhood of $\phi^{-1}(\{y\}, \{x\})$, and again, by properness there is a neighborhood of $U_y \times U_x$ of (y, x) such that $\phi^{-1}(U_y, U_x) = \Xi(U_x, U_y) \times U_x \subseteq K \times M$. So, $\Xi(U_x, U_y)$ is contained in a compact and is therefore relatively compact. Vice-versa, suppose that for all $x, y \in M$, there are U_x, U_y as before with $\Xi(U_x, U_y)$ relatively compact in G. Take $\{g_n\}_{n\in\mathbb{N}} \subset G$, $\{x_n\}_{n\in N} \subset M$, such that $x_n \longrightarrow x$ and $\Psi(g_n, x_n) \longrightarrow y$. Then, definitely, we have $x_n \in U_x$ and $\Psi(g_n, x_n) \in U_y$, and hence $g_n \in \Xi(U_x, U_y)$. Therefore, up to subsequences, $g_{k_n} \longrightarrow g \in \overline{\Xi(U_x, U_y)} \subseteq G$. □

Remark 1.2.5. It is clear that if G is compact, then every action by G on any manifold M is proper. On the other hand, every action by non-compact groups on compact manifolds cannot be proper. Furthermore, from the previous Proposition 1.2.4 we see that every proper action is such that for all $x \in M$ there exists an open neighborhood U_x of x such that the subset $\Xi(U_x, U_x) \subseteq G$ is relatively compact. Such actions are called *Cartan actions*. These actions are important because they form the broadest class of actions for which important structural results hold (e.g. Slice Theorem, see [91]). In summary, we have a sequence of proper (see again [91]) inclusions.

$$\{\text{actions of compact groups}\} \subset \{\text{proper actions}\} \subset \{\text{Cartan actions}\} \ .$$

A first important consequence of the properness of the action is that the immersion map of any orbit is an embedding, i.e. the group orbits are embedded submanifolds of M.

Proposition 1.2.6. *Let $\Psi : G \circlearrowleft M$ be proper action. Then for all $x \in M$, the map $\varphi : G/G_x \to M$, $\varphi([g]) := \Psi(g, x)$, is an embedding.*

Proof. Let us first observe that since $T_e(G/G_x) \simeq \mathfrak{g}/\mathfrak{g}_x$, $(\mathrm{d}\varphi)_{[e]}([\xi]) = (\mathrm{d}\Psi^x)_e(\xi) = \xi_M(x) = 0$ if and only if $\xi \in \mathfrak{g}_x$, i.e., if and only if $[\xi] = 0$. Therefore, $(\mathrm{d}\varphi)_{[e]}$ is injective. Let now $[g] \in G/G_x$ and recall that $T_{[g]}(G/G_x) = ((\mathrm{d}L_g)_e(\mathfrak{g})) \,/\, ((\mathrm{d}L_g)_e(\mathfrak{g}_x))$, then, as we have already seen, $(\mathrm{d}\varphi)_{[g]}([(\mathrm{d}L_g)_{[e]}\xi]) = (\mathrm{d}\Psi_g)_x(\xi_M(x))$. Therefore, $(\mathrm{d}\varphi)_{[g]}$ is injective for all g, and since φ is clearly injective, it is an injective immersion. To prove that it is a global diffeomorphism with the image, we simply need to check that φ is proper. This comes from the fact that if $(g, x) \mapsto (\Psi(g, x), x)$ is proper, then $g \mapsto \Psi(g, x)$ is proper and also $[g] \mapsto \Psi(g, x)$ is. □

As we announced, another fundamental result for proper actions is that the quotient space is a good topological space. More precisely, we have the following proposition.

Proposition 1.2.7. *If Ψ is a proper action of a Lie group G on a smooth manifold M, then the orbit space is a Hausdorff and second countable topological space.*

Proof. A standard fact tells us that the claim is equivalent to proving that the diagonal $\Delta_{M/G} := \{([x], [y]) \in (M/G) \times (M/G) \text{ } s.t. \text{ } [x] = [y]\}$ is closed in $(M/G) \times (M/G)$. We first observe that the image of the map $(x, g) \mapsto (x, \Psi(g, x)) =: F(x, g)$ is closed, because we have just seen that $g \mapsto \Psi(g, x)$ is proper, and then $(M \times M) \setminus \mathrm{Im}(F)$ is open. Therefore, since π is open and $(\pi \times \pi)((M \times M) \setminus \mathrm{Im}(F)) = ((M/G) \times (M/G)) \setminus \Delta_{M/G}$, we conclude that the diagonal is closed. The fact that M/G is second countable is automatic since M is second countable. □

An important class of proper actions are the so-called *properly discontinuous* actions. These are defined as follows. An action $\Psi \circlearrowleft M$ is properly discontinuous if for all $x \in M$ there exists a neighborhood U_x such that for all $g \in G$, $g \neq e$, we have $g(U_x) \cap U_x = \emptyset$ and for all $x, y \in M$, not in the same orbit, there are respective neighborhoods $U_x, U_y \subset M$ such that for all $g \in G$, $g(U_x) \cap U_y = \emptyset$.

In particular, if G is a discrete group, an action by G is properly discontinuous if and only if it is proper and free. The proof of this fact is as follows. First, a properly discontinuous action must clearly be free, because if $g \in G$, $g \neq e$, is such that $\Psi(g, x) = x$, then for all neighborhoods U_x of x, $\{x\} \subseteq g(U_x) \cap U_x \neq \emptyset$. We show that it is also proper. Indeed let $x, y \in M$ such that $y \in G(x)$, then $y = \Psi(g, x)$, because the group is discrete, and $g \in G$ is unique, because the action is free. Then, there is a neighborhood of U_x of x such that $g(U_x) \cap U_x = \emptyset$ for all $g \in G$, $g \neq e$. But then $U_y := g(U_x)$ is a neighborhood of y such that $\Xi(U_x, U_y) = \{e\}$, which is compact. If on the contrary $y \neq G(x)$, then there are U_x, U_y such that $\Xi(U_x, U_y) = \emptyset$. In both cases there are neighborhoos such that $\Xi(U_x, U_y)$ is compact, hence the action is proper. Vice-versa, suppose the action is proper and free. Then, for all $x \in M$ there is a neighborhood U_x of x such that $\Xi(U_x, U_x)$ is compact, hence it is finite because G is discrete. Let $\Xi(U_x, U_x) = \{e, g_1, \cdots, g_N\}$. Then, since the action is free, $\Psi(g_i, x) \neq x$, and we can find $\tilde{U}_x$ separating x from $\Psi(g_1, x), \cdots, \Psi(g_N, x)$ such that $\Xi(\tilde{U}_x, \tilde{U}_x) \subseteq \{e, g_1, \cdots, g_N\}$. Then $\hat{U}_x := \tilde{U}_x \setminus \left(\bigcup_{i=1}^N U_i\right)$, where U_i are open subsets $U_i := \tilde{U}_x \cap g_i(\tilde{U}_x)$, is an open neighborhood of x such that $g(\hat{U}_x) \cap \hat{U}_x = \emptyset$ for all $g \in G$, $g \neq e$. Suppose now $x, y \in M$, are not in the same orbit, since M/G is Hausdorff, there are open neighborhoods $V_{\pi(x)}$, $V_{\pi(y)}$ separating $\pi(x)$ from $\pi(y)$. Then $U_x := \pi^{-1}(V_{\pi(x)})$ and $U_y := \pi^{-1}(V_{\pi(y)})$ satisfy $g(U_x) \cap U_y = \emptyset$ for all $g \in G$.

The following elementary example is of fundamental importance.

Example 1.2.8. Let us consider the *right* action on G of a subgroup H by right translations. $G \times H \to G$, $(g, h) \mapsto gh$. This action is free since $gh = g$ implies $h = e$. This action is also proper: take $\{h_n\}_{n\in\mathbb{N}} \subset H$ sequence, and $\{g_n\}_{n\in\mathbb{N}}$ converging sequence such that $\{g_n h_n\}_{n\in\mathbb{N}}$ converges. We want to extract a converging subsequence $\{h_{k_n}\}_{k\in\mathbb{N}}$. We can write $h_n = g_n^{-1} g_n h_n$ which is actually convergent. Since H is closed this limit is in H. Another way of seeing this is to consider the very definition of properness. First, the map $G \times G \to G \times G$, $(g_1, g_2) \mapsto (g_1 g_2, g_2)$ is a diffeomorphism, hence it is a proper map. Moreover, the inclusion $H \subseteq G$ is proper by definition. Therefore, the composition $G \times H \to G \times G \to G \times H$, $(g, h) \mapsto (g, h) \mapsto (gh, h)$ is proper.

1.2.1 Proper and Free actions

An action that is proper and free gives rise to a *principal fiber bundle structure*. To match with the classical notion of principal fiber bundle, we consider G acting on M *from the right*. For clarity, we briefly recall the notion of a fiber bundle and of a principal fiber bundle.

Definition 1.2.9. Let E, B, F be topological spaces. A surjection $\pi : E \to B$ is called a *fiber bundle* with *characteristic fiber* F, *base space* M and *total space* E if B admits an open cover $\{U_\alpha\}$ and fiber-preserving homeomorphisms $\{\chi_\alpha\}$

$$\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \ ,$$

i.e. if $\pi_\alpha : U_\alpha \times F \to U_\alpha$ is the projection on the first factor, we have $\pi_\alpha \circ \chi_\alpha = \pi|_{\pi^{-1}(U_\alpha)}$.

If $\pi : E \to B$ and $\tilde{\pi} : \tilde{E} \to \tilde{B}$ are two fiber bundles, we call a continuous map $\Phi : E \to \tilde{E}$ a *fiber map* if there exists a continuous map $\phi : B \to \tilde{B}$ such that $\tilde{\pi} \circ \Phi = \phi \circ \pi$. In the following, all the maps will be considered to be smooth and we will denote a fiber bundle as a sequence of maps

$$F \hookrightarrow E \twoheadrightarrow B \ ,$$

where $F \hookrightarrow E$ is the inclusion of the generic fiber which is diffeomorphic to F and $E \twoheadrightarrow B$ is the projection on the base.

Definition 1.2.10. We call a fiber bundle $\pi : P \to B$ a *principal bundle* (PFB) if the fiber is a Lie group G and

i) there is a free right action $G \circlearrowright P$, $(p, g) \mapsto p.g$;

ii) the local trivializations are G-equivariant, where the right action on the products $U \times G$, is induced by the right translation in the group, i.e. $(x, h).g := (x, hg)$

Remark 1.2.11. Local trivializations together with the equivariance condition also determine the so-called *transition functions* of the principal bundle. Let $U_\alpha, U_\beta \subseteq M$ be open subsets trivializing P with $U_\alpha \cap U_\beta \neq \emptyset$. Let $p \in \pi^{-1}(U_\alpha \cap U_\beta)$, with $x := \pi(p)$. Then there are $h(p), k(p) \in G$ such that $\chi_\alpha(p) = (x, h(p))$, $\chi_\beta(p) = (x, k(p))$. Since G acts on itself by left translations, there is a unique element $g_{\alpha\beta}(p) \in G$ such that

$k(p) = g_{\alpha\beta}(p)h(p)$. We now see that the function $g_{\alpha\beta}$ depends only on x. In fact, let $q = p.g$, then

$$g_{\alpha\beta}(p.g) = k(p.g)h(p.g)^{-1} = k(p)g(h(p)g)^{-1} = k(p)gg^{-1}h(p)^{-1} = g_{\alpha\beta}(p) \ .$$

Finally, we remark that one can also define a principal bundle as a fiber bundle whose trivializations on any intersecting trivializing open sets $U_\alpha \cap U_\beta$ satisfy $\chi_\alpha \circ \chi_\beta^{-1} : (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$, $(x, h) \mapsto (x, g_{\alpha\beta}(x)h)$, where $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \to G$ is a smooth function. It is easy to see that using transition maps, we can define a right action of G on P.

From the above discussion, we conclude that one can see the base space of a principal bundle as the orbit space P/G of the right action $G \circlearrowright P$. Using the local trivializations, one easily sees that the right G-action is also proper. Conversely, it turns out that we can recover a principal bundle from a right action which is proper and free.

Theorem 1.2.12. *Let $\Psi : M \times G \to M$ be a proper and free right action. Then the orbit space M/G is a smooth manifold and has the structure of a principal fiber bundle $G \hookrightarrow M \twoheadrightarrow M/G$ where the projection map is the quotient map.*

Proof. Given $p \in M$, suppose S is a submanifold of M containing p such that $T_pM \simeq T_pS \oplus (\mathrm{d}\Psi_p)_e(\mathfrak{g})$ and let $\phi : S \times G \to M$, $\phi := \Psi|_{S\times G}$. Clearly, $\phi(s, e) = s$ for all $s \in S$ and hence $(\mathrm{d}\phi^e)_s = \mathrm{id}_{T_sS}$. Moreover, since the action is free, $(\mathrm{d}\phi_s)_e$ has trivial kernel. In conclusion, the map ϕ defines a local diffeomorphism in a neighborhood of $S \times \{e\}$. Now, we have that $\phi(s, g) = \Psi(\Psi(s, e), g) = \Psi^g(\phi(s, e))$ and hence

$$(\mathrm{d}\phi)_{(s,g)}(X, Y) = (\mathrm{d}\Psi^g)_s((\mathrm{d}\phi)_{(s,e)}(X, Y)) \ .$$

Therefore, since $(\mathrm{d}\Psi^g)_s$ is an isomorphism, the map ϕ is a local diffeomorphism near each point of $S \times G$. In order to prove that this is a diffeomorphism with the image, we only need to check that ϕ is injective. We claim that this is true up to shrinking S. By contradiction, suppose this is not possible. Then there exist $\{g_n\} \subset G$, $\{\tilde{g}_n\} \subset G$ and $\{s_n\} \subset S$, $\{\tilde{s}_n\} \subset S$ such that $s_n \longrightarrow s$, $\tilde{s}_n \longrightarrow s$, $s_n \neq \tilde{s}_n$ and $\phi(s_n, g_n) = \phi(\tilde{s}_n, \tilde{g}_n)$. Therefore, $\phi(s_n, g_n\tilde{g}_n^{-1}) = \phi(\tilde{s}_n, e) \longrightarrow s$, and by properness of the action, we conclude that, up to subsequences, $h_n := g_n\tilde{g}_n^{-1} \longrightarrow h \in G$ such that $\phi(s, h) = s$, that is, $h = e$, because the action is free. Finally, we found that $\phi(s_n, h_n) = \phi(\tilde{s}_n, e)$ in an arbitrarily small neighborhood of $S \times \{e\}$, which is a contradiction.

We have already seen that M/G is Hausdorff and second countable. The charts are provided by the charts of M as follows. Take $U \subseteq M$ open with $U \cap S \neq \emptyset$ with chart ψ. The map $\pi|_S$ has a well defined inverse, namely $(\pi|_S)^{-1}(b) = \phi(s, e)$, with $\pi(s) = b$, which is clearly a homeomorphism. Therefore, a chart for M/G on the open $\pi(U)$ is given by $(\psi|_S) \circ (\pi|_S)^{-1}$. The map ϕ also provides the local bundle trivializations. Indeed, consider two different submanifolds S_i, $i = 1, 2$, such that $\pi(S_1) \cap \pi(S_2) \neq \emptyset$, and set $\pi_i := \pi|_{S_i}$, $\phi_i := \Psi|_{S_i\times G}$ diffeomorphisms with the images. Let us also denote the inverses as $\phi_i^{-1}(x) =: (\pi_i^{-1}(\pi(x)), \theta_i(x))$ and open neighborhood U_i of S_i such that $U_1 \cap U_2 \neq \emptyset$. Now we define local trivializations $\chi_i : \pi(S_i) \times G \to \pi^{-1}(\pi(S_i))$, $\chi_i(b, g) := \phi_i(\pi_i^{-1}(b), g)$. The inverse map is given by $\chi_i^{-1}(p) = (\pi(p), \theta_i(p))$. Then, if $b \in (\pi(U_1) \cap \pi(U_2))$, we have

$$(\chi_2^{-1} \circ \chi_1)(b, g) = \chi_2^{-1}(\phi_1(\pi_1^{-1}(b), g)) =$$

$$= (\pi(\phi_1(\pi_1^{-1}(b), g)), \theta_2(\phi_1(\pi_1^{-1}(b), g))) = (b, \theta_2(\phi_1(\pi_1^{-1}(b), g))) \ .$$

Now, $h = \theta_2(\phi_1(\pi_i^{-1}(b), g))$ is such that $\Psi^h(\pi_2^{-1}(b)) = \phi_1(\pi_1^{-1}(b), g) = \Psi^g(\pi_1^{-1}(b))$. But then there is a unique $k(p) \in G$ such that $\pi_1^{-1}(b) = \Psi^{k(p)}(\pi_2^{-1}(b))$, because ϕ_1 is a diffeomorphism. Therefore, $\Psi^h(\pi_2^{-1}(b)) = \Psi^g(\Psi^{k(p)}(\pi_2^{-1}(b))) = \Psi^{k(p)g}(\pi_2^{-1}(b))$, because we use a right action. In conclusion, $\theta_2(\phi_1(\pi_i^{-1}(b), g)) = k(p)g$ and

$$(\chi_2^{-1} \circ \chi_1)(b, g) = (b, k(p)g) \ ,$$

i.e. changes of trivializations are provided by left translations. □

Remark 1.2.13. Note that the smooth structure on M/G is uniquely determined by asking that $\pi : M \to M/G$ is smooth and that for all $f : M/G \to \mathbb{R}$, f is smooth, if and only if $\pi^* f : M \to \mathbb{R}$ is smooth.

Corollary 1.2.14. *Let G be a Lie group and H a closed subgroup. Let H act on G from the right by right translations: $(g, h) \mapsto gh$. Then the orbit space G/H, called the space of right cosets, is a PFB $H \hookrightarrow G \twoheadrightarrow G/H$. Moreover if H is also a normal subgroup, G/H is a group and $\pi : G \to G/H$ is a Lie group homomorphism.*

Proof. We have seen in Example 1.2.8 that the action of a closed subgroup by multiplication of the group is proper and free, and therefore we can conclude, by the previous theorem, that $H \hookrightarrow G \twoheadrightarrow G/H$ is a PFB. For the second part of the statement, we recall that the quotient by a normal subgroup is a group. Indeed, the natural multiplication $gH.kH = gkH$ is well defined, for if $gh_1.kh_2 := gh_1kh_2$ we have,

$$gh_1kh_2 = gkk^{-1}h_1kh_2 \in gkH \ .$$

Similarly, the inverse is well defined. It is a standard fact that G/H is also a Lie group (see, e.g., [13]). Finally, $\pi : G \to G/H$, $g \mapsto gH$, is a group homomorphism by construction. □

Let $G \hookrightarrow P \twoheadrightarrow B$ be a principal fiber bundle and let F be a smooth manifold. Let $R : P \times G \to P$ be the right action of G on P and let $\Psi : G \times F \to F$ be a left action.

Definition 1.2.15. We define the *twisted space* $P \times_G F$ to be the orbit space of the left diagonal action $\delta : G \times (P \times F) \to (P \times F)$ defined as

$$\delta(g, (p, f)) := (R_{g^{-1}}p, \Psi_g f) \ .$$

We denote the equivalence classes in $P \times_G F$ with square parenthesis $[p, f] \in P \times_G F$.

Proposition 1.2.16. *Consider a principal bundle $G \hookrightarrow P \twoheadrightarrow B$, F a left G-manifold and the twisted space $P \times_G F$. Then $F \hookrightarrow P \times_G F \twoheadrightarrow B$ is a fiber bundle with structural group $G \hookrightarrow$ Diff(F), called the associated bundle of P.*

Proof. Let U be a G-invariant neighborhood of $R_G(p)$ for $p \in P$, and S be a submanifold of P transverse to $R_G(p)$. Then there is a diffeomorphism $\phi : S \times F \to U \times_G F$ defined as $\phi(s, f) := [s, f]$. From this fact we can find smooth charts for the total space $P \times_G F$. The local trivializations are built as follows. Let S_i, $i = 1, 2$, be two different transverse submanifolds and $\pi_i : S_i \to B$, $\pi : P \to B$, $\pi_i := \pi|_{S_i}$ diffeomorphisms. We claim that $\psi_i : \pi(S_i) \times F \to U \times_G F$ with $\psi_i(b, f) := [\pi_i^{-1}(b), f]$ is a local trivialization. If $b \in \pi(S_1) \cap \pi(S_2)$ we know that there exists $g_{12}(b) \in G$ such that $\pi_2^{-1}(b) = R_{g_{12}(b)}(\pi_1^{-1}(b))$. But then, we have $\psi_i^{-1}([p, f]) = (\pi(p), \Psi_{h(p)} f)$ where $R_{h_i(p)^{-1}} p = \pi_i^{-1}(\pi(p))$, because then

$$\psi_i(\psi_i^{-1}([p, f])) = \psi_i((\pi(p), \Psi_{h_i(p)} f)) = [\pi_i^{-1}\pi(p), \Psi_{h_i(p)} f] = [R_{h_i(p)^{-1}} p, \Psi_{h_i(p)} f] = [p, f] \ ,$$

where in the last equality we use the definition of $U \times_G F$. Similarly,

$$\psi_i^{-1}(\psi_i(b, f)) = \psi_i^{-1}([\pi_i^{-1}(b), f]) = [\pi(\pi_i^{-1}(b)), f] = (b, f) ,$$

because in this case $h_i(\pi_i^{-1}(b)) = e$. Now we have

$$(\psi_2^{-1} \circ \psi_1)(b, f) = \psi_2^{-1}([\pi_1^{-1}(b), f]) = (\pi(\pi_1^{-1}(b)), \Psi_{h_2(\pi_i^{-1}(b))} f) = (b, \Psi_{g_{12}(b) f}) .$$

Because, $R_{h_2(\pi_1^{-1}(b))}(\pi_1^{-1}(b)) = \pi_2^{-1}(\pi_1^{-1}(b)) = \pi_2^{-1}(b)$, hence $h_2(\pi_1^{-1}(b)) = g_{12}(b)$. □

Proposition 1.2.16 can be summarized using the following commuting diagram, called *Cartan mixing diagram*, where π_P and π_F are the projections on the respective factors of $P \times F$, and ρ, τ, π are the natural projections on the respective orbit spaces.

$$\begin{tikzcd}
P \arrow[d, "\rho"] & P \times F \arrow[l, "\pi_P"'] \arrow[r, "\pi_F"] \arrow[d, "\tau"] & F \arrow[d, "\pi"] \\
B & P \times_G F \arrow[l, "\alpha"'] \arrow[r, "\beta"] & F/G
\end{tikzcd}$$

We have just proved that $\alpha : P \times_G F \to B$ is a fiber bundle with characteristic fiber F. We point out that, in general, the map $\beta : P \times_G F \to F/G$ is only a map of topological spaces. More precisely, given any $[f] \in F/G$, we have $\beta^{-1}([f]) \simeq P/G_f$. In fact, recall that $[p, f] := \{(p.g, g^{-1}.f) \in P \times F \ : \ g \in G\}$. We clearly have $\beta^{-1}([f]) = \{[p, f] \ : \ p \in P\}$ and hence $[p, f] = [\tilde{p}, f]$ if and only if there exists $g \in G$ such that $(p, f) = (\tilde{p}.g, g^{-1}.f)$, i.e., if and only if $g \in G_f$ and $p = \tilde{p}.g$.

Example 1.2.17. We list some examples of associated vector and principal bundles.

i) Let $G \hookrightarrow P \twoheadrightarrow B$ be a principal fibre bundle. Let $\rho : G \to GL(V)$ be a representation of G on a vector space V. Then the associated bundle $V \hookrightarrow P \times_G V \twoheadrightarrow B$ with $\delta : G \times (P \times V) \to P \times V$,

$$\delta(g(p, v)) := (R_{g^{-1}} p, \rho(g) v) ,$$

is called the *associated vector bundle* of P with fiber V. Such a vector bundle, by construction, has structure group given by $\rho(G) \subseteq GL(V)$.

ii) Let $GL(V) \hookrightarrow FE \twoheadrightarrow M$ be the frame bundle of a vector bundle $V \hookrightarrow E \twoheadrightarrow M$ with characteristic fiber a vector space V and base M. As we could expect, the twisted space $FE \times_{GL(V)} V$ is canonically identified with E. In fact, to an element $[\mathbf{e}_x, v] \in FE \times_{GL(V)} V$ we can associate the element $(x, \mathbf{e}_x v) \in E$. To see that this map is well defined, suppose that $[\tilde{\mathbf{e}}_x, \tilde{v}] = [\mathbf{e}_x, v]$. This means that there exists a $g \in GL(V)$ such that $(\mathbf{e}_x g^{-1}, g v) = (\tilde{\mathbf{e}}_x, \tilde{v})$. But then

$$[\tilde{\mathbf{e}}_x, \tilde{v}] \mapsto (x, \tilde{\mathbf{e}}_x \tilde{v}) = (x, \mathbf{e}_x g^{-1} g v) = (x, \mathbf{e}_x v) \leftarrow\!\shortmid [\mathbf{e}_x, v] .$$

iii) Consider the principal fiber bundle $G \hookrightarrow G \times G \twoheadrightarrow (G \times G)/\Delta_{G \times G}$, obtained using the right diagonal action by G, $((g_1, g_2), g) \mapsto (g_1 g, g_2 g)$ which is proper and free. Notice that $(G \times G)/\Delta_{G \times G} \simeq G$. In fact, there is a natural map $(G \times G)/\Delta_{G \times G} \to G$, $(g_1, g_2) \to g_1 g_2^{-1}$ which is a well defined diffeomorphism. Thus, we can see G as the associated principal bundle $G \times_G G$ to the trivial principal bundle $G \hookrightarrow G \twoheadrightarrow e$

We end this subsection with the following example, which will be a central ingredient in Chapter 4.

Remark 1.2.18. Let $K_1, \ldots, K_N$ be compact Lie subgroups of G and $H \subset K_i$ a common Lie subgroup of the K_i. The group H acts from the left and from the right on each K_i by left and right translations, respectively, and we can define a *right action* of $H^{N-1} := H \times \overset{(N-1)\text{-times}}{\ldots} \times H$ on $K_1 \times \ldots \times K_{N-1}$ as follows

$$(k_1, k_2, \ldots, k_{N-1}).(h_1, h_2, \ldots, h_{N-1}) = (k_1 h_1, h_1^{-1} k_2 h_2, \ldots, h_{N-2}^{-1} k_{N-1} h_{N-1}) \ .$$

This action is proper and free. Moreover, H^{N-1} also acts on the left on K_N/H as

$$(h_1, \ldots, h_{N-1}).(k_N H) = h_{N-1} k_N H \ .$$

Therefore, we can construct the fiber product

$$\left(K_1 \times \ldots \times K_{N-1} \times (K_N/H)\right)/H^{N-1} = (K_1 \times \ldots \times K_{N-1}) \times_{H^{N-1}} (K_N/H) \ .$$

We will often use the following somewhat imprecise notation

$$K_1 \times_H K_2 \times_H \ldots \times_H K_{N-1} \times_H (K_N/H) := (K_1 \times \ldots \times K_{N-1}) \times_{H^{N-1}} (K_N/H) \ .$$

1.2.2 Slices and Tubular Neighborhoods

Proper actions have the property that we can find an embedded submanifold around each point $x \in M$, called *slice at* x, which is invariant under the G_x-subaction. More precisely, we give the following definition.

Definition 1.2.19. Let $\Psi : G \times M \to M$ be an action. We define a *slice* at $x \in M$ relative to the action Ψ to be an embedded submanifold S_x containing x, such that the following conditions hold.

i) $T_x M = (\mathrm{d}\Psi^x)_e(\mathfrak{g}) \oplus T_x S_x$ and $T_y M = (\mathrm{d}\Psi^y)_e(\mathfrak{g}) + T_y S_x$ for all $y \in S_x$;

ii) If $y \in S_x$ and $g \in G_x$, then $\Psi_g(y) \in S_x$;

iii) If $y \in S_x,\ g \in G,$ and $\Psi_g(y) \in S_x,$ then $g \in G_x$.

Theorem 1.2.20. *(Slice Theorem). If $\Psi : G \times M \to M$ is a proper action and $x \in M$, then there exists a slice S_x at x.*

Proof. Let $x \in M$. Since the action is proper, we have that G_x is compact. Therefore, if $\mathbf{g}$ is any Riemannian metric on M we can build a G_x-invariant metric $\mathbf{h}$ on M as follows

$$\mathbf{h}_y(v, w) := \int_{G_x} \mathbf{g}_{\Psi_g(y)}((\mathrm{d}\Psi_g)_y(v), (\mathrm{d}\Psi_g)_y(v))\ \mathrm{d}g \ ,$$

for all $y \in M$ and $v, w \in T_y M$. We set for $\varepsilon > 0$

$$S_x := \exp_x\left(B_\varepsilon(0) \cap (T_x(G(x))^\perp)\right) \ ,$$

where exp is the Riemannian exponential of $(M, \mathbf{h})$, $B_\varepsilon(0) := \{v \in T_x M \ : \ \mathbf{h}_x(v, v) \leq \varepsilon\} \subset T_x M$ and $(T_x(G(x))^\perp)$ is the orthogonal with respect to $\mathbf{h}$. In this way the subaction of G_x is by isometries. We have already observed that $(\mathrm{d}\Psi_g)_x$ leaves invariant $T_x(G(x))$ for all $g \in G_x$. Therefore, $(\mathrm{d}\Psi_g)_x$ also leaves the $\mathbf{h}$-orthogonal complement invariant. This implies that S_x is G_x-invariant. In fact, for all $g \in G_x$ and $y \in S_x$ we have

$$\Psi_g(y) = \Psi_g(\exp_x(v)) = \exp_x((\mathrm{d}\Psi_g)_x(v)) ,$$

where $v \in (T_x(G(x)))^\perp$ is such that $\exp_x(v) = y$ and we have just said that $(\mathrm{d}\Psi_g)_x(v) \in (T_x(G(x)))^\perp$. This proves *ii*). Suppose that for all $\varepsilon > 0$, *iii*) is not satisfied. This means that there is $x_n \in S_x$, $g_n \notin G_x$ such that $\Psi(g_n, x_n) \longrightarrow x \longleftarrow x_n$ with $\Psi_{g_n}(x_n) \in S_x$. By properness, we have that there is $g_{k_n} \longrightarrow g \in G$. Setting $\tilde{g}_k := g^{-1}g_{k_n}$ we get $\tilde{g}_k \longrightarrow e$ with $\tilde{g}_k \notin G_x$ and $S_x \ni \Psi(\tilde{g}_k, x_k) \longrightarrow x$. But, near the identity we have a diffeomorphism $G \simeq_{loc} G_x C$, with $C \subset G$, $\mathfrak{g} = \mathfrak{g}_x \oplus T_e C$, for which $\tilde{g}_k = h_k c_k$ for $h_k \in G_x$ and $c_k \in C$. Since $\tilde{g}_k \notin G_x$ it must be $c_k \neq e$. We have clearly that $\Psi(h_k, x_k) \in S_x$. But since $C \times S_x \to M$, $(c, s) \mapsto \Psi(c, s)$, is a diffeomorphism with the image, we have that $\Psi(c_k, \Psi(h_k, x_k)) = \Psi(\tilde{g}_k, x_k) \notin S_x$, which is a contradiction. □

Since the isotropy representation preserves the orthogonal splitting $T_xM = T_xG(x) \oplus (T_xG(x))^\perp$, we can consider its restriction to $(T_xG(x))^\perp$, i.e. $G_x \longrightarrow \mathrm{O}((T_xG(x))^\perp)$, $g \mapsto (\mathrm{d}\Psi_g)_x|_{(T_xG(x))^\perp}$, which is called the *slice representation.*

Definition 1.2.21. Let $\Psi : G \times M \to M$ be a proper action and let S_x be a slice at $x \in M$. We define the *tubular neighborhood* of the orbit $G(x)$ to be the image of S_x under the action of G, i.e. the subset of M

$$\mathfrak{T}(G(x)) := G(S_x) .$$

Theorem 1.2.22. *(Tubular Neighborhood Theorem). Let $\Psi : G \times M \to M$ be a proper action. For every $x \in M$ there exists a G-equivariant diffeomorphism between $\mathfrak{T}(G(x))$ and $G \times_{G_x} S_x$, which is the total space of the fiber bundle*

$$S_x \hookrightarrow G \times_{G_x} S_x \twoheadrightarrow G/G_x ,$$

where $G_x \hookrightarrow G \twoheadrightarrow G/G_x$ is the principal bundle of the right cosets and the left action of G_x on S_x is the restriction of the G-action to $G_x \times S_x$. Moreover, there is a natural left G-action on $G \times_{G_x} S_x$, given by $g.[h, s] := [gh, s]$.

Proof. The candidate diffeomorphism is clearly $\phi : G \times_{G_x} S_x \to \mathfrak{T}(G(x))$, $[g, s] \mapsto \Psi_g(s)$. This map is well defined and manifestly G-equivariant. In fact, if $[g_1, s_1] = [g_2, s_2]$, then there is $h \in G_x$, such that $g_2 = g_1 h^{-1}$ and $s_2 = \Psi_h(s_1)$. Then $\Psi_{g_2} s_2 = \Psi_{g_1 h^{-1}} \Psi_h s_1 = \Psi_{gh^{-1}h} s_1 = \Psi_{g_1} s_1$. This map is surjective by definition and is also injective because, if $\Psi_{g_1}(s_1) = \Psi_{g_2}(s_2)$, then $\Psi_{g_2^{-1}g_1}(s_1) = s_2$, hence $h := g_2^{-1} g_1 \in G_x$, because S_x is a slice. Therefore, $g_2 = g_1 h^{-1}$ and $s_2 = \Psi_h(s_1)$, i.e., $[g_1, s_1] = [g_2, s_2]$. We prove that ψ is a local diffeomorphism. Clearly, ϕ is the composition of the map $\psi := \Psi|_{G \times S_x} : G \times S_x \to \mathfrak{T}(G(x))$ and the natural projection $\pi : G \times S_x \to G \times_{G_x} S_x$, hence is smooth. We have already seen that $(\mathrm{d}\psi)_{(g,s)}$ is always surjective and $(\mathrm{d}\pi)_{(g,s)}$ is surjective because it is the projection of a fiber bundle. Therefore, $(\mathrm{d}\psi)_{(g,s)} = (\mathrm{d}\phi)_{[g,s]} \circ (\mathrm{d}\pi)_{(g,s)}$ tells us that also $(\mathrm{d}\phi)_{[g,s]}$ is surjective. We also observe that from the definitions of $G \times_{G_x} S_x$ and $\mathfrak{T}(G(x))$ we have

$$\begin{aligned} \dim(G \times_{G_x} S_x) &= \dim(G) + \dim(S_x) - \dim(G_x) = \\ &= \dim(G) + \dim(M) - \dim(G) + \dim(G_x) - \dim(G_x) = \\ &= \dim(M) = \dim(\mathfrak{T}(G(x))) , \end{aligned}$$

and then $(\mathrm{d}\phi)_{[g,s]}$ is injective for dimensional reasons. □

One important consequence of the Tubular Neighborhood Theorem is that even if M/G is not in general a manifold, if the action is proper M/G is locally compact, Hausdorff and second countable, and hence paracompact.

Remark 1.2.23. Another consequence of Theorem 1.2.22 is that the codimension of a G-orbit of any point $y \in \mathfrak{T}(G(x))$, is equal to the codimension of the G_x-orbit in S_x of any point in $S_x \cap G(y)$. In fact, we have $\dim(G(y)) = \dim(G/G_x) + \dim(G_x(y))$. Moreover, $\operatorname{codim}_M(G(y)) = \dim(M) - \dim(G(y)) = \dim(M) - \dim(G/G_x) - \dim(G_x(y)) = \dim(S_x) - \dim(G_x(y)) = \operatorname{codim}_{S_x}(G_x(y))$.

1.2.3 Isometric actions

From now on, we will consider M to be a Riemannian manifold with Riemannian metric $\mathbf{g}$, and we shall denote it as the pair $(M, \mathbf{g})$.

Definition 1.2.24. We call an action $\Psi : G \circlearrowleft M$ *isometric* with respect to a Riemannian metric $\mathbf{g}$ on M if for all $g \in G$, $\Psi_g \in \mathrm{Iso}(M, \mathbf{g})$, the group of isometries of $(M, \mathbf{g})$.

In other words, an isometric action determines a group homomorphism $G \to \mathrm{Iso}(M, \mathbf{g})$. A classical, profound theorem of Myers and Steenrod tells us that any closed subgroup of $\mathrm{Iso}(M, \mathbf{g})$ in the compact open topology, is a Lie group, hence $\mathrm{Iso}(M, \mathbf{g})$ itself is a Lie group. In particular, if M is compact, $\mathrm{Iso}(M, \mathbf{g})$ is a compact Lie group.

If $(M, \mathbf{g})$ is complete, at the infinitesimal level, we have that the Lie algebra of $\mathrm{Iso}(M, \mathbf{g})$ is given by the *Killing* vector fields $\mathfrak{Iso}(M, \mathbf{g})$, i.e. vector fields X such that, if ∇ is the Levi-Civita connection of $(M, \mathbf{g})$, solve the *Killing equation*

$$\mathbf{g}(\nabla_Y X, Z) + \mathbf{g}(\nabla_Z X, Y) = 0 \ ,$$

for all Y, Z vector fields on M. Isometric and proper actions are linked by the following fundamental theorems.

Theorem 1.2.25. *Let $(M, \mathbf{g})$ be a Riemannian manifold, and let G be a closed subgroup of $Iso(M, \mathbf{g})$. Then the action $(g, x) \mapsto g(x)$ is proper.*

Proof. By [88], we can restrict to the case in which $(M, \mathbf{g})$ is complete. By Proposition 1.2.2 we only need to prove that given sequences $\{x_n\} \subseteq M$, $x_n \longrightarrow x$ and $\{g_n\} \subseteq G$ such that $g_n(x_n) \longrightarrow y$, there exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \longrightarrow g \in G$. We can choose $N > 0$ big enough such that, if d is the distance induced by $\mathbf{g}$, we have $d(y, g_n(x_n)) \leq \varepsilon/2$ and $d(x, x_n) < \varepsilon/2$ for small $\varepsilon > 0$. It follows that for $n > N$, given a fixed radius $R > 0$, $g_n(B_R(x)) \subset B_{R+\varepsilon}(y)$, where for $r \in \mathbb{R}$ and $z \in M$, we denote as $B_r(z) := \{x \in M \ : \ d(z, x) \leq r\}$ the open ball in the metric d of radius r and center in z. In fact, by triangular inequality, for all $z \in B_R(x)$ we have

$$d(g_n(z), y) \leq d(g_n(z), g_n(x_n)) + d(y, g_n(x_n)) <$$
$$< d(z, x_n) + \varepsilon/2 \leq d(z, x) + d(x, x_n) + \varepsilon/2 < \varepsilon + R \ .$$

Since g_n is an isometry, the family $\{g_n\}$ is equicontinuous in the compact open topology. Then since $(M, \mathbf{g})$ is complete by Hopf–Rinow, the closures of the metric balls are compact in M. Then by the inequality above the family $\{g_n\}$ is also equibounded. By the classical Ascoli–Arzelà Theorem, there exists a subsequence $\{g^0_{n_k}\}$ that converges uniformly on $\overline{B_R}(x)$ to a continuous map $g : \overline{B_R}(x) \to M$. Using this argument inductively, for $R_i > R_{i-1} > \ldots > R_0 = R$, we come up with a grid of subsequences $\{g^i_{n_k}\}$ of $\{g_n\}$ each converging uniformly on $\overline{B_{R_i}}(x)$. Then the diagonal subsequence $\{g^i_{n_i}\}$ converges to a continuous map $g : M \to M$. By the theorem of Myers and Steenrod, $g \in G$ is an isometry. $\square$

Theorem 1.2.26. *Let M be a manifold, and let G be a Lie group. If $\Psi : G \times M \to M$ is a proper action, there exists a G-invariant metric $\boldsymbol{g}$ on M such that $\{\Psi_g \,:\, g \in G\}$ is a closed subgroup of Iso$(M, \boldsymbol{g})$.*

Proof. The result follows essentially from the existence of G-invariant partition of unity and from the compactness of the isotropies. More precisely, let $\{U_\alpha\}$ be an open covering of M consisting of G-invariant open subsets. The Tubular Neighborhood Theorem and, consequently, the paracompactness of M/G, prove that there exists a G-invariant partition of unity $f_j : M \to \mathbb{R}$ subordinate to $\{U_\alpha\}$. This means that each f_j is a smooth G-invariant function, $f_j \geq 0$ and the support of f_j is contained in some U_α. Moreover, $\{\mathrm{supp}(f_j)/G\}$ forms a locally finite family of compact subsets of M/G, and $\sum_j f_j = 1$. Note that such partitions of unity can be used to glue together arbitrary G-invariant structures.

Let $\mathbf{g}$ be any Riemannian metric on M, let $x_\alpha \in U_\alpha$ and let S_{x_α} be a slice at x_α. Then $\mathbf{h}^\alpha$ defined as

$$\mathbf{h}^\alpha_x(v, w) := \int_{G_{x_\alpha}} \mathbf{g}_{\Psi_g(x)}((\mathrm{d}\Psi_g)_x(v), (\mathrm{d}\Psi_g)_x(w)) \, \mathrm{d}g \ ,$$

where $x \in S_{x_\alpha}$, $v, w \in T_x(S_{x_\alpha})$ and $\mathrm{d}g$ is a right-invariant volume form on G_{x_α}, is a G_{x_α}-invariant Riemannian metric on S_{x_α}. Then we define a G-invariant metric $\mathbf{g}^\alpha$ on U_α as

$$\mathbf{g}^\alpha_{\Psi_g(x)}((\mathrm{d}\Psi_g)_x(v), (\mathrm{d}\Psi_g)_x(w)) := \mathbf{h}^\alpha_x(v, w) \ .$$

Finally, $\sum_\alpha f_\alpha \mathbf{g}^\alpha$ is a G-invariant metric on M.

Assume now that $\{\Psi_{g_n}\} \subseteq \mathrm{Iso}(M, \mathbf{g})$ converges uniformly on each compact subset of M to an isometry $f : M \to M$. In particular, for all $x \in M$, $\Psi_{g_n}(x) \longrightarrow y = f(x)$. Then, by properness, there exists a subsequence $g_{n_k} \longrightarrow g \in G$ and by continuity, $\Psi_{g_{k_n}}$ converges uniformly to $\Psi_g = f$. □

An important observation is that, in general, given a proper action $G \circlearrowleft M$, the G-invariant metric on M whose existence is guaranteed by the previous theorem is not necessarily complete. However, as noted by Podestà and Spiro (see [98]), if M/G is connected, we can always find one that is also complete.

Theorem 1.2.27. *Let $G \circlearrowleft M$ be a proper action such that M/G is connected. Then there exists a complete Riemannian metric $\boldsymbol{g}$ on M, such that G acts isometrically with respect to it.*

Proof. We may assume, without loss of generality, that M is connected. Indeed, let M' be a connected component of M and consider $G_{M'} := \{g \in G \,:\, gM' = M'\}$. It is well known that the identity component of G, that we denote with G^o, is a closed normal subgroup of G. Since $G^o \subseteq G_{M'} \subseteq G$, it follows that $G_{M'}$ is a closed subgroup of G and hence the $G_{M'}$-action on M' is proper. Assume there exists a $G_{M'}$-invariant, complete Riemannian metric $\mathtt{h}'$ on M'. Let M'' be another connected component of M. Since M/G is connected there exists $g_o \in G$ such that $g_o M'' = M'$. Therefore, $G_{M''} = g_0^{-1} G_{M'} g_0$ and $\mathtt{h}'' := g_0^* \mathtt{h}'$ is a $G_{M''}$-invariant, complete Riemannian metric on M''.

Let $\mathtt{h}$ be the G-invariant metric on M provided by Theorem 1.2.26 and let d be the induced distance. For any $x \in M$ we define $r(x)$ to be the supremum of the positive real numbers $r \in \mathbb{R}$ such that the open ball $B_r(x) := \{y \in M \,:\, d(x, y) < r\}$ is relatively compact. If $r(x) = +\infty$ for some $x \in M$, then any open ball is relatively compact. By the Hopf–Rinow Theorem $(M, \mathtt{h})$ is a complete Riemannian manifold [28]. Hence, we may assume $r(x) < +\infty$ for any $x \in M$.

Let $x, y \in M$ be such that $y \in B_{r(x)}(x)$. Then $r(y) \geq r(x) - d(x,y)$ and so

$$|r(x) - r(y)| \leq d(x,y).$$

Nomizu and Ozeki in [88] proved the existence of a positive smooth real function $f : M \longrightarrow \mathbb{R}$ satisfying $f(x) \geq \frac{1}{r(x)}$ for any $x \in M$ and $(M, f^2\mathtt{h})$ is complete. Indeed, if we denote by d' the distance induced by $\mathtt{g} = f^2\mathtt{h}$, then

$$\left\{y \in M \, : \, d'(x,y) < \frac{1}{3}\right\} \subset \left\{y \in M \, : \, d(x,y) \leq \frac{r(x)}{2}\right\}.$$

for any $x \in M$. This implies that any open ball of radius $\frac{1}{3}$ with respect to $\mathtt{g}$ is relatively compact and so any Cauchy sequence with respect to d' admits a convergent subsequence. By the Hopf–Rinow Theorem $(M, f^2\mathtt{h})$ is a complete Riemannian manifold [28].

Since $\mathtt{h}$ is G-invariant it follows that the function r is a G-invariant continuous function. Our result follows form the existence of a G-invariant function $f : M \longrightarrow \mathbb{R}$ such that $f(x) \geq \frac{1}{r(x)}$ for any $x \in M$.

Let $x \in M$. By the Tubular Neighborhood Theorem, we may think a G-invariant neighborhood of $G(x)$ as the homogeneous vector bundle $G \times_{G_x} \nu_x G(x)$. Let $B_n = \{v \in \nu_x G(x) : \parallel v \parallel < n\}$. The open covering $\mathcal{U} = \{GB_n\}_{n \in \mathbb{N}}$ of $G \times_{G_x} \nu_x G(x)$ is G-invariant. Let $\{h_i : G \times_{G_x} \nu_x G(x) \longrightarrow [0,1]\}_{i \in I}$ be a G-invariant partition of unity subordinate to $\mathcal{U}$. The function

$$\tilde{f} = \sum_{i \in I} \frac{h_{n(i)}}{r_{n(i)}},$$

where $n(i) \in \mathbb{N}$ is such that $\operatorname{supp} h_i \subset GB_{i(n)}$ and $r_n = \inf_{z \in GB_n} r$, satisfies $\tilde{f} \geq \frac{1}{r}$. Now, using the G-invariant partition of unity subordinated to $\{G \times_{G_x} \nu_x G(x)_x\}_{x \in M}$, one can construct a G-invariant smooth function f satisfying

$$f(y) \geq \frac{1}{r(y)},$$

for any $y \in M$, concluding the proof. □

We observe that in Theorem 1.2.26, the subgroup $\{\Psi_g \; : \; g \in G\} \subseteq \operatorname{Iso}(M, \mathbf{g})$ is isomorphic to G if and only if the action is effective. We also observe the following.

Remark 1.2.28. If we have a proper isometric effective action $G \circlearrowleft M$, then, if $n = \dim(M)$, we have $\dim(G) \leq \frac{n(n+1)}{2}$. Let $x \in M$, then first of all $\dim(G) - \dim(G_x) = \dim(G(x)) \leq \dim(M)$. We also have that for all $g \in G_x$, $(\mathrm{d}\Psi_g)_x : T_xM \to T_xM$, $(\mathrm{d}\Psi_g)_x \in SO(T_xM, \mathbf{g}_x) \simeq SO(n)$. Suppose $(\mathrm{d}\Psi_g)_x = \mathrm{Id}_{T_xM}$, then by a classical theorem of Cartan, $\Psi_g = \mathrm{Id}_M$, and $g = e$ by effectiveness. Therefore, we have a smooth injective, closed map from G_x to $SO(n)$, hence $\dim(G_x) \leq \frac{n(n-1)}{2}$, and $\dim(G) \leq n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Metric aspects of M/G

If we have an isometric action $G \circlearrowleft M$, on a complete Riemannian manifold $(M, \mathbf{g})$, the quotient M/G is a metric space, where the distance is given by

$$d_{M/G}([x],[y]) := \inf\{d_M(g.x, h.y) \; : \; g, h \in G\} \; .$$

where d_M is the distance on M induced by the metric, and $\pi : M \to M/G$ is the natural projection such that $\pi(x) = [x], \; \pi(y) = [y]$. We suppose that M/G is connected, e.g.

when M itself is connected. In this case, $d_{M/G}$ is always finite. In general, we shall say that the points in different connected components of a manifold are at “infinite distance”. The distances d_M and $d_{M/G}$ are well behaved under π.

Theorem 1.2.29. *Let $G \circlearrowleft M$ isometrically on the complete Riemannian manifold $(M, \boldsymbol{g})$. Then $(M/G, d_{M/G})$, defined below, is a metric space and $\pi : M \to M/G$ is a submetry, i.e. $\pi(B^d_\varepsilon(x)) = B^{d_{M/G}}_\varepsilon([x])$, where B^d and $B^{d_{M/G}}$ are the open metric balls with respect to the distances d and $d_{M/G}$ respectively. Moreover, $(M/G, d_{M/G})$ is also complete.*

Proof. First, we verify that $d_{M/G}$ is an actual distance. Suppose $d_{M/G}([x],[y]) = 0$, then we have a minimizing sequence g_n such that $d(g_n.x, y) \longrightarrow 0$. Then $g_n.x \longrightarrow y$. By properness of the action there is $g_{k_n} \longrightarrow g$ and therefore $y = g.x$, i.e. $[x] = [y]$. It is symmetric because d is symmetric and $d_{M/G}$ is a minimum. We finally need to prove the triangle inequality. Let $[x],[y],[z] \in M/G$, then $d_{M/G}([x],[y]) \le d(g.x, h.y) \le d(g.x, k.z) + d(k.z, h.y)$ for all $g, h, k \in G$. Therefore, by taking the infimimum of such an inequality, we get the triangular inequality for $d_{M/G}$.

To show that π is a submetry we proceed as follows. Let $y \in B^d_\varepsilon(x)$, then clearly $d_{M/G}([x],[y]) \le \varepsilon$, and hence $\pi(B^d_\varepsilon(x)) \subseteq B^{d_{M/G}}_\varepsilon([x])$. Suppose now that $[y] \in B^{d_{M/G}}_\varepsilon([x])$, then there is a $g \in G$ such that $d(x, g.y) \le \varepsilon$. Hence, in conclusion, $\pi(B^d_\varepsilon(x)) = B^{d_{M/G}}_\varepsilon([x])$

To show the completeness, we take a Cauchy sequence in M/G, i.e., take $[x_n] \in M/G$ such that for all $\varepsilon > 0$, there is N such that for all $n, m \ge N$, $d_{M/G}([x_m],[x_n]) \le \varepsilon$. Therefore, there are $g_m, g_n \in G$ such that $d(g_m x_m, g_n.x_n) \le \varepsilon$. It follows that $g_n.x_n$ is a Chauchy sequence. By Hopf–Rinow theorem (M, d) is a complete metric space, then we have a convergent subsequence $g_{k_n}.x_{k_n} \longrightarrow y$. Hence, definitely $\mathrm{d}_{M/G}([x_{k_n}],[y]) \le d(g_{k_n}.x_{k_n}, y) \le \varepsilon$, showing that $[x_{k_n}] \longrightarrow [y]$.

□

Vertical and Horizontal directions

Given $\Psi : G \circlearrowleft (M, \mathbf{g})$ isometric action, we can define the *vertical space* as

$$\mathcal{V}M := \bigsqcup_{x \in M} \mathcal{V}_x M := \bigsqcup_{x \in M} \ker((\mathrm{d}\pi)_x) \subseteq TM \ .$$

Moreover, we can use the metric to define the *horizontal space* as

$$\mathcal{H}M := \bigsqcup_{x \in M} \mathcal{H}_x M := \bigsqcup_{x \in M} \ker((\mathrm{d}\pi)_x)^\perp \subseteq TM \ ,$$

where the orthogonal is taken with respect to $\mathbf{g}_x$. It is clear that at each point $\mathcal{V}_x M = T_x(G(x))$. Note that these are not distributions since $\dim(G(x))$ varies with x. At each point then $T_x M = \mathcal{V}_x M \oplus \mathcal{H}_x M$ orthogonally with respect to $\mathbf{g}_x$.

Definition 1.2.30. We call a geodesic $\gamma : \mathbb{R} \to M$ a *horizontal geodesic* if $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)} M$ for all $t \in \mathbb{R}$.

We can easily obtain the following proposition.

Proposition 1.2.31. *Let $\gamma : \mathbb{R} \to M$ be a geodesic such that $\dot\gamma(0) \in \mathcal{H}_{\gamma(0)} M$. Then γ is a horizontal geodesic.*

Proof. Let ξ_M, for $\xi \in \mathfrak{g}$, be a vertical vector field. We have

$$\frac{d}{dt}\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \xi_M(\gamma(t)) = \mathbf{g}_{\gamma(t)}(\dot\gamma(t), (\nabla_{\dot\gamma(t)}\xi_M)(\gamma(t))) = -\mathbf{g}_{\gamma(t)}(\dot\gamma(t), (\nabla_{\dot\gamma(t)}\xi_M)(\gamma(t))) ,$$

because ξ_M is a Killing vector field. Therefore, $\frac{d}{dt}\mathbf{g}_{\gamma(t)}(\dot\gamma(t), \xi_M(\gamma(t)) = 0$ for all t and if $\mathbf{g}_{\gamma(0)}(\dot\gamma(0), \xi_M(\gamma(0)) = 0$, γ remains orthogonal to the orbits. □

By the formula for the first variation of the energy (see, e.g., [28]), a geodesic connecting $p \in M$ with any submanifold $L \subseteq M$ and having minimal length is orthogonal to L. Then from Proposition 1.2.31 we get the following corollary.

Corollary 1.2.32. *Let $\gamma : [0,1] \to M$ be a geodesic realizing the distance $d(\gamma(0), G(\gamma(1)))$. Then γ is a horizontal geodesic.*

One last fact we are going to recall is the following result by Kleiner, see [13] or Kleiner's PhD thesis [64]. Let $p, q \in M$ and let $\gamma : [0, l] \to M$ be a geodesic with $\gamma(0) \in G(p)$ and $\gamma(l) \in G(q)$. We say that γ is a *Kleiner geodesic* from $G(p)$ to $G(q)$ if its length $L(\gamma)$ equals the distance $d\,(G(p), G(q))$; it is obvious that a Kleiner geodesic is minimizing, hence horizontal. Kleiner's result tells us the following.

Lemma 1.2.33. *Given any two points $p, q \in M$, there exists a Kleiner geodesic $\gamma : [0, l] \to M$ from $G(p)$ to $G(q)$. Moreover, for any such Kleiner geodesic we have $G_{\gamma(t)} = G_\gamma$ for any $t \in (0, l)$, where*

$$G_\gamma := \{g \in G \ :\ g\gamma(s) = \gamma(s) \quad \textit{for all} \ \ s \in [0, l]\}$$

Proof. Let $x_n \in G(p)$ and $y_n \in G(q)$ be such that

$$\lim_{n\to+\infty} d(x_n, y_n) = d(G(p), G(q)) = l.$$

By the definition of orbital distance, we may assume that $x_n = p$ and so y_n lies in the same connected component of p. By the Hopf-Rinow Theorem there exists a minimizing geodesic γ_n, parameterized by arc length, joining p and y_n. The geodesic $\gamma_n : [0, t_n] \longrightarrow M$ is given by $\gamma_n(t) = \exp_p(t\xi_n)$, where ξ_n has unit norm and $\gamma(t_n) = y_n$. In particular, $L(\gamma_n) = t_n$ and $\lim_{n\to+\infty} t_n = d(G(p), G(q)) = l$. Up to passing to a subsequence, we may assume that the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ converges to some unit vector $v \in T_pM$. Now, keeping in mind that $G(q)$ is closed, it is easy to check that the geodesic $\gamma(t) := \exp_p(tv)$ is a minimizing geodesic joining p and $\gamma(l) \in G(q)$ and satisfying $L(\gamma) = d(G(p), G(q))$.

Let $t \in (0, l)$ and let $g \in G_{\gamma(t)}$. The curve

$$\tilde\gamma(s) = \begin{cases} g\gamma(s) & 0 \le s \le t \\ \gamma(s) & t < s \le l \end{cases}$$

joins $g(\gamma(0))$ and $\gamma(l)$. Since $L(\tilde\gamma) = L(\gamma) = d(G(p), G(q))$ it follows that $\tilde\gamma$ is a minimizing geodesic, hence it is smooth. This implies $(\mathrm{d}g)_{\gamma(t)}(\dot\gamma(t)) = \dot\gamma(t)$. Since $g(\gamma(t)) = \gamma(t)$, it follows that $g \in G_\gamma := \bigcap_{s\in[0,l]} G_{\gamma(s)}$, concluding the proof. □

1.3 Orbit types and Principal Orbit Theorem

In this section, we will show how the local structure of a G-manifold M has global implications, namely, the *Principal Orbit Theorem* and the *stratification* of M by *orbit types*.

Definition 1.3.1. Let $\Psi : G \circlearrowleft M$ be a proper action. Given a compact subgroup $K \subseteq G$, we denote by (K) the conjugacy class of K in G, i.e., the class of subgroups of G that are conjugate to K. We denote the set of all these conjugacy classes by $\mathcal{I}_G$ and we let $\mathcal{I}_\Psi := \{(G_x) \in \mathcal{I}_G \ : \ x \in M\}$. Given $x \in M$, we call the *orbit type of* x the class $(G_x) \in \mathcal{I}_\Psi$.

We have already observed that if $y \in G(x)$, then $y = g.x$ for some $g \in G$ and $G_y = gG_xg^{-1}$. Therefore, points belonging to the same orbit have the same orbit type. Moreover, a necessary and sufficient condition for two points $x, y \in M$ to have the same orbit type is that there exists a G-equivariant diffeomorphism between $G(x)$ and $G(y)$. To see this, suppose that $(G_x) = (G_y)$, with $G_y = gG_xg^{-1}$. Then there is a bijection $G_x \to G_y$, $h \mapsto ghg^{-1}$, and a natural G-equivariant map $G/G_x \to G/G_y$. But we have already seen that $G(x)$ is G-equivariantly diffeomorphic to G/G_x and similarly $G(y)$ to G/G_y. The converse statement is also easy to check.

Given $(K) \in \mathcal{I}_\Psi$, we denote by $M_{(K)}$ the set of points with orbit type (K). From the above it follows that $M_{(K)}$ is a G-invariant subset of M.

Remark 1.3.2. Let $\Psi : G \circlearrowleft M$ and let $I := \bigcap_{x \in M} G_x$ be the ineffective kernel. We have already seen that the quotient action $\tilde{\Psi} : (G/I) \circlearrowleft M$ is an effective, proper action. It follows that for all $x, y \in M$, $G(x)$ has the same orbit type as $G(y)$ if and only if $(G/I)(x)$ has the same orbit type as $(G/I)(y)$.

The set $\mathcal{I}_G$ can be partially ordered by the following relation. Let $c, c' \in \mathcal{I}_G$, then we write $c \preceq c'$ if and only if there exists $K \in c$, $K' \in c'$ such that $K' \subseteq K$. To see that $\preceq$ defines a partial ordering, we have to prove that it is reflexive, transitive, and antisymmetric. Reflexiveness and transitivity are clear. To see that also the antisymmetry is satisfied, we observe that given a compact subgroup $K \subseteq G$, we have that if $gKg^{-1} \subseteq K$ for some element $g \in G$, then $gKg^{-1} = K$. This is because conjugation by an element defines an injective map from the set C of connected components of K to itself. Since C is finite, because of the compactness of K, this map is also surjective. Then if $c \preceq c'$ and $c' \preceq c$, there are compact subgroups $H \in c$, $H' \in c'$ with $H' \subseteq H$ and $gHg^{-1} \subseteq H'$ for some $g \in G$. Hence, $H = H'$.

Definition 1.3.3. We say that an element c of $\mathcal{I}_G$ is *maximal* if $c \preceq c'$ for all $c' \in \mathcal{I}_G$. If $c \in \mathcal{I}_\Psi$ is maximal in $\mathcal{I}_\Psi$, we call c a *maximal orbit type* or a *principal orbit type*.

The sets $\mathcal{I}_G$ and $\mathcal{I}_\Psi$ have the property that every sequence $\cdots \preceq c_{n-1} \preceq c_n \preceq c_{n+1} \preceq \cdots$, where $c_n \in \mathcal{I}_G$ or $c_n \in \mathcal{I}_\Psi$ for all $n \in \mathbb{N}$, is stationary. This means that there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$, $c_n = c_{n+1}$. One can show that since the c_n are conjugacy classes of compact subgroups, it follows that every such sequence converges to a stationary value. Therefore, both $\mathcal{I}_G$ and $\mathcal{I}_\Psi$ always admit a maximal element. In fact:

Proposition 1.3.4. *Let $\Psi : G \circlearrowleft M$ be a proper action. The map $M \to \mathcal{I}_\Psi$, $x \mapsto (G_x)$ is lower semicontinuous, in the sense that for all $x \in M$, there is a G-invariant open neighborhood U of x such that for all $y \in U$ we have $(G_x) \preceq (G_y)$.*

Proof. Let S_x be a slice at x and let $U := GS_x$. By the property of a slice, we have that for any $z \in S_x$, $G_z \subseteq G_x$. Take $y \in U$, then there exists $g \in G$ such $g.y \in S_x$. Therefore, $G_{g.y} = gG_yg^{-1} \subseteq G_x$ and hence $(G_x) \preceq (G_y)$. □

We can now prove the Principal Orbit Theorem.

Theorem 1.3.5. *(Principal Orbit Theorem). Let $\Psi : G \circlearrowleft M$ be a proper action and suppose M/G is connected. Then $\mathcal{I}_\Psi$ admits a unique maximal element c. Moreover, M_c is open and dense in M and M_c/G is connected.*

Definition 1.3.6. We call the points of M_c *principal points* and their orbits *principal orbits*. Moreover, in the following we will use the notation $M_{\text{princ}} := M_c$.

Proof. We recall that $\mathcal{I}_\Psi$ admits a maximal element c and by Proposition 1.3.4, M_c is open. We prove the theorem by induction on the dimension of M. If $\dim(M) = 0$ there is nothing to prove.

Suppose the theorem is valid for $\dim(M) = n$ and let c be a maximal element of $\mathcal{I}_\Psi$. Let $U \subseteq M_c$ be a G-invariant open and closed subset of M_c. We claim that U is dense in M. To see this, let $x \in U$ and consider a tubular neighborhood $\mathfrak{T}(G(x)) := G \times_{G_x} S_x$ around $G(x)$, where S_x is a slice at x. We can also use the linearization of $\mathfrak{T} := \mathfrak{T}(G(x))$ given by $T := G \times_{G_x} V \simeq \mathfrak{T}$, where $V = (T_x(G(x))^\perp$, and the orthogonal is taken with respect to some G-invariant metric on M. Under this identification, $x \in M$ corresponds to $[(e, 0)] \in T$. Note that it is sufficient to show that $U \cap \mathfrak{T}$ is dense for any $x \in U$.

There are two cases. The first case is when G_x acts trivially on V. Then $\mathfrak{T} \simeq (G/G_x) \times V$, and hence $M_c/G = \mathfrak{T}/G \simeq V$, which is connected. Moreover, U/G is open and closed in $\mathfrak{T}/G$ and hence $U/G = \mathfrak{T}$. Since U is G-invariant, we get that $U = \mathfrak{T}$.

If the action of G_x on V is non-trivial, we consider the unit sphere $S := S(V)$ of V. Note that S/G_x is always connected. In fact, if $\dim(S) = 0$, the sphere is not connected but G_x acts non-trivially, hence S/G_x is a single point; if $\dim(S) \geq 1$ the sphere is connected, and so is S/G_x. Let $\mathfrak{S} \simeq G \times_{G_x} S$ be the image of $G \times_{G_x} S \subset G \times_{G_x} V = T$ under the G-equivariant diffeomorphism $T \simeq \mathfrak{T}$. Then $\mathfrak{S}$ is a G-invariant, codimension-1 submanifold of $\mathfrak{T}$ and such that $\mathfrak{S}/G \simeq S/G_x$ is connected. By the induction hypothesis there exists a unique maximal orbit type b for the subaction $G \circlearrowleft \mathfrak{S}$, $\mathfrak{S}_b$ is open and dense in $\mathfrak{S}_b$ and $\mathfrak{S}_b/G$ is connected. Then consider the homothetic action of $\mathbb{R}^+$ on T given by $(\lambda, [(g, v)]) \mapsto [(g, \lambda v)]$, and the action that it induces under the identification $T \simeq \mathfrak{T}$, $\mathbb{R}^+ \circlearrowleft \mathfrak{T}$. Note that by Kleiner's Lemma, points in the same $\mathbb{R}^+$-orbits have the same G-orbit type for the action Ψ. Therefore, $\mathbb{R}^+(\mathfrak{S}) \subseteq M_b$ is open and dense in $\mathfrak{T}$. Since M_c is open, $M_c \cap M_b \neq \emptyset$, and hence, $c = b$. In particular, M_c is dense in $\mathfrak{T}$.

Now, $(\mathbb{R}^+\mathfrak{S})/G \simeq \mathbb{R}^+ \times (\mathfrak{S}/G)$ is connected and is also open and dense in M_c/G. Therefore, also M_c/G is connected and U/G is open and closed. By G-invariance, $U = M_c$ and again, M_c is dense in $\mathfrak{T}$.

From the above, M_c is dense in M. To show that M_c/G is connected, let $\tilde{U}$ be any open and closed subset of M_c/G. Then, if $\pi : M \to M/G$ is the canonical projection, $U := \pi^{-1}(\tilde{U})$ is open and dense in M. By what we have just seen, U is also dense in M and therefore, $U = M_c$. Hence, M_c/G is connected. By Proposition 1.3.4, since M_c is dense, c is the unique maximal element of $\mathcal{I}_\Psi$. $\square$

Definition 1.3.7. Let $G \circlearrowleft M$ be a proper action. We define the *cohomogeneity* of the action as the dimension of the principal orbits, and we denote it as $\mathrm{cohom}(M, G)$.

We observe that from what we have seen so far, M_{princ} is an open submanifold of M and $\pi : M_{\text{princ}} \to M_{\text{princ}}/G$ is a fiber bundle with characteristic fiber G/G_x, where $x \in M_{\text{princ}}$ is any principal point. Note that M_{princ}/G is a manifold of dimension $\mathrm{cohom}(M, G)$.

1.3.1 Properties of Principal Orbits

We now give an alternative definition of principal orbit and we will see some characterization.

Let $\Psi : G \times M \to M$ be a proper action, which, without loss of generality, we suppose to be isometric with respect to some Riemannian metric. It follows that, given $x \in M$, we can define the normal slice representation of G_x on the orthogonal $(T_xG(x))^\perp$ of $T_xG(x)$ taken with respect to the metric.

Definition 1.3.8. We call an orbit $G(x)$, $x \in M$, a *principal orbit* and x a *principal point* if there exists an open neighborhood $U \subseteq M$ of x such that for all $y \in U$, we have $G_x \subseteq G_{\Psi(g,y)}$ for some $g \in G$.

It is clear that if $x \in M$ is principal in this sense, it is also principal in the sense of Definition 1.3.6 and vice versa. We prefer Definition 1.3.8 because it is more convenient for proving the following characterization.

Proposition 1.3.9. *Let $\Psi : G \times M \to M$ be a proper action and $x \in M$. Then the following are equivalent.*

i) $G(x)$ is a principal orbit;

ii) for any slice S_x at x, we have $G_x = G_y$ for all $y \in S_x$;

iii) the slice representation of G_x is trivial.

Proof. Suppose $G(x)$ is a principal orbit, then take U a neighborhood of x and $y \in U \cap S_x$ such that for some $g \in G$, we have $G_x \subseteq G_{\Psi(g,y)}$. Then, by definition of slice, $G_y \subseteq G_x$. Therefore, since $G_{\Psi(g,y)} = gG_yg^{-1}$, we have

$$g^{-1}G_xg \subseteq G_y \subseteq G_x \ .$$

Since G_x is compact (because the action is proper), we conclude that $g^{-1}G_xg = G_y = G_x$. For the converse, take $z \in \mathfrak{T}(G(x))$. Then, $G(z)$ intersects S_x in at least one point y. Then $z = \Psi(g, y)$ for some g and hence $G_y = g^{-1}G_zg$. By hypothesis $G_x = G_y = g^{-1}G_zg = G_{\Psi(g^{-1},z)}$.

Suppose again that $G(x)$ is principal, then there is an $\varepsilon > 0$ and $\gamma : [0, \varepsilon] \to M$ with $\dot\gamma(0) = v \in \nu_x(G(x))$, such that $\gamma([0, \varepsilon]) \subseteq S_x$. We have that $\Psi_g \circ \gamma = \gamma$ for all $g \in G_x$, because $G_x = G_{\gamma(t)}$. Then by differentiating at $t = 0$, we have

$$(\mathrm{d}\Psi_g)_x(v) = \frac{d}{dt}\Psi_g(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\gamma(t)\Big|_{t=0} = v \ .$$

Hence $(\mathrm{d}\Psi_g)_x = \mathrm{id}_{\nu_x(G(x))}$ for all $g \in G_x$. Vice-versa if $(\mathrm{d}\Psi_g)_x = \mathrm{id}_{\nu_x(G(x))}$ for all $g \in G_x$, take $y \in S_x$. Since S_x is a normal slice, we can write $y = \exp_x(v)$ for some $v \in \nu_x(G(x))$. Then, since Ψ_g is an isometry, we have that for all $g \in G_x$

$$\Psi_g(y) = \Psi_g(\exp_x(v)) = \exp_x((\mathrm{d}\Psi_g)_x(v)) = \exp_x(v) = y \ .$$

Therefore $g \in G_y$ and $G_x = G_y$. $\square$

It follows from Theorem 1.2.22 and Proposition 1.3.9 that if $x \in M_{\text{princ}}$, we have

$$\dim(M/G) = \dim(\mathfrak{T}(G(x))/G) = \dim(G \times_{G_x} S_x) =$$

$$= \dim(G/G_x \times S_x) = \dim((T_xG(x))^\perp) = \mathrm{cohom}(M, G) \ .$$

The following proposition describes some of the geometric properties of the principal orbits.

Proposition 1.3.10. *Let $\Psi : G \times M \to M$ be a proper and isometric action on a complete Riemannian manifold and let $x \in M$. If $G(x)$ is a principal orbit, the following facts hold true:*

i) For $n \in \nu_x(G(x))$, $\hat{n}_{\Psi(g,x)} := (d\Psi_g)_x(n)$ is a well defined orthogonal vector field along $G(x)$, called equivariant normal vector field;

ii) If $\mathcal{S}_N$ is the shape operator of $G(x)$ relative to a normal vector field N to $G(x)$, we have

$$\mathcal{S}_{\hat{n}(\Psi(g,x))} = (d\Psi_g)_x \mathcal{S}_{\hat{n}(x)} (d\Psi_g)_x^{-1} ;$$

iii) The principal curvature of the shape operator of $G(x)$ relative to an equivariant normal vector field to $G(x)$ is constant;

iv) The subset $\{\exp_y(\hat{n}(y)) \ : \ y \in G(x)\}$ is an orbit of Ψ.

Proof. For i) consider points $\Psi_g(x) = \Psi_h(x)$, so that $h^{-1}g \in G_x$. Then, since $G(x)$ is principal, we have that $(\mathrm{d}\Psi_{h^{-1}g})_x(n) = n$, therefore, $\hat{n}_{\Psi(g,x)} = (\mathrm{d}\Psi_g)_x(n) = (\mathrm{d}\Psi_h)_x(n) = \hat{n}_{\Psi(h,x)}$. For ii) we need to recall that $\mathcal{S}$ is the symmetric endomorphism associated to the second fundamental form, i.e. for $v, w \in T_x(S(x))$, n as before and X, Y vector fields tangent to $G(x)$ extending respectively v, w, $\langle\cdot,\cdot\rangle$ Riemannian metric of M and ∇ the Levi-Civita connection for such a metric,

$$\langle \mathcal{S}_n(v), w\rangle_x = -\langle \nabla_X \hat{n}, w\rangle_x.$$

In our case we have

$$\langle (\mathrm{d}\Psi_g)_x^{-1} \mathcal{S}_{\hat{n}(\Psi(g,x))} (\mathrm{d}\Psi_g)_x v, w\rangle = -\langle \Psi_g^* \nabla_{(\Psi_g)_* X} \hat{n}, w\rangle_x =$$

$$= -\langle \Psi_g^* (\Psi_g)_* \nabla_X \Psi_g^* \hat{n}, w\rangle_x = -\langle \nabla_X \hat{n}, w\rangle_x = \langle \mathcal{S}_n(v), w\rangle_x \ .$$

For iii) consider a principal curvature direction v, i.e. $\mathcal{S}_n(v) = \lambda v$. Then,

$$\lambda v = \left((\mathrm{d}\Psi_g)_x^{-1} \mathcal{S}_{\hat{n}(\Psi(g,x))} (\mathrm{d}\Psi_g)_x \right)(v) \ ,$$

in other words $\left(\mathcal{S}_{\hat{n}(\Psi(g,x))}\right)((\mathrm{d}\Psi_g)_x(v)) = \lambda(\mathrm{d}\Psi_g)_x(v)$. Finally, to prove *iv*), note that $\exp_{\Psi(g,x)}(\hat{n}(\Psi(g,x)) = \exp_{\Psi(g,x)}((\mathrm{d}\Psi_g)_x n) = \Psi_g(\exp_x(n))$. □

Remark 1.3.11. Item *iv*) of Proposition 1.3.10 can be seen as a consequence of a more general phenomenon, called *equifocality*, which occurs also for singular Riemannian foliations (see [13, 100]). In fact, the meaning of *iv*) is that we can recover all the orbits of a G-action by following the endpoint map of the equivariant normal vector field $\hat{n}$. This amounts to saying that the orbits are found as parallel submanifolds to a principal orbit.

1.3.2 Other Orbit Types and Stratification

We recall the following definitions.

Definition 1.3.12. Let $G \circlearrowleft M$ be a proper action and $x \in M$. Let M_{princ} be the collection of principal points. We call the orbit $G(x)$

i) *regular* if $\dim(G(x)) = \dim(G(y))$ for some $y \in M_{\mathrm{princ}}$;

ii) *singular* if $\dim(G(x)) < \dim(G(y))$ for some $y \in M_{\mathrm{princ}}$;

iii) *exceptional* if $G(x)$ is regular but $x \notin M_{\mathrm{princ}}$.

We also denote as M_{reg} the collection of points of M whose orbit is regular.

Exceptional orbits have the same dimension as the principal ones, but the isotropies of the normal orbits have a bigger number of connected components than the principal ones. It follows that principal orbits are finite cover of the exceptional ones.

Example 1.3.13. Let $SO(n)$ act on $\mathbb{S}^n$ as the group of rotations around a fixed axis $p \in \mathbb{S}^n$. Since this action commutes with the antipodal map $\mathbb{S}^n \ni q \mapsto -q \in \mathbb{S}^n$, for $q \in \mathbb{S}^n$, there is also an induced action $SO(n) \circlearrowleft \mathbb{R}P^n$. This action has a unique singular orbit given by $[p] \in \mathbb{R}P^n$ which is a fixed point of the action. The principal isotropy is isomorphic to $SO(n-1)$. Consider the intersection $\mathbb{S}^n \cap p^\perp$, of $\mathbb{S}^n$ with the hyperplane orthogonal to the line through p, then $[\mathbb{S}^n \cap p^\perp] \subset \mathbb{R}P^n$ is the unique exceptional orbit of $SO(n) \circlearrowleft \mathbb{R}P^n$, with isotropy isomorphic to $S(O(n-1) \times \mathbb{Z}_2)$.

Remark 1.3.14. Let us list some useful remarks.

i) Let $x \in M$ and $\mathfrak{T} := \mathfrak{T}(G(x))$ be the tube around $G(x)$. It follows from Remark 1.2.23 that there is a bijection between the set of orbit types of the G-action on $\mathfrak{T}$ and the set of orbit types of the G_x-action on S_x. The bijection is given by the natural map
$$(G_y)_G \mapsto (G_z)_{G_x} \ ,$$
where $(G_y)_G$ is the conjugacy class of G_y in G and $(G_z)_{G_x}$ is the conjugacy class of G_z in G_x with z being any point in the intersection $G(y) \cap S_x$. In particular, principal orbits of the G-action on $\mathfrak{T}$ correspond to the principal orbits of the G_x-action on S_x.

ii) Since the tube $\mathfrak{T} = \mathfrak{T}(G(x))$ is G-equivariantly diffeomorphic to the linear tube $G \times_{G_x} (T_x G(x))^\perp$ (see [98]), the set of orbit types $\mathcal{I}_\Psi$ of the restricted action $\Psi : G \circlearrowleft \mathfrak{T}$ is in bijection with the set of orbit types $\mathcal{I}_\psi$ of the linearized action of G_x, $\psi : G_x \circlearrowleft V$. Since ψ is a linear action, it follows that given $b \in \mathcal{I}_\Psi$, we have
$$\{y \in \mathfrak{T}(G(x)) \setminus G(x) \ : \ (G_y) = b\} = G(C_b) \ ,$$
where $C_b \subseteq V$ is a cone without the vertex.

iii) Always from Remark 1.2.23 and from the discussion above, we can easily see that for all $x \in M$,
$$\mathrm{cohom}(M, G) = \mathrm{cohom}((T_x G(x))^\perp, G_x) \ .$$
In particular if $x \in M_{\mathrm{princ}}$, we retrieve $\mathrm{cohom}(M, G) = \dim((T_x G(x))^\perp) = \dim(M) - \dim(G/G_x) = \dim(M/G)$.

The bijection between the orbit types of the G-action on $\mathfrak{T}(G(x))$ and those of the slice representation of G_x on $(T_x G(x))^\perp$ leads us to the following.

Theorem 1.3.15. $\mathfrak{T}(G(x))$ *contains only finitely many different orbit types. Therefore, M contains at most countably many orbit types. In particular, if M is compact, the number of orbit types is finite.*

Proof. The problem reduces to proving that a linear orthogonal action by a compact subgroup K of $O(n)$ acting on $\mathbb{R}^n$ has a finite number of orbit types. By induction, if $n = 1$, then either $K = \{\mathrm{Id}_{\mathbb{R}^n}\}$ or $K = \mathbb{Z}_2$. In both cases, the claim is clearly true. Suppose that the claim is valid for $n > 1$. The group K acts on the unit sphere $\mathbb{S}^n$

of $\mathbb{R}^{n+1}$. By the inductive hypothesis applied to the slice representation at any point $p \in \mathbb{S}^n$ and item i) of Remark 1.3.14, we conclude that the K-action on $\mathbb{S}^n$ has a finite number of orbit types. We conclude by item ii) of Remark 1.3.14. The remaining claims of the theorem easily follow since M is second countable. □

We now prove a structural result for singular orbits.

Theorem 1.3.16. *Let $s \in \mathcal{I}_\Psi$ be a singular orbit type for a proper action $\Psi : G \circlearrowleft M$. Then the following hold*

i) any connected component F of M_s is a submanifold of M;

ii) if H is the ineffective kernel of $\Psi|_{G\times F}$, then

$$\mathit{cohom}(F, G/H) < \mathit{cohom}(M, G)\ .$$

Moreover, $M_s = (M_s)_{\text{princ}}$ and $\pi : M_s \to M_s/G$ is a fibration.

Proof. We start by proving item *i*). Let $x \in M_s$, then by the Tubular Neighborhood Theorem, there exists a G-invariant neighborhood $\mathfrak{T}$ of $G(x)$, G-equivariantly diffeomorphic to $G \times_K V$, where $V = \nu_x G(x)$ and $K := G_x$, with $(G_x) = s$. Then, by the discussion above, $\mathfrak{T} \cap M_s \simeq G \times_K (V)^K = G/H \times V^K$, and any connected component of M_s is then an embedded submanifold of M. Moreover, if H is the ineffective kernel of the G-action on F, we see that $\operatorname{cohom}(F, K/H) = \dim(V^K)$.

To prove item *ii*), notice that since s is a singular isotropy, if $x \in M_s$, we have $\nu_x G(x) = V = V^K \oplus W$, where $W \neq \{0\}$. Then

$$\begin{aligned}&\operatorname{cohom}(M, G) = \operatorname{cohom}(V, K) \geq \operatorname{cohom}(V^K, K) + \operatorname{cohom}(W, K) > \\ &> \operatorname{cohom}(V^K, K) = \dim(V^K) = \operatorname{cohom}(F, K/H)\ .\end{aligned}$$

Finally, we observe that $G \circlearrowleft M_s$ has a unique orbit type and hence $M_s = (M_s)_{\text{princ}}$. □

The above proposition also tells us that any connected component F of M_s determines a fiber bundle $G/K \hookrightarrow F \twoheadrightarrow F/G$ with structural group $N(K)/K$, where $K = G_x$, for $x \in F$. In particular, F/G is a submanifold of dimension $\dim(V^K)$ inside M/G. F is usually called the stratum of a *local orbit type*, corresponding to s. In fact, by definition, $x, y \in M$ have the same local orbit type if and only if there exists a G-equivariant diffeomorphism between two tubular neighborhoods of x and y. Clearly, if x and y have the same local orbit type, x has the same orbit type as y. Therefore, the connected components of the same dimension of M_s are organized into local orbit types; for more details see [13, 29].

Remark 1.3.17. Let L be a principal orbit. Another consequence of equifocality (see Remark 1.3.11) is the fact that L-Jacobi fields along a horizontal geodesic γ span the tangent space of the orbits $L_{\gamma(t)}$ met by γ at every time t (see [100]). In particular, if $\gamma(t)$ is a singular point, then it is also a focal point for L along γ (see Appendix). Actions (or singular Riemannian foliations) for which the converse of this holds are called *variationally complete* (see Definition 4.3.9).

We present without proof the classical stratification result for the orbit space of a proper action (see [13, 29, 103]). We begin by recalling the following definition.

Definition 1.3.18. Let M be a manifold. A *stratification* of M is a collection of embedded submanifolds $\{M_i\}_{i\in I}$, where each $M_i \subseteq M$ is called a *strata*, such that the following conditions hold

i) the partition $\{M_i\}_{i\in I}$ is locally finite, i.e. for all M_i, $M_i \cap M_j \neq \emptyset$ only for j in a finite subset of I;

ii) for all $i \in I$, there exists $I_i \subseteq I \setminus \{i\}$ such that $\overline{M}_i = M_i \cup_{j\in I_i} M_j$;

iii) for all $j \in I_i$, we have $\dim(M_j) < \dim(M_i)$.

Proposition 1.3.19. *Let $\Psi : G \circlearrowleft M$ be a proper action. Then $\{F_c^1, \dots, F_c^{N(c)}\}_{c\in\mathcal{I}_\Psi}$, where by $F_c^1, \dots, F_c^{N(c)}$ we denote the connected components of M_c is a stratification of M.*

We note that the local finiteness condition is satisfied because of the same arguments as in the proof of Theorem 1.3.15.

Remark 1.3.20. By [29], the above stratification is also a Whitney stratification. Moreover, we point out that $\{F_c^1, \dots, F_c^{N(c)}\}_{c\in\mathcal{I}_\Psi}$ induces a stratification of M/G given by $\{F_c^1/G, \dots, F_c^{N(c)}/G\}_{c\in\mathcal{I}_\Psi}$.

1.4 Polar Actions

Let $G \circlearrowleft M$ be a proper action which is isometric with respect to some complete Riemannian metric $\mathbf{g}$ on M. We call a connected, closed, injectively immersed submanifold $\Sigma \subseteq M$ a *section* for the G-action if it meets all the G-orbits orthogonally.

We immediately see that if Σ is a section for a G-action on a manifold M, then $g\Sigma$ is also a section for all $g \in G$. Moreover, we have the following.

Proposition 1.4.1. *Let Σ be a section for $G \circlearrowleft M$ as above. Then*

i) cohom$(M, G) = \dim(\Sigma)$;

ii) $M_{reg} \cap \Sigma$ is dense in Σ;

iii) Σ is totally geodesic;

iv) if $x \in \Sigma$ and S_x is a slice at x, there is a neighborhood U of x such that $\Sigma \cap U \subseteq S_x \cap U$.

Proof. Let $x \in M_{\text{princ}} \cap \Sigma \neq \emptyset$. Then item *i)* easily follows. Also item *ii)* is a clear consequence of the density of M^{reg} and the fact that $M^{\text{reg}} \cap \Sigma \neq \emptyset$. A curve $\gamma : [0, 1] \to \Sigma$ is a geodesic for both M and Σ equipped with the induced metric if and only if the normal part of the acceleration $\nabla^{\perp}_{\dot{\gamma}_t} \dot{\gamma}_t$ vanishes for all t. Let $\{\xi^i\} \in \mathfrak{g}$ be a basis for $\mathfrak{g}$, then $\{\xi^i_M(\gamma(t))\}$ is a set of generators of $\nu_{\gamma(t)}\Sigma$ for all $t \in [0, 1]$. Then for all i, we have

$$\mathbf{g}(\nabla^{\perp}_{\dot{\gamma}}\dot{\gamma}, \xi^i_M) = \mathbf{g}(\nabla_{\dot{\gamma}}\dot{\gamma}, \xi^i_M) = -\mathbf{g}(\nabla_{\xi^i_M}\dot{\gamma}, \dot{\gamma}) = -\frac{1}{2}\xi^i_M(\mathbf{g}(\dot{\gamma}, \dot{\gamma})) = 0 \ ,$$

where we used that ξ^i_M are Killing fields for the metric. Item *iv)* follows from the construction of the slice at x and from the fact that by item *iii)*, Σ is invariant under the exponential map at x. □

The following proposition tells us what is the candidate section for an isometric action.

Proposition 1.4.2. *Let $G \circlearrowleft M$ be as above, let $x \in M$ and define $\Sigma_x := \exp_x((T_xG(x))^{\perp})$. We have* *i)* Σ_x *meets all the* G*-orbits;*

ii) if Σ *is a section through* $x \in M_{\mathrm{princ}}$ *then* $\Sigma = \Sigma_x$ *is the unique section through* x*.*

Proof. (Idea). The complete, detailed proof of this proposition can be found in [35]. For item i): by completeness, given $q \in M$ there is a minimizing geodesic joining x to $G(q)$. Such a geodesic is perpendicular to $G(q)$ by the first variation formula of the Energy. Then such geodesic must be perpendicular also in x. For item ii) we can apply Proposition 1.4.1, item iv) and rescale the radius of the Riemannian ball defining the slice. $\square$

Example 1.4.3. Classical examples of actions admitting sections are provided by symmetric spaces. We refer to the comprehensive treatment [58] for the details and the proofs. A very effective treatment is also given in [17]. Let $M = G/K$ be a symmetric space, where K is a compact Lie subgroup of G. There exists an Ad_K-invariant splitting of the Lie algebra $\mathfrak{g}$ of G

$$\mathfrak{g} = \mathfrak{k} \oplus p \ ,$$

the so called *Cartan decomposition* of $\mathfrak{g}$, and an involution, $\iota : \mathfrak{g} \to \mathfrak{g}$, such that $\iota|_{\mathfrak{k}} = \mathrm{id}_{\mathfrak{k}}$ and $\iota|_p = -\mathrm{id}_p$. One can prove that the following fundamental relations hold.

$$\begin{cases} [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \\ [\mathfrak{k}, p] \subseteq p \\ [p, p] \subseteq \mathfrak{k} \ . \end{cases} \tag{1.1}$$

The manifold M supports a natural G-action given by the left translations $(g, g'K) \mapsto gg'K$. There is also a clear identification $p \simeq T_{eK}(G/K)$, where e is the identity of G, under which the isotropy representation of K on $T_{eK}M$ corresponds to the adjoint action of K on p. Then every G-invariant Riemannian metric on M uniquely determines an Ad_K-invariant metric on p, by restriction. It follows that the tangent space to the K-orbit through an element $A \in p$ is given by $\{[\xi, A] \ : \ \xi \in \mathfrak{k}\}$. On the other hand, there is an orthogonal decomposition

$$p = [\mathfrak{k}, A] \oplus p_A \ , \tag{1.2}$$

where $p_A := \{B \in p \ : \ [A, B] = 0\}$. To see this, we prove $p_A = [\mathfrak{g}, A]^\perp$, where $\perp$ denotes the orthogonal complement in p. Let $\langle \cdot, \cdot \rangle$ denote the Ad_K-invariant metric on G. Take $B \in p_A$ then for all $[\xi, A] \in [\mathfrak{k}, A]$,

$$\langle B, [\xi, A] \rangle = \langle [A, B], \xi \rangle = 0 \ .$$

So $p_A \subseteq [\mathfrak{k}, A]^\perp$. To prove the other inclusion, $p_A \supseteq [\mathfrak{k}, A]^\perp$, let $B \in [\mathfrak{k}, A]^\perp$, for all $\xi \in \mathfrak{k}$ we have

$$0 = \langle B, [\xi, A] \rangle = \langle [A, B], \xi \rangle \ ,$$

This means that $[A, B]$, which by (1.1) is a vector in $\mathfrak{k}$, is the zero vector. Hence $B \in p_A$.

It is possible to prove that totally geodesic submanifolds of M are in bijective correspondence with the so called Lie triple systems, i.e. $\mathfrak{s} \subseteq p$ such that $[[\mathfrak{s}, \mathfrak{s}], \mathfrak{s}] \subseteq \mathfrak{s}$. This correspondence is the obvious one: given S a totally geodesic submanifold of M, set $S \mapsto \mathfrak{s} := T_{eK}S \subseteq T_{eK}M \simeq p$. Recalling that the curvature tensor of $(M, \langle \cdot, \cdot \rangle)$ at eK is given by the following nice formula

$$\mathcal{R}_{eK}(X, Y)Z = -[[X, Y], Z], \quad X, Y, Z \in p \ ,$$

one can also see that a totally geodesic submanifold S is flat if and only if the corresponding triple system $\mathfrak{s}$ is an Abelian Lie algebra. One can show that the following definition is well posed.

Definition 1.4.4. The *rank* of the symmetric space G/K is the maximal dimension of a totally geodesic submanifold, or, equivalently, the dimension of a maximal Abelian subalgebra of p. Such an algebra is called the *Cartan subalgebra.*

Moreover, if $\mathfrak{a}, \mathfrak{a}'$ are two different Cartan subalgebras, it is possible to prove that there exists $k \in K$ such that $\mathrm{Ad}_k \mathfrak{a} = \mathfrak{a}'$. Such an element k is also the one mapping S to S', S and S' being the totally geodesic submanifolds corresponding to $\mathfrak{a}$ and $\mathfrak{a}'$, respectively. One can also prove that for each $x \in M = G/K$ there exists a totally geodesic submanifold S containing x and eK. Therefore, one candidate manifold to be a section for the K-action on M is any flat totally geodesic submanifold S of M. We only need to check that the intersection between the K-orbits and S is orthogonal. To do this, it is equivalent to check that the orbits of the adjoint K-action on p are orthogonal to the Cartan subalgebra $\mathfrak{a}$ corresponding to S. Let $A \in \mathfrak{a}$, then by (1.2), we must have $\mathfrak{a} \subseteq p_A$, i.e., $\mathfrak{a}$ must be orthogonal to the adjoint orbit through A.

Example 1.4.5. Let G be a compact connected Lie group. A maximal abelian subgroup of G is called a *maximal torus* of G and its dimension is called the *rank* of G. It turns out that given any $g \in G$, there exists a maximal torus $T \ni g$ and that any two maximal tori are conjugate; in particular, the rank of G is well defined. A maximal torus T is a section for the conjugation action $G \times G \to G$, $(g, h) \mapsto ghg^{-1}$. We can see this as follows. First, T intersects every orbit. Indeed, given $h \in G$, there is a maximal torus $T' \ni h$. But T is conjugate to T' and hence there is $g \in G$ such that $T = gT'g^{-1}$. Then $ghg^{-1} \in T$. Second, the intersection is orthogonal. In fact, let $h \in T$. The tangent space of $G(h)$ at h is given by

$$T_h(\mathrm{Ad}_G(h)) = \{(\mathrm{d}R_h)_e\xi - (\mathrm{d}L_h)_e\xi \ : \ \xi \in \mathfrak{g}\} \ .$$

On the other hand, $T_hT = (\mathrm{d}L_h)_e(\nu)$, with $\nu \in \mathfrak{t} := \mathrm{Lie}(T)$. Then, if $\langle \cdot, \cdot \rangle$ is an Ad_G-invariant metric on G, we have for all $\xi \in \mathfrak{g}$ and $\nu \in \mathfrak{t}$ that

$$\begin{aligned}\langle (\mathrm{d}L_h)_e(\nu), (\mathrm{d}R_h)_e\xi - (\mathrm{d}L_h)_e\xi \rangle_h &= \langle (\mathrm{d}L_h)_e(\nu), (\mathrm{d}R_h)_e\xi \rangle_h - \langle (\mathrm{d}L_h)_e(\nu), (\mathrm{d}L_h)_e\xi \rangle_h = \\ &= \langle \mathrm{Ad}_h\nu, \xi \rangle_e - \langle \nu, \xi \rangle_e = \langle \nu, \xi \rangle_e - \langle \nu, \xi \rangle_e = 0 \ .\end{aligned}$$

Actually, Example 1.4.5 is a particular case of 1.4.3. Indeed, recall that we can always see a compact connected Lie group G as the quotient space $(G \times G)/K$, where $K = \Delta_{G \times G}$. Under the natural map $G \simeq (G \times G)/K$, we can see that G has the structure of a symmetric space of compact type.

From the previous examples, it is clear that actions admitting sections are particularly important. They deserve a definition.

Definition 1.4.6. An isometric action $G \circlearrowleft M$ is called *polar* if it admits a section Σ. In particular, if Σ is flat with respect to the induced metric, the action is called *hyperpolar.*

Note that the actions in examples 1.4.3 and 1.4.5 are actually hyperpolar actions.

Remark 1.4.7. Since a section is connected, it follows that G must act transitively on the connected components of M. Moreover, if G^0 is the identity component of G, we clearly have that the G-action is polar if and only if the G^0-subaction on any connected component of M is polar.

We shall also introduce the following infinitesimal version of Definition 1.4.6, which turns out to be particularly important, since, for example, it is sufficient for M/G to have a local Riemannian orbifold structure [72].

Definition 1.4.8. An isometric action $G \circlearrowleft M$ is called *infinitesimally polar* if all the slice representations are polar.

We shall prove the following well known result.

Proposition 1.4.9. *If an isometric action $G \circlearrowleft M$ is polar, then it is also* infinitesimally polar*.*

For the proof of the above, we will need the following general lemma.

Lemma 1.4.10. *Let $A : G \circlearrowleft M$ be an isometric action, let $x \in M$, let $w \in V := (T_x(G(x)))^\perp$ and let $\xi \in \mathfrak{g}_x$. Then, for all $t \neq 0$ if ξ_V is the infinitesimal generator of the slice representation of G_x on V, we have*

$$(d\exp_x)_{tw}(\xi_V(w)) = \frac{\xi_M(\exp_x(tw))}{t} \ . \tag{1.3}$$

Proof. By a direct computation, given $\xi \in \mathfrak{g}_x$, for all $t \in \mathbb{R}$ we have

$$(\mathrm{d}\exp_x)_{tw}(\xi_V(tw)) = (\mathrm{d}\exp_x)_{tw}\left(\frac{d}{ds}(\mathrm{d}A_{\exp(s\xi)})_x(tw)\right)\Big|_{t=0} =$$

$$= \frac{d}{ds}\exp_x\Big((\mathrm{d}A_{\exp(s\xi)})_x(tw)\Big)\Big|_{t=0} = \frac{d}{ds}\exp_{A_{\exp(s\xi)}(x)}\Big((\mathrm{d}A_{\exp(s\xi)})_x(tw)\Big)\Big|_{t=0} =$$

$$= \frac{d}{ds}A_{\exp(s\xi)}(\exp_x(tw)) = \xi_M(\exp_x(tw)) \ .$$

By linearity, we also obtain

$$(\mathrm{d}\exp_x)_{tw}(\xi_V(tw)) = (\mathrm{d}\exp_x)_{tw}(t\xi_V(w)) = t(\mathrm{d}\exp_x)_{tw}(\xi_V(w)) \ .$$

Then, for $t \neq 0$, we get equation (1.3).

$\square$

Remark 1.4.11. Let $x \in M$ and $w \in (T_xG(x))^\perp$. Then $\gamma(t) := \exp_x(tw)$ is a horizontal geodesic and by Lemma 1.4.10, $\xi_M \circ \gamma$ is a transversal Jacobi field along γ. For a proof, see Corollary 2.5 of [28].

Proof of Proposition 1.4.9. Let $x \in M$, let $V = (T_x(G(x))^\perp$ and let $G_x \circlearrowleft V$ be the slice representation. If $\Sigma \ni x$ is a section through x, we have that $\Sigma = \exp_x(W)$ for $W \subseteq V$ a vector subspace of V. It is clear that W intersects all the G_x-orbits in V. Let $w \in W$ and let $\xi_V(w)$, for $\xi \in \mathfrak{g}_x$, be the infinitesimal generator of the slice representation at x. By equation (1.3) we have

$$\xi_V(w) = \lim_{t\to 0}\frac{\xi_M(\exp_x(tw))}{t} \ .$$

Consider any $v \in W$ and let $v(t)$ be any smooth extension of v along the horizontal geodesic $\gamma(t) := \exp_x(tw)$, such that $v(t) \in T_{\gamma(t)}\Sigma$ for all t and $v(0) = v$. Then, since the action is polar,

$$\mathbf{g}_x(v, \xi_V(w)) = \lim_{t\to 0}\frac{\mathbf{g}_{\gamma(t)}(v(t), \xi_M(\gamma(t)))}{t} = 0 \ .$$

$\square$

1.4.1 Generalized Weyl Group

Let $G \circlearrowleft M$ be an action. We define the *normalizer* of any subset $X \subseteq M$ as the subgroup of G given by $N(X) := \{g \in G \ : \ g(X) \subseteq X\}$ and the *centralizer* of X as $Z(X) := \{g \in G \ : \ gx = x, \ \forall x \in X\}$. Note that $Z(X)$ is normal in $N(X)$.

Definition 1.4.12. Let $G \circlearrowleft M$ be a polar action and let Σ be a section. We define the *generalized Weyl group* of Σ as the group

$$W(\Sigma) := N(\Sigma)/Z(\Sigma) \ .$$

We observe that if $x \in \Sigma$ is a principal point with principal isotropy H, we have $H \subseteq Z(\Sigma)$ and by the very definition of $Z(\Sigma)$, we have $Z(\Sigma) \subseteq H$. Then $Z(\Sigma) = H$. Moreover, by the property of a section we have $N(\Sigma) \subseteq N(H)$, hence $W(\Sigma) \subseteq N(H)/H$.

Proposition 1.4.13. *Let $G \circlearrowleft M$ be a polar action, Σ a section and $W := N(\Sigma)/Z(\Sigma)$ its generalized Weyl group. Then W is a finite discrete group. Moreover, if Σ' is another section of the G-action, with Weyl group W', then W' is isomorphic to W.*

Proof. Let $x \in \Sigma$ be a principal point, then there exists $\varepsilon > 0$ such that $S_x := \Sigma \cap B_\varepsilon(x)$ is a slice through x. Hence, for all $y \in S_x$, and $g \in G$, $gy \in S_x$ if and only if $g \in G_x = Z(\Sigma)$. Therefore, $Z(\Sigma)$ is open in $N(\Sigma)$, hence W is discrete (and finite since $N(\Sigma)$ is compact). If Σ' is another section, then there exists $g \in G$ such that $\Sigma' = g\Sigma$. Then, $N(\Sigma') = N(g\Sigma) = gN(\Sigma)g^{-1}$ and $Z(\Sigma') = Z(g\Sigma) = gZ(\Sigma)g^{-1}$ and hence $W' = N(\Sigma')/Z(\Sigma') \simeq N(\Sigma)/Z(\Sigma) = W$. □

Example 1.4.14. Let $G \circlearrowleft G$ be the conjugation action of a compact connected Lie group G on itself considered in Example 1.4.5. A section is provided by any maximal torus T and hence $Z(T) = T$. It follows that the generalized Weyl group coincides with the usual Weyl group $N(T)/Z(T) = N(T)/T$.

In practice, let $G = U(n)$. Then, by the Spectral Theorem, for every element $u \in U(n)$ there exists a unitary matrix $p \in U(n)$ such that $pup^{-1} = pup^\dagger = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, with $\lambda_1, \dots, \lambda_n \in U(1)$. Then $T = U(1)^n$ is a maximal torus and $N(T)$ is the set of matrices that swap the diagonal elements in T. Then $N(T)/T \simeq \mathfrak{S}_n$ is the permutation group of n elements.

1.5 Symplectic and Hamiltonian actions

Let (M, ω) be a symplectic manifold, i.e. ω is a non-degenerate, closed 2-form on M.

Definition 1.5.1. We say that an action $\Psi : G \circlearrowleft M$ on a symplectic manifold (M, ω) is *symplectic* if for all $g \in G$, the diffeomorphism Ψ_g is a *symplectomorphism*, i.e., for all $g \in G$ we have $\Psi_g^* \omega = \omega$.

We note that if ξ_M, for $\xi \in \mathfrak{g}$, is an infinitesimal generator of a symplectic G-action on (M, ω), then $\mathcal{L}_{\xi_M}\omega = 0$, i.e., ξ_M is a *symplectic vector field* (see Appendix, Section 5.4).

Definition 1.5.2. Let (M, ω) be a symplectic manifold. If $f \in \mathcal{C}^\infty(M)$, we define the *symplectic gradient of* f (also called the *Hamiltonian vector field associated with* f) as the unique vector field X_f on M such that

$$\mathrm{d}f = \mathrm{i}_{X_f}\omega \ .$$

The *Poisson bracket* of two functions $f, g \in \mathcal{C}^\infty(M)$ is defined as the function $\{f, g\} \in \mathcal{C}^\infty(M)$ given by

$$\{f, g\} := \omega(X_f, X_g) \ ,$$

where X_f and X_g are the symplectic gradients of f and g, respectively.

Remark 1.5.3. It is well known that the pair $(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra. Given an action $G \circlearrowleft M$, we can consider the G-invariant functions on M, denoted as $\mathcal{C}^\infty(M)^G$, which thanks to the fact that symplectomorphisms preserve Poisson brackets, is a Lie subalgebra of $(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$ (for a brief account of these facts, see Appendix 5.4).

In the following, we will also need the fundamental definitions below.

Definition 1.5.4. Let $x \in M$ and $V \subseteq T_xM$ a vector subspace. We define the *symplectic orthogonal* of V as the vector subspace

$$V^\S := \{w \in T_xM \ : \ \omega_x(v, w) = 0 \ \forall v \in V\} \subseteq T_xM \ .$$

If $S \subseteq M$ is a submanifold of M, we call S

i) *isotropic* if for all $x \in S$, $T_xS \subseteq (T_xS)^\S$;

ii) *coisotropic* if for all $x \in S$, $T_xS \supseteq (T_xS)^\S$;

iii) *Lagrangian* if for all $x \in S$, $T_xS = (T_xS)^\S$;

iv) *symplectic* if for all $x \in S$, $T_xS \cap (T_xS)^\S = \{0\}$.

Proposition 1.5.5. *Let $G \circlearrowleft (M, \omega)$ be a symplectic action, and let $f \in \mathcal{C}^\infty(M)^G$, then for all $x \in M$, $X_f(x) \in (T_xG(x))^\S$.*

Proof. Let $\xi \in \mathfrak{g}$, then using the hypothesis on f and the definition of symplectic gradient, we have

$$0 = \frac{d}{dt} f(\exp(t\xi).x)\Big|_{t=0} = (\mathrm{d}f)_x(\xi_M(x)) = \omega_x(X_f(x), \xi_M(x)) \ .$$

Since $\mathrm{span}(\{\xi_M(x) \ : \ \xi \in \mathfrak{g}\}) = T_xG(x)$, the proposition follows. □

Hamiltonian Actions

We call a symplectic action Ψ *Hamiltonian* if Ψ admits a *momentum map*. In the following, we denote by $\langle\cdot|\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the canonical duality.

Definition 1.5.6. Given a symplectic action $\Psi : G \circlearrowleft M$ on (M, ω), a momentum map for Ψ is a map $\mu : M \to \mathfrak{g}^*$ that satisfies the following conditions for all $\xi \in \mathfrak{g}$, $x \in M$, $v \in T_xM$, $g \in G$

i) if $\mu^\xi : M \to \mathbb{R}$, $\mu^\xi(x) := \langle\mu(x)|\xi\rangle$, then $(\mathrm{d}\mu^\xi)_x(v) = \omega_x(\xi_M(x), v)$, i.e.

$$\mathrm{i}_{\xi_M}\omega = \mathrm{d}\mu^\xi \ ;$$

ii) μ is a G-equivariant map where the action on $\mathfrak{g}^*$ is the inverse coadjoint action, $\mu(\Psi_g(x)) = (\mathrm{Ad}^*_{g^{-1}})(\mu(x))$, i.e.

$$\mu \circ \Psi_g = \mathrm{Ad}^*_{g^{-1}} \circ \mu \ .$$

Observe that the infinitesimal generator ξ_M of a Hamiltonian action is a Hamiltonian vector field associated with the function μ^ξ.

Remark 1.5.7. If the symplectic form is exact, write $\omega = \mathrm{d}\theta$ for some 1-form θ, and $\mathcal{L}_{\xi_M}\theta = 0$, we can write the momentum map as $\langle \mu(x)|\xi\rangle = -\theta_x(\xi_M(x))$ up to a constant. This is the case of the momentum map in Hamiltonian Mechanics, where $M = T^*Q$, Q is a given manifold, M is equipped with the canonical symplectic form $\omega = \mathrm{d}\theta$, with θ the Liouville 1-form, and the action $\Psi : G \circlearrowleft T^*Q$ is the cotangent lift of an action $\psi : G \circlearrowleft Q$, i.e., $\Psi(g,(q,p)) := (\psi_g(q), (\mathrm{d}\psi_{g^{-1}})^*_q p)$. Recall that the Liouville 1-form is defined as $\theta_{(q,p)}(v) := p((\mathrm{d}\pi)_{(q,p)}(v))$ for all $(q,p) \in T^*Q$, $v \in T_{(q,p)}(T^*Q)$ and $\pi : T^*Q \to Q$ is the canonical projection. Then, for all $g \in G$ we have $\Psi_g^*\theta = \theta$. In fact, if $v \in T_{(q,p)}(T^*Q)$

$$(\Psi_g^*\theta)_{(q,p)}(v) = \theta_{\Psi_g((q,p))}((\mathrm{d}\Psi_g)_{(q,p)}(v)) = \theta_{(\psi_g(q),(\mathrm{d}\psi_{g^{-1}})^*_q p)}((\mathrm{d}\Psi_g)_{(q,p)}(v)) =$$

$$= ((\mathrm{d}\psi_{g^{-1}})^*_q p)((\mathrm{d}\pi)_{((\Psi_g(q),(\mathrm{d}\psi_{g^{-1}})^*_q p)}((\mathrm{d}\Psi_g)_{(q,p)}(v))) = ((\mathrm{d}\psi_{g^{-1}})^*_q p)((\mathrm{d}\psi_g)_q((\mathrm{d}\pi)_{(q,p)}(v))) =$$

$$= p((\mathrm{d}\psi_{g^{-1}})_{\psi_g(q)}(\mathrm{d}\psi_g)_q((\mathrm{d}\pi)_{(q,p)}(v)))) = p((\mathrm{d}\pi)_{(q,p)}(v)) = \theta_{(q,p)}(v) \; .$$

In conclusion, $\mathcal{L}_{\xi_M}\theta = 0$ for all $\xi \in \mathfrak{g}$.

We now prove a useful identity that we shall use in the following.

Proposition 1.5.8. *For all $\xi, \eta \in \mathfrak{g}$ and $x \in M$ we have*

$$\langle \mu(x)|[\xi,\eta]\rangle = \omega_x(\xi_M(x), \eta_M(x)) \; .$$

Proof. Indeed,

$$\langle \mu(x)|[\xi,\eta]\rangle = \frac{d}{dt}\langle \mu(x)|\mathrm{Ad}_{\exp(t\xi)}\eta\rangle\Big|_{t=0} = \frac{d}{dt}\langle \mathrm{Ad}^*_{\exp(t\xi)}\mu(x)|\eta\rangle\Big|_{t=0} =$$

$$= \frac{d}{dt}\langle \mu(\exp(-t\xi).x)|\eta\rangle\Big|_{t=0} = \langle (\mathrm{d}\mu)_x(-\xi_M(x))|\eta\rangle = -\omega_x(\eta_M(x), \xi_M(x)) \; .$$

□

We define for $\xi \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$, $\mathrm{ad}^*_\xi\alpha := \frac{d}{dt}\mathrm{Ad}^*_{\exp(t\xi)}\alpha\Big|_{t=0}$. Hence, $\langle \mathrm{ad}^*_\xi\alpha|\eta\rangle = -\langle\alpha|\mathrm{ad}_\xi\eta\rangle$, for all $\eta \in \mathfrak{g}$. Therefore, we have also proved that

$$\mathrm{ad}^*_\xi(\mu(x)) = (\mathrm{d}\mu)_x(\xi_M(x)) \; .$$

The Lie-Poisson 2-form. Given a Lie group G, the *coadjoint orbit* through $\alpha \in \mathfrak{g}^*$ is the subset $\{\beta \in \mathfrak{g}^* \;:\; \beta = \mathrm{Ad}^*_g(\alpha),\; g \in G\} \subseteq \mathfrak{g}^*$, denoted with $\mathrm{Ad}^*_G(\alpha)$. Provided $\mathrm{Ad}^*_G(\alpha)$ is a smooth manifold and Ad^* a smooth map with differential at the identity ad^*, then

$$\omega_{|\beta}(\xi,\eta) := \beta([v,w]) \; ,$$

where $\beta \in \mathrm{Ad}^*_G(\alpha)$, $\xi, \eta \in T_\beta\mathrm{Ad}^*_G(\alpha)$, and $v, w \in \mathfrak{g}$ are such that $\mathrm{ad}^*_v(\beta) = \xi$, $\mathrm{ad}^*_w(\beta) = \eta$, is a well defined symplectic form on $\mathrm{Ad}^*_G(\alpha)$, i.e. it is a non degenerate and closed 2-form. In order to see this, consider first a different element $\tilde{v} \in \mathfrak{g}$ such that $\mathrm{ad}^*_{\tilde{v}}(\beta) = \xi$. By the properties of the coadjoint representation

$$\beta([\tilde{v},w]) = \beta(\mathrm{ad}_{\tilde{v}}(w)) = (\mathrm{ad}^*_{\tilde{v}}(\beta))(w) = (\mathrm{ad}^*_v(\beta))(w) = \beta([v,w]) \; ,$$

hence $\omega|_\beta$ is well defined in the first argument. Since the bracket $[\cdot,\cdot]$ is antisymmetric, ω is a well defined differential 2-form. For the non degeneracy let $\xi \in \mathfrak{g}^*$ be such that $\omega_{|\beta}(\xi,\eta) = 0\ \forall \eta \in T_\beta \mathrm{Ad}^*_G(\alpha)$. The tangent space of the coadjoint orbit at β is found via the differential with respect to the group factor of Ad^* viewed as the action: $G \times \mathfrak{g}^* \to \mathfrak{g}^*,\ (g,\beta) \mapsto \mathrm{Ad}^*_g(\beta),\ \beta \in \mathfrak{g}^*$. Then $T_\beta \mathrm{Ad}^*_G(\alpha) \simeq \mathfrak{g}/\ker(\mathrm{ad}^*(\beta))$. By definition if $\mathrm{ad}^*_v(\beta) = \xi,\ \mathrm{ad}^*_w(\beta) = \eta$, then by hypotesis $0 = \omega_{|\beta}(\xi,\eta) = \beta([v,w]) = \beta(\mathrm{ad}_v(w)) = (\mathrm{ad}^*_v(\beta))(w) = \xi(w),\ \forall w \in \mathfrak{g}/\ker(\mathrm{ad}^*(\beta))$. But since $\mathfrak{g} \simeq \ker(\mathrm{ad}^*(\beta)) \oplus \mathfrak{g}/\ker(\mathrm{ad}^*(\beta))$, and $\xi(u) = (\mathrm{ad}^*_v(\beta))(u) = -(\mathrm{ad}^*_u(\beta))(v) = 0, \forall u \in \ker(\mathrm{ad}^*(\beta))$, we conclude that ξ, thought as a linear form, is null on $\mathfrak{g}$, hence it is null.

In the following we interpret the map $\mathrm{ad}^*_v : \mathrm{Ad}^*_G(\alpha) \to \mathfrak{g}^*,\ v \in \mathfrak{g}$ as a vector field on the coadjoint orbit through α.

The fact that the 2-form ω is closed follows from its invariance under the coadjoint action. Indeed, by the Cartan homotopy formula, for all $X \in \mathfrak{X}(\mathrm{Ad}^*_G(\alpha))$ one has

$$0 = \mathcal{L}_X\omega = i_X \mathrm{d}\omega + \mathrm{d}i_X\omega = i_X\mathrm{d}\omega + \mathrm{d}X = i_X\mathrm{d}\omega + \mathrm{d}(\mathrm{d}X) = i_X\mathrm{d}\omega\ ,$$

where X is regarded as a linear map, then $\mathrm{d}\omega = 0$. Therefore, ω is a symplectic form.

The invariance is shown as follows. Let $g \in G$, we have

$$\left[\left(\mathrm{Ad}^*_g\right)^*\omega\right]_{|\beta}(\xi,\eta) = \omega_{|\mathrm{Ad}^*_g(\beta)}(\mathrm{d}(\mathrm{Ad}^*_g)_\beta(\xi), \mathrm{d}(\mathrm{Ad}^*_g)_\beta(\eta)) =$$

$$= \omega_{|\mathrm{Ad}^*_g(\beta)}(\mathrm{Ad}^*_g(\xi), \mathrm{Ad}^*_g(\eta)) = \omega_{|\mathrm{Ad}^*_g(\beta)}(\mathrm{Ad}^*_g(\mathrm{ad}^*_v(\beta)), \mathrm{Ad}^*_g(\mathrm{ad}^*_w(\beta)))\ ,$$

where in the last passage the linearity of Ad^*_g was used and the previous definitions of v and w were maintained. Since[1]

$$\mathrm{Ad}^*_g(\mathrm{ad}^*_v(\beta)) = \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}(v)}(\mathrm{Ad}^*_g(\beta))\ ,$$

one can conclude

$$\left[\left(\mathrm{Ad}^*_g\right)^*\omega\right]_{|\beta}(\xi,\eta) = \omega_{|\mathrm{Ad}^*_g(\beta)}(\mathrm{Ad}^*_g(\mathrm{ad}^*_v(\beta)), \mathrm{Ad}^*_g(\mathrm{ad}^*_v(\beta))) =$$

$$= \omega_{|\mathrm{Ad}^*_g(\beta)}(\mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}(v)}(\mathrm{Ad}^*_g(\beta)), \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}(w)}(\mathrm{Ad}^*_g(\beta))) =$$

$$= [\mathrm{Ad}^*_g(\beta)]([\mathrm{Ad}_{g^{-1}}(v), \mathrm{Ad}_{g^{-1}}(w)]) = \beta(\mathrm{Ad}_g([\mathrm{Ad}_{g^{-1}}(v), \mathrm{Ad}_{g^{-1}}(w)])) =$$

$$= \beta([\mathrm{Ad}_g(\mathrm{Ad}_{g^{-1}}(v)), \mathrm{Ad}_g(\mathrm{Ad}_{g^{-1}}(w))]) = \beta([\mathrm{Ad}_{gg^{-1}}(v), \mathrm{Ad}_{gg^{-1}}(w)]) =$$

$$= \beta([\mathrm{Ad}_e(v)), \mathrm{Ad}_e(w)]) = \beta([v,w]) = \omega_{|\beta}(\xi,\eta)\ .$$

We now prove the following set of properties of the momentum map which will be useful in Chapter 3.

Proposition 1.5.9. *Let $\mu : M \to \mathfrak{g}^*$ be a momentum map for $\Psi : G \circlearrowleft (M,\omega)$. Then for all $x \in M$ we have:*

[1]Indeed:$[\mathrm{Ad}^*_g(\mathrm{ad}^*_v(\beta))](w) = \beta(\mathrm{ad}_v(\mathrm{Ad}_g(w))) = \beta([v, \mathrm{Ad}_g(w)]) = \beta(\mathrm{Ad}_g([\mathrm{Ad}_{g^{-1}}(v), w])) =$ $= [\mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}(v)}(\mathrm{Ad}^*_g(\beta))](w)$, because Ad_g is a differential and $[\cdot,\cdot]$ is the Lie bracket; this fact will be used also in the following.

i) $ker((d\mu)_x) = (T_x(G(x)))^{\S}$ *;*

ii) $Im((d\mu)_x) = \{\alpha \in \mathfrak{g}^* \ : \ \alpha(v) = 0,$ *for all* $v \in \mathfrak{g}_x\} =: \mathfrak{g}_x^0$ *;*

iii) $T_x(G(x)) \cap (T_x(G(x)))^{\S} = T_x(G_{\mu(x)}(x))$ *;*

iv) *If* $x \in M_{\text{princ}}$*, then* $\dim(G_{\mu(x)}) - \dim(G_x) \leq cohom(G, M)$*;*

v) $\omega|_{G(x)} = (\mu^*\sigma)|_{G(x)}$*, where* σ *is the Lie–Poisson 2-form on* $Ad_G^*(\mu(x))$*.*

In particular, if we look at the subaction $G_{\mu(x)} \circlearrowleft \mu^{-1}(\mu(x))$, we have that for all $x \in \mu^{-1}(\mu(x)) =: \tilde{M}$, $T_x\tilde{M} = \ker((\mathrm{d}\mu)_x)$, and by $i)$, $(T_x\tilde{M})^{\S} = ((T_x(G(x)))^{\S})^{\S} = T_x(G(x))$. Therefore, by $iii)$ we conclude

$$T_x\tilde{M} \cap (T_x\tilde{M})^{\S} = T_x(G_{\mu(x)}(x)) \ .$$

Proof. For any $v \in T_x(G(x))$, then $v = \xi_M(x)$ for some $\xi \in \mathfrak{g}$. Then if $w \in \ker((\mathrm{d}\mu))_x$, we have $0 = \langle (\mathrm{d}\mu)_x(w)|\xi\rangle = \omega_x(\xi_M(x), w)$, i.e. $w \in (T_x(G(x)))^{\S}$. Vice-versa, let $w \in (T_x(G(x)))^{\S}$. We get that for all $\xi \in \mathfrak{g}$, $\langle (\mathrm{d}\mu)_x(w)|\xi\rangle = 0$, i.e. $(\mathrm{d}\mu)_x(w) = 0$. Given $0 \neq w \in T_xM$, we have that for all $\xi \in \mathfrak{g}_x$, $\langle (\mathrm{d}\mu)_x(w)|\xi\rangle = \omega_x(\xi_M(x), w) = 0$, hence, $(\mathrm{d}\mu)_x(w) \in \mathfrak{g}_x^0$. We have proved $\mathrm{Im}((\mathrm{d}\mu)_x) \subseteq \mathfrak{g}_x^0$, to prove equality, we see that $\dim(\mathrm{Im}((\mathrm{d}\mu)_x)) = \dim(M) - \dim(\ker((\mathrm{d}\mu)_x)) = \dim(M) - \dim((T_x(G(x)))^{\S}) = \dim(M) - \dim(M) + \dim(G) - \dim(G_x) = \dim(G) - \dim(G_x) = \dim(\mathfrak{g}_x^0)$. This proves $i)$ and $ii)$. To prove $iii)$, let $\xi_M(x) \in T_x(G(x)) \cap (T_x(G(x)))^{\S}$, then

$$\frac{d}{dt}\mathrm{d}\mu(\exp(\xi).x)\Big|_{t=s} = \frac{d}{dr}\mathrm{d}\mu(\exp((r+s)\xi).x)\Big|_{r=0} = \frac{d}{dr}\mathrm{d}\mu(\exp((s\xi)\exp(r\xi).x)\Big|_{r=0} =$$

$$= \mathrm{Ad}^*_{\exp(-s\xi)}\frac{d}{dr}\mathrm{d}\mu(\exp(r\xi).x)\Big|_{r=0} = \mathrm{Ad}^*_{\exp(-s\xi)}(\mathrm{d}\mu)_x(\xi_M(x)) \ .$$

Therefore, for all $\eta \in \mathfrak{g}$ we have

$$\left\langle \frac{d}{dt}\mu(\exp(t\xi).x)\Big|_{t=s}\Big|\eta\right\rangle = \langle \mathrm{Ad}^*_{\exp(-s\xi)}(\mathrm{d}\mu)_x(\xi_M(x))|\eta\rangle =$$

$$\langle (\mathrm{d}\mu)_x(\xi_M(x))|\mathrm{Ad}_{\exp(s\xi)}\eta\rangle = \omega_x((\mathrm{Ad}_{\exp(s\xi)}\eta)_M(x), \xi_M(x)) = 0 \ .$$

We conclude that $\mu(\exp(t\xi).x)$ is constant and hence $\exp(t\xi) \in G_{\mu(x)}$. The same computation shows that if $v \in T_x(G_{\mu(x)}(x)) \subseteq T_x(G(x))$, hence $v = \xi_M(x)$ for some $\xi \in \mathfrak{g}_x$, then $v \in (T_x(G(x)))^{\S}$. Item $iv)$ can be proved as follows. If $x \in M_{princ}$, $\mathrm{cohom}(G, M) = \dim(M) - \dim(G)$, and from $i)$ we have

$$\dim(G_{\mu(x)}) - \dim(G_x) = \dim(T_x(G_{\mu(x)}(x))) = \dim(T_x(G(x)) \cap (T_x(G(x)))^{\S}) \leq$$

$$\leq \dim((T_x(G(x)))^{\S})) = \dim(\ker((\mathrm{d}\mu)_x)) = \dim(M) - \dim(G) \ .$$

Finally, by Proposition 1.5.8, $\langle \mu(x)|[\xi, \eta]\rangle = \omega_x(\eta_M(x), \xi_M(x))$, hence if σ denotes the Lie-Poisson 1-form of $\mathrm{Ad}^*_G(\mu(x))$ we have for $v, w \in T_x(G(x))$, such that $v = \xi_M(x)$ and $w = \eta_M(x)$

$$(\mu^*\sigma)_x(v, w) = \sigma_{\mu(x)}((\mathrm{d}\mu)_x(v), (\mathrm{d}\mu)_x(w)) = \sigma_{\mu(x)}((\mathrm{d}\mu)_x(\xi_M(x)), (\mathrm{d}\mu)_x(\eta_M(x))) =$$

$$= \sigma_{\mu(x)}(\mathrm{ad}^*_\xi(\mu(x)), \mathrm{ad}^*_\eta(\mu(x))) = \langle \mu(x)|[\xi, \eta]\rangle = \omega_x(\xi_M(x), \eta_M(x)) \ .$$

This completes the proof of $v)$.

□

1.5.1 Marsden–Meyer–Weinstein reduction

A fundamental result that we will use later is the following theorem, by Marsden–Meyer–Weinstein; see, for an effective treatment, Part IX of the book [22], or the original articles [74], [77].

Theorem 1.5.10. *Let $\Psi : G \circlearrowleft M$ be a proper symplectic Hamiltonian action on (M,ω), $\mu : M \to \mathfrak{g}^*$ be a momentum map for Ψ and $\alpha \in \mathfrak{g}^*$ a regular value of μ. We set $\tilde{M} := \mu^{-1}(\alpha)$ and denote as G_α the stabilizer of α with respect to the coadjoint action. Then if G_α acts freely on $\tilde{M}$, $\tilde{\tilde{M}} := \tilde{M}/G_\alpha$ is a symplectic manifold with symplectic form $\tilde{\tilde{\omega}}$ such that, if $\pi : \tilde{M} \to \tilde{\tilde{M}}$, is the natural projection, $\pi^*\tilde{\tilde{\omega}} = \omega|_{\tilde{M}}$.*

Proof. Since the action on $G_\alpha \circlearrowleft \tilde{M}$ is proper and free, $\tilde{M}$ is a principal fiber bundle $\pi : \tilde{M} \to \tilde{\tilde{M}}$. If $j : \tilde{M} \to M$ is the inclusion, $\tilde{\omega} := j^*\omega = \omega|_{\tilde{M}}$ is a closed 2-form. Such form is clearly G_α invariant because ω is G-invariant. Then there is a unique closed 2-form $\tilde{\tilde{\omega}}$ on $\tilde{\tilde{M}}$ such that $\pi^*\tilde{\tilde{\omega}} = j^*\omega$. We show that $\tilde{\tilde{\omega}}_{[x]}$ is non-degenerate for all $[x] \in \tilde{\tilde{M}}$. The map $(\mathrm{d}\pi)_x : T_x\tilde{M} \to T_{[x]}\tilde{\tilde{M}}$ is surjective, then for all $a, b \in T_{[x]}\tilde{\tilde{M}}$, there are $u, v \in T_x\tilde{M}$ such that $a = (\mathrm{d}\pi)_x(u)$ and $b = (\mathrm{d}\pi)_x(v)$. Then if for all $v \in T_x\tilde{M}$ we have

$$(\pi^*\tilde{\omega})_x(u,v) = \tilde{\tilde{\omega}}_{[x]}((\mathrm{d}\pi)_x(u),(\mathrm{d}\pi)_x(u)) = \tilde{\tilde{\omega}}_{[x]}(a,b) = 0 \ ,$$

this means that $u \in ((T_x\tilde{M})^{\S} \cap (T_x\tilde{M})) = T_x(G_\alpha(x)) = \ker((\mathrm{d}\pi)_x)$, by the previous proposition. Hence $a = 0$.

□

We remark that the dimension of the reduced space $\tilde{\tilde{M}}$ is equal to $\dim(\tilde{\tilde{M}}) = \dim(M) - 2\dim(G) + \dim(\mathrm{Ad}^*_G(\mu))$, which is apparently even, because we just showed that the coadjoint orbits are symplectic.

Remark 1.5.11. In general, the action of G_α is not free, and there is no automatic way to retrieve a “reduced” symplectic manifold. However, by the work of [12], there is a stratification of the set $M_\alpha := \mu^{-1}(G(\alpha))/G$ into smaller symplectic manifolds. We will use this fact in the following. For further relations with the Meyer–Marsden–Weinstein reduction, see [8].

1.5.2 Almost Kähler actions

It is possible to prove that given a symplectic manifold (M,ω), we can always find an almost complex structure J on M, such that $\mathbf{g}(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric. The three structures $\omega, J, \mathbf{g}$ are called *compatible*. Moreover, $\omega, \mathbf{g}$ are also *J-invariant*, i.e., $J^*\omega = \omega$, $J^*\mathbf{g} = \mathbf{g}$ (see [12] or Appendix, Section 5.4). We call the collection $(M,\omega,J,\mathbf{g})$ an *almost Kähler manifold*. If $\Psi : G \circlearrowleft (M,\omega)$ is a proper symplectic action, we can similarly find an almost complex structure J which is, by construction, G-equivariant. The Remannian metric $\mathbf{g}$ is then G-invariant. In fact, for all $X, Y \in \mathfrak{X}(M)$ and $g \in G$, we have

$$\begin{aligned}\Psi_g^*\mathbf{g}(X,Y) &= \mathbf{g}(\Psi_{g*}(X),\Psi_{g*}(Y)) = \omega(\Psi_{g*}(X),(J\circ\Psi_{g*})(Y)) = \\ &= \omega(X,(\Psi_{g*}^{-1}\circ J\circ\Psi_{g*})(Y)) = \omega(X,J(Y)) = \mathbf{g}(X,Y)\ .\end{aligned}$$

If $(M,\omega,J,\mathbf{g})$ is an almost Kähler manifold, and $\Psi : G \circlearrowleft (M,\omega,J,\mathbf{g})$ preserves all the three structures, we call Ψ an *almost Kähler action*. Therefore, we have just proved the following.

Proposition 1.5.12. *Given a symplectic manifold* (M, ω) *and a proper symplectic action* $\Psi : G \circlearrowleft (M, \omega)$, Ψ *is also an almost Kähler action* $\Psi : G \circlearrowleft (M, \omega, J, \boldsymbol{g})$, *where* J *is any* G*-equivariant almost complex structure associated to* ω *and* $\boldsymbol{g}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ *is a* G*-invariant Riemannian metric.*

Chapter 2

Complex structures on Product Manifolds

In the classical paper by Calabi and Eckmann [21] it is shown that the compact manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ admits a complex structure which is constructed by considering the natural $U(1)$-action on $\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}$. For $m > 0$ this provided the first known example of a compact, simply connected complex manifold which is not algebraic. In fact, by a simple cohomological computation, one obtains that the manifold $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ cannot be Kähler, and therefore not even algebraic. For example, the second de Rham cohomology group $H^2_{dR}(\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}) = \{0\}$, precisely when $n \geq 1$ or $m \geq 1$, which is incompatible with the Kähler condition (see Subsection 5.4.3 of the Appendix). We start this chapter by reviewing the original construction in [21], where we point out how it can be interpreted in terms of symplectic and Kähler reduction. Then we present a natural generalization of this construction (see Definition 2.2.1) and prove our main results, namely Theorem 2.2.2 and Theorem 2.2.4. Lastly, we will see some applications. All contents starting from Section 2.2 are based on the published work [1].

2.1 Idea and generalization

We briefly review the construction by Calabi and Eckmann in [21]. Let $U(1) \circlearrowleft \mathbb{C}^{n+1}$, $(\lambda, z) \mapsto \lambda z$, be the multiplication by a complex number of unit norm. Then this action can be restricted to the unit sphere of $\mathbb{C}^{n+1}$, which we denote by $\mathbb{S}^{2n+1}$. The action $U(1) \circlearrowleft \mathbb{S}^{2n+1}$ is proper and free. Hence, the projection $\pi_n : \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/U(1) = \mathbb{C}P^n$ is a principal fiber bundle. Similarly, we can consider $\pi_m : \mathbb{S}^{2m+1} \to \mathbb{C}P^m$. Then the map $\Pi := (\pi_n, \pi_m) : \mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1} \to \mathbb{C}P^n \times \mathbb{C}P^m$ is a principal fiber bundle with typical fiber diffeomorphic to a torus $U(1) \times U(1) = \mathbb{T}^2$. The general strategy is to put a convenient complex structure on such fibers that together with the natural complex structure on $\mathbb{C}P^n \times \mathbb{C}P^m$ provides a complex structure on $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$.

Let $[z_n^0 : \ldots : z_n^n]$ and $[z_m^0 : \ldots : z_m^m]$ be homogeneous coordinates for $\mathbb{C}P^n$ and $\mathbb{C}P^m$, respectively and consider the open subsets $V_{\alpha\beta}$ of $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ defined by $z_n^\alpha z_m^\beta \neq 0$, for $\alpha = 0, \ldots, n$ and $\beta = 0, \ldots, m$. Then

$$ {}_\alpha w_n^j := \frac{z_n^j}{z_n^\alpha}, \qquad {}_\beta w_m^k := \frac{z_m^k}{z_n^\beta} \ , $$

where $j = 0, \ldots, n$ and $k = 0, \ldots, m$, are coordinates on the open subsets $\Pi(V_{\alpha\beta})$

covering $\mathbb{C}P^n \times \mathbb{C}P^m$. Then, for any $\tau \in \mathbb{C}$ with $\mathcal{Im}(\tau) \neq 0$, one sets

$$t^\tau_{\alpha\beta} := \frac{1}{2\pi i}(\log(z^\alpha_n) + \tau \log(z^\beta_m)) \bmod(1,\tau) \ ,$$

where $\bmod(1,\tau)$ means that we are taking the quotient by the lattice Λ generated by 1 and τ. In the paper [21] it is explicitly shown that the $(n+m+1)$-complex functions ${}_\alpha w^j_n, {}_\beta w^j_m, t^\tau_{\alpha\beta}$ map diffeomorphically $V_{\alpha\beta}$ into the complex manifold $\mathbb{C}^{n+m} \times \mathbb{T}^2_\Lambda$, where $\mathbb{T}^2_\Lambda$ is the complex torus equipped with the quotient complex structure of $\mathbb{C}/\Lambda$. Moreover, ${}_\alpha w^j_n, {}_\beta w^j_m, t^\tau_{\alpha\beta}$ also constitute a complex atlas for $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$. Finally, on the open $V_{\alpha\beta}$ the projection Π can be written as $\Pi({}_\alpha w^j_n, {}_\beta w^j_m, t^\tau_{\alpha\beta}) = ({}_\alpha w^j_n, {}_\beta w^j_m)$ and on any overlap $V_{\alpha\beta} \cap V_{\delta\gamma} \neq \emptyset$, setting ${}_\alpha w^\alpha_n = {}_\gamma w^\gamma_n = {}_\beta w^\beta_m = {}_\delta w^\delta_m = 1$ we have the relations

$${}_\gamma w^j_n = \frac{{}_\alpha w^j_n}{{}_\alpha w^\gamma_n}, \quad {}_\delta w^k_m = \frac{{}_\beta w^k_m}{{}_\beta w^\delta_m} \qquad \text{for } j \in \{0,\dots,n\},\ k \in \{0,\dots,m\},$$

$$t^\tau_{\delta\gamma} = t^\tau_{\alpha\beta} + \frac{1}{2\pi i}(\log({}_\alpha w^\gamma_n) + \tau \log({}_\beta w^\gamma_m)) \bmod(1,\tau) \ ,$$

which show that the transition functions of the principal fiber bundle $\Pi : \mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1} \to \mathbb{C}P^n \times \mathbb{C}P^m$ are holomorphic.

Remark 2.1.1. Let us consider the infinitesimal generator of the action in local holomorphic coordinates over $V_{\alpha\beta} \simeq \pi_n(V_\alpha) \times \pi_m(V_\beta) \times \mathbb{C}$, where $V_\alpha \subseteq \mathbb{S}^{2n+1}, V_\beta \subseteq \mathbb{S}^{2m+1}$ are defined by $z^\alpha_n \neq 0 \neq z^\beta_m$. Let $\xi := 2\pi i \in \mathfrak{u}^1 = \mathrm{Lie}(U(1)) \simeq i\mathbb{R}$ and let $p = (p_n, p_m) = ({}_\alpha w^j_n, {}_\beta w^j_m, t^\tau_{\alpha\beta}) \in V_{\alpha\beta}$, then

$$\begin{aligned}\xi_{V_{\alpha\beta}}(p) &= \frac{d}{dt} e^{2\pi i t}({}_\alpha w^j_n, {}_\beta w^j_m, t^\tau_{\alpha\beta})\Big|_{t=0} = \Big({}_\alpha w^j_n, {}_\beta w^j_m, 1+\tau\Big)_p = \\ &= \Big({}_\alpha w^j_n, 0, 1\Big)_p + \Big(0, {}_\beta w^j_m, \tau\Big)_p = \xi_{V_\alpha}(p_n) + \xi_{V_\beta}(p_m) \ ,\end{aligned}$$

where we used the isomorphisms $T_p(\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}) \simeq (T_{p_n}\mathbb{S}^{2n+1}) \oplus (T_{p_m}\mathbb{S}^{2m+1})$ and $T_p(\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}) \simeq (T_{\pi_n(p_n)}(\mathbb{C}P^n)) \oplus (T_{\pi_m(p_m)}(\mathbb{C}P^n)) \oplus T_p(\mathbb{T}^2_\Lambda)$. In particular, if J_Λ is the complex structure of $\mathbb{T}^2_\Lambda$, we observe that

$$J_\Lambda(\xi_{V_\alpha}) = \xi_{V_\beta}, \quad \text{and} \quad J_\Lambda(\xi_{V_\beta}) = -\xi_{V_\alpha} \ . \tag{2.1}$$

A natural generalization of this construction can be guessed by noticing that the action $U(1) \circlearrowleft \mathbb{C}^{n+1}$ is symplectic and Hamiltonian with respect to the standard symplectic structure of $\mathbb{C}^{n+1}$. The momentum map of this action is given by $\mu : \mathbb{C}^{n+1} \to \mathrm{Lie}(U(1))^* \simeq i\mathbb{R}$

$$z \mapsto \mu(z) = \frac{iz^\dagger z}{2\pi} \ .$$

Then we simply notice that the spheres are level sets of the momentum map and $\mathbb{C}P^n$ is the Meyer–Marsden–Weintein quotient of $\mathbb{C}^{n+1}$. Moreover, the reduced simplectic form is the natural Kähler form of $\mathbb{C}P^{n+1}$. This is an example of *Kähler reduction*.

2.1.1 Kähler reduction

Following the above discussion, consider two Kähler manifolds $(M_i, h_i, J_i, \omega_i)$, $i = 1, 2$, where h_i is the Riemannian metric, J_i the complex structure, ω_i the compatible symplectic structure and letting G be a compact Lie group acting on each M_i by *Kähler isometries*. This means that for all $g \in G$ we have

$$g^*h_i = h_i, \qquad g_* \circ J_i = J_i \circ g_*, \qquad g^*\omega_i = \omega_i \ ,$$

where g^* and g_* denote the pull-back and the push-forward associated to the diffeomorphism induced by g, respectively. We assume that each action admits a *momentum map* $\mu_i : M_i \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ denotes the dual of the Lie algebra of G, and we let $N_i := \mu_i^{-1}(0)$ be a regular level set of μ_i. We further suppose that the action of G on N_i is free and consider the orbit space $P_i := N_i/G$. We denote with $\pi_i : N_i \to P_i$ the natural projection from N_i onto its orbit space. Note that since G is compact and the action is free, $\pi_i : N_i \to P_i$ is a principal fiber bundle with base P_i and characteristic fiber G. We recall that the manifold P_i is called the Meyer–Marsden–Weinstein quotient and, as we have seen in Theorem 1.5.10, it is a symplectic manifold. Guillemin and Sternberg in [46] proved that on P_i there exist a Riemannian metric $\tilde{h}_i$, a complex structure $\tilde{J}_i$ and a symplectic structure $\tilde{\omega}_i$, such that $(P_i, \tilde{h}_i, \tilde{J}_i, \tilde{\omega}_i)$ is a Kähler manifold. We call the process of passing from the manifold $(M_i, h_i, J_i, \omega_i)$ to the manifold $(P_i, \tilde{h}_i, \tilde{J}_i, \tilde{\omega}_i)$, a process of *Kähler reduction*.

We now describe more precisely the structure of TN_i and briefly review the reduction process. Since the action is free, the projection π_i defines a distribution inside TN_i, the so called *vertical distribution*, $V_i := \bigsqcup_{p_i \in N_i} V_{p_i}$, given point-wise by $V_{p_i} := \ker\left((\mathrm{d}\pi_i)_{p_i}\right)$. This means that V_i is integrable by definition, the integral leaves being the fibers $\pi_i^{-1}(\pi_i(p_i))$, which correspond to the orbits of the action. In particular, it follows that at each point p_i, V_{p_i} coincides with the tangent space to the G-orbit of p_i. We further notice that V_i is a trivial sub-bundle of TN_i, whose fiber is isomorphic to $\mathfrak{g}$. We then define a complementary subspace to V_{p_i} at $p_i \in N_i$ inside $T_{p_i}N_i$ as

$$H_{p_i} := \{v \in T_{p_i}N_i \ : \ (h_i)_{p_i}(v, w) = 0, \quad \forall w \in V_{p_i}\} \ .$$

Therefore, we have a $(h_i)_{p_i}$-orthogonal splitting $T_{p_i}N_i = H_{p_i} \oplus V_{p_i}$. The collection of the H_{p_i} defines a distribution $H_i := \cup_{p_i \in N_i} H_{p_i} \subset TN_i$, which is called *horizontal distribution*. Since the action of G is isometric, we also have that for all $g \in G$

$$g_* H_i = H_i \ .$$

This tells us that $H_i \subset TN_i$ is a *principal connection* for the principal fiber bundle $\pi_i : N_i \to P_i$; for a general treatment of principal bundles and principal connections, see Chapter II of the book [65]. We note that the distribution H_i is J_i-invariant, i.e.

$$J_i(H_i) \subseteq H_i \ .$$

To see this, we observe that for all $p_i \in N_i$, given $v \in H_{p_i}$, for all $w \in V_{p_i}$

$$(h_i)_{p_i}((J_i)_{p_i} v, w) = (\omega_i)_{p_i}(v, w) = 0 \ ,$$

because $(\omega_i)_{p_i}$ is null along V_{p_i}, see point *ii)* of the lemma at page 123 in [74]. Therefore, $(J_i)_{p_i} v \in H_{p_i}$.

Observe now that since $(\mathrm{d}\pi_i)_{p_i} : H_{p_i} \to T_{\pi_i(p_i)}P_i$ is an isomorphism, and since J_i is G-equivariant, there exists a unique almost complex structure $\tilde{J}_i$ on P_i such that $\pi_{i*} \circ (J_i|_{H_i}) = \tilde{J}_i \circ (\pi_{i*}|_{H_i})$. As it is proved in [46], we have that $\tilde{J}_i$ is actually a complex structure; see also the very clear treatment given in Section 2 of [33]. Similarly, from the G-invariance of ω_i and h_i, we find a unique Riemannian metric $\tilde{h}_i$ and a symplectic form $\tilde{\omega}_i$ on P_i such that $h_i|_{N_i} = \pi_i^* \tilde{h}_i$, and $\omega_i|_{N_i} = \pi_i^* \tilde{\omega}_i$. The compatibility between h_i, J_i, ω_i, easily descends to $\tilde{h}_i, \tilde{J}_i, \tilde{\omega}_i$, making $(P_i, \tilde{h}_i, \tilde{J}_i, \tilde{\omega}_i)$ a Kähler manifold.

2.2 Almost complex structure on $N_1 \times N_2$

In the following, we will use the decomposition $TN_i \simeq H_i \oplus V_i$ for both $i = 1, 2$ and hence the tangent bundle of $N_1 \times N_2$ will be split as

$$T(N_1 \times N_2) \simeq (H_1 \oplus V_1) \oplus (H_2 \oplus V_2) \ . \tag{2.2}$$

One last fact we are going to use in the computations is that the action $G \times N_i \to N_i$ defines a Lie algebra anti-homomorphism[1] between the Lie algebra of G and the Lie algebra of smooth vector fields on N_i. By Proposition 1.1.10, we have for all $\xi, \eta \in \mathfrak{g}$ that

$$[\xi, \eta]_{N_i} = -[\xi_{N_i}, \eta_{N_i}] \ .$$

In contrast, the bracket between sections of H_i is not automatically a section of H_i, that is, in general H_i is not an integrable distribution.

We give the following definition, which generalizes (2.1).

Definition 2.2.1. We define the following endomorphism J of $T(N_1 \times N_2)$ as

$$J := (J_1 \oplus T_1) \oplus (J_2 \oplus T_2) \ ,$$

where we have used the splitting (2.2), and we have set for $\xi \in \mathfrak{g}$

$$T_1(\xi_{N_1}) := \xi_{N_2} \qquad T_2(\xi_{N_2}) := -\xi_{N_1} \ .$$

Our first result is the following.

Theorem 2.2.2. ($\star$) *The endomorphism J defines an almost complex structure on $N_1 \times N_2$, i.e., $J^2 = -id_{T(N_1 \times N_2)}$. Furthermore, J is integrable, i.e. complex, if and only if G is Abelian.*

By the celebrated Newlander–Nirenberg Theorem [87], the integrability of the almost-complex structure J on $N_1 \times N_2$ is equivalent to the vanishing of the Nijenhuis tensor associated to J. This tensor is defined as

$$N_J(X, Y) := [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] \ ,$$

where X, Y are smooth vector fields on $N_1 \times N_2$.

We then study the action by the Abelian group $(\mathbb{C}, +)$ on $N_1 \times N_2$ defined as

$$\Psi : \mathbb{C} \times (N_1 \times N_2) \to (N_1 \times N_2), \quad (a + ib, (p_1, p_2)) \mapsto (e^{2\pi i(a+b)}.p_1, e^{2\pi i(a-b)}.p_2) \ , \tag{2.3}$$

where the dot denotes a free $U(1)$-action on the two factors. We prove the following proposition, telling us that, up to a sign, the map Ψ is holomorphic with respect to the complex structure J of Definition 2.2.1 associated with the $U(1)$-action.

Proposition 2.2.3.
1. *The map $\Psi_{(p_1,p_2)} : \mathbb{C} \to (N_1 \times N_2)$, $\Psi_{(p_1,p_2)}(a+ib) := \Psi(a+ib, (p_1, p_2))$ is holomorphic with respect to the complex structure of $\mathbb{C}$ and the complex structure $-J$;*
2. *The map $\Psi_{a+ib} : (N_1 \times N_2) \to (N_1 \times N_2)$, $\Psi_{a+ib}(p_1, p_2) := \Psi(a + ib, (p_1, p_2))$ is holomorphic with respect to the complex structure J.*

[1] We are following the sign conventions of [5].

Let now

$$\Lambda := \left\{ n\left(\frac{1+i}{2}\right) + m\left(\frac{1-i}{2}\right) \ : \ n, m \in \mathbb{Z} \right\} . \tag{2.4}$$

Then $\mathbb{C}/\Lambda \simeq \mathbb{T}^2 = U(1) \times U(1)$ is a one-dimensional complex torus. This is a compact Abelian group, and the quotient action by $\mathbb{T}^2$ is free. Without using Theorem 2.2.2, we are able to prove the following theorem.

Theorem 2.2.4. ($\star$) *Given the action (2.3), the natural projection $\pi : (N_1 \times N_2) \to (P_1 \times P_2)$ is a principal holomorphic bundle with characteristic fiber $\mathbb{T}^2$. Moreover, the construction can be easily generalized to the case of an action of $\mathbb{C}^n/\Lambda^n = \mathbb{T}^{2n}$.*

2.2.1 Proofs of the results

Proof. (**Theorem 2.2.2.**)
We start by proving that J is an almost complex structure. Consider for $(p_1, p_2) \in (N_1 \times N_2)$ a basis element for $T_{(p_1,p_2)}(N_1 \times N_2) \simeq (H_{p_1} \oplus \mathfrak{g}) \bigoplus (H_{p_2} \oplus \mathfrak{g})$, denoted as (v_1, ξ_1, v_2, ξ_2), where $v_i \in H_{p_i}$ and $\xi_i \in \mathfrak{g}$ are nonzero elements.

Since J_i is a complex structure on H_{p_i}, we have that

$$J^2(v_1, \xi_1, v_2, \xi_2) = J(J_1 v_1, -\xi_2, J_2 v_2, \xi_1) =$$

$$= (J_1^2 v_1, -\xi_1, J_2^2 v_2, -\xi_2) = -(v_1, \xi_1, v_2, \xi_2).$$

We conclude $J^2 = -\mathrm{id}_{T(N_1 \times N_2)}$.

To prove integrability, we show the vanishing of the Nijenhuis tensor of J, denoted as N_J. We consider each case comprised in the splitting

$$T(N_1 \times N_2) \simeq (H_1 \oplus V_1) \oplus (H_2 \oplus V_2) .$$

The integrability of J_i implies $N_J(X_i, \tilde{X}_i) = 0$ for all sections $X_i, \tilde{X}_i$ of H_i.
Let $Y_1 := \xi_{N_1}$, $Y_2 := \eta_{N_2}$ be vertical vector fields, we have

$$N_J(Y_1, Y_2) = [JY_1, JY_2] - [Y_1, Y_2] - J[JY_1, Y_2] - J[Y_1, JY_2] =$$

$$= -J[JY_1, Y_2] - J[Y_1, JY_2] = -J([\xi_{N_2}, \eta_{N_2}] - [\xi_{N_1}, \eta_{N_1}]) =$$

$$= -J([\xi, \eta]_{N_1} - [\xi, \eta]_{N_2}) .$$

Therefore, $N_J(Y_1, Y_2) = 0$ for all $\xi, \eta \in \mathfrak{g}$ if and only if G is Abelian.
Now we check that $N_J(X_1, Y_i) = 0$. For $i = 1$,

$$N_J(X_1, Y_1) = -[X_1, Y_1] - J[JX_1, Y_1] = -[X_1, Y_1] - J^2[X_1, Y_1] = [X_1, Y_1] - [X_1, Y_1] = 0 .$$

For $i = 2$,

$$N_J(X_1, Y_2) = [JX_1, JY_2] - J[X_1, JY_2] = J[X_1, JY_2] - J[X_1, JY_2] = 0 .$$

In fact, for X_i horizontal and $Y_i = \xi_{N_i}$ vertical we have

$$[JX_i, Y_i] = [J_i X_i, Y_i] = J_i[X_i, Y_i] = J[X_i, Y_i] .$$

This is because $J_i \circ g_* = g_* \circ J_i$ implies $\mathcal{L}_{\xi_{N_i}} J_i = 0$ for all $\xi \in \mathfrak{g}$, and hence

$$[Y_i, J_i X_i] = \mathcal{L}_{Y_i}(J_i X_i) = (\mathcal{L}_{Y_i} J_i)(X_i) + J_i(\mathcal{L}_{Y_i} X_i) = J_i(\mathcal{L}_{Y_i} X_i) = J_i[Y_i, X_i] .$$

Let now X_i be a section of H_i. Then, since $[H_1, H_2] = 0$, we have $N_J(X_1, X_2) = 0$. Lastly, we consider $Y_i = \xi_{N_i}$, $\tilde{Y}_i = \eta_{N_i}$.

$$N_J(Y_1, \tilde{Y}_1) = [J\xi_{N_1}, J\eta_{N_1}] - [\xi_{N_1}, \eta_{N_1}] - J[J\xi_{N_1}, \eta_{N_1}] - J[\xi_{N_1}, J\eta_{N_1}] =$$

$$= [\xi_{N_2}, \eta_{N_2}] - [\xi_{N_1}, \eta_{N_1}] - J[\xi_{N_2}, \eta_{N_1}] - J[\xi_{N_1}, \eta_{N_2}] = [\xi, \eta]_{N_1} - [\xi, \eta]_{N_2} \ .$$

We get $N_J(Y_1, \tilde{Y}_1) = 0$ for all $\xi, \eta \in \mathfrak{g}$ if and only if G is Abelian. Similarly, $N_J(Y_2, \tilde{Y}_2) = 0$ if and only if G is Abelian. The remaining cases are comprised in the previous ones because of the skew-symmetry of N_J.

Thus, we have proved that if G is Abelian, $N_J = 0$, and thanks to the Newlander-Nirenberg theorem, $N_1 \times N_2$ is a complex manifold with the holomorphic structure induced by J.

□

Proof. (**Proposition 2.2.3.**)

1. Consider the curve $t \mapsto (a + t + ib) \in \mathbb{C}$, so that $\frac{d}{dt}(a + t + ib)|_{t=0} = 1 \in T_{(a+ib)}\mathbb{C}$. Therefore, setting $(p_1', p_2') := \Psi_{a+ib}(p_1, p_2)$, we have

$$\left(\mathrm{d}\Psi_{(p_1,p_2)}\right)_{a+ib}(1) = \frac{d}{dt}\Psi_{(p_1,p_2)}(a + t + ib)\Big|_{t=0} =$$

$$= \frac{d}{dt}(e^{2\pi i(a+t+b)}.p_1, e^{2\pi i(a+t-b)}.p_2)\Big|_{t=0} =$$

$$= \frac{d}{dt}(e^{2\pi it}.p_1', e^{2\pi it}.p_2')\Big|_{t=0} =$$

$$= (\xi_{N_1}(p_1'), \xi_{N_2}(p_2')) \ ,$$

where $\xi = 2\pi i \in \mathfrak{u}(1)$, and ξ_{N_i} is the infinitesimal generator of the $U(1)$-action on N_i. For the tangent vector i, consider the curve $t \mapsto (a + i(t + b)) \in \mathbb{C}$, so that $\frac{d}{dt}(a + i(t + b))|_{t=0} = i \in T_{(a+ib)}\mathbb{C}$.
Now,

$$\left(\mathrm{d}\Psi_{(p_1,p_2)}\right)_{a+ib}(1) = \frac{d}{dt}\Psi_{(p_1,p_2)}(a + i(b + t))\Big|_{t=0} =$$

$$= \frac{d}{dt}(e^{2\pi i(a+t+b)}.p_1, e^{2\pi i(a-t-b)}.p_2)\Big|_{t=0} =$$

$$= \frac{d}{dt}(e^{2\pi it}.p_1', e^{-2\pi it}.p_2')\Big|_{t=0} =$$

$$= (\xi_{N_1}(p_1'), -\xi_{N_2}(p_2')) = \ ,$$

$$= -J_{(p_1',p_2')}(\xi_{N_1}(p_1'), \xi_{N_2}(p_2')) \ .$$

The result follows by observing that $i = i \cdot 1$ is the action of the complex structure of $\mathbb{C}$ on the tangent vector 1.

2. We use the decomposition of $TN_i \simeq H_i \oplus V_i$, for $i = 1, 2$. Let us write $\Psi_{a+ib} = (\psi^1_{a+ib}, \psi^2_{a+ib})$. It follows that for $i = 1, 2$, one has
$$(\psi^i_{a+ib})_* \circ J_i = J_i \circ (\psi^i_{a+ib})_*|_{H_i} \ .$$
For the vertical vector fields ξ_{N_1}, η_{N_2}, since they are invariant under $(\psi^1_{a+ib})_*$ and $(\psi^2_{a+ib})_*$, respectively, one gets
$$J((\Psi_{a+ib})_*)(\xi_{N_1},\eta_{N_2}))=J((\psi^1_{a+ib})_*\xi_{N_1},(\psi^2_{a+ib})_*\eta_{N_2})=J(\xi_{N_1},\eta_{N_2})=(-\eta_{N_1},\xi_{N_2}) \ .$$
We also have
$$(\Psi_{a+ib})_*(J(\xi_{N_1},\eta_{N_2}))=(\Psi_{a+ib})_*(-\eta_{N_1},\xi_{N_2})=(-(\psi^1_{a+ib})_*\eta_{N_1},(\psi^2_{a+ib})_*\xi_{N_2})=(-\eta_{N_1},\xi_{N_2}) \ .$$
In conclusion,
$$(\Psi_{a+ib})_* \circ J = J \circ (\Psi_{a+ib})_* \ .$$

□

Remark 2.2.5. We observe that, in general, given $A \in GL(2, \mathbb{R})$, the action $\Psi : \mathbb{C} \times (N_1 \times N_2) \to (N_1 \times N_2)$ defined as $\Psi(a+ib, (x, y)) := (e^{2\pi i(A_{11}a+A_{21}b)}.x, e^{2\pi i(A_{12}a+A_{22}b)}.y)$ is holomorphic with respect to J, and induces a free holomorphic action of $\mathbb{C}/\Lambda_A$, where
$$\Lambda_A := \left\{ n \left(\frac{A_{22} - iA_{12}}{\det(A)} \right) + m \left(\frac{-A_{21} + iA_{11}}{\det(A)} \right) \ : \ n, m \in \mathbb{Z} \right\} \ .$$
We retrieve the lattice (2.4) by letting $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Proof. (**Theorem 2.2.4.**)
We find holomorphic trivializations of $\pi : (N_1 \times N_2) \to (P_1 \times P_2)$ for the case of $U(1)$-actions, and the corresponding $(\mathbb{C}, +)$-action defined in (2.3) of page 3.

Since the quotient action by $\mathbb{C}/\Lambda$ is proper and free, we know that the natural projection $\pi : N_1 \times N_2 \to P_1 \times P_2$ is a principal $\mathbb{T}^2$-bundle. In particular, given $U \subseteq P_1 \times P_2$ open subset, there is a local trivialization
$$\phi : \pi^{-1}(U) \to U \times \mathbb{T}^2 \ .$$
Let us define $\Sigma := \phi^{-1}(U \times \{e\})$, where e is the identity of $\mathbb{T}^2$, and let $s := \phi^{-1}(x, e) \in \Sigma$, where $x \in U$. Since ϕ is a local trivialization, it easily follows that the map
$$\Phi : \Sigma \times \mathbb{T}^2 \to N_1 \times N_2 \ , (s, g) \mapsto g.s \ ,$$
where $g.s$ now denotes the $\mathbb{C}/\Lambda \simeq \mathbb{T}^2$-action on $N_1 \times N_2$, is a smooth open embedding. The map Φ is also equivariant, where we consider the $\mathbb{T}^2$-action on the second factor of $\Sigma \times \mathbb{T}^2$, given by a left translation. By Proposition 2.2.3, since Σ is complex (it is biholomorphic to U), this map is also holomorphic, hence a biholomorphism. Together with the holomorphic charts of $P_1 \times P_2$ and of $\mathbb{T}^2$, the map Φ allows us to construct holomorphic charts for $N_1 \times N_2$.

We now look for the transition maps. Let $\tilde{U} \subseteq P_1 \times P_2$ be another open subset, with $\tilde{U} \cap U \neq \emptyset$ and let
$$\tilde{\phi} : \pi^{-1}(\tilde{U}) \to \tilde{U} \times \mathbb{T}^2 \ ,$$

be the corresponding trivialization. We define $\tilde{\Sigma} := \tilde{\phi}^{-1}(\tilde{U} \times \{e\})$. Again, we have an analogous map $\tilde{\Phi} : \tilde{\Sigma} \times \mathbb{T}^2 \to N_1 \times N_2$, which is a biholomorphism with the image. Define maps $\Theta, \tilde{\Theta}$ as follows

$$\Phi^{-1} : \Phi(\Sigma \times \mathbb{T}^2) \to \Sigma \times \mathbb{T}^2,\ p \mapsto (\Pi^{-1}(\pi(p)), \Theta(p)) \ ,$$

$$\tilde{\Phi}^{-1} : \tilde{\Phi}(\tilde{\Sigma} \times \mathbb{T}^2) \to \tilde{\Sigma} \times \mathbb{T}^2,\ p \mapsto (\tilde{\Pi}^{-1}(\pi(p)), \tilde{\Theta}(p)) \ ,$$

where $\Pi := \pi|_\Sigma$ and $\tilde{\Pi} := \pi|_{\tilde{\Sigma}}$ are smooth, invertible, biholomorphic maps. Then we can write (inverse) trivializations as

$$H(x, g) := \Phi(\Pi^{-1}(x), g) \ .$$

Similarly,

$$\tilde{H}(x, g) := \tilde{\Phi}(\tilde{\Pi}^{-1}(x), g) \ .$$

Therefore, we have that $H^{-1}(p) = (\pi(p), \Theta(p))$. Now,

$$(H^{-1} \circ \tilde{H})(x,g) = H^{-1}(\tilde{\Phi}(\tilde{\Pi}^{-1}(x),g)) = (\pi(\tilde{\Phi}(\tilde{\Pi}^{-1}(x),g)), \Theta(\tilde{\Phi}(\tilde{\Pi}^{-1}(x),g))) = (x, \Theta(\tilde{\Phi}(\tilde{\Pi}^{-1}(x),g))) \ ,$$

which shows that $H^{-1} \circ \tilde{H}$ is a biholomorphism. We can also see that the transition maps are provided by a right translation in the group. Indeed, let $h(x)$ be such that $\tilde{\Phi}(\tilde{\Pi}^{-1}(x), e) = h(x).\Phi(\Pi^{-1}(x), e)$. Then, by equivariance,

$$(H^{-1} \circ \tilde{H})(x, g) = (x, \Theta(\tilde{\Phi}(\tilde{\Pi}^{-1}(x), g))) = (x, \Theta(g.\tilde{\Phi}(\Pi^{-1}(x), e))) =$$

$$= (x, \Theta(gh(x).\Phi(\Pi^{-1}(x), e))) = (x, \Theta(\Phi(\Pi^{-1}(x), gh(x)))) = (x, gh(x)) \ .$$

In summary, we have proved that the transition functions of the principal fiber bundle $\pi : (N_1 \times N_2) \to (P_1 \times P_2)$ are holomorphic. Hence $N_1 \times N_2$ is a holomorphic principal bundle. It also follows that the natural complex structure induced by the complex atlas is, in fact, J. Moreover, if we realize $P_1 \times P_2$ as Meyer–Marsden–Weinstein reduction by an action of $\mathbb{S}^1 \times \overset{k-\text{times}}{\cdots} \times \mathbb{S}^1$, then the same argument proves that $\pi : (N_1 \times N_2) \to (P_1 \times P_2)$ is a $\left(\mathbb{T}^2 \times \overset{k-\text{times}}{\cdots} \times \mathbb{T}^2\right)$-principal holomorphic bundle. As before $N_1 \times N_2$ admits a complex atlas and J is precisely the complex structure induced by the atlas.

□

2.3 Applications

2.3.1 Complex Stiefel manifolds

Consider the action of the unitary group in complex dimension k, $U(k)$, on the space of linear maps $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, for $n \geq k$.

$$\Psi : U(k) \times \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \to \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \ ,$$

$$\Psi(u, A) := A \circ u^\dagger \ ,$$

where we denoted with the superscript † the complex conjugate transposed. This is a proper left action[2]. We identify the tangent space at any point of $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ with

[2] The action is proper since U(k) is compact, and is a left action since $\Psi(u_2 u_1, A) = A \circ (u_2 u_1)^\dagger = A \circ u_1^\dagger \circ u_2^\dagger = \Psi(u_2, \Psi(u_1, A))$.

$\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ itself, and let $J : T(\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)) \to T(\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n))$ correspond to the scalar multiplication by $i \in \mathbb{C}$ defined on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ as a complex vector space. The action Ψ is isometric with respect to the Hermitian metric on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ given by

$$\mathrm{Tr}(A \circ B^\dagger) \ .$$

In fact, for all $u \in U(k)$ we have

$$(\Psi_u^*)(\mathrm{Tr}(A \circ B^\dagger)) = \mathrm{Tr}(A \circ u^\dagger \circ u \circ B^\dagger) = \mathrm{Tr}(A \circ B) \ .$$

The associated J-invariant Riemannian metric is $\mathfrak{Re}(\mathrm{Tr}(A \circ B^\dagger))$. The associated J-invariant symplectic form is $\omega(A, B) := -\mathcal{I}\mathfrak{m}(\mathrm{Tr}(A \circ B^\dagger))$. ω is exact and equal to the differential of the 1-form $\theta|_A(B) := -\frac{1}{2}\mathcal{I}\mathfrak{m}(\mathrm{Tr}(A \circ B^\dagger))$. To make this clear, we change the notation. If z denotes the natural holomorphic coordinates on $\mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$, in matrix notation, we verify that we can write $\omega = \frac{i}{2}\mathrm{Tr}\left((\mathrm{d}z) \wedge (\mathrm{d}z^\dagger)\right)$. In fact,

$$\begin{aligned}
\omega &= \frac{i}{2}\mathrm{Tr}\left((\mathrm{d}z) \wedge (\mathrm{d}z^\dagger)\right) = \frac{i}{2}\mathrm{Tr}\left((\mathrm{d}x + i\mathrm{d}y) \wedge (\mathrm{d}x^T - i\mathrm{d}y^T)\right) = \\
&= \frac{i}{2}\mathrm{Tr}\left((\mathrm{d}x \wedge \mathrm{d}x^T - \mathrm{d}y \wedge \mathrm{d}y^T + i(\mathrm{d}y \wedge \mathrm{d}x^T - \mathrm{d}x \wedge \mathrm{d}y^T))\right) = \\
&= \frac{i}{2}\mathrm{Tr}\left(-2i\mathrm{d}x \wedge \mathrm{d}y^T\right) = \mathrm{Tr}(\mathrm{d}x \wedge \mathrm{d}y^T) \ ,
\end{aligned}$$

which is the standard symplectic structure of $\mathbb{C}^{kn} \simeq \mathbb{R}^{2kn}$. In particular, if $A = A_x + A_y$ and $B = B_x + B_y$ are tangent vectors, we have

$$\omega(A, B) = \mathrm{Tr}(A_x B_y^T - B_x A_y^T) = -\mathcal{I}\mathfrak{m}(\mathrm{Tr}(A \circ B^\dagger)) \ .$$

We now verify that $\theta = \frac{i}{4}\mathrm{Tr}\left(z(\mathrm{d}z^\dagger) - (\mathrm{d}z)z^\dagger\right) = \frac{1}{2}\mathrm{Tr}\left(x\mathrm{d}y^T - y\mathrm{d}x^T\right)$, which is clearly a primitive for ω. In fact, for any point $A = A_x + A_y$ and tangent vector $B = B_x + B_y$

$$\theta|_A(B) = \frac{1}{2}\mathrm{Tr}(A_x B_y^T - A_y B_x^T) = -\frac{1}{2}\mathcal{I}\mathfrak{m}(\mathrm{Tr}(A \circ B^\dagger)) \ .$$

The infinitesimal generator of the action at $A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ for $\xi \in \mathfrak{u}(k)$ is given by

$$\frac{d}{dt}\left(A \circ \exp t\xi\right)\Big|_{t=0} = -A \circ \xi \ .$$

Note that $\Psi_u^*\theta = \theta$, then, by Remark 1.5.7, we can compute the momentum map as follows

$$2\langle\mu(A)|\xi\rangle = -2\theta_A(-A\circ\xi) = -\mathcal{I}\mathfrak{m}(\mathrm{Tr}(A\circ\xi^\dagger\circ A^\dagger)) = \mathcal{I}\mathfrak{m}(\mathrm{Tr}(A^\dagger\circ A\circ\xi)) = -i\mathrm{Tr}(A^\dagger\circ A\circ\xi) \ .$$

Hence, $\langle\mu(A)|\xi\rangle = -\frac{i}{2}\mathrm{Tr}(A^\dagger \circ A \circ \xi)$. This means that

$$\mu(A) = -\frac{i}{2}(A^\dagger \circ A) \in \mathfrak{u}(k)^* \ .$$

The momentum map is $\mathrm{Ad}^*_{U(k)}$-equivariant, indeed

$$\mu(A \circ u^\dagger) = -\frac{i}{2}(u \circ A^\dagger \circ A \circ u^\dagger) = u \circ \mu(A) \circ u^\dagger \ .$$

Note that

$$\mu^{-1}\left(-\frac{i}{2}\mathbb{I}\right) = \{A \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n) \ : \ A^\dagger \circ A = \mathbb{I}\}$$

is equal to the set of unitary k-frames in $\mathbb{C}^n$, the so called Stiefel manifold $V_k(\mathbb{C}^n)$, and that $U(k)$ acts on $V_k(\mathbb{C}^n)$ freely.

Then, since $-\frac{i}{2}\mathbb{I}$ is central, we can perform a Kähler reduction. We also note that

$$V_k(\mathbb{C}^n)/U(k) = \mathrm{Gr}_k(\mathbb{C}^n) \ .$$

This shows in particular that the complex Grassmannians are Kähler manifolds.

Thanks to our construction, the product $V_k(\mathbb{C}^n) \times V_k(\mathbb{C}^n)$ carries a non integrable almost complex structure.

We can also consider a p-dimensional torus in $U(k)$, $p \leq k$, and obtain a free action by $\mathbb{C}^p/\Lambda$ on $V_k(\mathbb{C}^n) \times V_k(\mathbb{C}^n)$, which is then foliated by tori. In particular, if $p = 1$, the orbits are J-holomorphic curves (see [75]).

2.3.2 Infinite Calabi-Eckmann manifolds

Consider a complex Hilbert space $\mathcal{H}$ and its unit sphere $S(\mathcal{H})$. Theorem 2.2.4 tells us that the product $\mathbb{S}^{2n+1} \times S(\mathcal{H})$ is holomorphic with complex structure J and then we have the following.

Theorem 2.3.1. ($\star$) *There cannot be a J-invariant symplectic structure ω on $\mathbb{S}^{2n+1} \times S(\mathcal{H})$ such that $\omega(\cdot, J\cdot)$ is Riemannian. In other words, $\mathbb{S}^{2n+1} \times S(\mathcal{H})$ cannot be Kähler with respect to the complex structure J.*

Proof. If such a symplectic form ω does exist, the complex submanifolds $j : \mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1} \hookrightarrow \mathbb{S}^{2n+1} \times S(\mathcal{H})$ would be Kähler with respect to $j^*\omega$. This is a contradiction since $\mathbb{S}^{2n+1} \times \mathbb{S}^{2m+1}$ cannot be Kähler, e.g., as we have already recalled, for cohomological reasons, when $n \geq 1$ or $m \geq 1$.

□

Notice that we cannot use the classical cohomological conditions to exclude the presence of a Kähler structure on $\mathbb{S}^{2n+1} \times S(\mathcal{H})$, because, as is well known, $S(\mathcal{H})$ is not compact; see Subsection 5.4.3 of the Appendix. As a matter of fact, $S(\mathcal{H})$ is contractible, so $H^*(\mathbb{S}^{2n+1} \times S(\mathcal{H}), \mathbb{R}) = H^*(\mathbb{S}^{2n+1}, \mathbb{R})$.

Chapter 3

Reduction Principles

Let the pair (M, G) denote a G-manifold M, and let $M' \subseteq M$ be a submanifold which is invariant under the restricted action of a subgroup $G' \subseteq G$ of G. We call the G'-manifold (M', G'), where $G' \circlearrowleft M'$ is the subaction of $G \circlearrowleft M$, a *reduction* of (M, G) if the inclusion $M' \hookrightarrow M$ induces an isomorphism of stratified spaces $M'/G' \to M/G$. This nomenclature is quite spread in the literature; e.g., see [35, 37, 38, 92] and [105]. In the case in which M is endowed with a Riemannian metric and the action by G is isometric with respect to it, it follows that also the subaction by G' is isometric with respect to the induced metric. Then, if the inclusion induces an isomorphism of stratified spaces, which is also an isometry with respect to the orbital distance, we call the reduction (M', G') a *metric reduction*.

Suppose to have a reduction (M', G') of (M, G) and that the action $G \circlearrowleft M$ satisfies a certain property $\mathcal{P}$. We say that the property $\mathcal{P}$ admits a *reduction principle* if the following holds true

$$\mathcal{P} \textit{ holds for } G \circlearrowleft M \textit{ if and only if } \mathcal{P} \textit{ holds for } G' \circlearrowleft M'.$$

It is important to note that a reduction principle depends on both the given reduction and the property considered. The following example shows that, in general, a given reduction is not suited to support a reduction principle, even for fundamental properties.

Example 3.0.1. Let $M = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and let $G = \mathbb{S}^1 \times \mathbb{R}$. Let $(\mathbb{R} \times \mathbb{S}^1) \circlearrowleft \mathbb{T}^2$ be the action defined by $((\tau, t), (\theta, \phi)) \mapsto (\theta + \tau, \phi + t)$. The submanifold $M' := \mathbb{S}^1 \times \{1\} \subset \mathbb{T}^2$ is invariant under the action by the subgroup $G' := \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times \mathbb{R}$, and the inclusion $M' \hookrightarrow M$ clearly induces an isomorphism of stratified spaces. It follows that (M', G') is a reduction of (M, G). However, it is clear that if $\mathcal{P}$ is the properness of the action, $\mathcal{P}$ does not admit a reduction principle.

One possible question we can ask about the reduction and the reduction principles of a G-manifold, is whether there exists a particular reduction for which many properties admit a reduction principle. The answer is provided by the *core* of the action. As we shall see, the reduction to the core is associated with many reduction principles.

In this chapter, we present reduction principles associated with the reduction to the core for different classes of actions: *polar*, *coisotropic*, *coisotropic Hamiltonian* and *infinitesimally almost homogeneous* actions.

All the results in the following sections are based on the preprint [2].

3.1 Reduction to the Core

Let M be a smooth manifold, and let $G \circlearrowleft M$ be a proper action such that M/G is connected. Thanks to Theorem 1.2.27, the action is also isometric with respect to some complete Riemannian metric $\mathbf{g}$. In the following, we will always refer to the metric structure on M induced by $\mathbf{g}$. Let $K \subset G$ be a subgroup of G.

Definition 3.1.1. We define the following spaces.

i) $M^K := \{x \in M \ : \ k.x = x, \text{ for all } k \in K\} = \{x \in M \ : \ G_x \supseteq K\}$;

ii) $M_K := \{x \in M \ : \ G_x = K\}$;

iii) $M_{(K)} := \{x \in M \ : \ (G_x) = (K)\}$.

We immediately see that $G(M_K) = M_{(K)}$, hence $M_{(K)}$ is G-invariant and contains all the points with orbit type (K). This spaces are all $N(K)$-invariant. This is obvious for $M_{(K)}$. For M^K take $x \in M^K$ and $n \in N(K)$, then for all $k \in K$, we have $k.(n.x) = kn.x = n((n^{-1}kn).x) = n.x$, because $n^{-1}kn \in H$. Similarly, the same argument holds for M_K. Note also that $M_K \subseteq M^K$ and $M_K \subseteq M_{(K)}$. It is not restrictive to ask that K is closed, because $M^K = M^{\overline{K}}$, $M^K = M_{\overline{K}}$, and $M_{(K)} = M_{(\overline{K})}$. This means that we can assume that the subactions $N(K) \circlearrowleft M^K$, $N(K) \circlearrowleft M_K$ are proper. Finally, we see that given $x \in M_K$, $G(x) \cap M_K = N(K)(x)$. In fact, let $g.x \in G(x) \cap M_K$, hence $G_{g.x} = gG_xg^{-1} = G_x = K$, therefore $g \in N(K)$.

Theorem 3.1.2. *The connected components of M^K, M_K, and $M_{(K)}$ are embedded submanifolds of possibly different dimension. In particular, M^K is closed and for all $x \in M^K$*

$$T_x(M^K) = (T_xM)^K := \{v \in T_xM \ : \ (d\Psi_k)_x(v) = v, \ \textit{for all } k \in K\} \ .$$

Moreover, the connected components of M^K are totally geodesic submanifolds of $(M, \mathbf{g})$.

Proof. Take $x \in M^K$ and $\varepsilon > 0$ such that $\exp_x : B_\varepsilon^{T_xM}(0) =: B \to M$ is a diffeomorphism with the image. Then $\exp_x((\mathrm{d}\Psi_k)_x(v)) = \Psi_k(\exp_x(v)) = \exp_x(v)$ if and only if $(\mathrm{d}\Psi_k)_x(v) = v$. Therefore, $\exp_x^{-1}$ provides submanifold charts for M^K. This also shows that M^K is a totally geodesic submanifold.

Let $x \in M_K$, S_x be a slice at x and let $\mathfrak{T}(S(x))$ be the tubular neighborhood of $G(x)$. Given $[g, s] \in \mathfrak{T}(G(x)) \cap M_K$, we have $G_{[g,s]} = gG_sg^{-1} = K$ if and only if $G_s = K$ and $g \in N(K)$. Therefore,

$$\mathfrak{T}(G(x)) \cap M_K = (N(K)) \times_K (S_x)^K \ . \tag{3.1}$$

Moreover,

$$\begin{aligned} \mathfrak{T}(G(x)) \cap M_{(K)} &= G(\mathfrak{T}(G(x)) \cap M_K) = \\ &= G((N(K)) \times_K (S_x)^K) = G \times_K (S_x)^K \ . \end{aligned} \tag{3.2}$$

Therefore, M_K and $M_{(K)}$ are embedded submanifolds of M.

∎

As one can expect, the topological and the differential information on the quotient $M_{(K)}/G$ are the same of $M_K/(N(K))$. In fact,

Proposition 3.1.3. *The inclusion $M_K \hookrightarrow M_{(K)}$ induces a diffeomorphism $M_K/(N(K)) \to M_{(K)}/G$. Moreover, there is a G-equivariant diffeomorphism $\rho : G \times_{N(K)} M_K \to M_{(K)}$, $[g, x] \mapsto g.x$.*

Proof. We have already observed that $G(x) \cap M_K = N(K)(x)$. It follows that the inclusion $M_K \hookrightarrow M_{(K)}$ induces a bijective map in the quotients $M_K/(N(K)) \to M_{(K)}/G$. Equations (3.1) and (3.2) imply that the map ρ is a local diffeomorphism, hence a diffeomorphism.

Always from equations (3.1) and (3.2) we conclude that the map ρ is a local diffeomorphism, and since $M_{(K)} = GM_K$, it is also surjective. We prove injectivity by contradiction. Suppose $\rho([g, x]) = \rho([h, y])$, for $x, y \in M_K$ and $g.h \in G$, i.e. $g.x = h.y$. Then $N(K)(x) \ni (h^{-1}g).x = y \in N(K)(x)$, hence $h^{-1}g \in N(K)$. Therefore, $[g, x] = [gg^{-1}h, h^{-1}g.x] = [h, y]$. Thus, we proved that ρ is a bijective local diffeomorphism, hence a diffeomorphism. □

We now focus on the principal stratum M_{princ}. Let us fix a principal isotropy $G_x = H$, for $x \in M_{\text{princ}}$ so that $M_{\text{princ}} = M_{(H)} = GM_H$. In the following, we will denote the normalizer $N(H)$ of H inside G simply as N. Since G_x acts trivially on the slice S_x, by Proposition 3.1.3, one has that M_{princ} is a homogeneous bundle over G/N with characteristic fiber M_H. Moreover, if M/G is connected, by Theorem 1.3.5, we know that M_{princ}/G is connected and by proposition 3.1.3 also M_H/N is. Therefore, the connected components of M_H have the same dimension. It turns out that the natural map $M_H/N \to M_{\text{princ}}/G$ induced by the inclusion is also an isometry with respect to the orbital distance. For future reference, we state this as a proposition.

Proposition 3.1.4. *The inclusion $M_H \hookrightarrow M_{(H)}$ induces an isometry $M_H/(N(H)) \to M_{(H)}/G$.*

Proof. Let $x, y \in M_H$. We need to prove that

$$d_{M_H/N}([x], [y]) := \inf_{n \in N} d(n.x, y) = \inf_{g \in G} d(g.x, y) = d_{M/G}([x], [y]) \ ,$$

where, with an abuse of notation, we use the same symbol for the equivalence classes of M_H/N and of M/G. Since N is a subgroup of G, we clearly have, from the very definition of orbital distance, that $d_{M_H/N}([x], [y]) \geq d_{M/G}([x], [y])$. We shall prove that the infimum over G is achieved inside N. Let $\gamma : [0, l] \to M$ a Kleiner geodesic connecting $G(x)$ with $G(y)$ such that $\gamma(0) = x$ and $\gamma(l) \in G(y)$. By Kleiner's Lemma (see Lemma 1.2.33), we have $G_x \supseteq G_\gamma \subseteq G_{\gamma(l)}$. But $G_x = H$ is principal and hence $G_\gamma = H$. On the other hand, we have $(G_y) = (G_{\gamma(l)}) = (H)$. It follows that $G_{\gamma(l)} = H$, i.e., $\gamma(l) \in G(y) \cap M_H$ and therefore $\gamma(l) \in N(y)$. The equality follows. □

The previous proposition also tells us that M_H could be a good candidate for a reduction of (M, G). The following definition was given in [43] and in [105] for the case of compact group actions.

Definition 3.1.5. Let $G \circlearrowleft M$ be a proper action. Let H be a principal isotropy. We define the *core* of the action as the closure of M_H inside M and we denote it as

$$_cM := \overline{M_H} \ .$$

We observe that the definition essentially does not depend on the particular principal isotropy chosen since, given $H' \in (H)$, the core associated with H' is clearly

G-equivariantly diffeomorphic with the one associated with H. We also notice that there are obvious inclusions

$$M_H \subseteq {}_cM \subseteq M^H \ ,$$

where all the three spaces are N-invariant.

The following will provide a characterization of the core in terms of M_H and M^H.

Proposition 3.1.6. *Let $F \subseteq M^H$ be a connected component of M^H that intersects M_H. Then for all $x \in F$ there exists $z \in S_x$ such that $G_z = H$.*

Proof. Let $G_F := \{g \in G \ : \ gF = F\}$. Clearly, G_F is a non-empty open, closed subgroup of N, which is a closed subgroup of G. Then the G_F-action on F is proper. Let $x \in F$. Since $F \cap M_H \neq \emptyset$, by Lemma 1.2.33, there is a Kleiner geodesic $\gamma : [0, l] \longrightarrow F$ from $G_F(x)$ to $G_F(y)$ with $\gamma(0) = x$ and $\gamma(l) = y \in F \cap M_H$. Since $G_y = H$ is principal, we have that $(G_F)_\gamma = (G_F)_y = H$. Now, γ is a horizontal geodesic. In particular, $v = \gamma'(0) \in (T_xG(x))^\perp$. Let S_x be a slice at x. It follows that for $\tau > 0$ small enough, $z = \exp_x(\tau v) \in S_x$. Hence, $G_z \subset G_x$. Thus, every element $g \in G$ fixing z must also fix x and since the action is by isometries, it must fix also $\gamma(l)$. We conclude that $G_z = H$. ☐

We immediately get the following corollary.

Corollary 3.1.7. *Let $F_j \subseteq M^H$ for $j \in J$ be the connected components of M^H such that $F_j \cap M_H \neq \emptyset$. Then,*

$$ {}_cM = \bigsqcup_{j \in J} F_j \ .$$

Moreover, we have

$$ {}_cM = \{x \in M \ : \ \forall S_x \text{ slice at } x \ \exists z \in S_x \text{ such that } G_z = H\} \ .$$

Remark 3.1.8. A notable consequence of Proposition 3.1.6 is that for all $x \in {}_cM$, we have

$$N(x) = G(x) \cap {}_cM \ .$$

In fact, let S_x be a slice at x and suppose $g.x \in {}_cM$. Then $S_{g.x} := gS_x$ is a slice at $g.x$. By Proposition 3.1.6, we have that there exists $z \in S_x$ such that $G_z = H$. Similarly, there exists $y \in S_{g.x}$ such that $y = kg.z$ for some $k \in G_{g.z}$ and $G_{kg.z} = H$. Let $n := kg$, hence $n.x = g.x$. Moreover,

$$nHn^{-1} = kgG_zg^{-1}k^{-1} = kG_{g.z}k^{-1} = G_{kg.z} = H \ .$$

By Corollary 3.1.7, the core is a closed, embedded and totally geodesic submanifold of M of dimension $\dim({}_cM) = \dim(M/G) + \dim(N/H)$ (i.e. the connected components of ${}_cM$ have all the same dimension). It follows that $({}_cM, \mathbf{g}|_{{}_cM})$ is a complete Riemannian submanifold of $(M, \mathbf{g})$, $({}_cM/N, d_{{}_cM/N})$ is a complete metric space, and $\pi : {}_cM \to {}_cM/N$ is a submetry. Moreover, we have the following proposition.

Proposition 3.1.9. *The inclusion ${}_cM \hookrightarrow M$ induces an isometry ${}_cM/N \to M/G$.*

Proof. M_H/N is open and dense in ${}_cM/N$. Similarly, M_{princ}/G is open and dense in M/G. By Proposition 3.1.4, we have that the induced map $\Theta : {}_cM/N \to M/G$ restricts to an isometry $\Theta\big|_{M_H/N} : M_H/N \to M_{\text{princ}/G}$. Since both $(M/G, d_{M/G})$ and $({}_cM/N, d_{{}_cM/N})$ are complete metric spaces, it is easy to check that Θ is the unique isometry that extends $\Theta\big|_{M_H/N}$. ☐

An important consequence of 3.1.9 is the following, from which we will derive the fact that $({}_cM, N)$ is a metric reduction of (M, G).

Lemma 3.1.10. *Let V be a real finite representation of a compact group G and let H be a principal isotropy. Then the fixed point set of G, V^G is equal to the fixed point set of N, V^N.*

Proof. Let N denote the normalizer of H in G. Since $N \subseteq G$, we have $V^G \subseteq V^N$. We shall prove the reverse inclusion. Since G is compact, we can find a G-invariant inner product. Then we can consider the subaction of G on the unit sphere of V, $M := S(V)$. Clearly, the core of $G \circlearrowleft M$ is given by the unit sphere of V^H, ${}_cM = S(V^H)$. Observe that if $x \in {}_cM$ is a fixed point of $N \circlearrowleft {}_cM$, also $-x \in {}_cM$ is fixed by the N-action. By Proposition 3.1.9, we have that the natural map ${}_cM/N \longrightarrow M/G$ is an isometry and hence

$$\pi = d_{{}_cM}(x, y) = d_{{}_cM/N}\Big([x], [y]\Big) = d_{M/G}\Big([x], [y]\Big) = \inf_{g \in G} d_M(g.x, y) \ .$$

Note that in the sphere, any two non-antipodal points have distance $< \pi$. It follows that x is a fixed point of the G-action on M and since $G \circlearrowleft V$ is a representation, it is a fixed point of the G-action on V. □

Recall that there is an isomorphism $\mathfrak{g}/\mathfrak{g}_x \cong T_xG(x)$. Then we have seen that the G_x-action on $T_xG(x)$ corresponds to the adjoint action of G_x on $\mathfrak{g}/\mathfrak{g}_x$. It follows that if $x \in {}_cM = \overline{M_H}$, then $\big(\mathfrak{g}/\mathfrak{g}_x\big)^H = \mathfrak{n}/\mathfrak{g}_x$, where $\mathfrak{n}$ is the Lie algebra of the normalizer $N = N(H)$. Hence $\Big(T_xG(x)\Big)^H = T_xN(x)$ and therefore, we get the following.

Proposition 3.1.11. *Let $x \in {}_cM$ and S_x be a slice for the G-action on M. Then $(S_x)^H$, the fixed point set for the H-action on S_x, is a slice at x for the N-action on ${}_cM$.*

In order to prove that $({}_cM, N)$ is a reduction of (M, G) we are left to prove that there is a bijection between the orbit strata of $G \circlearrowleft M$ and $N \circlearrowleft {}_cM$.

Let $x \in M$, and let $K := G_x$. The Tubular Neighborhood Theorem implies that S_x/K is an open neighborhood of $[x] \in M/G$. We have that given $y \in S_x$, $[y]$ belongs to the same orbit type stratum of x if and only if $y \in (S_x)^K$, i.e. if y is a fixed point for the K-action on S_x. Now assume $x \in {}_cM$, and recall that the K-action on S_x is equivalent to the slice representation $K \to \mathrm{O}(V)$, where $V = (T_xG(x))^\perp$. As discussed above, the slice representation for the N-action on ${}_cM$ at x is the subrepresentation $(N_K(H))_x \to \mathrm{O}(V^H)$. Then, Lemma 3.1.10 implies that $[y] \in {}_cM/N$ belongs to the same N-orbit type stratum of $[x] \in {}_cM$ if and only if $[y] \in M/G$ belongs to the same G-orbit type stratum of $[x] \in M/G$.

We have thus proved the main theorem of this section.

Theorem 3.1.12. ($\star$) *Let $G \circlearrowleft M$ be a proper action, $H \subseteq G$ a principal isotropy and ${}_cM$ the core associated with H. Then $({}_cM, N)$ is a metric reduction of (M, G). In particular,*

$$({}_cM)_{\text{princ}} = M_{\text{princ}} \cap {}_cM \ .$$

Another important property of the core is that it carries all information about G-invariant functions on M. This is the content of the following result, which can be seen as a generalization to the context of proper group actions of the celebrated Chevalley Restriction Theorem (see [68, 92, 102]).

Theorem 3.1.13. ($\star$) *Let $G \circlearrowleft M$ be a proper action, $H \subseteq G$ a principal isotropy and ${}_cM$ the core associated with H. Then the inclusion $\iota : {}_cM \hookrightarrow M$ induces an isomorphism between the smooth G-invariant functions on M and the N-invariant functions on ${}_cM$*

$$\iota^* : C^\infty(M)^G \to C^\infty({}_cM)^N \ .$$

The proof is based on the following observation (see also [29]).

Remark 3.1.14. Let $G \circlearrowleft M$ be a proper action. The set of smooth G-invariant functions on M, $C^\infty(M)^G$ is in bijection with the smooth functions on M/G, $C^\infty(M/G) := \{f : M/G \to \mathbb{R} \ : \ \exists \tilde{f} \in C^\infty(M), \pi^* f = \tilde{f}\}$, where $\pi : M \to M/G$ is the canonical projection. The stratification of M into orbit types $\{M_i\}_{i \in I}$ induces a stratification of M/G into smooth manifolds $\{M_i/G\}_{i \in I}$. Then $C^\infty(M/G)$ can be seen as the space of continuous functions $\{f : M/G \to \mathbb{R}\}$ such that the restrictions to each strata of M/G is smooth, i.e., $f|_{M_i} \in C^\infty(M_i)$.

Proof. (Theorem 3.1.13) By Theorem 3.1.12, we have a homeomorphism $M/G \simeq {}_cM/N$ which is an isomorphism of stratified spaces. This induces an isomorphism $C^\infty(M/G) \simeq C^\infty({}_c(M)/N)$. We conclude by Remark 3.1.14 that we have isomorphisms

$$C^\infty(M)^G \simeq C^\infty(M/G) \simeq C^\infty({}_cM/N) \simeq C^\infty({}_cM)^N \ .$$

□

Remark 3.1.15. It is clear that H is the ineffective kernel of the N-action on ${}_cM$. If we set $L := N/H$ we get an effective action of L on ${}_cM$ and all the previous results hold also for the L-action. This is a slight further improvement of the reduction process in the sense that, in principle, we are considering a smaller group. Moreover, it is well known that the connected component of the identity, N^0, of N coincides with the connected component of the identity of $Z(H)H$, where $Z(H)$ is the centralizer of H in G. It follows that the connected components of the L-orbits in ${}_cM$ are the same as the $Z(H)^0$-orbits in ${}_cM$.

3.2 Polar Actions

Let $G \circlearrowleft M$ be a proper isometric polar action on a complete Riemannian manifold M. A reduction principle for polar actions appears to be known to experts (see [45]), but since no proof seems to be available, we give a proof here.

In the following, we will fix a principal isotropy H and consider the respective core ${}_cM = \overline{M}_H$. We denote the normalizer of H in G simply by N and set $L := N/H$.

Theorem 3.2.1. ($\star$) *Let G be a Lie group acting properly and isometrically on M. Then the G-action on M is polar (hyperpolar, respectively) if and only if the L-action on ${}_cM$ is polar (hyperpolar, respectively).*

Proof. Since L and N have the same orbits, we can equivalently prove the theorem for the N-action. Suppose that the G-action is polar. Let $x \in M^{\text{princ}}$, so that by Proposition 1.4.2 we have that $\Sigma := \exp_x(\nu_x(G(x))$ is the unique section through x. Since x is principal, H acts trivially on $\nu_x(G(x))$ and hence also on Σ. Therefore, $\Sigma \subseteq {}_cM$. Moreover, since for all $y \in {}_cM$, $G(y) \cap {}_cM = N(y)$, we have $T_y\Sigma \subseteq (T_y(G(y)))^\perp \subseteq (T_y(N(y)))^\perp$. In conclusion, Σ meets all the N-orbits orthogonally.

Assume now that N acts polarly on ${}_cM$. Let $x \in ({}_cM)^{\text{princ}} = M_H$ and let Σ be a section through x. We notice that Σ is totally geodesic in M because it is totally geodesic in ${}_cM$ and ${}_cM$ is totally geodesic in M. Moreover, by Proposition 1.4.2, $\Sigma := \exp_x(\nu_x(N(x)) \subseteq T_x({}_cM))$ is the unique section through x. By reduction to the core, ${}_cM$ is locally diffeomorphic to $L \times (S_x)^H = L \times S_x$, $S_x = \exp_x(\nu_x(G(x)) \cap B_\varepsilon(x))$. From this we see that $\Sigma = \exp_x(\nu_x(G(x)))$ and hence Σ intersects the G-orbits orthogonally. By hypothesis Σ meets all the N-orbits, and hence all the G-orbits. Finally, since ${}_cM$ is totally geodesic, all the previous arguments also hold for the case of a hyperpolar action. □

Together with Remark 1.4.7, Theorem 3.2.1 implies the following.

Corollary 3.2.2. *Let F be a connected component of ${}_cM$. Then N^0 acts polarly, respectively hyperpolarly, on F if and only if G acts polarly, respectively hyperpolarly, on M.*

We can also prove that the infinitesimally polar action admits a reduction principle (see Definition 1.4.8).

Theorem 3.2.3. ($\star$) *A proper isometric action $G \circlearrowleft M$ is infinitesimally polar if and only if the action $L \circlearrowleft {}_cM$ is infinitesimally polar.*

Proof. The claim follows immediately from a result of Lytchak and Thorbergsson (see [72]) where they prove that an action is infinitesimally polar if and only if the quotient is a Riemannian orbifold. By the reduction to the core (Theorem 3.1.12), we have that $M/G \simeq {}_cM/L$, and the result follows. However, we can also provide a direct proof as follows. Let $x \in {}_cM$. First, we have seen that the slice representation of $N_x := N \cap G_x$ on $\nu_x(N(x)) \cap T_x({}_cM)$ is the core of the slice representation of G_x on $\nu_x(G(x))$. Then, by Theorem 3.1.12, one of these two representations is polar if and only if the other is. We can conclude that this condition actually holds for every $y \in M$ since ${}_cM$ intersects every G-orbit and because if the slice representation at x is polar, then the slice representation at every $y \in G(x)$ is polar. □

Not only the polar condition undergoes a reduction principle, but also the following invariant, called the *copolarity* of the action, which was introduced in [37]. This quantity tells us how far the action is from being polar.

Definition 3.2.4. Let G act on M properly and isometrically and let k be a nonnegative integer. A k-section for the G-action on M is a connected, complete submanifold Σ of M such that the following conditions hold:

i) Σ is totally geodesic;

ii) Σ intersects all the G-orbits;

iii) for all $x \in \Sigma \cap M^{\text{princ}}$, $(T_xG(x))^\perp \subseteq T_x\Sigma$ is a subspace of codimension k in $T_x\Sigma$;

iv) for all $x \in \Sigma \cap M^{\text{princ}}$, we have that if $gx \in \Sigma$, then $g\Sigma = \Sigma$.

The integer copolarity of the action is defined as the integer

$$\operatorname{copol}(M, G) := \min\{k \in \mathbb{N} \ : \ \text{there exists a } k\text{-section } \Sigma \subseteq M\}$$

It is clear that for a polar action $G \circlearrowleft M$, we have $\operatorname{copol}(M, G) = 0$. Moreover, as we have seen, ${}_cM \subseteq M$ provides a k-section with $k = \dim(N/H)$, from which we can conclude that

$$0 \le \operatorname{copol}(M, G) \le \dim(N/H)\ . \tag{3.3}$$

Theorem 3.2.5. ($\star$) *Let G act on M properly and isometrically. Let ${}_cM$ be the core associated with a principal isotropy H and let $L = N/H$. Then* $\operatorname{copol}(M, G) = \operatorname{copol}({}_cM, L)$.

Proof. Let $x \in {}_cM$ and let Σ be a k-section for the L-action on ${}_cM$ passing through x. Then Σ is totally geodesic in M and by Remark 3.1.8, Σ intersects all the G-orbits. Moreover, since $G(x) \cap {}_cM = L(x)$ implies $\operatorname{codim}(L(x), {}_cM) = \operatorname{codim}(G(x), M)$, item *iii*) is satisfied. We only need to check item *iv*). So take $y \in M_{\text{princ}}$ and $g \in G$ such that $gy \in \Sigma$ then, since $G(y) \cap {}_cM = L(y)$, there exists $n \in N$ such that $g = nh$ for some $h \in H$ and hence $g\Sigma \subseteq \Sigma$.

Conversely, let Σ be a k-section for the G-action on M. Then, by inequality (3.3), we have $\Sigma \subseteq {}_cM$. Since Σ intersects all G-orbits, it intersects all the the L-orbits. Moreover, again since $G(y) \cap {}_cM = L(y)$ implies that $\operatorname{codim}(L(x), {}_cM) = \operatorname{codim}(G(x), M)$ we get item *iii*). Item *iv*) is clear since $({}_cM)_{\text{princ}} = M_{\text{princ}} \cap {}_cM$. ☐

3.3 Coisotropic actions

Consider a symplectic manifold (M, ω) and $\Psi : G \circlearrowleft (M, \omega)$ a proper symplectic action. Then Ψ is also an almost Kähler action $\Psi : G \circlearrowleft (M, \omega, J, \mathbf{g})$, where $\mathbf{g}$ is a G-invariant complete Riemannian metric and J is the almost complex structure associated with ω and $\mathbf{g}$.

We start with the following central definition.

Definition 3.3.1. We call a symplectic action $G \circlearrowleft (M, \omega)$ *coisotropic* if there exists an open and dense subset $U \subseteq M$ such that for all $x \in U$, $G(x)$ is a coisotropic submanifold of (M, ω).

We also give another definition, which was first introduced in [43], where a motivation for the name is given in terms of representation theory.

Definition 3.3.2. We call a symplectic action $G \circlearrowleft (M, \omega)$ *multiplicity free* if the Lie subalgebra $(\mathcal{C}^\infty(M)^G, \{\cdot, \cdot\})$ is Abelian.

The following proposition was known in the case of G compact (see e.g., [62]).

Proposition 3.3.3. ($\star$) *Let $G \circlearrowleft (M, \omega)$ be a proper symplectic action. Then Ψ is coisotropic if and only if it is multiplicity free.*

Proof. Suppose $G \circlearrowleft M$ is coisotropic. Let $U \subseteq M$ be open and dense, such that $G(x)$ is coisotropic for all $x \in U$. Then for all $f, h \in \mathcal{C}^\infty(M)^G$, we have

$$\{f, h\}(x) = \omega_x(X_f(x), X_h(x)) = 0 \ ,$$

because, by Proposition 1.5.5, $X_f(x), X_h(x) \in (T_x(G(x))^\S \subseteq T_x(G(x))$. Then $\{f, h\}(x) = 0$ for all $x \in M$.

Vice versa, suppose $G \circlearrowleft M$ is multiplicity free. Take $x \in M_{\text{princ}}$. Then by the Tubular Neighborhood Theorem, there exists a G-invariant open neighborhood of $G(x)$,

$$U \simeq G \times_{G_x} (T_x(G(x)))^\perp \simeq (G/G_x) \times (T_x(G(x)))^\perp \ .$$

We take $f_1, \cdots, f_k \in \mathcal{C}^\infty(U)^G$, $k = \dim((T_x(G(x)))^\perp) = \operatorname{cohom}(G, M)$, such that $(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k)|_{G(x)} \neq 0$ and $G(x) = \{y \in U \ : \ f_j(y) = 0, \ j = 1, \cdots k\}$. Note that these functions can be the coordinate functions on $(T_x(G(x)))^\perp$ pulled-back to the product $(G/G_x)\times(T_x(G(x)))^\perp$ by the projection on the second factor. Using smooth cutoff functions we can extend the f_j to $\tilde{f}_j \in \mathcal{C}^\infty(M)^G$ such that $\tilde{f}_j|_{W\subseteq U} = f_j$, where W is G-invariant. Then $X_{\tilde{f}_j} \in T_y(G(x))$, $\forall y \in G(x)$, because $0 = \{\tilde{f}_i, \tilde{f}_j\}(y) = (\mathrm{d}\tilde{f}_i)_y(X_{\tilde{f}_j}(y))$. By construction, $\mathrm{span}(\{X_{\tilde{f}_j}(y)\}_{1\leq i\leq k}) = T_y(G(x))$ and $X_{\tilde{f}_j}(y) \in (T_y(G(x)))^\S$. We conclude that if $x \in M_{\mathrm{princ}}$, then $G(x)$ is coisotropic, and hence $G \circlearrowleft M$ is coisotropic. $\square$

We actually proved that $G \circlearrowleft M$ is coisotropic if and only if $G(x)$ is coisotropic for all $x \in M_{\mathrm{princ}}$. We can prove more.

Proposition 3.3.4. *If $G \circlearrowleft M$ is coisotropic, then $G(x)$ is coisotropic for all $x \in M_{reg}$.*

Proof. Let $G(x)$ be a regular orbit. If S_x is a slice at x, there is a geodesic segment $\gamma : [0,1] \to M$, $\gamma(t) := \exp_x(tv)$, with $v \in (T_x(G(x)))^\perp$ such that $\gamma(t) \in M_{\mathrm{princ}} \cap S_x$, $\forall t \in (0,1]$. Then $G(\gamma(t))$ is coisotropic $\forall t \in (0,1]$. Let $Y_1, \cdots, Y_k$ vector fields along γ such that $\mathrm{span}(\{Y_j(t)\}_{1\leq j\leq k}) = (T_{\gamma(t)}G(\gamma(t)))^\perp$, $\forall t \in [0,\varepsilon]$, $k = \mathrm{cohom}(G,M)$.

$$\mathrm{span}(\{J_{\gamma(t)}Y_j(t)\}_{1\leq j\leq k}) = (T_{\gamma(t)}G(\gamma(t)))^\S, \quad \forall t \in [0,\varepsilon] \ .$$

Take $v \in (T_x(G(x)))^\S$, there are $\alpha^1, \cdots, \alpha^k$ such that $v = \alpha^i Y_i(0)$ and for all $1 \leq j \leq k$, we have

$$\mathbf{h}_x(v, Y_j(0)) = \lim_{t\to 0} \mathbf{h}_{\gamma(t)}(\alpha^i J_{\gamma(t)} Y_i(t), Y_j(t)) = 0 \ .$$

We conclude $v \in T_xG(x)$, because $(T_{\gamma(t)}G(\gamma(t)))^\S \subseteq T_{\gamma(t)}G(\gamma(t))$, $\forall t \in (0,1]$. $\square$

Reduction Principle

Let H be a principal isotropy and let ${}_cM$ be the associated core. We start with the following observation.

Proposition 3.3.5. *The core ${}_cM$ is a symplectic submanifold of (M,ω)*

Proof. Let $x \in {}_cM$ and let $v \in T_x({}_cM) \cap (T_x({}_cM))^\S$. We have $v \in T_x({}_cM)$, if and only if for all $h \in H$, $(\mathrm{d}h)_x v = v$. By equivariance of J, we have for all $h \in H$

$$(\mathrm{d}h)_x(J_x v) = J_x((\mathrm{d}h)_x v) = J_x v \ .$$

Therefore, $J_x v \in T_x({}_cM)$. Recall that we can write $(T_x({}_cM))^\S = J_x(T_{x\,c}M)^\perp$ and hence there exists $w \in (T_x({}_cM))^\perp$ such that $v = J_x w$. Finally, we check that

$$\mathbf{h}_x(v,v) = \mathbf{h}_x(v, J_x w) = -\mathbf{h}_x(J_x v, w) = 0 \ ,$$

from which $v = 0$. $\square$

The previous proposition tells us that $({}_cM, \omega|_{{}_cM}, J|_{{}_cM}, \mathbf{h}|_{{}_cM})$ is an almost Kähler manifold, and $N(H) \circlearrowleft {}_cM$ is a proper and almost Kähler action. We further recall that we can equivalently study the action by $L := N(H)/H \circlearrowleft {}_cM$, which is the effective action associated with $N(H) \circlearrowleft {}_cM$.

The following property of coisotropic actions will be refined when we will talk about Hamiltonian actions.

Proposition 3.3.6. *Let $G \circlearrowleft (M,\omega)$ be a symplectic coisotropic action and H a principal isotropy. Then $dim(N(H)/H) \geq cohom(G,M)$.*

Proof. Let $x \in M_H$, hence $G(x)$ is coisotropic. The slice representation at x is trivial and commutes with J_x, hence H fixes $J_x(T_xG(x))^\perp = (T_xG(x))^\S \subseteq T_xG(x)$, i.e.,

$$((T_xG(x))^\perp)^H = (T_xG(x))^\perp \subseteq J_x(T_xG(x)) \ .$$

By the tubular neighborhood theorem, $(T_xG(x))^H = T_x(N(H)(x))$, hence $(T_xG(x))^\perp \subseteq (J_x(T_xG(x)))^H$, and

$$\mathrm{cohom}(G, M) = \dim(T_xG(x))^\perp \leq \dim(N(H)(x)) = \dim(N(H)/H) \ .$$

□

For the proof of the reduction principle, we will need the following lemma.

Lemma 3.3.7. *Let H be a principal isotropy and $L := N(H)/H$ the closed subgroup acting properly on ${}_cM$, and $x \in ({}_cM)_{princ}$. Then $L(x)$ is coisotropic in ${}_cM$ if and only if $G(x)$ is coisotropic in M.*

Proof. By the tubular neighborhood theorem, if $x \in ({}_cM)_{\mathrm{princ}}$, there is a G-invariant neighborhood of $G(x)$ such that $U \simeq (G/H) \times (T_x(G(x)))^\perp$. Then,

$$U \cap ({}_cM) \simeq L \times (T_x(G(x)))^\perp \ .$$

Hence $T_xL(x) = (T_xG(x))^H$. We have $(T_xG(x))^\S = J((T_xG(x))^\perp)$, and since the slice representation is trivial, $(T_xG(x))^\S \subseteq (T_xM)^H$. Therefore, $(T_xG(x))^\S = J((T_xG(x))^\perp) \subseteq T_xG(x)$ if and only if

$$(T_xL(x))^\S = J((T_xL(x))^\perp) \subseteq (T_xG(x))^H = T_xL(x) \ .$$

This concludes the proof. □

The reduction principle then easily follows thanks to the previous remark. We are ready to prove the reduction principle for coisotropic actions.

Theorem 3.3.8. ($\star$) *Let $G \circlearrowleft (M, \omega)$ proper symplectic action such that M/G is connected. Let H be a principal isotropy and $L = N(H)/H$. Then $G \circlearrowleft M$ is coisotropic if and only if $L \circlearrowleft ({}_cM)$ is coisotropic.*

Proof. Recall that $G({}_cM)_{\mathrm{princ}} = M_{princ}$. Then by Lemma 3.3.7, we have that given $x \in ({}_cM)_{\mathrm{princ}}$, $L(x)$ is coisotropic in $({}_cM)_{\mathrm{princ}}$ if and only if $G(x)$ is coisotropic in M_{princ}. Since we observed that an action is coisotropic if and only if its principal orbits are coisotropic, we conclude.

□

The proof of the reduction principle can be also seen as a consequence of the Theorem 3.1.13 and the following lemma, whose proof is an adaptation of the same claim for M_H (see Proposition 4.2.9 of [90]).

Lemma 3.3.9. *Let $A : G \circlearrowleft (M, \omega)$ be a proper symplectic action, H a principal isotropy and ${}_cM$ the core relative to H. Then the inclusion $\iota : {}_cM \hookrightarrow M$ satisfies*

$$\iota^*\{f, g\} = \{\iota^* f, \iota^* g\}_{cM} \quad \forall f, g \in \mathcal{C}^\infty(M)^G \ ,$$

where $\{\cdot, \cdot\}_{cM}$ is the restriction of the Poisson bracket to ${}_cM$.

Proof. Take $f \in \mathcal{C}^\infty(M)^G$, then if $x \in {}_cM$, $X_f(x) \in T_x({}_cM)$. In fact, for all $v \in T_xM$ and $h \in H$, we have $\omega_x((\mathrm{d}A_h)_x(X_f(x)), v) = \omega_x(X_f(x), (\mathrm{d}A_h)_x^{-1}(v)) = (\mathrm{d}f)_x((\mathrm{d}A_h)_x^{-1}(v)) = (\mathrm{d}f)_x(v) = \omega_x(X_f(x), v)$. Now $\forall x \in {}_cM$ and $v \in T_x({}_cM)$ we have

$$(\iota^*\mathrm{d}f)_x(v) = (\mathrm{d}(\iota^* f))_x(v) = (\mathrm{d}f)_x((\mathrm{d}\iota)_x(v)) = \omega_x(X_f(x), (\mathrm{d}\iota)_x(v)).$$

On the other hand, we can write also

$$(\iota^*\mathrm{d}f)_x(v) = (\mathrm{d}(\iota^* f))_x(v) = (\iota^*\omega)_x(X_{\iota^* f}(x), v) = \omega_x((\mathrm{d}\iota)_x(X_{\iota^* f}(x)), (\mathrm{d}\iota)_x(v)) \ .$$

Therefore, $\omega_x(X_f(x) - (\mathrm{d}\iota)_x(X_{\iota^* f}(x)), w) = 0$, $\forall w \in T_x({}_cM)$. We use the fact that ${}_cM$ is symplectic, i.e. $\forall x \in {}_cM$, $T_xM = T_x({}_cM) \oplus (T_x({}_cM))^{\S}$. Hence $\omega_x(X_f(x) - (\mathrm{d}\iota)_x(X_{\iota^* f}(x)), v) = 0$, $\forall v \in T_xM$, and

$$X_f(x) = (\mathrm{d}\iota)_x(X_{\iota^* f}(x)) \ .$$

Take now $f, g \in \mathcal{C}^\infty(M)^G$ and therefore $\iota^* f, \iota^* g \in \mathcal{C}^\infty({}_cM)^{N(H)}$. We have $\forall x \in {}_cM$,

$$\{\iota^* f, \iota^* g\}_{cM}(x) = (\iota^*\omega)_x(X_{\iota^* f}(x), X_{\iota^* g}(x)) =$$

$$= \omega_x((\mathrm{d}\iota)_x(X_{\iota^* f}(x)), (\mathrm{d}\iota)_x(X_{\iota^* g}(x))) =$$

$$= \omega_x(X_f(x), X_g(x)) = \{f, g\}(x) = \iota^*\{f, g\}(x) \ .$$

This concludes the proof. □

The proof of the reduction principle then easily follows.

Proof of Theorem 4 using Theorem 5. $G \circlearrowleft M$ is coisotropic if and only if $\mathcal{C}^\infty(M)^G$ is Abelian. But $\mathcal{C}^\infty(M)^G \xrightarrow{\iota^*} \mathcal{C}^\infty({}_cM)^{N(H)}$ is an isomorphism of Poisson algebras, because ι preserves Poisson brackets of invariant functions. Hence $\mathcal{C}^\infty(M)^G$ is Abelian if and only if $\mathcal{C}^\infty({}_cM)^{N(H)}$ is, and therefore if and only if $N(H) \circlearrowleft {}_cM$ is coisotropic.

We can summarize the previous results adding some other characterization in the following Equivalence theorem. We denote as G^0 and $N(H)^0$ the identity components of G and $N(H)$ respectively.

Theorem 3.3.10. ($\star$) *Let $G \circlearrowleft (M, \omega)$ be a proper symplectic action. Assume M/G connected. Then the following conditions are equivalent.*

1. *$G \circlearrowleft (M, \omega)$ is coisotropic;*
2. *$\forall x \in M_{princ}$, $G(x)$ is coisotropic;*
3. *$(\mathcal{C}^\infty(M)^G, \{\cdot, \cdot\})$ is Abelian;*
4. *If H is a principal isotropy, $N(H) \circlearrowleft ({}_cM, \omega|_{cM})$ is coisotropic;*
5. *G^0 acts coisotropically on M or on any connected component of it;*
6. *$N(H)^0$ acts coisotropically on ${}_cM$ or on any connected component of it.*

Proof. We have already proved $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$. We prove the missing ones. Suppose G^0 acts coisotropically. We have $(\mathcal{C}^\infty(M)^{G^0}, \{\cdot,\cdot\})$ is Abelian, and $\mathcal{C}^\infty(M)^{G^0} \supseteq \mathcal{C}^\infty(M)^G$, so $(\mathcal{C}^\infty(M)^G, \{\cdot,\cdot\})$ is Abelian too; Suppose G acts coisotropically on M. We have that $\forall x \in M$, $T_x(G(x)) = T_x(G^0(x))$, so $G(x)$ is coisotropic if and only if $G^0(x)$ is. Suppose G^0 acts coisotropically on a connected component M' of M. Since M/G is connected, for any other connected component M'' there is $g \in G$ such that $gM' = M''$. Then, for all $y \in M''$, there is $x \in M'$ such that $G^0(y) = G^0(g.x) = gG^0(x)$. So, if $U \subseteq M'$ is open and dense, with $G^0(x)$ coisotropic, $G(U)$ is open and dense and $\forall z \in G(U)$, $G(z)$ is coisotropic. The inverse implication is obvious. Similar arguments shows that $N(H)^0$ acts coisotropically on ${}_cM$ or on any connected component of it if and only if $N(H)$ acts coisotropically on M. We conclude using $1 \Leftrightarrow 4$.

□

3.3.1 Infinitesimally Almost-Homogeneous Actions

We can also prove a reduction principle for some slightly more general type of actions. Let (M, g, J, ω) be an almost Kähler manifold and let G be a lie group acting properly on M and preserving the three structures. We also assume that M/G is connected.

Definition 3.3.11. The G-action on M is called *infinitesimally almost homogeneous* if there exists $x \in M$ such that $T_xM = T_xG(x) + J(T_xG(x))$.

Note that from the orthogonal splitting $T_xM = T_xG(x) \oplus T_xG(x)^\perp$, it follows that condition $T_xM = T_xG(x) + J(T_xG(x))$ holds if and only if $T_xG(x)^\perp \subseteq T_xG(x) + J(T_xG(x))$.

Remark 3.3.12. If the G-action on M is coisotropic, then it is infinitesimally almost homogeneous. In fact, let $x \in M^{\mathrm{princ}}$. Then $G(x)$ is coisotropic, and so $J(T_xG(x)^\perp) \subseteq T_xG(x)$. This implies $T_xG(x)^\perp \subseteq J(T_xG(x))$ concluding the argument.

Let G^0 denote the identity component of G. The following lemma is an immediate consequence of Definition 3.3.11, since for all $x \in M$ we clearly have $T_xG^0(x) = T_xG(x)$.

Lemma 3.3.13. *The G-action on M is infinitesimally almost homogeneous if and only if the G^o-action on M is infinitesimally almost homogeneous. If M' is a connected component of M then the G-action on M is infinitesimally almost homogeneous if and only if the G^o-action on M' is infinitesimally almost homogeneous.*

Since the condition $T_xG(x) + J(T_xG(x)) = T_xM$ is an open condition, we get the following result.

Proposition 3.3.14. *The G-action on M is infinitesimally almost homogeneous if and only if there exists $x \in M^{\mathrm{princ}}$ such that $T_xM = T_xG(x) + J(T_xG(x))$.*

We are ready to prove a reduction principle for almost homogeneous actions.

Theorem 3.3.15. ($\star$) *Let G be a Lie group acting properly on (M, ω) by symplectomorphisms. Let H be a fixed principal isotropy, let ${}_cM$ be the core relative to H, $N := N(H)$ and $L := N(H)/H$. The G-action on M is infinitesimally almost homogeneous if and only if the L-action on ${}_cM$ is infinitesimally almost homogeneous.*

Proof. It suffices to prove the theorem for the N-action on ${}_cM$. Let $x \in ({}_cM)_{\mathrm{princ}}$. Then $(T_xN(x))^\perp \cap T_x({}_cM) = (T_xG(x))^\perp$. Since $(T_xG(x))^H = T_xN(x)$ we have

$$(T_xG(x) + J(T_xG(x))^H = T_xN(x) + J(T_xN(x)).$$

The subspace $T_xG(x)^\perp$ is fixed point-wise by H. This implies $(T_xG(x))^\perp \subseteq T_xG(x) + J(T_xG(x))$ if and only if $(T_xG(x))^\perp \subseteq T_xN(x) + J(T_xN(x))$ and so if and only if the orthogonal complement of $T_xN(x)$ in $T_x({}_cM)$ is contained in $T_xN(x) + J\,(T_xN(x))$. □

By Lemma 3.3.13 also the following result holds.

Theorem 3.3.16. *If F is a connected component of the ${}_cM$, then the L^o-action on F is infinitesimally almost homogeneous if and only if the G-action on M is infinitesimally almost homogeneous.*

Remark 3.3.17. The definition 3.3.11 is a generalization of the definition of *almost homogeneous action*; see [61]. Let Z be a compact complex manifold, and U a subgroup of the group of holomorphic automorphisms of Z. In this setting, the action of U is called almost homogeneous if U has an open orbit in Z. In our setting, if we restrict to a compact group G, we can consider its complexification $G^{\mathbb{C}}$, which, in general, does not even act on M (e.g. if M is not compact). Note that if $G^{\mathbb{C}}$ acts on M then the Definition 3.3.11 precisely means that the $G^{\mathbb{C}}$-orbit is open.

3.4 Coisotropic Hamiltonian Actions

Let G be a Lie group acting properly and symplectically on (M, ω) with M/G connected. Throughout this section, we suppose that the G-action is Hamiltonian, with momentum map $\mu : M \to \mathfrak{g}^*$, $\mathfrak{g}^*$ being the dual of the Lie algebra of G.

The momentum map gives a criterion for a G-orbit being symplectic or coisotropic.

Proposition 3.4.1. *Let $x \in M$. The following assertions hold true:*

1. *$G(x)$ is a symplectic submanifold of M if and only if $(G_x)^0 = (G_{\mu(x)})^0$ and so if and only if $\mu : G(x) \to G(\mu(x))$ is a covering map;*
2. *If $x \in \mathrm{princ}$, then $G(x)$ is coisotropic if and only if $\dim(G_{\mu(x)}) - \dim(G_x) = cohom(G, M)$.*

Proof. By Lemma 1.5.9 item *iii*), we have that $G(x)$ is symplectic if and only if $T_xG_{\mu(x)}(x) = 0$ and so if and only if $\mathfrak{g}_x = \mathfrak{g}_{\mu(x)}$. This proves item *(1)*

Since $\dim((T_xG(x))^\omega) = \mathrm{cohom}(G, M)$, we have that item *(2)* follows from Lemma 1.5.9, item *iv*). □

From now on, we will again fix a principal isotropy H, and consider its associated core ${}_cM$, carrying the action of the normalizer N of H, or the effective action of $L = N/H$. We recall a result by Bates and Lerman that we will use in the following. The result is from [12], Theorem 16, page 220, where it is stated for M_H. However, the same claim still holds true for ${}_cM = \overline{M_H}$.

Theorem 3.4.2. *If G-action on M is Hamiltonian, then the L-action on ${}_cM$ is Hamiltonian.*

From now on, we will assume that the coadjoint orbits in $\mathfrak{g}^*$ are embedded submanifolds (e.g., locally closed). In this hypothesis, the momentum map of the action can be written in the so called *normal form*. An extensive and more precise treatment can be found in [12]; here we recall the facts that will be relevant to us.

Let $(M, \omega, J, \mathtt{h})$ be the almost-Kähler manifold associated with (M, ω). Let $x \in M$ and define $\beta := \mu(x) \in \mathfrak{g}^*$. We will fix an $\mathrm{Ad}^*_{G_x}$-invariant metric in $\mathfrak{g}^*$ which will also define an Ad_{G_x}-invariant metric on $\mathfrak{g}$. We can find such metrics because G_x is compact.

Proposition 3.4.3. *The following facts hold true.*

i) $\mathfrak{g}_x$ *and* $\mathfrak{g}_\beta$ *are* G_x*-invariant;*

ii) $V_x := (T_xG_\beta(x))^\perp \cap (T_xG(x))^\S$*, is a symplectic vector subspace of* T_xM*;*

iii) there is a G_x*-invariant orthogonal decomposition of* $\mathfrak{g}$ *given by*

$$\mathfrak{g} = \mathfrak{g}_\beta \oplus \mathfrak{s} = \mathfrak{g}_x \oplus \mathfrak{m} \oplus \mathfrak{s} \ ;$$

iv) $\mathfrak{s}_x := \{\xi_M(x) \ : \ \xi \in \mathfrak{s}\} \subseteq T_xM$ *is a symplectic vector subspace of* T_xM*;*

v) $V_x \cap \mathfrak{s}_x = \emptyset$*, and hence*

$$T_xM = (V_x \oplus \mathfrak{s}_x) \oplus (V_x \oplus \mathfrak{s}_x)^\S \ .$$

Proof. Let $\xi \in \mathfrak{g}_x$, then we have seen that $(\mathrm{Ad}_g\xi)_M(x) = (\mathrm{d}g)_x\xi_M(g^{-1}x) = (\mathrm{d}g)_x\xi_M(x) = 0$, because $g \in G_x$. For $\xi \in \mathfrak{g}_\beta$, we have that $\mu(\exp(t\xi)) = \mathrm{Ad}^*_{\exp(-t\xi)}\beta = \beta$, from which it follows that $\mathrm{ad}^*_\xi\beta = 0$. Then, since $g \in G_x$, $\mathrm{Ad}^*_{g^{-1}}\beta = \mathrm{Ad}^*_{g^{-1}}\mu(x) = \mu(gx) = \mu(x) = \beta$, and we have

$$\mathrm{ad}^*_{\mathrm{Ad}_g\xi}\beta = \mathrm{Ad}^*_g(\mathrm{ad}^*_\xi(\mathrm{Ad}_{g^{-1}}\beta)) = \mathrm{Ad}^*_g(\mathrm{ad}^*_\xi\beta) = 0 \ ,$$

which concludes the proof of item i). To prove item ii), recall that $T_xG_\beta(x) = (T_xG(x)) \cap (T_xG(x))^\S$ from which $(T_xG(x))^\S = T_xG_\beta(x) \oplus V$, hence $V = (T_xG(x))^\S/((T_xG(x)) \cap (T_xG(x))^\S)$ and therefore $(V, \omega|_V)$ is a symplectic subspace of T_xM. Item iii) simply follows from the existence of a G_x-invariant metric and since $\mathfrak{g}_x \subseteq \mathfrak{g}_\beta$. By item v) of Proposition 1.5.9, $\omega|_{\mathfrak{s}_x}$ is nondegenerate, hence $\mathfrak{s}_x$ is symplectic. To prove v) we can simply observe that $\mathfrak{s}_x \subseteq T_xG_\beta(x)$. Therefore, $V_x \oplus \mathfrak{s}_x$ is symplectic. □

Note that $T_xG_\beta(x)$ is a null space in $(V_x \oplus \mathfrak{s}_x)^\S$ and has half the dimension, i.e., it is a Lagrangian subspace. In particular, since $T_xG_\beta(x) \simeq \mathfrak{g}_\beta/\mathfrak{g}_x$, we have $(V_x \oplus \mathfrak{s}_x)^\S \simeq (\mathfrak{g}_\beta/\mathfrak{g}_x) \oplus (\mathfrak{g}_\beta/\mathfrak{g}_x)^*$. We conclude that there is a G_x-invariant splitting

$$(T_xM, \omega_x) \simeq (V_x, \omega_x|_{V_x}) \oplus (\mathfrak{s}_x, \mu^*(\omega|_{G(x)}))\, ((\mathfrak{g}_\beta/\mathfrak{g}_x) \oplus (\mathfrak{g}_\beta/\mathfrak{g}_x)^*, \varsigma) \ ,$$

where ς is the standard symplectic structure on the direct sum of a vector space with its dual (see Appendix 5.4).

We also observe that $V_x \oplus (\mathfrak{g}_\beta/\mathfrak{g}_x^*)$ is a G_x-invariant, complementary subspace for $T_xG(x)$ inside T_xM. From this we can find a G-invariant neighborhood $\mathfrak{T} = \mathfrak{T}(G(x))$ of $G(x)$ such which is G-equivariantly diffeomorphic to a G-equivariant neighborhood of the zero section of the associated vector bundle to the principal bundle $G \mapsto G/G_x$ given by $Y := G \times_{G_x} (V_x \oplus (\mathfrak{g}_\beta/\mathfrak{g}_x)) \to G/G_x$. Using the identification $\varphi : Y \to \mathfrak{T}$, the momentum map can be written in the following form.

$$\varphi^*\mu|_{\mathfrak{T}} =: \mu_Y([(g, v, m)]) = \mathrm{Ad}^*_g(\beta + i(\mu_V(v)) + j(m)) \ ,$$

where $i : \mathfrak{g}_x^* \to \mathfrak{g}^*$ is the inclusion, $j : (\mathfrak{g}_\beta/\mathfrak{g}_x)^* \simeq \mathfrak{m} \to \mathfrak{g}^*$ is the natural map, and $\mu_V : V_x \to \mathfrak{g}_x^*$ is the momentum map of the linear G_x-action on the symplectic vector space $(V_x, \omega_x|_{V_x})$.

3.4.1 Symplectic Orbits

Let $X \in M$. Assume now that $G(x)$ is a symplectic submanifold, i.e., $T_xM = T_xG(x) \oplus (T_xG(x))^{\S}$. From Proposition 1.5.9, item *iii*), it follows that $\mathfrak{g}_x = \mathfrak{g}_\beta$. Hence, by what we have just seen, $Y = G \times_{G_x} V_x$ is diffeomorphic to a G-invariant neighborhood of $G(x)$. Moreover, recalling that J is G-equivariant, and $(T_xG(x))^{\S} = J_x(T_xG(x))^{\perp}$, the linear action of G_x on V, which is called *symplectic slice representation*, is equivalent to the normal slice representation of G_x on $(T_xG(x))^{\perp}$. The symplectic slice representation controls the coisotropy of the G-action on Y. In fact, we have:

Proposition 3.4.4. *Let $G(x)$ be a symplectic orbit of a Hamiltonian action $G \circlearrowleft M$. Then the G-action on $Y = G \times_{G_x} V_x$ is coisotropic if and only if the symplectic slice representation of G_x on V_x is multiplicity free.*

Proof. Let G act coisotropically on Y. Let $v \in V_x^{\mathrm{princ}}$, then $y = [e, v] \in Y^{\mathrm{princ}}$, and so $G(y)$ is coisotropic by Proposition 3.3.4 and the discussion above. By Proposition 1.5.9, item *i*), we have $(T_vG_x(v))^{\S} = \ker(\mathrm{d}\mu_V)_v$. By the local normal form of the momentum map, we have

$$\mu_Y([g, v]) = \mathrm{Ad}_g^*(\beta + i(\mu_V(v))) \ .$$

since Ad_g is linear, we have $(\mathrm{d}\mu_Y)_y([0, w]) = (\mathrm{d}\mu_V)_v(w) = 0$. Therefore, always using Proposition 1.5.9, item *i*), $[0, w] \in \ker(\mathrm{d}\mu_Y)_y = (T_yG(y))^{\S} \subseteq T_yG(y)$. Now, V_x defines a slice for the G-action at $x = [0, e]$ and therefore, $(T_yG(y)) \cap T_y(V_x) = T_yG_x(y)$. It follows that $w \in T_vG_x(v)$, and hence we conclude $(T_vG_x(v))^{\S} \subseteq T_vG_x(v)$. For the converse implication, let $v \in V_x^{\mathrm{princ}}$, so that by the reduction to the core, $y = [e, v] \in Y^{\mathrm{princ}}$. By item *ii*) of Proposition 3.4.1 we have

$$\dim((G_x)_{\mu_v(v)}) - \dim((G_x)_v) = \mathrm{cohom}(V_x, G_x) = \mathrm{cohom}(M, G) \ .$$

It is also clear that since $G_x \subseteq G_\beta$, we have $(G_x)_{\mu_v(v)} \subseteq G_\beta$ and hence, because $\dim((G_x)_v) = \dim(G_y)$, we conclude

$$\dim(G_\beta) - \dim(G_y) \geq \mathrm{cohom}(M, G) \ .$$

By item *iv*) of Proposition 1.5.9, we also have that $\dim(G_\beta) - \dim(G_y) \leq \mathrm{cohom}(M, G)$, from which, again from item *ii*) of Proposition 3.4.1, we conclude that $G(y)$ is coisotropic. Finally, since by the reduction to the core, $Y^{\mathrm{princ}} = GV_x^{\mathrm{princ}}$, the result follows. □

Let now G be a compact connected Lie group. In [62] the *rank* of a G-action on M at $x \in M$ is defined as the codimension of a maximal torus of G_x inside a maximal torus of G. For convenience, we use a similar definition introduced in [99], called the *homogeneity rank* of the G-action, denoted with $\mathrm{homrank}(M, G)$, and defined as

$$\mathrm{homrank}(M, G) = \mathrm{rank}(G) - \mathrm{rank}(G_x) - \mathrm{cohom}(M, G) \ .$$

Note that $\mathrm{homrank}(M, G)$ is constant on an open and dense subset of M. The interest in this particular invariant is due to the fact that the coadjoint G-orbit in $\mathfrak{g}^*$ through $\mu(x)$ is the image of $\mu|_{G(x)}$, and hence is the centralizer of a torus and, in particular, it contains a maximal torus of G. If the G-action is such that $\mathrm{homrank}(M, G) = 0$, M is called a *spherical* G-manifold. Always in [62], it is proved that a spherical G-manifold has coisotropic principal orbit, i.e. the G-action is coisotropic. Putting together this result with Proposition 3.4.4, we get the following.

Corollary 3.4.5. *Let $G \circlearrowleft M$ be a proper Hamiltonian action. Let $x \in M$ be such that $G(x)$ is symplectic. Then the G-action on $Y = G \times_{G_x} V_x$ is coisotropic if and only if* $\mathrm{homrank}((T_xG(x))^\perp, G_x) = 0$. *Moreover, if G is compact and connected, G acts coisotropically if and only if* $\mathrm{homrank}((T_xG(x))^\perp, G_x) = 0$.

Proof. Proposition 3.4.4 together with the G-equivariance of J_x, tells us that the $G \circlearrowleft Y$ is coisotropic if and only if $G_x \circlearrowleft (T_xG(x))^\perp$ is coisotropic. We conclude by [62], Theorem 3, page 267. For the second claim, recall that since $G(x)$ is symplectic, $\mathfrak{g}_x = \mathfrak{g}_{\mu(x)}$, $G_x \subseteq G_{\mu(x)}$ and that G_μ is connected, we conclude $G_x = G_{\mu(x)}$. Therefore, by what we have already observed, G_x contains a maximal torus of G. After a simple computation, we get

$$\mathrm{homrank}((T_xG(x))^\perp, G_x) = \mathrm{homrank}(M, G) \ ,$$

from which we conclude, again, by [62], Theorem 3, page 267. □

3.4.2 Symplectic Stratification

Let $G \circlearrowleft M$ be a proper Hamiltonian action on an almost-Kähler manifold $(M, \omega, J, \mathtt{h})$, let $\mu : M \to \mathfrak{g}^*$ be a momentum map for the action and let $\alpha \in \mathfrak{g}^*$. One defines the so called *reduced space* relative to the value α as the orbit space

$$M_\alpha := \mu^{-1}(G(\alpha))/G \ .$$

The space M_α has the structure of a stratified space, where the strata, called *symplectic pieces*, are symplectic manifolds. There is also a Poisson structure on M_α, $(C^\infty(M_\alpha), \{\cdot,\cdot\}_\alpha)$, where

$$C^\infty(M_\alpha) := C^\infty(M)^G\Big|_{\mu^{-1}(G(\alpha))} \ ,$$

that is, $C^\infty(M_\alpha)$ consists of the restrictions of G-invariant smooth functions on M to the subset $\mu^{-1}(G(\alpha))$. The Poisson bracket $\{\cdot,\cdot\}$ on M_α is the one induced by the Poisson bracket on $C^\infty(M)$.

More precisely, the symplectic pieces of M_α can be described as follows, where we follow [8, 12, 90]. Let $H \subset G$ be an isotropy group of the G-action on M. Then, we recall that any connected component C of $M_{(H)} := \{x \in M \ : \ G_x \text{ is conjugate to } H\}$ is a smooth submanifold of M. Since we assume that the coajoint orbits are locally closed, it follows by the *Symplectic Slice Theorem* (see [90]), that the subset $\mu^{-1}(G(\alpha)) \cap C$ are submanifolds of M such that the topological quotient

$$(M_\alpha)_C := \Big((\mu^{-1}(G(\alpha)) \cap C\Big) \Big/ G \ ,$$

has the structure of a smooth symplectic manifold (see [12]) such that the restriction of the projection $\pi : M \to M/G$ to $\mu^{-1}(G(\alpha)) \cap C \to (M_\alpha)_C$ is a smooth map and the inclusion $(M_\alpha)_C \hookrightarrow M_\alpha$ is Poisson map.

We can now state a first characterization for coisotropic Hamiltonian proper actions.

Lemma 3.4.6. *Let (M, ω) be a symplectic manifold and let G be a Lie group acting in a Hamiltonian fashion on M. Then the G-action on M is coisotropic if and only if M_α is discrete for every $\alpha \in \mathfrak{g}^*$.*

Proof. Suppose that the G-action is coisotropic. Then, by Theorem 3.3.10, $(C^\infty(M)^G, \{\cdot,\cdot\})$ is Abelian. Therefore, also $(C^\infty(M_\alpha), \{\cdot,\cdot\}_\alpha)$ is Abelian. Let $S \subseteq M_\alpha$ be a symplectic piece of M_α. Since $S \hookrightarrow M_\alpha$ is a Poisson map, also the Poisson structure of S is Abelian. Recall that the Poisson structure on S is coming from a symplectic structure, then S is forced to be zero dimensional, i.e., consists of a discrete collection on points.

Conversely, suppose that M_α is discrete. Let $f, g \in C^\infty(M)^G$. Let $x \in M_\alpha$ and C be a connected component of $M_{(G_x)}$ such that $x \in C \cap \mu^{-1}(\alpha)$ and $(M_\alpha)_C$ is a piece of M_α. In particular, $(M_\alpha)_C$ is discrete. Then $\{f, g\}_\alpha(x) = 0$ and since x was arbitrary, $\{f, g\} = 0$ everywhere. □

Let H be a principal isotropy, N its normalizer in G and $L := N/H$. As a corollary of Lemma 3.4.6, we find the following.

Corollary 3.4.7. *Let (M,ω) be a symplectic manifold and let G be a Lie group acting in a Hamiltonian fashion on M. Then the G-action on M is coisotropic if and only if $({}_cM)_\alpha$ is discrete for every $\alpha \in \mathfrak{l}^*$, where $\mathfrak{l}$ is the Lie algebra of L*

Proof. By the Equivalence Theorem 3.3.10, the G-action on M is coisotropic if and only if the L-action on ${}_cM$ is coisotropic. By Theorem 3.4.2 also the L-action is Hamiltonian and the coadjoint orbits of L in $\mathfrak{l}^*$ are locally closed because we are assuming that the coadjoint orbits of G in $\mathfrak{g}^*$ are locally closed. Then the result follows from Lemma 3.4.6. □

We can summarize the previous results on coisotropic and coisotropic Hamiltonian actions in the following Equivalence Theorem.

Theorem 3.4.8. $(\star)$ *Let (M,ω) be a symplectic manifold on which a Lie group G acts properly and in a Hamiltonian fashion, with momentum map $\mu : M \to \mathfrak{g}^*$. Assume also that the coadjoint orbits of G are locally closed and M/G is connected. Then the following conditions are equivalent.*

(1)-(6) of Theorem 3.3.10;

(7) for every $\alpha \in \mathfrak{g}^*$, the reduced space M_α is discrete;

(8) for any $x \in M^{\text{princ}}$, we have that

$$\dim(G_{\mu(x)}) - \dim(G_x) = \operatorname{cohom}(G, M);$$

(9) let $\mathfrak{l}$ be the Lie algebra of L. For every $\alpha \in \mathfrak{l}^*$, the reduced space $({}_cM)_\alpha$ is discrete.

We present an application of the theory developed so far in a concrete example.

Example 3.4.9. Consider the diagonal action of $U(n)$ on $M = \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{m \text{ times}}$ with $n \geq 2$. Consider the functions $f, g : M \to \mathbb{R}$ given by

$$f(z_1, \ldots, z_m) = \mathfrak{Re}(z_1^\dagger z_2), \qquad g(z_1, \ldots, z_m) = \mathfrak{Re}(z_2^\dagger z_3);$$

where $\mathfrak{Re}(\cdot)$ denotes the real part and $(\cdot)^\dagger$ denotes the complex conjugate transposed. The functions f and g are both $U(n)$-invariant. Moreover, it is easy to see that $\{f, g\} \neq 0$. Hence, for $m \geq 3$, we have that this action is not coisotropic. If $m = 1$, we have that the principal $U(n)$-orbits in $\mathbb{C}^n$ are spheres. Since every codimension 1 submanifold is coisotropic, we have that the $U(n)$ action on $\mathbb{C}^n$ is coisotropic.

Now, consider the action of $U(n)$ on $\mathbb{C}^n \oplus \mathbb{C}^n$. The principal isotropy is $U(n-2)$. Then $\left(\mathbb{C}^n \oplus \mathbb{C}^n\right)^{U(n-2)} = \mathbb{C}^2 \oplus \mathbb{C}^2$ and the $(N(U(n-2))/U(n-2))$-action on $\mathbb{C}^2 \oplus \mathbb{C}^2$ is equivalent to the diagonal $U(2)$ action on $\mathbb{C}^2 \oplus \mathbb{C}^2$. The cohomogeinity of the latter is 4 and the principal isotropy is trivial. The momentum map is given by

$$\mu(z, w) = -\frac{i}{2}(zz^\dagger + ww^\dagger).$$

Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$. Then the orbit of $x_0 = (e_1, e_1 + e_2) \in \mathbb{C}^2 \oplus \mathbb{C}^2$ is principal. Moreover,

$$\mu(x_0) = \frac{i}{2}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence $\dim U(2)_{\mu(x_0)} < 4$. By Theorem 3.4.8, the $U(n)$-action on $\mathbb{C}^n \oplus \mathbb{C}^n$ is not coisotropic. In summary, we have proved that the diagonal action of $U(n)$ on $\underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{m\text{-times}}$ is coisotropic if and only if $m = 1$. This example shows that if a principal G-orbit is coisotropic, then the G-action is not in general coisotropic. Indeed, let $(e_1, e_2) \in \mathbb{C}^2 \oplus \mathbb{C}^2$. Then

$$\mu(e_1, e_2) = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and so $\dim(\mathrm{U}(2)_{\mu(e_1,e_2)}) = 4 = \mathrm{cohom}(U(2), \mathbb{C}^4)$. By Proposition 3.4.1 the $U(2)$-orbit throught (e_1, e_2) is coisotropic. Hence the orbit $U(n)((e_1, e_2))$ is coisotropic, but the $U(n)$-action on $\mathbb{C}^n \oplus \mathbb{C}^n$ is not coisotropic. Note that the diagonal action of $U(n)$ on $\mathbb{C}^n \oplus \mathbb{C}^n$ is infinitesimally almost homogeneous by Proposition 3.3.14 and Theorem 3.3.15.

Lemma 3.4.10. *Let C be a closed, G-invariant, symplectic submanifold of M. If the G-action on M is coisotropic, then the restricted G-action on C is coisotropic.*

Proof. It is clear that if $\mu : M \to \mathfrak{g}^*$ is a momentum map associated with the G-action on M, the restriction $\mu|_C$ is a momentum map for the restricted action on C. Then it is also clear that $C_\alpha \subseteq M_\alpha$ and we conclude by Theorem 3.4.8. □

As a corollary, we find the following.

Corollary 3.4.11. *Let G be an Abelian Lie group acting properly, coisotropically and in a Hamiltonian fashion on M and let $K \subseteq G$ be a compact subgroup. Then $M^K \subseteq M$ is a G-invariant submanifold and the restricted G-action to M^K is coisotropic.*

Proof. Recall that M^K is $N(K)$-invariant and since G is abelian $N(K) = G$. Recalling that $T_x M^K = \{v \in T_x M \ : \ (\mathrm{d}g)_x v = v \ \forall g \in K\}$ and that $(T_x M^K)^\S = J_x(T_x M^K)^\perp$, since J is G-equivariant, M^K is a symplectic submanifold of M. Then we can conclude by Lemma 3.4.10. □

In a similar way, we get an analogous result for the fixed set of the action.

Corollary 3.4.12. *Let G be a Lie group acting properly and coisotropically on M. Then M^G is discrete.*

Consider now K a compact subgroup of G, N its normalizer in G and let $L := N/K$. It is clear that any connected component F of M_K is a symplectic submanifold of M and that the L-action on F is Hamiltonian, proper and free. Let $x \in F$ and $\alpha = \mu(x)$ be the value of the momentum map of the G-action. In [12] it is shown that there exists $\tilde{\alpha} \in \mathfrak{l}^*$ such that the reduced space $F_{\tilde{\alpha}}$ is symplectomorphic to the symplectic piece

$$(M_\alpha)_F := \left(\mu^{-1}(G(\alpha)) \cap F\right)\Big/G \ .$$

Therefore, as a consequence of Lemma 3.4.6, we find the following.

Proposition 3.4.13. *Let L, M_K be as above. Then if the G-action on M is coisotropic, the L-action on M_K is coisotropic.*

Assume now that G is compact and acts coisotropically on M and hence, by [62], $\mathrm{homorank}(G, M) = 0$. Let H be a principal isotropy and let $N = N(H)$ be its normalizer in G. By Theorem 3.4.8, the N-action on any connected component F of ${}_cM$ is coisotropic and hence $\mathrm{homorank}(N, F) = 0$. Since $\mathrm{cohom}(G, M) = \mathrm{cohom}(N^0, F)$, we thus obtain that $\mathrm{rank}(G) = \mathrm{rank}(N^0)$. This proves the following.

Proposition 3.4.14. *Let G be a compact Lie group acting coisotropically on (M, ω). Let H be a principal isotropy. Then $N(H)^0$ contains a maximal torus of G.*

3.4.3 Simultaneous Reduction

Let us now consider a Lie group $G = G_1 \times G_2$, where G_1 and G_2 are closed Lie subgroups. We assume that the coadjoint orbits of G_1 and of G_2 are locally closed in $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$, respectively. Obviously $\mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$, and the momentum map $\mu : M \to \mathfrak{g}^*$ decomposes as $\mu = \mu_1 + \mu_2$, where μ_i is the corresponding momentum map for the G_i-action on M for $i = 1, 2$.

Let $\alpha_2 \in \mathfrak{g}_2^*$. The G_1-action on the pieces of M_{α_2} is symplectic and Hamiltonian. Thus, we may obtain a continuous map

$$\mu_{12} : M_{\alpha_2} \to \mathfrak{g}_1^*, \quad [x] \mapsto \mu_1(x).$$

which we also call momentum map, whose restriction to each piece gives the momentum map corresponding to the respective G_1-action.

Definition 3.4.15. We say that the G_1-action on M_{α_2} is *multiplicity free* if the ring of G_1-invariant smooth functions on M_{α_2} forms an abelian Poisson subalgebra of $C^\infty(M_{\alpha_2})$

For any $\alpha_1 \in \mathfrak{g}_1^*$, we consider the reduced space with respect to the G_1-action on M_{α_2} defined as

$$(M_{\alpha_2})_{\alpha_1} = \mu_{12}^{-1}(G_1(\alpha_1))/G$$

We make similar remarks and definitions for the G_2-action on M_{α_1}. We point out that the G_1-action on M_{α_2} preserves the pieces of M_{α_2}. We finally note that if we write $\alpha = \alpha_1 + \alpha_2 \in \mathfrak{g}^* = \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$, then $(M_{\alpha_2})_{\alpha_1}$ are both $(M_{\alpha_1})_{\alpha_2}$ homeomorphic to M_α via the natural map $[[x]] \mapsto [x]$.

Theorem 3.4.16. *Let (M, ω) be a symplectic manifold and let $G = G_1 \times G_2$ be a Lie group acting in a Hamiltonian fashion on M. Assume that the coadjoint orbits of G_1 and G_2 are, respectively, locally closed submanifolds. The following are equivalent*

1. *The G-action on M is multiplicity free;*

2. *For every $\alpha_2 \in \mathfrak{g}_2^*$, the G_1 action on M_{α_2} is multiplicity free;*

3. *For every $\alpha_1 \in \mathfrak{g}_1^*$, the G_2 action on M_{α_1} is multiplicity free.*

Proof. By symmetry, it suffices to show that (1) $\Leftrightarrow$ (2). Let $\alpha = \alpha_1 + \alpha_2 \in \mathfrak{g}^*$ and let S be a piece of M_{α_2}. For any $\alpha_1 \in \mathfrak{g}_1^*$ we have that $S \cap \mu_{12}^{-1}(G_1(\alpha_1))/G_1$ is a stratified space with symplectic strata

$$\left(S \cap \mu_{12}^{-1}(G_1(\alpha_1))\right)/G_1 = \bigcup_{i \in I} P_i. \tag{3.4}$$

Each of the P_i's are pieces of M_α.

Let us now first assume that M is a multiplicity free G-space. By Lemma 3.4.6, we see that every P_i consists of points. Again, using Lemma 3.4.6 we conclude that the G_1-action on every piece of M_{α_2} is multiplicity free. Now, if $f, g \in C^\infty(M_{\alpha_2})^{G_1}$, we have that the restriction of $\{f, g\}$ to any piece is identically zero. Thus, $\{f, g\} = 0$ and M_{α_2} is a multiplicity free G_1-space.

Conversely, assume that the G_1-action on M_{α_2} is multiplicity free for all α_2. Write $\alpha = \alpha_1 + \alpha_2$ as above with $\alpha_1 \in \mathfrak{g}_1^*$ arbitrary. By (3.4), each piece of M_α is a symplectic stratum of the reduced space at α_1 for the G_1-action on a piece S of M_{α_2}. By hypothesis, and using again Lemma 3.4.6, we have that for each piece S of M_{α_2}, the reduced space $\left(S \cap \mu_{12}^{-1}(G_1(\alpha_1))\right)/G_1$ is discrete. Hence, all the pieces of M_α are discrete and M is a multiplicity free G-space. $\square$

3.4.4 Hamiltonian actions on Kähler manifolds

Let $(M, \omega, J, \mathtt{h})$ be a Kähler manifold and let G be a compact Lie group acting by holomorphic isometries with M/G connected. We assume that the G-action on M extends to a holomorphic action of the complexification $G^{\mathbb{C}}$ on M and the G-action on M is Hamiltonian and so there exists a momentum map $\mu : M \longrightarrow \mathfrak{g}^*$. We fix $\langle \cdot, \cdot \rangle$ an $\mathrm{Ad}(G)$-invariant scalar product on $\mathfrak{g}$ and so we may think of the momentum map as a $\mathfrak{g}$-valued map $\mu : M \longrightarrow \mathfrak{g}$ by means of $\langle \cdot, \cdot \rangle$. The following result is well-known and it follows from the equvariance property of the momentum map.

Lemma 3.4.17. *Let $x \in M$. Then $\mu(x) \in \mathfrak{g}^{G_x}$. In particular, $[\mu(x), X] = 0$ for any $X \in \mathfrak{g}_x$.*

Definition 3.4.18. A point $x \in M$ is called:

i) *stable* if $\mathfrak{g}_x^{\mathbb{C}} = \{0\}$ and $G^{\mathbb{C}}(x) \cap \mu^{-1}(0) \neq \emptyset$;

ii) *semistable* if $\overline{G^{\mathbb{C}}(x)} \cap \mu^{-1}(0) \neq \emptyset$;

iii) *polystable* if $G^{\mathbb{C}}(x) \cap \mu^{-1}(0) \neq \emptyset$.

We denote by M^s, M^{ss}, M^{ps} the set of the stable points, respectively semistable, polystable points. These sets are obviously G-invariant. Therefore, we talk about stable, semistable, and polystable G-orbit. The results of Heinzner-Huckleberry-Loose [52, 53, 54, 55], see also Sjamaar [104], show that

Theorem 3.4.19. *The subsets M^s, M^{ss} are open in M. The closure in M^{ss} of a G-orbit contains a unique polystable orbit. Moreover, there exists a good quotient $\pi : M^{ss} \longrightarrow Q$ with the properties:*

i) the induced map $\mu^{-1}(0)/G \longrightarrow Q$ is an homemorphism.

ii) two semistable G-orbits have the same image in Q if and only if their closures in M^{ss} are not disjoint, and this happens if and only if the polystable orbits in their closures in M^{ss} coincide.

Therefore, the set of the semistable points admits a good quotient, which can be identified as a topological space with the corresponding (possibly singular) symplectic quotient. There is no compactness or completeness condition needed.

Let H be a principal isotropy and let F be a connected component of the core ${}_cM$. Let G_F denote the subgroup of G which preserves F. Then $N(H)^o \subseteq G_F \subseteq N(H)$, F is a Kähler manifold, and the G_F-action on F is by holomorphic isometries. Since F is Kähler it follows that F is $G_F^{\mathbb{C}}$-invariant. By Lemma 3.4.17, keeping in mind that $F \subseteq M^H$, we have $\mu(F) \subseteq \mathfrak{z}(H)$, where $\mathfrak{z}(H)$ denotes the Lie algebra of the centralizer of H. Since $Z(H) \subseteq N(H)$ we get the following result

Proposition 3.4.20. *The G_F-action on F is Hamiltonian and a momentum map is given by the restriction of the momentum map μ to F.*

Summing up, the G_F-action on F is Hamiltonian with a momentum map $\mu : F \longrightarrow \mathfrak{z}(H)$ and the G_F-action on F extends to a holomorphic action of the complexification $G_F^{\mathbb{C}}$ on F. Although G_F is not connected, Theorem 3.4.19 holds for the $G_F^{\mathbb{C}}$-action on F (see, for instance, [56, Theorem 1.1]). Theorem 3.1.12 shows that the inclusion

$$\mu_{|_F}^{-1}(0) = \mu^{-1}(0) \cap F \hookrightarrow \mu^{-1}(0)$$

induces a homeomorphism

$$\mu^{-1}(0) \cap F/G_F \longrightarrow \mu^{-1}(0)/G.$$

and so we get the following result.

Theorem 3.4.21. *The good quotient of the semistable points of M is homemorphic to the set of semistable points of F.*

Let $x \in M$, let $\beta \in \mathfrak{g}$ and let $\mu^\beta : M \longrightarrow \mathbb{R}$ denote the contraction of the momentum along β, that is, $\mu^\beta(x) := \langle \mu(x), \beta \rangle$. Then $\operatorname{grad} \mu^\beta = J(\beta_M)$ and the function

$$\lambda(x, \beta, t) = \langle \mu(\exp(t\mathbf{i}\beta)x, \beta \rangle,$$

increases, and so the limit

$$\lambda(x, \beta) = \lim_{t \to +\infty} \lambda(x, \beta, t)$$

exists.

Definition 3.4.22. The maximal weight of $x \in M$ in the direction of β is the numerical value

$$\lambda(x, \beta) = \lim_{t \to +\infty} \lambda(x, \beta, t) \in \mathbb{R} \cup \{+\infty\}$$

Let $x \in M$ and let $\beta \in \mathfrak{g}$. Let $c_\beta(t) = \exp(t\mathbf{i}\beta)x$. The energy of c_β is given by

$$E(c_\beta) = \int_0^{+\infty} \mathtt{h}(\operatorname{grad} \mu^\beta(c_\beta(t)), \operatorname{grad} \mu^\beta(c_\beta(t)))\mathrm{dt},$$

where $\mathtt{h}$ denotes the Kähler metric. The following definition was introduced by Teleman [108].

Definition 3.4.23. A G-action on M is called *energy complete* if for any $x \in M$ and for any $\beta \in \mathfrak{g}$, if $E(c_\beta) < +\infty$ then $\lim_{t \mapsto +\infty} \exp(t\mathbf{i}\beta)x$ exists.

A Theorem of Teleman [108, Theorem 3.3], see also [14, 84], shows that

Theorem 3.4.24. *If the G-action on M is energy complete, then $x \in M$ is semistable if and only if $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{g}$.*

Since M^{ss} is $G^{\mathbb{C}}$-invariant, it follows that $M^{ss} = G^{\mathbb{C}}(M^{ss} \cap F)$.

Theorem 3.4.25. ($\star$) *If M is energy complete, then $M^{\mathrm{ss}} \cap F$ is the set of semistable points of F relative to the $G_F^{\mathbb{C}}$-action on F.*

Proof. Let $x \in F^{ss}$. Since the momentum map of the G_F-action on F is the restriction of the momentum map of the G-action on M, it follows that $x \in M^{ss}$.

Conversely, let $x \in M^{ss} \cap F$. By Theorem 3.4.24 we have $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{g}$ and so for any $\beta \in \mathfrak{n}(\mathfrak{h})$. Applying Theorem 3.4.24, we get $x \in F^{ss}$, concluding the proof. $\square$

Let ν denote the norm square momentum map, that is, $\nu(p) = \frac{1}{2} \parallel \mu(p) \parallel^2$, where $\parallel \cdot \parallel$ is the norm relative to the $\mathrm{Ad}(G)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. ν restricted to F is the norm squared momentum map relative to the G_F-action on F and will be denoted by ν_F. Since $GF = M$ it follows that

$$\inf_{x \in M} \nu = \inf_{x \in F} \nu_F, \qquad \sup_{x \in M} \nu = \sup_{x \in F} \nu_F.$$

By [57, Corollary 6.12 page 178] we get the following result

Proposition 3.4.26. *If ν has a local maximum at $x \in F$ then both $G(x)$ and $G_F(x)$ are complex.*

Proposition 3.4.27. *If $x \in F$ and $G(x)$ is complex, then $G_F(x)$ is complex as well.*

Proof. Since $G(x) = G^{\mathbb{C}}(x)$ and $G(x) \cap F = G_F(x)$, it follows that $G_F^{\mathbb{C}}(x) = G_F(x)$, concluding the proof. $\square$

From now on, we assume that M is compact.
Let $\mathcal{C}_M$ denote the set of critical points of μ. By the G-invariance of the norm squared gradient map, the set of critical points $\mathcal{C}_M$ of ν is G-invariant and so $\mathcal{C}_M = G(\mathcal{C}_M \cap F)$.

Lemma 3.4.28. *$\mathcal{C}_M = G\mathcal{C}_F$, where $\mathcal{C}_F$ is the set of the critical points of ν_F.*

Proof. The gradient of the norm squared gradient map at x is given by $(\mu(x))_M(x)$. By Corollary 3.4.20 if $x \in F$ then $(\mu(x))_M(x) \in T_pF$. This proves $\mathcal{C}_F = \mathcal{C}_M \cap F$ concluding the proof. $\square$

Let $\mathcal{B}_M = \mu(\mathcal{C}_M)$, respectively $\mathcal{B}_F = \mu(\mathcal{C}_F)$ be the set of the singular values of ν, respectively the set of the singular values of ν_F. By the above Lemma we get $\mathcal{B}_M = G\mathcal{B}_F$.

Let $\beta \in \mathcal{B}_M$ and Let $T_\beta := \overline{\exp(\mathbb{R}\beta)}$. Since $N = \{x \in M : \beta_M(x) = 0\} = M^{T_\beta}$ it follows that any connected components of N is a totally geodesic submanifold of M. The function μ^β is constant on any connected component of N.

Let $Z_\beta := \{x \in M : \mu^\beta(x) = \parallel \beta \parallel^2\}$ and let $C_\beta := G(Z_\beta \cap \mu^{-1}(\beta))$. Let $Y_\beta := \{x \in M : \lim_{t \mapsto -\infty} \exp(t\mathbf{i})\beta)x \in Z_\beta\}$. The stratum S_β associated to β is a G-invariant neighborhood of C_β in GY_β [63]. A proof of the the following result is given in [57, 63].

Proposition 3.4.29. *Let $\beta_1, \beta_2 \in \mathcal{B}_M$. The following are equivalent:*

i) $C_{\beta_1} = C_{\beta_2}$*;*

ii) $\beta_1 \in \mathrm{Ad}(G)(\beta_2)$*;*

iii) $S_{\beta_1} = S_{\beta_2}$*.*

Hence, the stratum S_β only depends on $\mathrm{Ad}(G)(\beta)$. Since M is compact, there are only finitely many strata (see, for instance, [57, 63]).

By the above proposition, we may assume $\beta \in \mathcal{B}_F$.
Let $x_0 \in M$ and let $x : \mathbb{R} \longrightarrow M$ be the negative gradient flow of ν through x_0. Since $\mathrm{grad}\,\nu(x) = (\mu(x))_M\,(x)$ it follows that the complexified orbits are invariant under the gradient flow. Duistermaat (see Lerman [67] and also [34]) noted that the Lojasiewicz gradient inequality holds for the norm square momentum map. In particular, the limit

$$x_\infty := lim_{t\to+\infty} x(t),$$

exists and satisfies $(\mu(x_\infty))_M\,(x_\infty) = 0$. Moreover, the G-orbit through x_∞ depends only of the $G^{\mathbb{C}}$-orbit through x_0, see [34, Theorem 6.4 page 43] and also [15, Theorem 3.3. page 12].

Proposition 3.4.30. *Let $\beta \in \mathcal{B}_F$. Then $S_\beta \cap F$ is the stratum associated to β relative to ν_F.*

Proof. Let $q \in S_\beta \cap F$. Since the momentum map of the G_F-action on F is the restriction of the momentum map μ to F it follows that the negative gradient flow of ν coincides with the negative gradient flow of ν_F. This implies that $S_\beta \cap F$ is contained in the stratum associated to β relative to ν_F.

Let $\tilde{S}_\beta$ be the stratum associated to β. By Corollary 7.6 p.24 in [57] we have

$$\tilde{S}_\beta = \{p \in F : \beta \in \mu(\overline{G_F^{\mathbb{C}} \cdot p}) \, \text{and} \, \parallel \beta \parallel \leq \parallel \mu(gp) \parallel \ \forall g \in G_F^{\mathbb{C}}\}$$

and so $\tilde{S}_\beta \subseteq S_\beta \cap F$, concluding the proof. □

In summary, we have proved the following result.

Theorem 3.4.31. $(\star)$ *Let $\nu_F : F \longrightarrow \mathbb{R}$ be the norm squared momentum map. Let $B_F = \mu(\mathcal{C}_F)$ be the critical value of ν_F. Let S_β^F denote the stratum of the Morse-like function ν_F associated to $\beta \in \mathcal{B}_F$. Then*

$$M = \bigsqcup_{\beta \in B_F} G^{\mathbb{C}} S_\beta^F,$$

is the smooth stratification of M induced to ν.

Chapter 4

Rational Hyperbolicity Problem

We start this chapter by recalling the general framework of the Rational Hyperbolicity Problem, which is the content of Conjecture 4.1.4 and Conjecture 4.1.5. After recalling some basic facts about Rational Homotopy Theory, we describe some counterexamples disproving the above-mentioned conjectures. Lastly, with Theorem 4.3.1 we provide additional sufficient conditions for such conjectures to hold true. In order to do so, we will need some technical tools from Morse Theory. To lighten the exposition, we refer the reader to Section 5.5 of the Appendix for a brief account of some of the technicalities involved. Most importantly, we point out that all the results of this chapter can be stated in the more general setting of Singular Riemannian Foliations; for an extensive exposition of the structure theory of singular Riemannian foliations, see [100]. However, to be coherent with the focus of the thesis work, we prefer to restrict our attention to Lie group actions.

4.1 Setting, conjectures and counterexamples

Let M be a compact, 1-connected manifold and let $\pi_i(M, x)$ denote the i-th homotopy group of M with base point $x \in M$. For simplicity, we will use the notation $\pi_i(M) := \pi_i(M, x)$. We start with the following definition.

Definition 4.1.1. M is called *rationally elliptic* if $\sum_{i>1} \dim(\pi_i(M) \otimes \mathbb{Q}) < +\infty$, otherwise M is called *rationally hyperbolic.*

By a theorem of Felix and Halperin, if M is rationally hyperbolic, $\sum_{i=1}^{k} \dim(\pi_i(M) \otimes \mathbb{Q})$ grows exponentially in k (see Theorem IV of [30]). Rational ellipticity is known to impose strong topological restrictions: e.g. sharp bounds on the total Betti numbers or nonnegativity of the Euler characteristic; see [32] and [31] for a vast account. In particular, many of these restrictions are only conjectured for non-negatively curved manifolds. From this point of view, the following conjecture by Bott and later generalized by Gorve and Halperin is of special interest.

Conjecture 4.1.2. (Bott–Grove–Halperin) Let M be compact simply connected manifold admitting a Riemannian metric of (almost) nonnegative curvature. Then M is rationally elliptic.

The solution of this conjecture is far out of reach; however, in many works, see e.g. [42, 44, 45, 96, 101], several geometric criteria are found to imply rational ellipticity. Most of these use the assumption that M supports an action of a compact group G of isometries. For example, we have:

Theorem 4.1.3. *Let M be a compact, simply connected manifold with an isometric action by a compact group G. Then:*

i) from [42]: if $\dim(M/G) = 1$ *then M is rationally elliptic;*

ii) from [45]: if the G-action is polar with flat or spherical section, then M is rationally elliptic;

iii) from [44]: if $\dim(M/G) = 2$ *(in which case M/G is a 2-dimensional orbifold by [72]) and M/G is an* elliptic orbifold*, then M is rationally elliptic.*

Here, by *elliptic orbifold* we mean that M/G is either a non-smooth manifold without boundary that is compact and simply connected, or it is covered (in the sense of orbifolds, see, e.g., [36]) by the 2-sphere or the Euclidean plane. In [44] it is also shown that any two dimensional compact orbifold is either elliptic or it is covered by the hyperbolic plane. In view of this last result, one can ask whether both items *ii*) and *iii*) of Theorem 4.1.3 can be sharpened in the following sense.

Rational Hyperbolicity Problem

Conjecture 4.1.4. [45, p. 309] Let M be a compact, simply connected manifold with a polar G-action by a compact group G. If the section is hyperbolic, then M is rationally hyperbolic.

Conjecture 4.1.5. [44, Problem 2.6] Let M be a compact, simply connected manifold with an isometric G-action by a compact group G, such that $\dim M/G = 2$ and M/G is covered by the hyperbolic plane. Then M is rationally hyperbolic.

In the next section, we will provide negative answers for both conjectures 4.1.4 and 4.1.5.

4.2 Counterexamples to Conjectures

To construct counterexamples, we need to briefly review some basic facts from Rational Homotopy Theory.

Let M be a simply connected manifold with finite homology. It is possible to prove that one can always construct from M the differential graded algebra of the so-called *polynomial differential forms* $A^*_{PL}(M, F)$, where F is any field of characteristic zero; for a precise definition, see the original article [107] or the book [31] and for a clear motivation of the name, see Remark 1 at page 122 of the same book. In particular, when $F = \mathbb{R}$, one has that $A^*_{PL}(M, \mathbb{R})$ is weakly equivalent to the differential graded algebra of smooth differential forms on M. Moreover, there is always an isomorphism

$$H^*(A^*_{PL}(M, F)) \simeq H^*(M, F) \ . \tag{4.1}$$

When $F = \mathbb{Q}$, it is possible to show that the functor $A^*_{PL}(\cdot, \mathbb{Q})$ defines a bijection between the set of rational homotopy types of closed simply connected manifolds and the set of weak equivalence classes of differential graded algebras over $\mathbb{Q}$. In fact, the rational homotopy type of M is determined by $A^*_{PL}(M, \mathbb{Q})$. For all these facts, we refer again to [31].

Minimal Models

We recall that given a differential graded algebra over a field F, $\left((A = \bigoplus_{i\geq 0} A_i, F), \mathrm{d}_A\right)$ equipped with an antiderivation of degree one d_A, a *minimal model* for A is a free differential graded algebra over F, $\left((\Lambda V = \bigoplus_{j\geq 0} V_j, F), \mathrm{d}_{\Lambda V}\right)$ such that

i) there exists a chain map $f : \Lambda V \to A$, i.e., $f \circ \mathrm{d}_A = \mathrm{d}_{\Lambda V} \circ f$, that induces isomorphisms in cohomology;

ii) the differential of any element of ΛV is either zero or decomposable, i.e., if $x \in \Lambda V$ is a generator, either $\mathrm{d}_{\Lambda V} x = 0$ or $\mathrm{d}_{\Lambda V} x \in \left(\bigoplus_{j>0} \Lambda V_j\right)\left(\bigoplus_{j>0} \Lambda V_j\right)$.

If a differential graded algebra $((A, F), \mathrm{d}_A)$ is 1-connected, i.e. $H^0(A, \mathrm{d}_A) \simeq F$ and $H^1(A, \mathrm{d}_A) \simeq 0$, it always admits a minimal model which is unique up to isomorphism (see for instance [31], or [107]). A celebrated theorem by Sullivan (see [26]) allows us to get information about the rational homotopy of M using the minimal model associated to $A^*_{PL}(M, \mathbb{Q})$. More precisely, we have the following theorem (see [18]).

Theorem 4.2.1. *If M is 1-connected, $A^*_{PL}(M, \mathbb{Q})$ is 1-connected. Moreover, $dim(\pi_k(M) \otimes \mathbb{Q})$ is equal to the number of generators in dimension k of the minimal model associated with $A^*_{PL}(M, \mathbb{Q})$.*

The following easy example will be of great use later on.

Example 4.2.2. Consider the case of a simply connected rational cohomology n-sphere, i.e., a simply connected manifold M with the rational cohomology of an n-sphere: $H^*(M, \mathbb{Q}) = H^*(\mathbb{S}^n, \mathbb{Q})$. Let ω be an integral volume form for $\mathbb{S}^n$ and consider its integral cohomology class $[\omega]$ as an element $[\omega] \in H^n(M, \mathbb{Q})$. It is well known that if $p^0, \ldots, p^n$, are coordinates for $\mathbb{R}^{n+1}$, and $X = p^i \partial_i$ is the radial vector field on $\mathbb{R}^{n+1}$, up to a normalization constant c_n, we can write ω in a polynomial form,

$$\omega = c_n \mathrm{i}_X(\mathrm{d}p^0 \wedge \ldots \wedge \mathrm{d}p^n) = c_n \sum_{i=0}^{n+1} (-1)^i p^i \mathrm{d}p^1 \wedge \ldots \wedge \widehat{\mathrm{d}p^i} \wedge \ldots \wedge \mathrm{d}p^{n+1} \ .$$

Therefore, $\omega \in A^*_{PL}(\mathbb{S}^n, \mathbb{Q})$, and ω can be seen as an element $\omega \in A^*_{PL}(M, \mathbb{Q})$ defining the class $[\omega] \in H^n(M, \mathbb{Q}) \simeq H^n(A^*_{PL}(M, \mathbb{Q}))$ (see isomorphism (4.1)).

Suppose $n = 2k+1$. Consider the graded algebra over $\mathbb{Q}$, generated by an element x in dimension $2k+1$, which we denote by $\Lambda(x)$, equipped with a trivial differential. Clearly, the map $x \mapsto \omega$ defines an isomorphism in cohomology. Therefore, $(\Lambda(x), 0)$ is a minimal model for $A^*_{PL}(M, \mathbb{Q})$.

Suppose now $n = 2k$. Following the idea of the previous case, we look for a chain map such that $x \mapsto \omega$. But now we need another element that takes care of the fact that $[\omega]^2 = 0$. We choose the differential graded algebra over $\mathbb{Q}$, denoted by $\left(\Lambda(x, y), \mathrm{d}_{\Lambda(x,y)}\right)$, generated by two elements x, y, where x is in dimension $2k$ and y is in dimension $4k-1$, and such that $\mathrm{d}_{\Lambda(x,y)} x = 0$ and $\mathrm{d}_{\Lambda(x,y)} y = x^2$. Then we consider the map defined by $x \mapsto \omega$ and $y \mapsto 0$. Again, we easily see that this map induces isomorphisms in cohomology and therefore, $\left(\Lambda(x, y), \mathrm{d}_{\Lambda(x,y)}\right)$ is a minimal model for $A^*_{PL}(M, \mathbb{Q})$.

The previous example, together with Theorem 4.2.1, allows us to state the following useful result.

Proposition 4.2.3. *A simply connected manifold M is a rational cohomology sphere if and only if M has the same rational homotopy as a sphere. In particular, it is rationally elliptic.*

For this reason, we refer to both rational homotopy and rational cohomology spheres simply as *rational spheres*.

In general, rational ellipticity is not preserved under elementary topological operations, e.g. surgeries. However, for rational spheres we have the following.

Proposition 4.2.4. *Let M_i, $i = 1, 2$, be a simply connected rational sphere of dimension n. Then the connected sum $M_1 \# M_2$ is a rational sphere of dimension n.*

Proof. Using Mayer–Vietoris for the natural covering of $M_1 \# M_2$, we have $H^k(M_1 \# M_2, \mathbb{Q}) \simeq H^k(M_1, \mathbb{Q}) \oplus H^k(M_2, \mathbb{Q})$ for $k \neq 0, n$ and $H^0(M_1 \# M_2, \mathbb{Q}) \simeq H^n(M_1 \# M_2, \mathbb{Q}) = \mathbb{Q}$. We conclude using Proposition 4.2.3. □

We note that if M is rationally elliptic and closed, then $\dim(\pi_*(M, \mathbb{Q})) \leq \dim(M)$ (see [32]). One immediate consequence is the following well known fact.

Proposition 4.2.5. *If M is closed and is not a rational sphere, then there exists $k \in \mathbb{N}$ such that the connected sum of k copies of M, denoted by $M^{\#k}$, is rationally hyperbolic.*

Since we will make a comparison between this proposition and our Theorem 4.3.1, we briefly recall its proof.

Proof. First note that, since M is closed and simply connected, if M is not a rational sphere, $n := \dim(M) \geq 4$. Then we can consider the lowest $j \neq 0, 1, n-1, n$, such that $b_j := \dim(H^j(M, \mathbb{Q})) \neq 0$. By Hurewicz Theorem $\dim(\pi_j(M) \otimes \mathbb{Q}) = b_j$. Using Mayer–Vietoris for the natural covering of $M \# M$, we have $H^j(M \# M, \mathbb{Q}) \simeq H^j(M, \mathbb{Q}) \oplus H^j(M, \mathbb{Q})$, and $H^l(M \# M, \mathbb{Q}) \simeq 0$ for $0 < l < j$. Applying iteratively this argument for k copies of M, we conclude

$$\dim(\pi_j(M^{\#k}) \otimes \mathbb{Q}) = k b_j \ .$$

Therefore, by choosing k such that $k b_j > n$, we see that $M^{\#k}$ must be rationally hyperbolic.

□

4.2.1 W/SO(3) vs SU(3)/SU(3)

As described in [45], the homogeneous space $W := SU(3)/SO(3)$, also called the *Wu manifold*, admits a cohomogeneity-2 polar action by $SO(3)$, whose quotient is a flat, equilateral triangle. It is well known that $SU(3)/SO(3)$ is a rational 5-sphere, and hence it is rationally elliptic. We notice, in particular, that this is in agreement with Theorem 4.1.3. The vertices of the triangle correspond to fixed points of the action. We consider one of such fixed points, $p \in W$, and its tubular neighborhood $\mathfrak{T}_\varepsilon(p)$, which corresponds to a Riemannian ball of radius ε. Then we can take the equivariant connected sum of k copies of W along $\partial \mathfrak{T}_\varepsilon(p)$ and get a new manifold $W^{\#k}$ that admits a cohomogeneity-2 polar action by $SO(3)$ and whose quotient is a hyperbolic polygon of $2 + k$ sides and with all angles equal to $\frac{\pi}{3}$. By Proposition 4.2.4, $W^{\#k}$ is still a rational 5-sphere. Then $W^{\#k}$ is rationally elliptic, and its quotient is a hyperbolic polygon for all $k \geq 2$. This is the desired counterexample. We state this as a theorem for future reference.

Theorem 4.2.6. ($\star$) *For any $\ell \geq 4$, there exists a compact, 1-connected, rationally elliptic 5-manifold $M_\ell = W^{\#\ell-2}$, with an $SO(3)$-action such that $M_\ell/SO(3)$ is isometric to a hyperbolic polygon with ℓ edges and all angles equal to $\pi/3$.*

Remark 4.2.7. Consider the conjugacy action of $SU(3)$ on itself. This is a cohomogeneity-2 polar action, and the quotient is a flat equilateral triangle. The three vertices correspond to fixed points, the sides to singular orbits diffeomorphic to $\mathbb{C}P^2$, and the interior of the triangle to principal orbits diffeomorphic to $SU(3)/T$, T being a maximal torus of $SU(3)$. We can take the equivariant connected sum of k copies of $SU(3)$ near the fixed points, as we did for W, and get a manifold $SU(3)^{\#k}$ with a cohomogeneity-2 polar action by $SU(3)$, whose quotient is isometric to $W^{\#k}/SO(3)$. Since $SU(3)$ is not a rational sphere, by Proposition 4.2.5, there exists $\tilde{k} \in \mathbb{N}$ such that $SU(3)^{\#\tilde{k}}$ is rationally hyperbolic.

Thanks to Remark 4.2.7, the counterexamples in Theorem 4.2.6 turn out to provide negative answer to the following:

Conjecture 4.2.8 (Radeschi–Samani–Wilhelm)**.** Assume M, M' are closed, simply connected manifolds, with isometric actions by Lie groups G on M, and G' on M', such that M/G is isometric to M'/G'. Then M is rationally elliptic if and only if M' is rationally elliptic.

We actually find infinite families of manifolds with isometric quotients but with different rational homotopy behaviors:

Theorem 4.2.9. ($\star$) *There exist infinite pairs of manifolds* $(M_\ell = W^{\#\ell}, G = SO(3))$, $(N_\ell = SU(3)^{\#\ell}, H = SU(3))$ *with* M_ℓ *rationally elliptic,* N_ℓ *rationally hyperbolic, and* M_ℓ/G *isometric to* N_ℓ/H*.*

Note that by Remark 4.2.7, such pairs of manifolds start when $\ell \geq \tilde{k}$. Actually, we will see that $\tilde{k} = 2$.

4.2.2 A Hyperbolic triangle

All previous counterexamples have hyperbolic quotients with at least four sides. We now show that there are also counterexamples with only three sides.

By [45], there exists an 8-dimensional closed, simply connected manifold M carrying a cohomogeneity-2 action, whose quotient is a hyperbolic triangle T. The precise polar data is described in [45], page 304.

To simplify computations in cohomology, we consider the same triangle with the universal covering groups of $SO(4)$ and $SO(3)$, i.e., $\mathbb{S}^3 \times \mathbb{S}^3$ and $\mathbb{S}^3$, respectively. We use the fact, due to an observation of Miyaoka [80], that the sides opposite to the vertices corresponding to fixed points retract to a principal fiber bundle $\mathbb{S}^3 \hookrightarrow L \to \mathbb{R}P^2$. We compute $H^*(L, \mathbb{Z})$ using the Gysin sequence (see [18]),

$$0 \to \mathbb{Z} \to H^0(L) \to 0 \to 0 \to H^1(L) \to 0 \to \mathbb{Z}_2 \to H^2(L) \to 0 \to 0 \to$$
$$\to H^3(L) \to \mathbb{Z} \to 0 \to H^4(L) \to 0 \to 0 \to H^5(L) \to \mathbb{Z}_2 \to 0\ .$$

Hence,

q	0	1	2	3	4	5	$\cdots$
$H^q(L, \mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$0 \cdots$

Table 4.1: Cohomology of L

We compute the singular cohomology of the side N with one of the vertices given by a fixed point using Mayer-Vietoris.

$$0 \to H^0(N) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to H^1(N) \to 0 \to 0 \to H^2(N) \to 0 \to \mathbb{Z}_2 \to$$

$$\to H^3(N) \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to H^4(N) \to 0 \to 0 \to H^5(N) \to 0 \to \mathbb{Z}_2 \to H^6(N) \to 0$$

We work in the three possible cases $\alpha = \mathrm{id}_{\mathbb{Z}}$, $\alpha = 0_{\mathbb{Z}}$, $\alpha = m\ \mathrm{id}_{\mathbb{Z}}$, $m \in \mathbb{Z}$. We get,

	q	0	1	2	3	4	5	6	$\cdots$
$\alpha = \mathrm{id}_{\mathbb{Z}}$	$H^q(N,\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	$0 \cdots$
$\alpha = 0_{\mathbb{Z}}$	$H^q(N,\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$0 \cdots$
$\alpha = m\ \mathrm{id}_{\mathbb{Z}}$	$H^q(N,\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_m$	0	$\mathbb{Z}_2$	$0 \cdots$

Table 4.2: Cohomology of N

To compute the cohomology of M, we use again Mayer-Vietoris.

$$0 \to H^0(M) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to H^1(M) \to 0 \to 0 \to H^2(M) \to 0 \to$$

$$\to 0 \to H^3(M) \to H^3(N) \to 0 \to H^4(M) \to H^4(N) \to 0 \to H^5(M) \to 0 \to$$

$$\to 0 \to H^6(M) \to \mathbb{Z}_2 \to 0 \to H^7(M) \to 0 \to \mathbb{Z} \to H^8(M) \to 0$$

In the three cases, we get Table 4.3, from which we can also compute the homology in Table 4.4.

	q	0	1	2	3	4	5	6	7	8	$\cdots$
$\alpha = \mathrm{id}_{\mathbb{Z}}$	$H^q(M,\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$0 \cdots$
$\alpha = 0_{\mathbb{Z}}$	$H^q(M,\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$0 \cdots$
$\alpha = m\ \mathrm{id}_{\mathbb{Z}}$	$H^q(M,\mathbb{Z})$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_m$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$0 \cdots$

Table 4.3: Cohomology of M

	q	0	1	2	3	4	5	6	7	8	$\cdots$
$\alpha = \mathrm{id}_{\mathbb{Z}}$	$H_q(M,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}$	$0 \cdots$
$\alpha = 0_{\mathbb{Z}}$	$H_q(M,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$0 \cdots$
$\alpha = m\ \mathrm{id}_{\mathbb{Z}}$	$H_q(M,\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_m$	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}$	$0 \cdots$

Table 4.4: Homology of M

Comparing Tables 4.3 and 4.4 we see that the case $\alpha = 0_{\mathbb{Z}}$ is excluded because M would not satisfy Poincaré duality. The case $\alpha = m\ \mathrm{id}_{\mathbb{Z}}$ satisfies Poincaré duality if and only if $m = 1$. Then the only admissible case is when $\alpha = \mathrm{id}_{\mathbb{Z}}$, and we see that the manifold M is a rational 8-sphere.

Remark 4.2.10. We can perform an equivariant connected sum of k copies of M to obtain a new 8-manifold $M^{\#k}$ which is again a rational 8-sphere with a cohomogeneity-2 $SO(4)$-action such that the quotient is a hyperbolic polygon. These polygons have two angles equal to $\pi/6$ and k angles equal to $\pi/3$. Note that all such $M^{\#k}$ have different torsions in integral cohomology.

4.3 Positive Results

In this section we are going to prove the following theorem that provides a positive answer to conjectures 4.1.4 and 4.1.5 under additional conditions.

Theorem 4.3.1. ($\star$) *Let $G \circlearrowleft M$ be a variationally complete action of a compact Lie group G on a compact, 1-connected manifold. If M/G supports a hyperbolic metric and one of the following holds*

i) all the singular isotropies are Abelian;

ii) a principal isotropy is connected,

then M is rationally hyperbolic.

For instance, the manifolds $C_2^{n,m} := (\mathbb{S}^n \times \mathbb{S}^m)\#(\mathbb{S}^n \times \mathbb{S}^m)$ naturally carry a cohomogeneity-2 $(SO(n)\times SO(m))$-action with hyperbolic section $(\mathbb{S}^1\times\mathbb{S}^1)\#(\mathbb{S}^1\times\mathbb{S}^1)$, and hence with hyperbolic quotient. By Theorem 4.3.10, this action is variationally complete and the principal isotropies are given by the trivial group. Then we can apply Theorem 4.3.1 and conclude that $C_2^{n,m}$ is rationally hyperbolic. It follows that also $C_k^{n,m} := (\mathbb{S}^n \times \mathbb{S}^m)^{\#k}$ is rationally hyperbolic for $k \geq 2$.

Another application is the case of the action $SU(3) \circlearrowleft SU(3)^{\#k}$ for $k \geq 2$, discussed in Remark 4.2.7. The quotient $SU(3)^{\#k}/SU(3)$ is a hyperbolic polygon, and hence we can again apply Theorem 4.3.10 to conclude that the action is variationally complete. A principal isotropy of this action is given by a maximal torus of $SU(3)$ and hence we can apply Theorem 4.3.1 with item *ii*).

Remark 4.3.2. We note that, in determining the rational hyperbolicity of $SU(3)^{\#k}$, our Theorem 4.3.1 works better than the classical Proposition 4.2.5, for which we can conclude that $SU(3)^{\#k}$ is rationally hyperbolic only when $k > 8$. In fact, $SU(3)$ is not a cohomology sphere, actually $H^*(SU(3),\mathbb{Q}) \simeq H^*(\mathbb{S}^3\times\mathbb{S}^5,\mathbb{Q})$ and $\dim(SU(3)) = 8$. Following the notation of the proof of Proposition 4.2.5, we have $j = 3$ and $b_j = 1$, therefore $k > 8$.

Theorem 4.3.1 also tells us that the counterexamples for conjectures 4.1.4 and4.1.5 in Theorem 4.2.6 are optimal in at least two ways. First, the dimension of the Lie group G is minimal, since if $\dim(G) < 3$, G is Abelian, and hence M is rationally hyperbolic by Theorem 4.3.1. On the other hand, it is known that if $\dim(M) \leq 4$ and M is (closed, simply connected) rationally elliptic, then $M \simeq \mathbb{S}^2$, $\mathbb{S}^3$, $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{S}^2\times\mathbb{S}^2$, $\mathbb{C}P^2\#\mathbb{C}P^2$, $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ (see [59] and [95]).

4.3.1 Fibration arguments

We will use the following important result, linking the rational homotopy behavior of M with the behavior of the rational homology of its loop space (see Theorem V in [30]).

Theorem 4.3.3. *Let M be a compact 1-connected manifold and let $\Omega(M)$ be its loop space. We have that*

i) if M is rationally elliptic, then $\dim(H_k(\Omega(M),\mathbb{Q}))$ grows polynomially in k;

ii) if M is rationally hyperbolic, then $\dim(H_k(\Omega(M),\mathbb{Q}))$ grows exponentially in k.

Consider an embedded submanifold $L \subseteq M$, $q \in L$, and let $p \in M$. Let $\Omega_{p,q}$ be the space of continuous paths in M from q to p, let $\Omega_{L,q}$ be the space of paths from L to p and let $\Omega(L) := \Omega(L, q)$ be the space of loops in L based at q. We have two natural fibrations

$$\Omega(L) \to \Omega_{q,p} \to \Omega_{L,p} \ , \tag{4.2}$$

$$\Omega_{q,p} \to \Omega_{L,p} \to L \ . \tag{4.3}$$

These fibrations come from the so called Puppe sequence (see [51]). In fact, let $\iota : L \hookrightarrow M$ be the inclusion. The mapping path space of the inclusion is given by

$$\{(\gamma, q) \in \mathcal{C}^0([0,1], M) \ : \ \gamma(0) = \iota(q) = q\} \simeq \{\gamma : [0,1] \to M \ : \ \gamma(0) \in L\} \ .$$

Then the map $\Omega_L := \{\gamma : [0,1] \to M \ : \ \gamma(0) \in L\} \to M$, $\gamma \mapsto \gamma(1)$ is a fibration and the typical fiber is $\Omega_{L,p}$

$$\Omega_{L,p} \to \Omega_L \to M \ .$$

In other words, $\Omega_{L,q}$ is the homotopy fiber of the inclusion. The Puppe sequence associated with this fibration is

$$\cdots \to \Omega^2(\Omega_L) \to \Omega^2(M) \to \Omega(\Omega_{L,p}) \to \Omega(\Omega_L) \to \Omega(M) \to \Omega_{L,p} \to \Omega_L \to M \ ,$$

where $\Omega(\cdot)$ means that we are taking the space of loops of the argument space, and each pair of adjacent maps is a fibration. Recall that Ω_L is homotopy equivalent to L and since M is connected, given any pair of points p, q, we have that $\Omega_{q,p}$ is homotopy equivalent to $\Omega(M)$. Then (4.2) is given by the fibration $\Omega(\Omega_L) \to \Omega(M) \to \Omega_{L,p}$ and (4.3) is given by the fibration $\Omega(M) \to \Omega_{L,p} \to \Omega_L$.

Proposition 4.3.4. *From fibrations (4.2) and (4.3), we obtain that*

i) if L is rationally elliptic (e.g., when L is a simply connected, homogeneous space), then M is rationally elliptic if $\dim(H_k(\Omega_{L,q}, \mathbb{Q}))$ *grows polynomially.*

ii) M is rationally hyperbolic if $\dim(H_k(\Omega_{L,q}, \mathbb{Q}))$ *grows exponentially.*

Proof. To simplify the notation in the following, $H_*(\cdot)$ will denote the homology in rational coefficients $H_*(\cdot, \mathbb{Q})$. To prove the first claim, suppose for a moment that the fibration $\Omega(L) \to \Omega_{q,p} \to \Omega_{L,p}$ is orientable, i.e. suppose that the fundamental group of the base acts trivially in the homology of the fiber. Then the elements in the second page of the Serre spectral sequence (see [51, 79]) of this fibration are given by

$$E^2_{h,k} = H_h(\Omega_{L,p}, H_k(\Omega(L))) \ .$$

The spectral sequence converges to the homology of $\Omega_{q,p}$, i.e., $\sum_{h+k=j} E^\infty_{h,k} = H_j(\Omega_{q,p}) = H_j(\Omega(M))$. It follows that

$$\dim(H_j(\Omega(M))) \leq \sum_{h+k=j} \dim(H_h(\Omega_{L,p})) \dim(H_k(\Omega(L))) \ . \tag{4.4}$$

If L is rationally elliptic, $\dim(H_k(\Omega(L))$ grows at most polynomially, and by (4.4), we conclude that M is rationally elliptic if $\dim(H_h(\Omega_{L,p}))$ grows polynomially. If L is not simply connected (which is the case in most situations of interest) the fibration we are considering is not orientable. However, there is a trick to retrieve the above argument (see [44]). Let $\hat{G} \supseteq G$ be a simply connected group and consider the $\hat{G}$-manifold $\hat{M} := \hat{G} \times_G M$. It can be shown that $\pi_1(\hat{M}) = \{0\}$, $\hat{M}/\hat{G}$ is isometric to M/G, that if $x \in M \subseteq \hat{M}$, $\hat{G}(x)$ has finite fundamental group and that M is rationally elliptic if and only if $\hat{M}$ is rationally elliptic. Then we can use Lemma 4.3 and Corollary 4.4 in [44] to conclude by the above Serre spectral sequence argument applied to the case of the manifold $\hat{M}$ and the orbit $\hat{L}$.

The Serre spectral sequence can also be applied to (4.3), because the action of $\pi_1(L)$ on the homology of $\Omega_{q,p}$ factors through $\pi_1(M) = 0$ and hence is trivial. Then, we have

$$E^2_{h,k} = H_h(L, H_k(\Omega_{p,q})) \ .$$

By the same argument as before, we conclude that

$$\dim(H_j(\Omega_{L,p})) \leq \sum_{h+k=j} \dim(H_h(L)) \dim(H_k(\Omega_{q,p})) \ . \tag{4.5}$$

Furthermore, if $C := \dim(H_*(L))$, we have

$$\dim(H_j(\Omega_{L,p})) \leq C \sum_{k=0}^{j} \dim(H_k(\Omega_{q,p})) \ .$$

Therefore, we conclude that if $\dim(H_j(\Omega_{L,p}))$ has exponential growth, $\dim(H_j(\Omega_{q,p}))$ must have exponential growth.

□

Our interest in 4.3.4 relies on the fact that we can control the rational homotopy behavior of M by looking at the homology of the path space $\Omega_{L,p}$. In the next subsections, we show how one can use Morse Theory to estimate the growth of $\dim(H_j(\Omega_{L,p}))$.

4.3.2 Energy Functional

Let $L \subseteq M$ be an embedded submanifold and let $p \in M$ be not focal for L. We define the following space of paths[1]:

$$\Omega_{p,L} := \{\gamma : [0,1] \to M \ : \ \gamma(0) = p, \ \gamma(1) \in L\} \ .$$

The critical points of the energy functional

$$E : \Omega_{p,L} \to \mathbb{R} \ , \qquad \gamma \mapsto E(\gamma) := \int_0^1 \|\dot{\gamma}(t)\|^2_{\mathbf{g}} \ \mathrm{d}t \ ,$$

are geodesics joining p and L that start orthogonal to L. We denote by $\mathrm{Crit}(E, \Omega_{p,L})$ the set of all critical points of E on $\Omega_{p,L}$. Moreover, $\mathrm{Crit}_k(E, \Omega_{p,L})$ will denote the subset of critical points with index k and $\mu_k(E, \Omega_{p,L}) := \#\{\mathrm{Crit}_k(E, \Omega_{p,L})\}$ will denote its cardinality. Let $r \in \mathbb{R}$, and define $\Omega^r_{p,L} := \{\gamma \in \Omega_{p,L} \ : \ E(\gamma) \leq r\}$. We know from Morse Theory that if r is a critical value of E and F is any field of coefficients, there exists $\varepsilon > 0$ such that $[r-\varepsilon, r)$ does not contain critical values of E and $\Omega^r_{p,L}$ has the homotopy type of $\Omega^{r-\varepsilon}_{p,L}$ with as many k-cells attached as the the number of critical points of E of index k and value r. From this, one can easily obtain the so-called *Morse inequalities*

$$\dim(H_k(\Omega_{p,L}, F)) \leq \mu_k(E, \Omega_{p,L}) \ .$$

Definition 4.3.5. E is called an F-perfect Morse function if for all k, all Morse inequalities are equalities:

$$\dim(H_k(\Omega_{p,L}, F)) = \mu_k(E, \Omega_{p,L}) \ .$$

[1]The two different choices of starting and ending point have been made because of different commodities in the present and in the previous sections. This is not a problem since there is an obvious homeomorphism, with respect to the C^0-topology, $\Omega_{p,L} \simeq \Omega_{L,p}$, given by $\gamma(t) \mapsto \gamma(1-t)$.

Remark 4.3.6. We notice that E is F-perfect if and only if in the long exact sequence in homology with coefficient in F for the couple $(\Omega^r_{p,L}, \Omega^{r-\varepsilon}_{p,L})$, the maps

$$H_k(\Omega^r_{p,L}) \longrightarrow H_k(\Omega^r_{p,L}, \Omega^{r-\varepsilon}_{p,L}) ,$$

are surjective. Equivalently, the connecting isomorphisms

$$H_k(\Omega^r_{p,L}, \Omega^{r-\varepsilon}_{p,L}) \longrightarrow H_{k-1}(\Omega^{r-\varepsilon}_{p,L}) ,$$

are all trivial. In particular, at the level of cells, this means that a k-dimensional cell, e^k, corresponding to a critical point $\gamma \in \Omega^r_{p,L} \setminus \Omega^{r-\varepsilon}_{p,L}$ of index k, is such that $[\partial e^k] = 0 \in H_{k-1}(\Omega^{r-\varepsilon}_{p,L})$ and for each such e^k, there exists a non-trivial element $[\lambda_\gamma] \in H_k(\Omega^r_{p,L})$, which is called a *linking cycle* for γ; see Subsection 5.5.1 of the Appendix for a more detailed exposition.

In particular, the previous remark proves the following.

Proposition 4.3.7. *$E : \Omega_{p,L} \to \mathbb{R}$ is an F-perfect Morse function if and only if for every $\gamma \in Crit_k(E, \Omega_{p,L})$ there exists a linking cycle $[\lambda_\gamma] \in H_k(\Omega_{p,L}, F)$.*

The perfectness of the Energy functional will provide us with a more manageable tool to control the rational homotopy behavior of M. More precisely, we have the following proposition.

Proposition 4.3.8. *Let $L \subseteq M$ be an embedded submanifold of a compact, 1-connected, complete Riemannian manifold and let $p \in M$ be not focal for L. Then*

i) if L is rationally elliptic and $\mu_k(E, \Omega_{p,L})$ grows polynomially in k, then M is rationally elliptic;

ii) if $E : \Omega_{p,L} \to \mathbb{R}$ is perfect, then M is rationally hyperbolic if $\mu_k(E, \Omega_{p,L})$ grows exponentially.

Proof. To prove i), we simply have to recall that by the Morse inequalities applied to E, $\dim H_k(\Omega_{L,p}) \leq \mu_k(E, \Omega_{L,p})$, and we conclude by the first part of Proposition 4.3.4. To prove item ii), since $\dim H_k(\Omega_{L,p}) = \mu_k(E, \Omega_{L,p})$, we conclude by the second part of proposition 4.3.4. □

Our strategy to prove Theorem 4.3.1 is then to show that under its hypothesis, the energy functional is perfect and that the number of critical points grows exponentially.

4.3.3 Variationally complete actions

A key ingredient that we will use is the following notion introduced in [17].

Definition 4.3.9. An isometric action $G \circlearrowleft M$ is called *variationally complete* if for every G-orbit, $L := G(q)$, $q \in M$, and every horizontal geodesic γ starting from L, every L-Jacobi field along γ, that is tangent to $G(\gamma(t))$ for some $t \neq 0$, remains tangent to all the orbits met by γ.

This notion has a natural generalization in the more general setting of singular Riemannian foliations; see [71], where they speak of *singular Riemannian foliations without horizontally conjugate points*. The reason for this name can be explained in our situation as follows. Suppose that the action is trivial, i.e., $G(p) = p$ for all $p \in M$, then Definition 4.3.9 is equivalent to saying that M has no conjugate points. In the general case, the idea is that focal points of the orbits of variationally complete actions lie on the singular stratum of the manifold, and hence horizontal geodesics project onto geodesics in M/G that have no conjugate points in the interior of M/G. This is made more precise by a theorem of Lytchak and Thorbergsson; see [72] where the same result is stated for singular Riemannian foliations and generalizes many previous works [27, 23, 24, 39, 71].

Theorem 4.3.10. *A proper, isometric action $G \circlearrowleft M$ on a complete Riemannian manifold M is variationally complete if and only if M/G is isometric to a quotient N/W where N is a complete Riemannian manifold with no conjugate points, and W is a discrete group of isometries.*

Noteworthy examples of variationally complete actions are provided by polar actions with flat or negatively curved sections. In that case, N is given by any section of the action and $W = W(N)$ is the generalized Weyl group. In this class of examples is comprised the case of symmetric spaces, $M = G/K$, where N is a flat, totally geodesic submanifold of G/K. This case was studied by Bott and Samelson in [17]. In particular, if we consider the conjugation action of a compact connected group G on itself, N is a maximal torus of G, and $W(N)$ is an actual Weyl group.

Remark 4.3.11. It is important to note that by Theorem 4.3.10, the actions considered in conjectures 4.1.4 and 4.1.5 are variationally complete.

By Proposition 9.3 of [17] (see also the more general results in [69] or in [89]), we have the following.

Proposition 4.3.12. *Let $G \circlearrowleft M$ be an isometric, variationally complete action on a complete Riemannian manifold M, then for any regular orbit $L = G(q)$, $q \in M^{\mathrm{reg}}$, and any horizontal geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(0) \in L$, we have that $\gamma(t)$ is focal of multiplicity $k > 0$ for L if and only if*

$$\dim(L) - \dim(G(\gamma(t))) = k \ .$$

In particular, the index $\mathrm{Ind}(c)$ of a critical point of the energy functional $E : \Omega_{L,p} \to \mathbb{R}$, where p is not focal for L, is given by

$$\mathrm{Ind}(c) = \sum_{i=1}^{N} \dim(L) - \dim(G(c(t_i))) \ ,$$

where $0 < t_1 < \ldots < t_N < 1$ are the times, that are finite in number, at which $c(t_i)$ lies on a singular orbit.

Reflection group

We recall that given a Riemannian manifold N, an isometry $\sigma : N \to N$ is called a *reflection* if $\sigma^2 = \mathrm{id}_N$ and if the points that are fixed by σ form a codimension-1 submanifold of N. If the manifold M in Theorem 4.3.10 is simply connected, the group W is actually generated by reflections:

Theorem 4.3.13. *Let $G \circlearrowleft M$ be a proper, variationally complete action on a simply connected, complete Riemannian manifold M. Then M/G is isometric to N/W, where N is simply connected and without conjugate points, and W is generated by reflections.*

Proof. By Theorem 4.3.10, M/G is isometric to N/W with N complete Riemannian manifold without conjugate points and W discrete group of isometries. By Theorem 1.4 of [72], since the G-orbits are closed, the action is infinitesimally polar and by Corollary 1.3 of [70], this is equivalent to have no exceptional leaves (see also [7]). By Theorem 1.8 of the same paper, this implies that N/W is a *Coxeter orbifold*, i.e. it is locally diffeomorphic to a quotient of the Euclidean space by finite Euclidean Coxeter group. We are left to verify that W is generated by reflections. By Lemma 3.3 of [36] this is equivalent to check that $C_0 := (N/W)\backslash\partial(N/W)$ has trivial orbifold fundamental group, $\pi_1^{\text{orb}}(C_0) = \{0\}$. However, since there are no exceptional leaves, C_0 is a manifold and $\pi_1^{\text{orb}}(C_0) = \pi_1(C_0)$. To see that $\pi_1(C_0)$ is trivial, consider a loop $\gamma : [0,1] \to C_0$ based at $p = \gamma(0) = \gamma(1)$ and any horizontal lift $\bar{\gamma} : [0,1] \to M$. Then if $\pi : M \to N/W$ is the natural projection, $\bar{\gamma}(0) \in \pi^{-1}(\bar{p}) = G(\bar{\gamma}(0)) \ni \bar{\gamma}(1)$. Since the endpoints lie on the same connected components of $G(\bar{\gamma}(0))$, there is a curve $\beta : [0,1] \to G(\gamma(0))$, such that $\beta(0) = \bar{\gamma}(0)$ and $\beta(1) = \bar{\gamma}(1)$. The concatenation $\eta := \gamma\beta^{-1}$ is then a loop in M. It follows that if f_s, $s \in [0,1]$, is a homotopy from η to the constant path $\eta(0)$, $\pi \circ f_s$ is a homotopy from $\pi \circ \eta = \gamma$ to the constant path $\gamma(0)$. We conclude that every loop in C_0 is contractible, and hence $\pi_1(C_0) = \{0\}$. □

Reflection groups on Riemannian manifolds have been studied in [6], where the following facts are proved.

Theorem 4.3.14. *Let N be a simply connected Riemannian manifold, and W a group acting on N generated by reflections. Then:*

i) Every reflection in W fixes a totally geodesic hypersurface called central hypersurface*. The complement of the union of central hypersurfaces is a disjoint union of open sets, called* open chambers*.*

ii) W acts freely and transitively on the set of chambers: fixing a chamber C, any other chamber is of the form $w \cdot C$ for a unique $w \in W$. In particular, any chamber is a fundamental domain*, i.e.*

$$w \cdot C_{\hat{p}} \cap C_{\hat{p}} = \emptyset \ \forall w \neq e \quad \text{and} \quad \bigcup_{w\in W} w \cdot \overline{C_{\hat{p}}} = N.$$

iii) Given a principal point $\hat{p} \in N$, the Dirichlet domain

$$C_{\hat{p}} = \{\hat{q} \in N \ : \ d(\hat{p},\hat{q}) < d(\hat{p}, w\cdot\hat{q}) \ \forall w \neq e \in W\}$$

is the chamber containing $\hat{p}$.

iv) The projection $N \to N/W$ restricts to a homeomorphism $\overline{C_{\hat{p}}} \to N/W$. The inverse $\varphi : N/W \to N$ thus identifies N/W with $C_{\hat{p}}$.

v) Every boundary component of N/W is sent to an open set of a wall.

We are ready to prove the following theorem.

Theorem 4.3.15. ($\star$) *Let $G \circlearrowleft M$ be a variationally complete action on a complete, simply connected manifold M. Let $L = G(q)$ be a principal orbit, let $p \in M^{\text{princ}}$ and let N be a simply connected manifold without conjugate points such that $M/G \simeq N/W$, where W is a discrete group of reflections of N. If the Energy functional $E : \Omega_{p,L} \to \mathbb{R}$ is perfect, M is rationally hyperbolic if and only if N has exponential volume growth.*

Proof. In the following, we will identify the points of M/G with the points of N/W. We first show that W is in bijection with the critical points of E. To do this, let $C \subseteq N$ be any chamber of the W-action and let $\hat{p}, \hat{q} \in C$ correspond to the points $p, q \in M$ through $\hat{p} := \varphi(\pi(p))$ and $\hat{q} = \phi(\varphi(q))$; see item iv) of Theorem 4.3.14. Then for each $w \in W$ there is a unique geodesic $\hat{\gamma}_w : [0,1] \to N$ connecting $\hat{p}$ to $w \cdot \hat{q}$. This geodesic projects to an orbifold geodesic $\gamma_w : [0,1] \to N/W$. Then we can consider its unique horizontal lift $\bar{\gamma}_w : [0,1] \to M$ starting from $p \in M$. Note that we have $\bar{\gamma}(1) \in G(q)$, and hence $\bar{\gamma}_w \in \mathrm{Crit}(E, \Omega_{p,L})$. Always using Theorem 4.3.14, this correspondence can easily be inverted.

We now want to estimate the index of a critical point c in terms of an appropriate metric in the group W. To do so, let $W \to \mathrm{Crit}(E, \Omega_{p,L})$, $w \mapsto c_w$, be the one-to-one correspondence described above and define the following functions on W

$$\ell(w) := \int_0^1 \|\dot{c}_w(t)\| \, \mathrm{d}t \ , \qquad \iota(w) = \mathrm{Ind}(c_w), \qquad \chi(w) = \#((\pi \circ c_w) \cap \partial(N/W)) \ ,$$

where ℓ gives the length of the critical point c_w, ι gives the index of c_w and χ gives the number of times the orbifold geodesic $\pi \circ c_w$ hits the boundary of N/W.

By Proposition 4.3.12, and the discussion below it, for a variationally complete action, we have $\mathrm{Ind}(c_w) = \sum_{t \in [0,1]} (\dim L - \dim G(c_w(t)))$, i.e., the index of c_w equals the number of times, counted with multiplicity, the geodesic c_w intersects a singular orbit. Since we can always take p not focal for L, this sum is finite. Moreover, since singular leaves map precisely to points in $\partial(N/W)$, $\chi(c_w)$ also counts the number of times the orbifold geodesic c_w hits a singular orbit in M. Thus, we get

$$\chi(w) \leq \iota(w). \tag{4.6}$$

Let $\Sigma_1, \ldots, \Sigma_N \subset N/W$ denote the boundary strata of N/W. Then, it follows from Theorem 4.3.14 that for each Σ_i, $\varphi(\Sigma_i)$ is contained in the fixed point set of some reflection $s_i \in W$. The set $S = \{s_1, \ldots s_N\}$ is a generating set for W (see [6]). Furthermore, if $\pi \circ c_w$ intersects, in order, the boundary strata $\Sigma_{\alpha(1)}, \ldots, \Sigma_{\alpha(m)}$, where $m := \chi(w)$, we have that

$$\hat{\gamma}_w(1) = \left(s_{\alpha(1)} s_{\alpha(2)} \cdots s_{\alpha(m)}\right) \cdot \hat{q}.$$

Therefore, the minimal length of w as a word in W written in the alphabet S, is less than or equal to $\chi(w)$. On the other hand, by Schwarz–Milnor Lemma (see Proposition 8.19 of [20]) , the length of w is bounded from below by $A\ell(w) + B$ for some A, B, and by rearranging the terms, we get

$$A\ell(w) + B \leq \chi(w) \ . \tag{4.7}$$

Given a horizontal geodesic $c \in \mathrm{Crit}(E, \Omega_{p,L})$, since L is a principal orbit, there exists only a finite number of focal points along it. Therefore, there are constants ℓ_0 and κ such that the number of L-focal points on any sub-interval $I_0 \subseteq [0,1]$, for which the length of $c|_{I_0}$ is less than ℓ_0, is at most κ. Therefore, if $\ell(c)$ is the length of c, the index of c is bounded from above by

$$\kappa \left(\left\lceil \frac{\ell(c)}{\ell_0} \right\rceil + 1 \right) \leq \frac{\kappa \ell(c)}{\ell_0} + 2\kappa$$

Applying this to the geodesics c_w, we get that there are positive constants A', B' such that

$$\iota(w) \leq A'\ell(w) + B' \tag{4.8}$$

By putting inequalities (4.6),(4.7) and (4.8) together, we obtain $A\ell(w) + B \leq \iota(w) \leq A'\ell(w) + B'$ and as a consequence

$$\#B_{A\ell(w)+B}(e) \leq \#\{w \in W \mid \iota(w) \leq k\} \leq \#B_{A'\ell(w)+B'}(e), \tag{4.9}$$

where $\#B_r(e)$ is the number of $w \in W$ that can be written as a word in the alphabet S with length less than or equal to r.

By another version of the Schwarz–Milnor Lemma (see Proposition 36 in [25]), since W acts cocompactly on N, the volume of any ball in N has the same growth rate as the growth rate of the group W. Then if N has exponential volume growth, by inequality (4.9), W has exponential growth rate. Therefore, $\sum_{i=1}^{k} \mu_i(E, \Omega_{p,L}) = \#\{w \in W \mid \iota(w) \leq k\}$ grows exponentially in k. By item $ii)$ of Proposition 4.3.8, M is rationally hyperbolic.

Conversely, suppose that M is rationally hyperbolic. Since L is rationally elliptic, by the estimate (4.4) in the proof of the first part of Proposition 4.3.4, $\dim(H_i(\Omega_{p,L}))$ must grow exponentially. Then, since E is perfect, also $\mu_i(E, \Omega_{p,L})$ grows exponentially and always by (4.9), N must have exponential volume growth. □

Remark 4.3.16. Notice that if M is rationally hyperbolic, we can actually conclude that N has exponential volume growth without asking that the Energy functional is perfect. In fact, by Morse inequalities, if $\dim(H_i(\Omega_{p,L}))$ grows exponentially, so does $\mu_i(E, \Omega_{p,L})$.

To prove Theorem 4.3.1, we are left to verify if its hypotheses are sufficient to ensure that the energy functional is $\mathbb{Q}$-perfect. The result will follow by Theorem 4.3.15.

4.3.4 Bott-Samelson cycles

We are going to prove that the energy functional is $\mathbb{Z}_2$-perfect by constructing for each critical point a linking cycle of special type. We start with the following definition.

Definition 4.3.17. Let X be a smooth manifold, $f : X \to \mathbb{R}$ be a smooth function and $x \in X$ be a non-degenerate critical point of index k for f. Let Y be a compact, connected manifold of dimension k, oriented over some coefficient ring R and let $\psi : Y \to X$ be a smooth map. We call the couple (Y, ψ) a cycle of *Bott-Samelson type* for f at x, over R, if $(f \circ \psi) : Y \to \mathbb{R}$ has a unique, nondegenerate, maximum at $y \in \psi^{-1}(x)$.

We will rely on the following theorem (for a proof, see Section 5.5 of the Appendix).

Theorem 4.3.18. *If (Y, ψ) is a Bott-Samelson cycle for $f : X \to \mathbb{R}$ at $x \in X$, over R, and if $[Y] \in H_k(Y, R)$ is the orientation class of Y, then $\psi_*([Y]) \in H_k(X, R)$ is a linking cycle for f.*

Putting this together with Proposition 4.3.7, we obtain the following lemma.

Lemma 4.3.19. *If for every $c \in Crit(E, \Omega_{p,L})$ there exists a cycle of Bott-Samelson type (Δ_c, ψ) over R, then E is an R-perfect Morse function.*

Construction of the cycles

We follow the classical construction by Bott and Samelson (see [17]). Let $L \subseteq M$ be a principal orbit and let $p \in M$ be not focal for L. Consider the Energy functional $E : \Omega_{p,L} \to \mathbb{R}$ and a critical point $c \in \mathrm{Crit}(E, \Omega_{p,L})$, such that $c(0) = p$ and $c(1) = q \in L$. Let $0 < t_1 < \ldots < t_N < 1$ be the times at which c meets a singular orbit, and let

$K_i := G_{c(t_i)}$, for $i = 1, \dots, N$, be the corresponding singular isotropies. Let $H := G_q$ be the principal isotropy at q. By Corollary 1.2.32, c is a horizontal geodesic, and hence H fixes c.

We define the map $\tilde{\psi}_c : K_1 \times \ldots \times K_N \to \Omega_{p,L}(M)$ by

$$\tilde{\psi}_c(k_1, \dots, k_N)|_{[t_i,t_{i+1}]} := (k_1 \cdots k_i).c|_{[t_i,t_{i+1}]} \ ,$$

where one sets $t_0 = 0,\ t_{N+1} = 1$ and $\tilde{\psi}_c(k_1, \dots, k_N)|_{[0,t_1]} := c|_{[0,t_1]}$. Note that $\tilde{\psi}_c$ is well defined since at the overlap of two adjacent intervals, $[t_{i-1}, t_i] \cap [t_i, t_{i+1}]$, we have

$$(k_1 \cdots k_{i-1}).c(t_i) = (k_1 \cdots k_{i-1}k_i).c(t_i) \ ,$$

because $k_i \in K_i$. It is easy to see that $\tilde{\psi}_c(k_1, \dots, k_N)$ is a horizontal polygonal curve with corners possibly at $t_1 < \ldots < t_N$ that joins p with the point $\tilde{\psi}_c(k_1, \dots, k_N)(1) = (k_1 \cdots k_N).q \in L$.

The isotropy $H \subseteq K_i$ acts on K_i both on the left and on the right by left and right translations, respectively, and by definition of $\tilde{\psi}_c$, we have

$$\tilde{\psi}_c(k_1, \dots, k_N) = \tilde{\psi}_c(k_1h_1, h_1^{-1}k_2h_2, \dots, h_{N-1}^{-1}k_Nh_N)$$

for all $(h_1, \dots, h_N) \in H^N := H \times \overset{N\text{-times}}{\cdots} \times H$. Therefore, there is a natural induced map

$$\psi_c : (K_1 \times \ldots \times K_N)/H^N \to \Omega_{p,L}(M) \ ,$$

which is injective. The following imprecise notation, which is reminiscent of an iterative procedure, is sometimes used (e.g. in [40]; see also Remark 1.2.18)

$$K_1 \times_H K_2 \times_H \ldots \times_H K_{N-1} \times_H (K_N/H) := (K_1 \times \ldots \times K_N)/H^N \ .$$

Definition 4.3.20. We define the *Bott-Samelson manifold* of c as the compact, connected manifold

$$\Delta_c := K_1 \times_H K_2 \times_H \ldots \times_H K_{N-1} \times_H (K_N/H) \ .$$

The manifold Δ_c has an iterated fiber bundle structure in the following sense. Let

$$\Delta_c^i := K_1 \times_H K_2 \times_H \ldots \times_H K_{i-1} \times_H (K_i/H) \ ;$$

in particular, $\Delta_c^N = \Delta_c$. To be precise, let $K_0 := H$ and note that we can write $\Delta_c^1 = K_1 \times_H (K_0/H) = K_1 \times_H \{e\} = K_1/H$ and we set $\Delta_c^0 := \{e\}$, e being the identity of G. Then for all $i = 1, \dots, N$

$$\pi_i : \Delta_c^i \to \Delta_c^{i-1},\ [(k_1, \cdots, k_i)] \mapsto [(k_1, \cdots, k_{i-1})]$$

is a fiber bundle with fiber K_i/H. In fact, it is easy to verify that $\Delta_c^i = (K_1 \times \cdots \times K_{i-1}) \times_{H^{i-1}} (K_i/H)$ and, by Proposition 1.2.16,

$$\pi_i : (K_1 \times \cdots \times K_{i-1}) \times_{H^{i-1}} (K_i/H) \to (K_1 \times \cdots \times K_{i-1})/H^{i-1} = \Delta_c^{i-1}$$

is a fiber bundle with fiber K_i/H. Note that for $i = 1$, $\pi_1 : K_1/H \to \{e\}$ is the constant map. In summary, we have an iterated fiber bundle structure

$$\Delta_c^N \to \Delta_c^{N-1} \to \cdots \to \Delta_c^1 \to \Delta_c^0 = \{e\} \ .$$

It also follows from Proposition 1.2.16 that the action of the structure group of $\pi_i : \Delta_c^i \to \Delta_c^{i-1}$ on K_i/H is equivalent to the left action of H on the right H-cosets of K_i.

Remark 4.3.21. In the next section, we will be interested in the following situations, regarding the orientability or the triviality of Δ_c:

i) if K_i is Abelian, the bundle $\Delta_c^i \to \Delta_c^{i-1}$ is trivial:

$$\Delta_c^i = \Delta_c^{i-1} \times (K_i/H) \ ;$$

ii) if all the K_i are Abelian,

$$\Delta_c = (K_1/H) \times \ldots \times (K_N/H) \ ;$$

iii) if all the K_i are Abelian and all the K_i/H are spheres,

$$\Delta_c = \mathbb{T}^N \ ;$$

iv) if H is connected, the bundles $\Delta_c^i \to \Delta_c^{i-1}$ are orientable. Moreover, if K_1/H is orientable, Δ_c is orientable as a manifold.

In case *iii*), the proof of which is obvious, the manifold Δ_c^N is clearly orientable. Furthermore, if all the K_i/H are spheres $\mathbb{S}^{m_i}$ with $m_i := \dim(K_i) - \dim(H) \geq 2$, then $\Delta_c = \Delta_c^N$ is simply connected. In fact, by induction, $\Delta_c^1 = \mathbb{S}^{m_1}$ is a simply connected sphere and if we assume that Δ_c^i is simply connected, by the homotopy exact sequence for the fiber bundle $\mathbb{S}^{m_{i+1}} \to \Delta_c^{i+1} \to \Delta_c^i$, we have

$$\cdots \to \pi_1(\mathbb{S}^{m_{i+1}}) = \{0\} \to \pi_1(\Delta_c^{i+1}) \to \pi_1(\Delta_c^i) = \{0\} \to \{0\} \ ,$$

from which we conclude that $\pi_1(\Delta_c^{i+1}) = \{0\}$.

To prove claim *iv*) of Remark 4.3.21, we need the following general proposition.

Proposition 4.3.22. *Let $F \hookrightarrow E \twoheadrightarrow B$ be the unit sphere bundle of a vector bundle over B. Then if B is orientable, $F \hookrightarrow E \twoheadrightarrow B$ is orientable if and only if E is orientable as a manifold.*

Proof. The claim follows from a careful examination of the Jacobians of the transition maps of E or, equivalently, of the transition maps of the respective vector bundle. These have an upper triangular block matrix form, where one diagonal block is given by the transition functions of the bundle, and the other diagonal block is the Jacobian of the transition map of the base. Then if the determinant of one of the blocks is positive, the determinant of the whole matrix is positive if and only if the determinant of the other block is. For a more geometric proof, let $\mathcal{H} \subseteq TE$ be an Ehresmann connection for the bundle. Let $\pi : E \to B$ be the projection and $j : F \hookrightarrow E$ be the injection of the typical fiber. There is a splitting $TE = \mathcal{H} \oplus \mathcal{V}$, where for all $e \in E$, $(\mathrm{d}\pi)_e(\mathcal{H}_e) \simeq T_{\pi(e)}B$ and $\mathcal{V}_e = \ker((\mathrm{d}\pi)_e)$. Note that for all $e \in E$ and $f \in F$ with $j(f) = e$, $V_e \simeq j_*(T_f F)$. Then we can compute the first Stiefel-Whitney class of TE using Whitney summation formula as

$$w_1(TE) = w_1(\mathcal{H} \oplus \mathcal{V}) = w_1(\pi^*(TB)) + w_1(\mathcal{V}) = \pi^*(w_1(TB) + w_1(E)) \in H_1(E, \mathbb{Z}_2) \ .$$

So, if B is orientable, we have $w_1(TB) = 0$ and then $w_1(TE) = 0$ if and only if $w_1(E) = 0$. Then the claim follows.

□

The item *iv*) follows from a simple iterative argument. First, since the action is isometric, the structure group of $\Delta_c^i \to \Delta_c^{i-1}$ acts as an orthogonal representation $H \to O(T_{[k]}(K_i/H))$, but since H is connected, the image of this homomorphism is in $SO(T_{[k]}(K_i/H))$ for any $[k] \in (K_i/H)$. Therefore, $\Delta_c^i \to \Delta_c^{i-1}$ is orientable. The starting manifold $\Delta_c^1 = K_1/H$ is by hypothesis orientable, $\pi_2 : \Delta_c^2 \to \Delta_c^1$ is orientable and then, by Proposition 4.3.22, Δ_c^2 is orientable as a manifold. Again, $\Delta_c^3 \to \Delta_c^2$ is an orientable bundle and by the previous step and Proposition 4.3.22, Δ_c^3 is orientable as a manifold too. We can repeat this argument up to $\Delta_c^N = \Delta_c$.

Proposition 4.3.23. *There exists $\varepsilon > 0$ and an injective continuous map $\psi_c^\varepsilon : \Delta_c \to \Omega_{p,L}$ such that the couple $(\Delta_c, \psi_c^\varepsilon)$ is a cycle of Bott-Samelson type for the energy functional $E : \Omega_{p,L} \to \mathbb{R}$ over the field $\mathbb{Z}_2$.*

Proof. First of all Δ_c is a compact connected manifold of dimension

$$\dim(\Delta_c) = \sum_{i=1}^{N} \dim(K_i) - \dim(H) \ .$$

However, by a simple rewriting,

$$\dim(\Delta_c) = \sum_{i=1}^{N} \dim(K_i) - \dim(H) \pm \dim(M) \pm \dim(G) =$$

$$= \sum_{i=1}^{N} \dim(L) - \dim(G(c(t_i))) = \mathrm{Ind}(c) \ ,$$

because of Proposition 4.3.12. Then Δ_c has actually the right dimension to be a candidate cycle of Bott-Samelson type for c. Moreover, every manifold is oriented over $\mathbb{Z}_2$ and hence there is a nonzero orientation class $[\Delta_c] \in H_{\mathrm{Ind}(c)}(\Delta_c, \mathbb{Z}_2)$.

Notice that the natural candidate (Δ_c, ψ_c) is not suited for being a cycle of Bott-Samelson type, because $E(\gamma) = E(c)$ for all $\gamma \in \psi_c(\Delta_c)$. In other words, the set $\psi_c(\Delta_c)$ lies entirely in the level set of c. To avoid this problem, we need to appropriately deform $\psi_c(\Delta_c)$. Let $\varepsilon_i > 0$, for $i = 1, \ldots, N$, be such that

i) $2\varepsilon_i < t_{i+1} - t_{i-1}$;

ii) the ball $B_{\varepsilon_i}(c(t_i))$ is convex;

iii) for all $x, y \in \overline{B_{\varepsilon_i}(c(t_i))}$, x and y can be joined by a unique minimizing geodesic.

Note that for all $g \in G$, $B_{\varepsilon_i}(g.c(t_i))$ is isometric to $B_{\varepsilon_i}(c(t_i))$ through g, so that every point in the orbit $y \in G(c(t_i))$ admits an ε_i-ball $B_{\varepsilon_i}(y)$ with the same property as $B_{\varepsilon_i}(c(t_i))$. Let $\varepsilon := \min_{1 \le i \le N}\{\varepsilon_i\}$. Let $k = ([k_1, \ldots, k_N]) \in \Delta_c$ and let r_i be the unique minimal geodesic joining $\psi_c(k)(t_i - \varepsilon)$ with $\psi_c(k)(t_i + \varepsilon)$. We observe that such geodesic exists because $\psi_c(k)(t_i - \varepsilon) \in \partial(B_\varepsilon(\psi_c(k)(t_i))) \ni \psi_c(k)(t_i - \varepsilon)$ and by convexity, $r_i((t_i - \varepsilon, t_i + \varepsilon)) \subset B_{\varepsilon_i}(\psi_c(k)(t_i))$.
Then, we define the deformation ψ_c^ε as

$$\Delta_c \ni k \mapsto \psi_c^\varepsilon(k)(t) := \begin{cases} \psi_c(k)(t) & \text{if} \quad t \in [0, t_1 - \varepsilon] \cup [t_N + \varepsilon, 1]; \\ r_i(t) & \text{if} \quad t \in [t_i - \varepsilon, t_i + \varepsilon] \ . \end{cases}$$

Then, if $L(\cdot)$ denotes the length of a curve, we have the following inequality

$$L(r_i|_{[t_i-\varepsilon, t_i+\varepsilon]}) = d(\psi_c(k)(t_i - \varepsilon), \psi_c(k)(t_i + \varepsilon)) \le$$

$$\leq d(\psi_c(k)(t_i-\varepsilon),\psi_c(k)(t_i))+d(\psi_c(k)(t_i),\psi_c(k)(t_i+\varepsilon))=L(\psi_c(k)|_{[t_i-\varepsilon,t_i+\varepsilon]})\ ,$$

i.e.,

$$L(r_i|_{[t_i-\varepsilon,t_i+\varepsilon]})\leq L(\psi_c(k)|_{[t_i-\varepsilon,t_i+\varepsilon]})\ .$$

Since for any geodesic $\gamma:[a,b]\to M$, $E(\gamma|_{[a,b]})=\frac{L(\gamma|_{[a,b]})^2}{b-a}$, from the previous inequality we get

$$E(r_i|_{[t_i-\varepsilon,t_i+\varepsilon]})\leq E(\psi_c(k)|_{[t_i-\varepsilon,t_i+\varepsilon]})\ .$$

In conclusion, we obtain that for all $k\in\Delta_c$,

$$E(\psi_c^\varepsilon(k))\leq E(\psi_c(k))\ ,$$

and that the only element for which equality holds is precisely $\psi_c([e,\dots,e])=c$. By construction, the map is continuous in the C^0-topology of $\Omega_{p,L}$. This map can also be shown to be smooth with respect to the smooth structure of any finite dimensional approximation of $\Omega_{p,L}$ and c is a non-degenerate maximum of $E\circ\psi_c^\varepsilon$; see proposition 10.2 and 11.1 in [17] and Subsection 5.5.3 of the Appendix.

□

As an immediate corollary, we find the following.

Corollary 4.3.24. *The energy functional $E:\Omega_{p,L}\to\mathbb{R}$ is $\mathbb{Z}_2$-perfect.*

Proof. For any critical point $c\in\mathrm{Crit}(E,\Omega_{p,L})$, by Proposition 4.3.23, the pair $(\Delta_c,\psi_c^\varepsilon)$ is a cycle of Bott–Samelson type for E. Then $(\psi_c^\varepsilon)_*([\Delta_c])\in H_{\mathrm{Ind(c)}}(\Omega_{p,L},\mathbb{Z}_2)$ is of linking type by Proposition 5.5.8. We conclude by Lemma 4.3.19 that E is $\mathbb{Z}_2$-perfect. □

Thanks to Remark 4.3.21, we can also point out some situations in which the energy functional is $\mathbb{Q}$-perfect.

Proposition 4.3.25. *The energy functional is $\mathbb{Q}$-perfect if one of the following conditions is satisfied*

i) all the K_i are Abelian and K_i/H are all spheres;

ii) the principal isotropy H is connected and K_i/H are all spheres;

iii) $\dim(K_i)-\dim(H)\geq 2$ for all $i=1,\dots,N$.

In fact, the cases $i),ii),iii)$ implies that Δ_c is an orientable manifold, and hence there is a non-zero fundamental class $[\Delta_c]\in H_{\mathrm{Ind}(c)}(\Delta_c,\mathbb{Q})$, such that $(\psi_c^\varepsilon)_*([\Delta_c])\in H_{\mathrm{Ind}(c)}(\Omega_{p,L},\mathbb{Q})$ is a linking cycle over $\mathbb{Q}$.

4.3.5 Proof of the Positive Result

We are now able to prove Theorem 4.3.1 using the following general lemma.

Lemma 4.3.26. *Let $G\circlearrowleft M$ be a proper isometric action. Let L' be a singular orbit, and let $s\in L'$. Then there exists $\epsilon>0$ such that, for every principal orbit $L\subset\mathfrak{T}_\varepsilon(L')$, the closest point projection $\pi:L\to L'$ is a fiber bundle of characteristic fiber G_s/G_p, where $p\in L$ is any point lying on a slice for L' at s. Moreover, if the codimension in M/G of the stratum M_{G_s}/G is equal to one, then G_s/G_p is a sphere of dimension $\dim(G_s)-\dim(G_p)$.*

Proof. Let $\varepsilon > 0$ small enough such that the tubular neighborhood $\mathfrak{T}_\varepsilon(L') \simeq G \times_{G_s} (B_\varepsilon(0) \cap (T_sL')^\perp)$. Let $p \in L$ lie on a slice of L' at s. This means that p corresponds to a point $[e, v]$, for $v \in B_\varepsilon(0) \cap (T_sL')^\perp$ through $p = \exp_s^\perp(v)$, and hence the closest point projection is $\pi(p) = s$. Moreover, by equivariance, $\pi(gp) = g\pi(p) = gs$. It follows that if $\phi : G/G_p \to G/G_s$ is the map $gG_p \mapsto gG_s$, the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "\sim"] \arrow[d, "\pi"] & G/G_p \arrow[d, "\phi"] \\ L' \arrow[r, "\sim"] & G/G_s \end{tikzcd} \tag{4.10}$$

Therefore, since $\phi : G/G_p \to G/G_s$ is clearly a fiber bundle with fiber G_s/G_p, and $L \simeq G/G_p$ and $L' \simeq G/G_s$ are G-equivariant diffeomorphisms, also the projection $\pi : L \to L'$ is a fiber bundle of typical fiber G_s/G_p. Suppose now that $\pi(L')$ lies on a codimension-1 stratum Σ of M/G. First, we can write $(T_sL')^\perp \simeq V_0 \oplus V$, where the slice representation of G_s is trivial on V_0. Then $T_{L'}\Sigma \simeq V_0$ and

$$T_{L'}(M/G) = (T_sL')^\perp/G_s \simeq V_0 \times (V/G_s) \simeq T_{L'}\Sigma \times \mathbb{R}^+ \ .$$

Therefore, $G_s \circlearrowleft V$ is transitive on the unit sphere $\mathbb{S}^{\dim(V)-1}$ of V, with isotropy G_p. Then $G_s/G_p \simeq \mathbb{S}^{\dim(V)-1}$ and, in particular, $\dim(V) - 1 = \dim(G_s) - \dim(G_p)$. □

Proof of Theorem 4.3.1. We start by observing that we can always move the point p in such a way that it is not focal for L and that the orbifold geodesic $\pi \circ c$ only touches codimension one singular strata of M/G. Then by the previous Lemma 4.3.26, the fibers of the Bott-Samelson cycle Δ_c are all spheres and thanks to item *i)* and *ii)* of Proposition 4.3.25, we conclude that the energy functional is $\mathbb{Q}$-perfect. Since we are assuming that M/G supports a hyperbolic metric, the manifold N provided by Theorem 4.3.10, has hyperbolic volume growth. Then, we conclude by Theorem 4.3.15 that M is rationally hyperbolic. □

4.3.6 Topological Entropy

Let (X, d) be a metric space and let $\phi : \mathbb{R} \times X \to X$ be any continuous complete flow on it. Then we can define for any $T \in \mathbb{R}$ a new metric on X as follows

$$d_T(x, y) := \max_{0 \leq t \leq T} d(\phi(t, x), \phi(t, y)) \ .$$

Then one defines the *topological entropy* of ϕ as

$$h_{\text{top}}(\phi) := \lim_{\varepsilon \to 0} \limsup_{T \to +\infty} \frac{\log(N_T^\varepsilon)}{T} \ ,$$

where N_T^ε is the minimum number of balls of radius ε in the metric d_T needed to cover X.

Let $(M, \mathbf{g})$ be a complete Riemannian manifold and let SM be its unit tangent bundle. The *topolgical entropy* of $(M, \mathbf{g})$, denoted by $h_{\text{top}}(\mathbf{g})$, is defined as the topological entropy of its geodesic flow $\phi_{\mathbf{g}} : \mathbb{R} \times SM \to SM$, $\phi_{\mathbf{g}}(t, (p, v)) := \exp_p(tv)$.

Remark 4.3.27. If two flows $\phi : \mathbb{R} \times X \to X$, $\psi : \mathbb{R} \times Y \to Y$ are semi-conjugate, i.e., there is a continuous surjective map, $f : X \to Y$ such that $f \circ \phi = \psi \circ f$, then it can be shown that $h_{\text{top}}(\phi) \geq h_{\text{top}}(\psi)$; see [50, 94]. Moreover, it is immediate to notice, from the very definition, that if $(\overline{M}, \overline{\mathbf{g}})$ is a Riemannian manifold and $\pi : M \to \overline{M}$ is a Riemannian submersion which is also a finite cover, then $h_{\text{top}}(\mathbf{g}) = h_{\text{top}}(\overline{\mathbf{g}})$.

It turns out that the topological entropy is bounded from below by the so called *fundamental group entropy*. In turn, this entropy is bounded from below by $\lim_{n\to+\infty} \frac{\gamma(n)}{n}$, where $\gamma(n)$ denotes the growth rate of the fundamental group; see [50] for the definitions and the proofs.

Example 4.3.28. Recall that the fundamental group of a compact surface Σ_g of genus $g > 1$ is free, non Abelian of rank $k > 1$. Then it has exponential growth rate and hence positive fundamental group entropy. It follows that also $S\Sigma_g$ has positive fundamental group entropy, and hence given any metric $\mathbf{k}$ on Σ_g, we have $h_{\mathrm{top}}(\mathbf{k}) > 0$.

Let us consider the following result of [96], linking the topological entropy with the rational type.

Theorem 4.3.29. *If M is a nilpotent manifold (e.g. simply connected) admitting a metric $\boldsymbol{g}$ with $h_{top}(\boldsymbol{g}) = 0$, then M is rationally elliptic.*

A natural question emerging from this can be phrased as the following conjecture.

Conjecture 4.3.30. Suppose that M is a simply connected manifold such that for every Riemannian metric $\mathbf{g}$ we have $h_{\mathrm{top}}(\mathbf{g}) > 0$. Then M is rationally hyperbolic.

The rationally elliptic manifolds $W^{\#k}$ described in Subsection 4.2.1 provide a partial negative proof of this conjecture in the case of an isometric group action. From the discussion above, if $\mathbf{g}$ is a G-invariant metric and $\mathbf{h}$ is the quotient metric, we have that $h_{\mathrm{top}}(\mathbf{g}) \geq h_{\mathrm{top}}(\mathbf{h})$. Then we can consider a finite cover of D, $\pi : \Sigma_g \to D$, given by a compact surface Σ_g of genus $g > 1$, with metric $\mathbf{k} := \pi^*\mathbf{h}$. It follows that

$$h_{\mathrm{top}}(\mathbf{g}) \geq h_{\mathrm{top}}(\mathbf{h}) = h_{\mathrm{top}}(\mathbf{k}) > 0 \ ,$$

because of Example 4.3.28.

Chapter 5

Calabi–Matsushima Decomposition

Let Z be a compact connected Kähler manifold and $U \subseteq U(n)$ a compact connected Lie subgroup of the unitary group, acting in a Hamiltonian fashion and by Kähler isometries on Z. As is well known, under these hypotheses, the action extends to a holomorphic action of the reductive group $U^{\mathbb{C}}$. We then consider a real compatible subgroup $G \subseteq U^{\mathbb{C}}$, and we prove that the Lie algebra of the stabilizer of a point $\mathfrak{g}_z \subseteq \mathfrak{g}$, $z \in Z$, relative to the holomorphic sub-action of G admits a decomposition into non-negative eigenspaces, $\mathfrak{g}_z = \bigoplus_{\lambda \geq 0} \mathfrak{g}_\lambda$, where $\mathfrak{g}_0$ is the reductive part of $\mathfrak{g}_z$.

5.1 Setting

We consider a compact connected Kähler manifold (Z, J, ω), where J is the complex structure and ω the symplectic form, and $U \subset U(n)$ a compact connected Lie subgroup of the unitary group, acting by Kähler isometries on Z, i.e., the action preserves J, ω and the Riemannian metric $Q(\cdot,\cdot) := \omega(\cdot, J\cdot)$. We also consider the *complexification* of U, i.e., the *reductive* group $U^{\mathbb{C}} = \{\exp(i\eta)u \ : \ \eta \in \mathfrak{u} = \mathrm{Lie}(U), \ u \in U\}$.

Remark 5.1.1. Notice that in order to be able to extend the U-action on Z to a holomorphic $U^{\mathbb{C}}$-action, the hypothesis that Z is compact is crucial. In fact, let $D := \{z \in \mathbb{C} \ : \ |z| < 1\}$ be the unit open disk in $\mathbb{C}$ centered at the origin. Then consider the natural $U(1)$-action given by the multiplication by a unitary complex number. One can easily see that $U(1)^{\mathbb{C}} = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but D is not invariant under the $\mathbb{C}^*$-action.

Proposition 5.1.2. *Let $\Psi : U \circlearrowleft Z$ be Kähler action and assume that Z supports an action of $U^{\mathbb{C}}$ by holomorphic isometries $U^{\mathbb{C}} \circlearrowleft M$. Then the infinitesimal generator of $\zeta = \xi + i\eta \in \mathrm{Lie}(U^{\mathbb{C}}) = \mathrm{Lie}(U) \oplus i\mathrm{Lie}(U) = \mathfrak{u} + i\mathfrak{u}$, for $\xi, \eta \in \mathfrak{u}$ is given by*

$$\xi_Z := \xi_Z + J\eta_Z \ ,$$

where ξ_Z, η_Z are the infinitesimal generators of the U-action.

Proof. Since the action is by holomorphic isometries, for all $z \in Z$, $(\mathrm{d}\Psi^z)_e \circ i = J_z \circ (\mathrm{d}\Psi^z)_e$. Then the result follows by linearity. □

In the following, we will work under the hypothesis that the action of U on Z is Hamiltonian, i.e., it admits an Ad^*_U-equivariant *momentum map* that we denote with $\mu : Z \to \mathfrak{u}^* =: \mathrm{Lie}(U)^*$.

Definition 5.1.3. We call a real Lie subgroup G of $U^{\mathbb{C}}$, *real compatible*, if it is compatible with the *Cartan decomposition* of $U^{\mathbb{C}}$, i.e. we have a decomposition $G = K\exp(p)$, with $K = U \cap G$ and $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{k} \oplus p$, with $p = i\mathfrak{u} \cap \mathfrak{g}$, $\mathrm{Lie}(K) = \mathfrak{k}$.

Definition 5.1.4. Let $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$. We define the map $\langle \cdot, \cdot \rangle : \mathfrak{u}^{\mathbb{C}} \times \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ as

$$\langle X, Y \rangle := \begin{cases} -\mathfrak{Re}(B(X,Y)) & \text{if} \quad X, Y \in \mathfrak{u} \\ \mathfrak{Re}(B(X,Y)) & \text{if} \quad X, Y \in i\mathfrak{u} \\ \mathfrak{Re}(B(X,Y)) & \text{if} \quad X \in \mathfrak{u}, Y \in i\mathfrak{u} \ , \end{cases}$$

where $B : \mathfrak{u}^{\mathbb{C}} \times \mathfrak{u}^{\mathbb{C}} \to \mathbb{C}$ is the bilinear form given by

$$B(X, Y) := \mathrm{Tr}(XY) \ .$$

Remark 5.1.5. Notice that $B(X, X) < 0$ for all $X \in \mathfrak{u}$ and $B(Y, Y) > 0$ for all $Y \in i\mathfrak{u}$, and that B is manifestly Ad_U-invariant because of the cyclicity of the trace. We say that such a bilinear form B is of *Euclidean type*. It follows that $\langle \cdot, \cdot \rangle$ is an Euclidean Ad_U-invariant scalar product on $\mathfrak{u}^{\mathbb{C}}$. In particular, we have that the splitting $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$.

Definition 5.1.6. Given $\mu : Z \to \mathfrak{u}^*$ an Ad^*_U-equivariant momentum map relative to the isometric action of U on Z, we call *gradient map* the map $\mu_p : Z \to p$ given by

$$\mu_p(z) := \pi(\# i\mu(z)) \ ,$$

where $z \in Z$, $\# : (\mathfrak{u}^{\mathbb{C}})^* \to \mathfrak{u}^{\mathbb{C}}$ is the duality induced by $\langle \cdot, \cdot \rangle$, and $\pi : \mathfrak{u}^{\mathbb{C}} \to p$ is the orthogonal projection onto p.

Proposition 5.1.7. (i) *The map μ_p is Ad_K-equivariant, i.e., for all $k \in K$ and $z \in Z$ we have*

$$\mu_p(k.z) = Ad_k(\mu_p(z)) \ ,$$

where $k.z$ denotes the action of $k \in K$ on $z \in Z$.

(ii) If $\xi \in p$, and $\xi_Z(z) := \frac{d}{dt} \exp(t\xi).z|_{t=0}$ is the infinitesimal generator we have

$$(\mathrm{d}\langle \mu_p, \xi \rangle)_z(\cdot) = Q|_z(\cdot, \xi_Z(z)) \ ,$$

that is equivalent to say

$$(\nabla \mu_p^{\xi})(z) = \xi_Z(z) \ ,$$

where we defined $\mu_p^{\xi} := \langle \mu_p(z), \xi \rangle$, and where the gradient[1] is relative to the Riemannian metric $Q(\cdot, \cdot) := \omega(\cdot, J\cdot)$.

Proof. (i) By definition $\mu_p := \pi \circ \# \circ i \circ \mu$. We have that $\mu \circ k. = \mathrm{Ad}^*_k \circ \mu$ by Ad^*_K-equivariance of μ. The multiplication by i clearly commutes with Ad^*_k and therefore we are left to show that $\pi \circ \# \circ \mathrm{Ad}^*_k = \mathrm{Ad}_k \circ \pi \circ \#$. For $\# \circ \mathrm{Ad}^*_k = \mathrm{Ad}_k \circ \#$ we have that, denoting the canonical duality of $\mathfrak{u}$ as $\langle \cdot | \cdot \rangle$, for all $\xi \in \mathfrak{u}$, $\langle \mathrm{Ad}^*_k \mu | \xi \rangle = \langle \mu | \mathrm{Ad}_{k^{-1}} \xi \rangle$. Therefore,

$$\langle \#(\mathrm{Ad}^*_k \mu), \xi \rangle = \langle \# \mu, \mathrm{Ad}_{k^{-1}} \xi \rangle = \langle \mathrm{Ad}_k(\# \mu), \xi \rangle \ ,$$

and hence $\# \circ \mathrm{Ad}^*_k = \mathrm{Ad}_k \circ \#$. Finally, Ad_k commutes with π, because π depends only on $\langle \cdot, \cdot \rangle$, which is Ad_U-invariant, and p is Ad_K-invariant.
(ii) Since μ is a momentum map we have that for all $\xi \in \mathfrak{u}$, $z \in Z$, and $v \in T_z Z$

$$(\mathrm{d}\langle \mu | \xi \rangle)_z(v) = \omega|_z(v, \xi_Z(z)) \ ,$$

[1]We define the gradient of $F : Z \to \mathbb{R}$ to be the vector field $\nabla F \in \mathfrak{X}(Z)$ such that for all $V \in \mathfrak{X}(Z)$ we have $Q(\nabla F, V) = \mathrm{d}F(Y)$.

where again $\langle\cdot|\cdot\rangle$ is the canonical duality of $\mathfrak{u}$. Now, since the action extends to a holomorphic action of $U^{\mathbb{C}}$ (because Z is compact), we have that $(i\xi)_Z = J\xi_Z$ and therefore,

$$(\mathrm{d}\langle i\mu|\xi\rangle)_z(v) = (\mathrm{d}\langle\mu|i\xi\rangle)_z(v) = \omega|_z(v,(i\xi)_Z(z)) = \omega|_z(v,J(\xi_Z(z))) = Q|_z(v,\xi_Z(z)) \ .$$

But by the definition of μ_p and since $\xi \in p$, we have $\langle i\mu|\xi\rangle = \langle\#(i\mu),\xi\rangle = \langle\pi(\#(i\mu)),\xi\rangle = \mu_p^\xi$.

□

Proposition 5.1.8. *Let $f : \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ be a smooth function and $\mu_p : Z \to p \subset \mathfrak{u}^{\mathbb{C}}$ be the gradient map. Then $z \in Z$ is a critical point of $f \circ \mu_p$ if and only if*

$$\#(df)_{\mu_p(z)} \in \mathfrak{g}_z \ ,$$

where $\mathfrak{g}_z$ is the Lie algebra of the stabilizer of z.

Proof. If $z \in Z$ is critical for $f \circ \mu_p$, we have that for all $v \in T_zZ$

$$0 = (\mathrm{d}(f\circ\mu_p))_z(v) = (\mathrm{d}f)_{\mu_p(z)}((\mathrm{d}\mu_p)_z(v)) = \langle\#(\mathrm{d}f)_{\mu_p(z)},(\mathrm{d}\mu_p)_z(v)\rangle \ ,$$

since we have identified $T^*_{\mu_p(z)}p$ with p^* and p^* with p with $\#$ induced by $\langle\cdot,\cdot\rangle$. We have already seen that

$$\langle\#(\mathrm{d}f)_{\mu_p(z)},(\mathrm{d}\mu_p)_z(v)\rangle = Q|_z(v,(\#(\mathrm{d}f)_{\mu_p(z)})_Z(z)) = 0 \ ,$$

for all $v \in T_zZ$, which tells us precisely that $\#(\mathrm{d}f)_{\mu_p(z)} \in \mathfrak{g}_z$.

□

Remark 5.1.9. We can consider the particular case of $F : \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ given by $F(\xi) := \frac{1}{2}\langle\xi,\xi\rangle$. Here $(\mathrm{d}F)_{\mu_p(z)} = \#^{-1}(\mu_p(z))$ and therefore $z \in Z$ is critical for $F \circ \mu_p$ if and only if

$$\#(\mathrm{d}F)_{\mu_p(z)} = \mu_p(z) \in \mathfrak{g}_z \ .$$

In the following, for simplicity of notation, we will write

$$(\nabla F)_\xi := \#(\mathrm{d}F)_\xi \ ,$$

which is a vector field on $\mathfrak{u}^{\mathbb{C}}$ computed at ξ.

5.2 Calabi-Matsushima decomposition

In this section, we are going to prove our main theorem (Theorem 5.2.2), which will be derived from two technical lemmas and makes use of the following definition.

Definition 5.2.1. Let $F : \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ be a smooth function. We say that F is $(\mathfrak{k},p)$-compatible if the Hessian of F is in block form at every point

$$\mathrm{H}F = \partial^2_{\mathfrak{k}^\perp\cap\mathfrak{u}}F + \partial^2_{\mathfrak{k}}F + \partial^2_{p}F + \partial^2_{p^\perp\cap(i\mathfrak{u})}F \ ,$$

where the orthogonal complements are taken with respect to $\langle\cdot,\cdot\rangle$, and all other mixed derivatives are zero.

Theorem 5.2.2. ($\star$) *Let $\mu_p : Z \to p$ be the gradient map associated with an Ad_U^*-equivariant momentum map $\mu : Z \to \mathfrak{u}^*$, and $F : \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ be an Ad_U-invariant strictly convex $(\mathfrak{k}, p)$-compatible function. If $z \in Z$ is a critical point of $F \circ \mu_p : Z \to \mathbb{R}$, we have a direct sum decomposition of $\mathfrak{g}_z$*

$$\mathfrak{g}_z = \bigoplus_{\lambda \geq 0} \mathfrak{g}_\lambda \ ,$$

where $\mathfrak{g}_\lambda := \{\xi \in \mathfrak{g}_z \ : \ ad_{(\nabla f)_{\mu_p(z)}} \xi = \lambda \xi\}$, with $f := F|_p$, and $\mathfrak{g}_0 = \mathfrak{k}_z \oplus p_z$ is the reductive part of $\mathfrak{g}_z$.

Definition 5.2.3. Let $F : \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ be a smooth Ad_U-invariant strictly convex $(\mathfrak{k}, p)$-compatible function. We define a positive definite inner product on $T_\zeta \mathfrak{u}^{\mathbb{C}} \simeq \mathfrak{u}^{\mathbb{C}}$ for every $\zeta \in \mathfrak{u}^{\mathbb{C}}$, that we denote as $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\zeta : \mathfrak{u}^{\mathbb{C}} \times \mathfrak{u}^{\mathbb{C}} \to \mathbb{R}$ as follows.

$$\langle\!\langle \xi, \eta \rangle\!\rangle_\zeta := \langle \xi, [\#(\mathrm{H}F)_\zeta]^{-1}(\eta) \rangle \ ,$$

where $(\mathrm{H}F)_\zeta$ is the Hessian of F computed at ζ, and $\#$ is the sharp isomorphism induced by $\langle \cdot, \cdot \rangle$.

Remark 5.2.4. The fact that for all $\zeta \in \mathfrak{u}^{\mathbb{C}}$ this is a scalar product comes from the fact that F is strictly convex, i.e. $(\mathrm{H}F)_\zeta$ is positive definite. This also allows us to invert the isomorphism of $\mathfrak{u}^{\mathbb{C}}$ given by $\# \circ (\mathrm{H}F)_\zeta$, still obtaining a scalar product. The splittings $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus (i\mathfrak{u})$, $\mathfrak{g} = \mathfrak{k} \oplus p$ are still orthogonal with respect to the new scalar product.

Remark 5.2.5. In the following, we will use the following identities.

$$\langle \mathrm{ad}_\xi^* \alpha | \eta \rangle = -\langle \alpha | \mathrm{ad}_\xi \eta \rangle \qquad \forall\, \xi, \eta \in \mathfrak{u}^{\mathbb{C}},\ \forall\, \alpha \in (\mathfrak{u}^{\mathbb{C}})^* \ ;$$

$$\langle \mathrm{ad}_\xi \zeta, \eta \rangle = -\langle \zeta, \mathrm{ad}_\xi \eta \rangle \qquad \forall\, \xi \in \mathfrak{u},\ \forall\, \eta, \zeta \in \mathfrak{u}^{\mathbb{C}} \ ;$$

and hence

$$\# \circ \mathrm{ad}_\xi^* = \mathrm{ad}_\xi \circ \# \quad \forall\, \xi \in \mathfrak{u} \ .$$

Lemma 5.2.6. *For all $\xi \in \mathfrak{u}$ and $\eta, \zeta \in \mathfrak{u}^{\mathbb{C}}$, F Ad_U-invariant, and every $t \in \mathbb{R}$, we have that*

$$\langle (dF)_{Ad_{\exp(t\xi)}\eta} | Ad_{\exp(t\xi)} \zeta \rangle = \langle (dF)_\eta | \zeta \rangle \ ,$$

i.e., the quantity on the LHS actually does not depend on t.

Proof. The property follows directly from Ad_U-invariance of F.

$$\langle (\mathrm{d}F)_{\mathrm{Ad}_{\exp(t\xi)}\eta} | \mathrm{Ad}_{\exp(t\xi)} \zeta \rangle = \frac{d}{ds} F(\mathrm{Ad}_{\exp(t\xi)}\eta + s\mathrm{Ad}_{\exp(t\xi)}\zeta)\Big|_{s=0} =$$

$$= \frac{d}{ds} F(\mathrm{Ad}_{\exp(t\xi)}(\eta + s\zeta))\Big|_{s=0} = \frac{d}{ds} F(\eta + s\zeta)\Big|_{s=0} = \langle (\mathrm{d}F)_\eta | \zeta \rangle \ .$$

□

Lemma 5.2.7. *For $z \in Z$ and $\xi, \eta \in \mathfrak{g}$, we have the following identity*

$$\Big\langle\!\Big\langle ad_{(\nabla f)_{\mu_p(z)}} \xi, \eta \Big\rangle\!\Big\rangle_{\mu_p(z)} = \langle ad_{\mu_p(z)} \xi, \eta \rangle.$$

Proof. Clearly, since $f = F|_p$, $(\nabla f)_{\mu_p(z)} \in p$ and, therefore, thanks to the properties of the Cartan decomposition and the orthogonality of the splitting with respect to the shifted scalar product, we get

$$\left\langle\!\left\langle \mathrm{ad}_{(\nabla f)_{\mu_p(z)}}\xi, \eta \right\rangle\!\right\rangle_{\mu_p(z)} = \left\langle\!\left\langle \mathrm{ad}_{(\nabla f)_{\mu_p(z)}}\xi^{\mathfrak{k}}, \eta^p \right\rangle\!\right\rangle_{\mu_p(z)} + \left\langle\!\left\langle \mathrm{ad}_{(\nabla f)_{\mu_p(z)}}\xi^p, \eta^{\mathfrak{k}} \right\rangle\!\right\rangle_{\mu_p(z)} .$$

Consider the term $\mathrm{ad}_{(\nabla f)_{\mu_p(z)}}\xi^{\mathfrak{k}} = -\mathrm{ad}_{\xi^{\mathfrak{k}}}(\nabla f)_{\mu_p(z)}$ and write $\zeta := \mu_p(z)$ for simplicity. We then observe that for all $\sigma \in \mathfrak{u}^{\mathbb{C}}$, thanks to Remark 5.2.5 and Lemma 5.2.6, we have

$$\langle \mathrm{ad}_{\xi^{\mathfrak{k}}}(\nabla f)_\zeta, \sigma\rangle = \langle \mathrm{ad}_{\xi^{\mathfrak{k}}}{}^*(\mathrm{d}f)_\zeta|\sigma\rangle = -\langle(\mathrm{d}f)_\zeta|\mathrm{ad}_{\xi^{\mathfrak{k}}}\sigma\rangle = -\langle(\mathrm{d}f)_\zeta|\frac{d}{dt}\mathrm{Ad}_{\exp(t\xi^{\mathfrak{k}})}\sigma\Big|_{t=0}\rangle =$$

$$= -\frac{d}{dt}\langle(\mathrm{d}f)_{\mathrm{Ad}_{\exp(t\xi^{\mathfrak{k}})}\zeta}|\mathrm{Ad}_{\exp(t\xi^{\mathfrak{k}})}\sigma\rangle\Big|_{t=0} + \langle\frac{d}{dt}(\mathrm{d}f)_{\mathrm{Ad}_{\exp(t\xi^{\mathfrak{k}})}\zeta}\Big|_{t=0}|\sigma\rangle =$$

$$= \langle\frac{d}{dt}(\mathrm{d}f)_{\mathrm{Ad}_{\exp(t\xi^{\mathfrak{k}})}\zeta}\Big|_{t=0}|\sigma\rangle = \langle(\mathrm{H}f)_\zeta(\mathrm{ad}_{\xi^{\mathfrak{k}}}\zeta)|\sigma\rangle = \langle\#(\mathrm{H}f)_\zeta(\mathrm{ad}_{\xi^{\mathfrak{k}}}\zeta),\sigma\rangle .$$

Therefore,

$$\mathrm{ad}_{(\nabla f)_{\mu_p(z)}}\xi^{\mathfrak{k}} = -\#(\mathrm{H}f)_\zeta(\mathrm{ad}_{\xi^{\mathfrak{k}}}\zeta) .$$

Using Definition 5.2.3, we conclude that

$$\left\langle\!\left\langle \mathrm{ad}_{(\nabla f)_{\mu_p(z)}}\xi^{\mathfrak{k}}, \eta^p \right\rangle\!\right\rangle_{\mu_p(z)} = \langle \mathrm{ad}_{\mu_p(z)}\xi^{\mathfrak{k}}, \eta^p\rangle .$$

Now we consider the term $\mathrm{ad}_{(\nabla f)_{\mu_p(z)}}i\xi^p = -\mathrm{ad}_{i\xi^p}(\nabla f)_{\mu_p(z)}$. Since $i\xi^p \in \mathfrak{u}$, we can repeat the same computation of before and get

$$\langle \mathrm{ad}_{i\xi^p}(\nabla f)_\zeta, \sigma\rangle = \langle \mathrm{ad}_{i\xi^p}{}^*(\mathrm{d}f)_\zeta|\sigma\rangle = -\langle(\mathrm{d}f)_\zeta|\mathrm{ad}_{i\xi^p}\sigma\rangle = -\langle(\mathrm{d}f)_\zeta|\frac{d}{dt}\mathrm{Ad}_{\exp(ti\xi^p)}\sigma\Big|_{t=0}\rangle =$$

$$= -\frac{d}{dt}\langle(\mathrm{d}f)_{\mathrm{Ad}_{\exp(t\xi^{\mathfrak{k}})}\zeta}|\mathrm{Ad}_{\exp(ti\xi^p)}\sigma\rangle\Big|_{t=0} + \langle\frac{d}{dt}(\mathrm{d}f)_{\mathrm{Ad}_{\exp(ti\xi^p)}\zeta}\Big|_{t=0}|\sigma\rangle =$$

$$= \langle\frac{d}{dt}(\mathrm{d}f)_{\mathrm{Ad}_{\exp(ti\xi^p)}\zeta}\Big|_{t=0}|\sigma\rangle = \langle(\mathrm{H}f)_\zeta(\mathrm{ad}_{i\xi^p}\zeta)|\sigma\rangle = \langle\#(\mathrm{H}f)_\zeta(\mathrm{ad}_{i\xi^p}\zeta),\sigma\rangle .$$

Therefore,

$$\mathrm{ad}_{(\nabla f)_{\mu_p(z)}}i\xi^p = -\#(\mathrm{H}f)_\zeta(\mathrm{ad}_{i\xi^p}\zeta) .$$

Again, using Definition 5.2.3, we conclude that

$$\left\langle\!\left\langle \mathrm{ad}_{(\nabla f)_{\mu_p(z)}}\xi^p, \eta^{\mathfrak{k}} \right\rangle\!\right\rangle_{\mu_p(z)} = -\left\langle\!\left\langle \mathrm{ad}_{(\nabla f)_{\mu_p(z)}}i\xi^p, i\eta^{\mathfrak{k}} \right\rangle\!\right\rangle_{\mu_p(z)} = -\langle \mathrm{ad}_{\mu_p(z)}i\xi^p, i\eta^{\mathfrak{k}}\rangle = \langle \mathrm{ad}_{\mu_p(z)}\xi^p, \eta^{\mathfrak{k}}\rangle .$$

In summary, we obtain the result. ☐

Proof. (Theorem 5.2.2.) Step 1. We first prove that the theorem is valid in particular for $F(\cdot) = \langle \cdot, \cdot \rangle$. In this case $\#(\mathrm{d}F)_{\mu_p(z)} = \mu_p(z) \in \mathfrak{g}_z \cap p$. In general, given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus p$, we know that for all $\beta \in p$ the endomorphism $\mathrm{ad}_\beta : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. Therefore, consider an eigenvector $\xi \in \mathfrak{g}_z$, $\mathrm{ad}_{\mu_p(z)}\xi = [\mu_p(z), \xi] = \lambda \xi$, and take the scalar product with ξ

$$\langle [\mu_p(z), \xi], \xi \rangle = \lambda \langle \xi, \xi \rangle \ .$$

Now, we can write $\xi = \xi^{\mathfrak{k}} + \xi^p$ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus p$, having $\xi^{\mathfrak{k}} \in \mathfrak{k} \cap \mathfrak{g}_z$ and $\xi^p \in p \cap \mathfrak{g}_z$. Since, in general, for a Cartan decomposition we have $[p, p] \subseteq \mathfrak{k}$ and $[p, \mathfrak{k}] \subseteq p$, and since the splitting is orthogonal with respect to $\langle \cdot, \cdot \rangle$, the only remaining terms in the LHS of the previous equation are

$$\langle [\mu_p(z), \xi^p], \xi^{\mathfrak{k}} \rangle + \langle [\mu_p(z), \xi^{\mathfrak{k}}], \xi^p \rangle = \lambda \left(\langle \xi^{\mathfrak{k}}, \xi^{\mathfrak{k}} \rangle + \langle \xi^p, \xi^p \rangle \right) \ .$$

Now, we study the two terms in the LHS separately.

$$\begin{aligned}
&\langle [\mu_p(z), \xi^p], \xi^{\mathfrak{k}} \rangle = \langle [(\pi \circ \#)(i\mu(z)), \xi^p], \xi^{\mathfrak{k}} \rangle = \\
&= \langle [\#(i\mu(z)), \xi^p], \xi^{\mathfrak{k}} \rangle = \langle [\#\mu(z), \xi^p], i\xi^{\mathfrak{k}} \rangle = \\
&= -\langle \mathrm{ad}_{\xi^p}(\#\mu(z)), i\xi^{\mathfrak{k}} \rangle = -\langle \#(\mathrm{ad}^*_{\xi^p}(\mu(z))), i\xi^{\mathfrak{k}} \rangle = \\
&= -\langle \mathrm{ad}^*_{\xi^p}(\mu(z)) | i\xi^{\mathfrak{k}} \rangle = -\frac{d}{dt} \langle \mathrm{Ad}^*_{\exp(t\xi^p)}(\mu(z)) | i\xi^{\mathfrak{k}} \rangle \Big|_{t=0} = \\
&= -\frac{d}{dt} \langle \mu(\exp(t\xi^p.z)) | i\xi^{\mathfrak{k}} \rangle \Big|_{t=0} = -\langle (\mathrm{d}\mu)_z(\xi^p_Z(z)) | i\xi^{\mathfrak{k}} \rangle = \\
&= -\omega|_z(\xi^p_Z(z), J(\xi^{\mathfrak{k}}_Z(z))) = -Q|_z(\xi^p_Z(z), \xi^{\mathfrak{k}}_Z(z)) \ .
\end{aligned}$$

For the other term, we have

$$\langle [\mu_p(z), \xi^{\mathfrak{k}}], \xi^p \rangle = -\langle \mathrm{ad}_{\xi^{\mathfrak{k}}}(\mu_p(z)), \xi^p \rangle = -\frac{d}{dt} \langle \mathrm{Ad}_{\exp(t\xi^{\mathfrak{k}})}(\mu_p(z)), \xi^p \rangle \Big|_{t=0} =$$

$$= -\frac{d}{dt} \langle \mu_p(\exp(t\xi^{\mathfrak{k}}).z, \xi^p \rangle \Big|_{t=0} = -\langle (\mathrm{d}\mu_p)_z(\xi^{\mathfrak{k}}_Z), \xi^p \rangle = -Q|_z(\xi^p_Z(z), \xi^{\mathfrak{k}}_Z(z)) \ .$$

Therefore, since $\xi^p_Z(z) + \xi^{\mathfrak{k}}_Z(z) = \xi_Z(z) = 0$, because $\xi \in \mathfrak{g}_z$

$$\langle [\mu_p(z), \xi], \xi \rangle = -2Q|_z(\xi^p_Z(z), \xi^{\mathfrak{k}}_Z(z)) = 2Q|_z(\xi^p_Z(z), \xi^p_Z(z)) \geq 0 \ .$$

This shows that $\lambda \geq 0$.

Step 2. Consider now a general F as in the hypothesis of Theorem 5.2.2. Again, since $f = F|_p$, $\mathrm{ad}_{(\nabla f)_{\mu_p(z)}} : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable, because $(\nabla f)_{\mu_p(z)} \in p$. Therefore, always from the computations in step 1, we conclude that if $\xi \in \mathfrak{g}_z$ is an eigenvector of $(\nabla f)_{\mu_p(z)}$ of eigenvalue λ, we have

$$\lambda \langle\!\langle \xi, \xi \rangle\!\rangle = \left\langle\!\!\left\langle \mathrm{ad}_{(\nabla f)_{\mu_p(z)}} \xi, \xi \right\rangle\!\!\right\rangle_{\mu_p(z)} = \langle \mathrm{ad}_{\mu_p(z)} \xi, \xi \rangle = 2Q|_z(\xi^p_Z(z), \xi^p_Z(z)) \geq 0 \ ,$$

showing that also in this case $\lambda \geq 0$.

□

Remark 5.2.8. We note that if $\xi \in \mathfrak{g}_z$ is in the kernel of $\mathrm{ad}_{(\nabla f)_{\mu_p(z)}}$, then $\xi^k_Z(z) = 0 = \xi^p_Z(z)$ and hence $\xi \in \mathfrak{k}_z \oplus p_z$.

Appendix

We are going to review some classical facts of Riemannian Geometry, Symplectic Geometry and Morse Theory that are used throughout the thesis. Throughout the process, we also fix some notation and convention. For further details, we refer to [28, 97], [5, 9, 22, 75, 109] and [16, 78, 86, 92]

5.3 Riemannian Geometry

The couple $(M, \mathbf{g})$ will denote a Riemannian manifold with Riemannian metric $\mathbf{g}$. Since there are different, but equivalent, approaches and conventions, we present briefly the one we follow and recall the relations to others frequently used. To start fixing signs and normalizations, if ∇ is a connection on TM we define its torsion as

$$\mathcal{T}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \in \mathcal{C}^\infty(M,TM) \ ,$$

and its curvature as

$$\mathcal{R}(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \in \mathcal{C}^\infty(M,TM),$$

for $X, Y, Z \in \mathcal{C}^\infty(M, TM)$ smooth vector fields.

If ∇ is the Levi-Civita connection of $\mathbf{g}$, i.e., the unique connection on TM such that $\mathcal{T} = 0$ and $\nabla \mathbf{g} = 0$, we call $\mathcal{R}$ the *Riemann curvature tensor* of $(M, \mathbf{g})$. We denote by R the fully covariant version of it

$$R(X,Y,Z,W) := \mathbf{g}(R(X,Y)Z, W) \ .$$

The *Ricci tensor*, is the covariant symmetric tensor field defined as the following trace

$$\mathrm{Ric}(X,Y) := \mathrm{Tr}(\mathcal{R}(X,\cdot)Y) \ .$$

We define the *scalar curvature* as the trace with respect to $\mathbf{g}$ of the Ricci tensor

$$\mathrm{Scal} := \mathrm{Tr}_{\mathbf{g}}(\mathrm{Ric}) \in \mathcal{C}^\infty(M) \ .$$

Finally, we define the *sectional curvature* as

$$K(X,Y) := \frac{R(X,Y,X,Y)}{\mathbf{g}(X,X)\mathbf{g}(Y,Y) - \mathbf{g}(X,Y)^2} \ .$$

It is important to recall that the sectional curvature is actually a function on the Grassmannian of 2-planes in each tangent space, $K_x : \mathrm{Gr}_2(T_x M) \to \mathbb{R}$. We notice that, in the literature, there are different sign and normalization conditions, but they all agree for the sectional curvature.

5.3.1 Isometric Immersions

Given $M^n \subseteq \overline{M}^{n+k}$ an n-dimensional immersed sub-manifold of the $(n+k)$-dimensional Riemannian manifold $(\overline{M}^{n+k}, \overline{\mathbf{g}})$, we consider the Levi-Civita connection $\overline{\nabla}$ of $(\overline{M}, \overline{\mathbf{g}})$. If we denote by $\iota : M \hookrightarrow \overline{M}$ the inclusion map, the induced metric $\iota^*\overline{\mathbf{g}}$ on M, i.e., the restriction of $\overline{\mathbf{g}}$, is the so called *first fundamental form* of M, which is the most immediate isometric invariant associated to M. To construct another isometric invariant associated to M, we shall study how the normal directions interact with the tangent ones. For any $x \in M$ we consider the orthogonal splitting $T_x\overline{M} = \nu_x M \oplus T_x M$, where $\nu_x M$ is the $\mathbf{g}$-orthogonal complement of $T_x M$ in $T_x\overline{M}$. We will denote for any $\bar{v} \in T_x\overline{M}$ its as $v = v^\perp + v^\top$ where $v^\perp \in \nu_x M$ and $v^\top \in T_x M$. We know that we can always locally extend $X, Y \in \mathcal{C}^\infty(M, TM)$ to vector fields of $\overline{M}$, $\overline{X}, \overline{Y} \in \mathcal{C}^\infty(\overline{M}, T\overline{M})$. Then, by the properties of the connection, we have $(\overline{\nabla}_{\overline{X}}\overline{Y})^\top = \nabla_X Y$, where ∇ is the Levi-Civita connection of $\iota^*\overline{\mathbf{g}}$. We define the *second fundamental form* of M at $x \in M$ as the map $\mathrm{I\!I}_x : T_x M \times T_x M \to \nu_x M$ given by

$$\mathrm{I\!I}_x(u, v) := (\overline{\nabla}_{\overline{X}}\overline{Y})^\perp = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y \ ,$$

where $X(x) = u,\ Y(x) = v$. This definition does not depend on the choice of X and Y and on the local extensions $\overline{X}$ and $\overline{Y}$. This map is also symmetric and bilinear. Given any $w \in \nu_x M$, we can also consider the map $\mathrm{I\!I}^w_x : T_x M \times T_x M \to \mathbb{R}$ given by $\mathrm{I\!I}^w_x(u, v) := \overline{\mathbf{g}}_x(w, \mathrm{I\!I}_x(v, u))$. We will also omit the evaluation at a particular point $x \in M$, meaning that $\mathrm{I\!I}$ and $\mathrm{I\!I}^w$ are fiber bundle maps. In particular, we define the *mean curvature vector* H as the trace of $\mathrm{I\!I}$, which is a section of νM. Sub-manifolds with $H = 0$ are called *minimal*, because they stationarize the n-dimensional volume functional.

An important phenomenon occurs when $\mathrm{I\!I} = 0$, or equivalently, when $\mathrm{I\!I}^w_x = 0 \in \mathbb{R}$ for all $x \in M,\ w \in \nu_x M$. Such immersed manifolds are called *totally geodesic*. This condition is equivalent to the fact that all the geodesics of $(M, \iota^*\overline{\mathbf{g}})$ are also geodesic of $(\overline{M}, \overline{\mathbf{g}})$. A totally geodesic sub-manifold is automatically minimal, not the converse.

To each symmetric bilinear form $\mathrm{I\!I}^w_x$ on $T_x M$ we can associate a self-adjoint operator $\mathcal{S}^w_x : T_x M \to T_x M$ defined by the relation

$$(\iota^*\overline{\mathbf{g}})_x(\mathcal{S}^w_x(u), v)) = \mathrm{I\!I}^w_x(u, v) \ ,$$

for all $v \in T_x M$. This is the so-called *shape operator*. The eigenvectors of it are called *principal directions* and the corresponding eigenvalues *principal curvatures*.

The relation between the Riemann curvature tensor $\overline{R}$ of $\overline{\nabla}$ and R of ∇ is given by the *Gauss* equation. Following the notation of before

$$R(X, Y, X, Y) = \overline{R}(\overline{X}, \overline{Y}, \overline{X}, \overline{Y}) + \overline{\mathbf{g}}(\mathrm{I\!I}(X, X), \mathrm{I\!I}(Y, Y)) - \overline{\mathbf{g}}(\mathrm{I\!I}(X, Y), \mathrm{I\!I}(X, Y)) \ .$$

The sectional curvatures $\overline{K}$ of $\overline{\nabla}$ and K of ∇ are called extrinsic and intrinsic curvatures of $(M, \iota^*\overline{\mathbf{g}})$. Clearly, they agree if M is totally geodesic.

5.3.2 Riemannian Submersions

Let $(\overline{M}, \overline{\mathbf{g}})$ and $(M, \mathbf{g})$ be Riemannian manifolds of dimensions $(n+k)$ and n respectively, and $\pi : \overline{M} \to M$ be a submersion. We call π a *Riemannian submersion* if π is an isometry. We can naturally define two sub-bundles of $T\overline{M}$ as follows $\mathcal{V}\overline{M} := \bigsqcup_{\bar{x} \in \overline{M}} \mathcal{V}_{\bar{x}}\overline{M}$ and $\mathcal{H}\overline{M} := \bigsqcup_{\bar{x} \in \overline{M}} \mathcal{H}_{\bar{x}}\overline{M}$, where

$$\mathcal{V}_{\bar{x}}\overline{M} := \ker(\mathrm{d}\pi_{\bar{x}}) \quad \text{and} \quad \mathcal{H}_{\bar{x}}\overline{M} := (\ker(\mathrm{d}\pi_{\bar{x}}))^\perp \ ,$$

$\perp$ denoting the orthogonal complement of $\ker(\mathrm{d}\pi_{\overline{x}}) \subseteq T_{\overline{x}}\overline{M}$ with respect to $\overline{\mathbf{g}}_{\overline{x}}$. $\mathcal{V}\overline{M}$ and $\mathcal{H}\overline{M}$ are called, respectively, the *vertical* and the *horizontal* distributions. We will write tangent vectors and vector fields on $\overline{M}$ according to this splitting as $\mathcal{C}^\infty(\overline{M}, T\overline{M}) \ni \overline{X} = \overline{X}^{\mathcal{V}} + \overline{X}^{\mathcal{H}}$, where $\overline{X}^{\mathcal{V}} \in \mathcal{C}^\infty(\overline{M}, \mathcal{V}\overline{M})$, the space of *vertical vector fields*, and $\overline{X}^{\mathcal{H}} \in \mathcal{C}^\infty(\overline{M}, \mathcal{H}\overline{M})$, the space of *horizontal vector fields*. We notice immediately that the vertical distribution is integrable by definition, the integral sub-manifolds being the fibers $\pi^{-1}(x)$ for $x \in M$. By the Frobenius Theorem this implies that the Lie bracket of two vertical vector fields is again vertical. The horizontal distribution is non-integrable in general. If X, Y are vector fields on M, we can uniquely lift them to horizontal vector fields $\overline{X}, \overline{Y} \in \mathcal{C}^\infty(\overline{M}, \mathcal{H}\overline{M})$. If $\overline{V} \in \mathcal{C}^\infty(\overline{M}, \mathcal{V}\overline{M})$ is a vertical vector field, then we readily see that $[\overline{V}, \overline{X}]$ is vertical and the Lie derivative $(\mathcal{L}_{\overline{V}}\overline{\mathbf{g}})(\overline{X}, \overline{Y}) = 0$. Moreover, we have

$$\overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla_X Y} = \frac{1}{2}[\overline{X}, \overline{Y}]^{\mathcal{V}} ,$$

where $\overline{\nabla_X Y}$ is the horizontal lift of $\nabla_X Y$. Finally, we notice that the bilinear map on $\mathcal{C}^\infty(\overline{M}, \mathcal{H}\overline{M})$ with values in $\mathcal{C}^\infty(\overline{M}, \mathcal{V}\overline{M})$, given by $(\overline{X}, \overline{Y}) \mapsto [\overline{X}, \overline{Y}]^{\mathcal{V}}$ is $\mathcal{C}^\infty(\overline{M})$-linear, and naturally measures the non-integrability of $\mathcal{H}\overline{M}$. Indeed, for any $f \in \mathcal{C}^\infty(\overline{M})$ we have $[\overline{X}, f\overline{Y}]^{\mathcal{V}} = (\overline{X}(f)\overline{Y} + f[\overline{X}, \overline{Y}])^{\mathcal{V}} = f[\overline{X}, \overline{Y}]^{\mathcal{V}}$. Moreover, we can associate to the regular fibers of π their first and second fundamental forms. The second fundamental form of a regular fiber $\pi^{-1}(x)$, $x \in M$, can be expressed in terms of the connection $\overline{\nabla}$ as the bilinear map on $\mathcal{C}^\infty(\overline{M}, \mathcal{V}\overline{M})$ with values in $\mathcal{C}^\infty(\overline{M}, \mathcal{H}\overline{M})$, defined by $\mathbf{II}(\overline{V}, \overline{W}) := (\overline{\nabla}_{\overline{V}}\overline{W})^{\mathcal{H}}$. As before, this map is also $\mathcal{C}^\infty(\overline{M})$-linear. If $\mathcal{H}\overline{M}$ is integrable, the integral leaves $S_{\overline{x}}$, $\overline{x} \in \overline{M}$, are totally geodesic, and we have that $\pi|_{S_{\overline{x}}}$ is a local invertible isometry.

A vector field $\overline{V}$ on $\overline{M}$ is said *projectable* if there exists a vector field V on M which is π-related to it, i.e. for all $\overline{x} \in \overline{M}$ we have $V(\pi(\overline{x})) = (\mathrm{d}\pi)_{\overline{x}}(\overline{V}(\overline{x}))$. We call a vector filed on $\overline{M}$ *basic*, if it is horizontal and projectable. We notice that $\mathcal{H}\overline{M}|_{\pi^{-1}(x)}$ is the normal bundle to the fiber $\pi^{-1}(x)$ for all $x \in M$. In such a bundle there is a natural notion of parallelism. A section $\overline{\eta} \in \mathcal{C}^\infty(\pi^{-1}(x), \mathcal{H}\overline{M}|_{\pi^{-1}(x)})$ is called π-*parallel* if $(\mathrm{d}\pi)_{\overline{y}}(\overline{\eta}(\overline{y})) = \eta \in T_x M$ for all $\overline{y} \in \pi^{-1}(x)$. This notion of parallelism is different from the one induced on $\nu(\pi^{-1}(x)) = \mathcal{H}\overline{M}|_{\pi^{-1}(x)}$ by the connection $\overline{\nabla}$, the normal connection, i.e., $\overline{\eta} \in \mathcal{C}^\infty(\pi^{-1}(x), \nu(\pi^{-1}(x)))$ is parallel if $(\overline{\nabla}\overline{\eta})^{\mathcal{H}} = 0$.

It is important to observe that π-parallelism corresponds to a flat connection with no holonomy and is independent of $\overline{\mathbf{g}}$. These two notions of parallelism coincide if the distribution $\mathcal{H}\overline{M}$ is integrable. We will be using the notation, $1 \leq i, j \leq n$, $n+1 \leq \alpha, \beta \leq n+k$, and $1 \leq A, B \leq n+k$.

Suppose $\mathcal{H}\overline{M}$ is integrable with $S_{\overline{x}}$ a leaf through $\overline{x}$. Then, if $\{\overline{e}_A\}$ is a local $\overline{\mathbf{g}}$-orthonormal frame of $T\overline{M}$ around $\overline{x}$, such that $\{\overline{e}_i\}$ is basic and $\{\overline{e}_\alpha\}$ is vertical, i.e. $\{\overline{e}_i\}$ is a local frame for $TS_{\overline{x}}$. We show that $\{\overline{e}_i\}$, which is by construction π-parallel is a parallel orthonormal frame for every fiber $\pi^{-1}(x)$. In fact, since the second fundamental form of each leaf $S_{\overline{x}}$ is zero, we have for all α, i, j that $\overline{\mathbf{g}}(\overline{e}_i, \overline{\nabla}_{\overline{e}_j}\overline{e}_\alpha) = -\overline{\mathbf{g}}(\overline{\nabla}_{\overline{e}_j}\overline{e}_i, \overline{e}_\alpha) = 0$, or equivalently, that $\overline{\nabla}_{\overline{e}_j}\overline{e}_\alpha$ is vertical. Therefore, since $[\overline{e}_j, \overline{e}_\alpha] = \overline{\nabla}_{\overline{e}_j}\overline{e}_\alpha - \overline{\nabla}_{\overline{e}_\alpha}\overline{e}_j$, and $[\overline{e}_j, \overline{e}_\alpha]$ is vertical, also $\overline{\nabla}_{\overline{e}_\alpha}\overline{e}_j$ is vertical. Therefore, $\overline{e}_j$ is parallel with respect to the normal connection of $\pi^{-1}(x)$, $(\overline{\nabla}\overline{e}_j)^{\mathcal{H}} = 0$.

Vice versa, suppose that $\overline{e}_j|_{\pi^{-1}(x)}$ is a parallel normal field for all $x \in M$. This means that $\overline{\nabla}_{\overline{e}_\alpha}\overline{e}_j$ is vertical. For the same reason of before, also $\overline{\nabla}_{\overline{e}_j}\overline{e}_\alpha$ is vertical. Therefore, for all α

$$\overline{\mathbf{g}}([\overline{e}_i, \overline{e}_j], \overline{e}_\alpha) = \overline{\mathbf{g}}(\overline{\nabla}_{\overline{e}_i}\overline{e}_j - \overline{\nabla}_{\overline{e}_j}\overline{e}_i, \overline{e}_\alpha) = \overline{\mathbf{g}}(\overline{e}_i, \overline{\nabla}_{\overline{e}_j}\overline{e}_\alpha) - \overline{\mathbf{g}}(\overline{e}_j, \overline{\nabla}_{\overline{e}_i}\overline{e}_\alpha) = 0;$$

and we see that $[\overline{e}_i, \overline{e_j}]$ is horizontal, hence $\mathcal{H}\overline{M}$ is integrable.

5.3.3 Focal points

The notion of *focal point* generalizes the notion of *conjugate point*. Let $(M, \mathbf{g})$ be a Riemannian manifold and $N \subset M$ an immersed submanifold, with inclusion map $\iota : N \hookrightarrow M$. We define a focal point of N to be a critical value of the *normal exponential map*, which is the restriction to $\nu(N) \subset j^*(TM)$ of the Riamannian exponential of $(M, \mathbf{g})$. We denote it point-wise as $\exp_x^\perp : \nu_x N \to M$, for $x \in N$. In particular, we call a critical point $w \in \nu_x N$ a *critical vector* and then the focal point will be the critical value $\exp_x^\perp(w)$. The *multiplicity* of a focal point will mean the dimension of the kernel of $(\mathrm{d}\exp_x^\perp)_w$.

As it happens for conjugate points, there is a characterization of focal points in terms of *Jacobi fields*. Let $\gamma : [0,1] \to M$ be a geodesic segment starting from N and normal to N, i.e. $\gamma(0) \in N$ and $\dot\gamma(0) \in \nu_{\gamma(0)} N$. To define the proper notion of Jacobi fields that is needed, consider a *variation* of γ, i.e. a smooth map $\Gamma : (-\varepsilon, \varepsilon) \times [0,1] \to M$, $\varepsilon > 0$, such that $\Gamma(0,t) = \gamma(t)$, $\Gamma(s, \cdot)$ is a geodesic for all $s \in (-\varepsilon, \varepsilon)$. We define the space of Jacobi fields along γ as the space $\mathcal{J}_\gamma$ of vector fields along γ that comes from a variation of γ, in the sense that $J \in \mathcal{J}_\gamma$ if and only if there exists a variation Γ such that $J(t) = \frac{\partial}{\partial s}\Gamma(s,t)|_{s=0}$. The space $\mathcal{J}_\gamma$ is also characterized as the space of solutions to the following equation

$$J'' + \mathcal{R}(\dot\gamma, J)\dot\gamma = 0 \ ,$$

where $J' = \frac{\nabla J}{\mathrm{d}t}$ is the second covariant derivative of J along γ. Then, we see that $\mathcal{J}_\gamma$ is a vector space of dimension $2\dim(M)$. One obvious solution to the equation is given by $J(t) = (at+b)\dot\gamma(t)$, with $a, b \in \mathbb{R}$. To define Jacobi fields for a geodesic normal to the submanifold N one introduces the space of N-Jacobi fields $\mathcal{J}_\gamma^N$, which is defined as the subspace of $\mathcal{J}_\gamma$ of fields coming from a variation Γ such that $\Gamma(s,0) \in N$ and $\frac{\partial}{\partial t}\Gamma(s,t)|_{t=0} \in \nu_{\Gamma(s,0)} N$, for all $s \in (-\varepsilon, \varepsilon)$. One can prove that $J \in \mathcal{J}_\gamma^N$ if and only if $J \in \mathcal{J}_\gamma$ is such that

$$i) \ J(0) \in T_{\gamma(0)} N \ ;$$

$$ii) \ J'(0) + \mathcal{S}_{\gamma(0)}^{\dot\gamma(0)}(J(0)) = 0 \ ,$$

where $\mathcal{S}$ is the shape operator of N. It is clear that such a space has dimension equal to $\dim(N)$.

Let γ be a normal geodesic segment to N, then $\gamma(t)$ is a focal point of N if and only if the space $\Lambda_\gamma^N(t) := \{J \in \mathcal{J}_\gamma^N \ : \ J(0) = 0 = J(t)\}$ of N-Jacobi fields vanishing at 0 and t is nonempty. Moreover, its multiplicity is precisely equal to $\dim(\Lambda_\gamma^N)$. This situation is a clear generalization of the theory of conjugate points, which corresponds to the case where the submanifold reduces to a single point $N = \{x\}$. It is also well known that the notion of focal point generalizes the notion of focal point in geometric optics.

We also have a Morse index theorem, which states that the index of a geodesic segment, defined as $\Lambda_\gamma^N := \sum_{t \leq 1} \Lambda_\gamma^N(t) < +\infty$. This tells us that the *regular points* of M, i.e., the points which are not focal points of N are everywhere dense in M (see the discussion in [16]).

5.4 Symplectic Geometry

In this section, we recall some basic facts of Symplectic Geometry. We generally refer to the classical treatment [5, 9]. We recall that the origin of the name Symplectic Geometry can be found in [110] and a clear exemplification is given in [10].

5.4.1 Symplectic Vector Spaces

Let (V, ω) be a symplectic vector space, i.e. V is a vector space and ω is a non degenerate skew-symmetric bilinear form.

Any symplectic vector space is even dimensional. Indeed, if $\langle \cdot, \cdot \rangle$ is any scalar product on V, there is a skew-symmetric endomorphism $A \in \mathrm{End}(V)$ such that $\omega(\cdot, \cdot) = \langle \cdot, A\cdot \rangle$. Then, we must have $0 \leq \det(A^2) = \det(AA^T) = \det(A)\det(A^T) = (-1)^{\dim(V)} \det(A)^2$, and we conclude.

Subspaces. Let $W \subseteq V$ be a vector subspace. We define the symplectic orthogonal of W as the vector subspace $W^\S := \{v \in V \ : \ \omega(v, w) = 0 \ \forall w \in W\}$. Then W is called

i) *isotropic* if $W \subseteq W^\S$;

ii) *coisotropic* if $W \supseteq W^\S$;

iii) *Lagrangian* if $W = W^\S$;

iv) *symplectic* if $W \cap W^\S = \{0\}$.

It is clear form the definition that if $\dim(V) = 2n$, $\dim(W) \leq n$ if W is isotropic, $\dim(W) \geq n$ if W is coisotropic and $\dim(W) = n$ if W is Lagrangian. If W is symplectic, then $\dim(W) = 2k$ for $k \leq n$.

Let $W \subseteq V$ be any subspace of a symplectic vector space (V, ω). We have that $\tilde{W} := W/(W \cap W^\S)$ is endowed with a natural symplectic structure, and hence $\tilde{W}$ is a symplectic vector space.

Proof. Let $[w_1], [w_2] \in \tilde{W}$, set $\tilde{\omega}([w_1], [w_2]) := \omega(w_1, w_2)$. This is a well defined skew-symmetric bilinear form on W. Indeed, if $w_1 \in [w_1] \ni w_1 + u$, where $u \in W \cap W^\S$, we have

$$\omega(w_1 + u, w_2) = \omega(w_1, w_2) = \tilde{\omega}([w_1], [w_2]) \ .$$

Suppose now that $\tilde{\omega}([w_1], [w_2]) = 0$ for all $[w_2] \in \tilde{W}$ then $\omega(w_1, w_2) = 0$ for all $w_2 \in W$, i.e., $w_1 \in W^\S$, but since $w_1 \in W$, we have $w_1 \in W \cap W^\S$, i.e., $[w_1] = 0 \in \tilde{W}$. We conclude that $\tilde{\omega}$ is also non degenerate. □

Example 5.4.1. Let V be any vector space and V^* its dual. Then $V \oplus V^*$ is naturally endowed with a symplectic structure given by

$$\varsigma((v, \alpha), (w, \beta)) := \beta(v) - \alpha(w) \ .$$

Darboux Basis. Given a $2n$-dimensional symplectic vector space (V, ω), there is always a special basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ such that $\omega(e_i, e_j) = 0 = \omega(f_i, f_j)$ and $\omega(e_i, f_j) = \delta_{ij}$, which we call a *Darboux basis*.

Proof. We prove this by induction on the dimension of the space. For $n = 1$, let $v \in V$ then, by non degeneracy, there exists $w \in V$ such that $\omega(v, w) \neq 0$ and, up to rescaling, $\omega(v, w) = 1$. Then, by setting $v = e_1, w = f_1$, $\{e_1, f_1\}$ is a Darboux basis. Suppose $\dim(V) = 2n$, then by the same argument we can find $W := \mathrm{span}\{e_1, f_1\}$ such that $\omega(e_1, f_1) = 1$. We have that $W^{\S}$ is $(2n-2)$-dimensional and hence we can apply the inductive hypothesis, and find $\{e_2, \dots, e_n, f_2, \dots, f_n\}$ a Darboux basis of $W^{\S}$. By construction, $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is a Darboux basis. If we let $\{e^1, \dots, e^n, f^1, \dots, f^n\}$ be the dual basis of $\{e_1, \dots, e_n, f_1, \dots, f_n\}$, we can write $\omega = \sum_i e^i \wedge f^i$. If we use the coordinate notation $v = q^i e_i + p^i f_i$, then $e^i = \mathrm{d}q^i$ and $f^i = \mathrm{d}p^i$ and we write

$$\omega = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p^i \ ,$$

which is called the *standard symplectic form* of $\mathbb{R}^{2n}$. ☐

Symplectic Group

Let $\sigma \in \mathrm{End}(V)$ be an endomorphism of V. We say that S is a symplectic map if $\omega(\sigma\cdot, \sigma\cdot) = \omega(\cdot, \cdot)$. Let $\{e_i, f_i\}$ be a Darboux basis for V. Then, if we use the standard realization of vectors and covectors in matrix notation, S is the matrix of a symplectic map σ if and only if $S^T \mathcal{J} S = \mathcal{J}$, where

$$\mathcal{J} = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \ .$$

We define the $2n$-dimensional *symplectic group* $Sp(2n, \mathbb{R})$ as the set $Sp(2n, \mathbb{R}) := \{S \in \mathbb{M}_{2n\times 2n} \ : \ S^T \mathcal{J} S = \mathcal{J}\}$. It is easy to check that $\omega \wedge \overset{n-times}{\cdots} \wedge \omega$ is a volume form for V. It follows that for all $S \in Sp(2n, \mathbb{R})$, $\det(S) = 1$. Observe that $S \in Sp(2n, \mathbb{R})$ sends a Darboux basis into a Darboux basis.

It is easy to prove that $Sp(2n, \mathbb{R})$ is a Lie group of dimension $n(2n+1)$. Its Lie algebra is given by the so called *Hamiltonian matrices*, $Ham(2n, \mathbb{R}) = \{H \in \mathbb{M}_{2n\times 2n} \ : \ H^T \mathcal{J} + \mathcal{J} H = 0_{2n}\}$.

Notice that $\mathcal{J}^2 = -1_{2n}$. Such an endomorphism is called a complex structure on V. Then V is also a $\mathbb{C}$-vector space in the following sense. Let $v \in V$, and $a + ib \in \mathbb{C}$, we set $(a + ib)v = av + b\mathcal{J}v$. Then V is isomorphic, as a complex vector space, to $\mathbb{C}^n$. Let $\langle \cdot, \cdot \rangle$ be the standard scalar product with respect to the Darboux coordinates, i.e., $\langle e_i, e_j \rangle = \delta_{ij} = \langle f_i, f_j \rangle$ and $\langle e_i, f_j \rangle = 0$, then if $v = a^i e_i + b^i f_i \mapsto z(v) := (a^i + b^i) z_i \in \mathbb{C}^n$, $\{z_i\}$ being the canonical basis of $\mathbb{C}^n$, we have that $\langle v, w \rangle = \bar{z}(v)_i z(w)^i$, where $\bar{z}_i z^i$ is the standard Hermitian product of $\mathbb{C}^n$. Recalling that there is a natural map $\mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_{2n}(\mathbb{R})$ given by

$$A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \ ,$$

it is possible to prove that

$$O(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R}) = O(2n, \mathbb{R}) \cap Gl(n, \mathbb{C}) = Sp(2n, \mathbb{R}) \cap Gl(n, \mathbb{C}) = U(n) \ ,$$

where $U(n) := \{X \in \mathbb{M}_n(\mathbb{C}) \ : \ X\bar{X}^T = 1_n\}$ is the unitary group, i.e., the group preserving the standard Hermitian product of $\mathbb{C}^n$.

5.4.2 Symplectic Manifolds

A symplectic manifold is a pair (M, ω), where M is an even dimensional manifold, $\dim(M) = 2n$, and ω, called the symplectic form, is a 2-form on M which is non-degenerate and closed. It follows that $\omega \wedge \overset{n-times}{\cdots} \wedge \omega$ is a volume form for M, which is called the *symplectic volume*; in particular, this means that M is orientable. It also follows from the definition that at every point $x \in M$, there is a non-degenerate bilinear, skew-symmetric form on T_xM, given by $\omega_x \in \Lambda^2(T_x^*M)$. In other words, the pair (T_xM, ω_x) is a symplectic vector space for all $x \in M$.

A diffeomorphism of M, $\phi \in \mathrm{Diff}(M)$, which preserves the simplectic structure, i.e. such that $\phi^*\omega = \omega$, is called a *symplectomorphism* of (M, ω). The set of all symplectomorphisms of (M, ω) is a subgroup of the group of all diffeomorphisms that we denote as $Sp(M, \omega) \subseteq \mathrm{Diff}(M)$. In particular, a symplectomorphism preserves the symplectic volume, since $\phi^*(\omega \wedge \ldots \wedge \omega) = \omega \wedge \ldots \wedge \omega$.

We call a vector field $X \in \mathcal{C}^\infty(M, TM)$ a *symplectic vector field* if $\mathcal{L}_X\omega = 0$, or, equivalently by Cartan homotopy formula, if $\mathrm{d}(\mathrm{i}_X\omega) = 0$. If, in particular, $\mathrm{i}_X\omega = \mathrm{d}F$ for some $F \in \mathcal{C}^\infty(M)$, X is called a *Hamiltonian vector field.* Note that symplectic vector fields are solenoidal, i.e. they have zero divergence. In fact, since $\omega \wedge \ldots \wedge \omega$ is a volume form, $0 = \mathcal{L}_X(\omega \wedge \ldots \wedge \omega) = \mathrm{div}(X)(\omega \wedge \ldots \wedge \omega)$. We denote the space of symplectic vector fields as $\mathfrak{sp}(M, \omega)$ and the space of Hamiltonian vector fields as $\mathfrak{ham}(M, \omega)$. We note that $\mathfrak{ham}(M, \omega)$ is an ideal of $\mathfrak{sp}(M, \omega)$ with respect to the Lie bracket of vector fields.

By a simple computation, the flow of a symplectic vector field is a symplectomorphism. Vice versa, given a 1-parameter group of symlectomorphisms $\{\phi_t\}_{t\in\mathbb{R}}$, the so-called *infinitesimal generator* of ϕ_t, i.e the vector field defined for every $x \in M$ by $X^\phi(x) := \frac{d}{dt}\phi_t(x)|_{t=0}$, is a symplectic vector field. This can be seen as a consequence of the fact that $Sp(M, \omega)$ is an infinite dimensional Lie group with Lie algebra given by $\mathfrak{sp}(M, \omega)$.

Given a function $H : M \to \mathbb{R}$, we call the *Hamiltonian vector field* associated to H the vector field $X_H \in \mathcal{C}^\infty(M, TM)$ defined by the relation $\mathrm{i}_{X_H}\omega = \mathrm{d}H$. In other words, thanks to the symplectic structure, we can define a map $\mathcal{C}^\infty(M) \to \mathfrak{ham}(M, \omega)$. We can also define a Lie bracket for functions for which this map is an algebra anti-homomorphism. Indeed, given $H, F \in \mathcal{C}^\infty(M)$ we define the so called *Poisson bracket* between H and F as

$$\{H, F\} := \omega(X_H, X_F) = (\mathrm{d}H)(X_F) = X_F(H) \ .$$

It is clear that $\{\cdot, \cdot\}$ is bilinear and skew-symmetric. We notice that, with this choice of signs, we have $X_{\{H,F\}} = -[X_H, X_F]$. In fact, for any Z vector field on M we have

$$\begin{aligned}
&\omega(X_{\{H,F\}},Z)=(\mathrm{d}\{H,F\})(Z)=Z(\omega(X_H,X_F))=(\mathcal{L}_Z\omega)(X_H,X_F)+\omega([Z,X_H],X_F)+\omega(X_H,[Z,X_F])=\\
&=(\mathrm{d}(\mathrm{i}_Z\omega))(X_H,X_F)-(\mathrm{d}F)([Z,X_H])+(\mathrm{d}H)([Z,X_F])=\\
&=X_H(\omega(Z,X_F))-X_F(\omega(Z,X_H))-\omega(Z,[X_H,X_F])-(\mathrm{d}F)([Z,X_H])+(\mathrm{d}H)([Z,X_F])=\\
&=-X_H(Z(F))+X_F(Z(H))-\omega(Z,[X_H,X_F])-Z(X_H(F))+X_H(Z(F))+Z(X_F(H))-X_F(Z(H))=\\
&=\omega([X_H,X_F],Z)+2Z(\omega(X_H,X_F)) \ .
\end{aligned}$$

Therefore, we get

$$Z(\omega(X_H, X_F)) = \omega([X_H, X_F], Z) + 2Z(\omega(X_H, X_F)) \ ,$$

i.e., for all Z

$$\omega(X_{\{H,F\}}, Z) = Z(\omega(X_H, X_F)) = \omega(-[X_H, X_F], Z) \ ,$$

and we conclude

$$X_{\{H,F\}} = -[X_H, X_F] \ .$$

Note that if we defined the Poisson bracket with the opposite sign, we would get a homomorphism instead. We can now easily see that the Poisson bracket also satisfies the Jacobi identity. In fact, let $F, H, K \in \mathcal{C}^\infty(M)$. Then

$$\begin{aligned}
&\{F, \{H, K\}\} = X_{\{H,K\}}(F) = -[X_H, X_K](F) = -X_H(X_K(F)) + X_K(X_H(F)) \ ; \\
&\{\{K, \{F, H\}\}\} = -X_K(X_H(F)) \ ; \\
&\{\{H, \{K, F\}\}\} = X_H(X_K(F)) \ .
\end{aligned}$$

Hence,

$$\{F,\{H,K\}\}+\{\{K,\{F,H\}\}\}+\{\{H,\{K,F\}\}\}=-X_H(X_K(F))+X_K(X_H(F))-X_K(X_H(F))+X_H(X_K(F))=0 \ .$$

If we have a symplectomorphism $f : M \to M$, then $f^*\{H, F\} = \{f^*H, f^*F\}$. Indeed, we first observe that for all Y vector fields

$$\omega(X_{f^*H}, Y) = (\mathrm{d}f^*H)(Y) = f^*(\mathrm{d}H)(Y) = (\mathrm{d}H)(f_*Y) = \omega(X_H, f_*Y) = \omega(f_*^{-1}X_H, Y) \ ,$$

i.e. $X_{f^*H} = f_*^{-1}X_H = f^*X_H$. Therefore,

$$\{f^*H,f^*F\}=\omega(X_{f^*H},X_{f^*F})=(\mathrm{d}(f^*H))(X_{f^*F})=f^*(\mathrm{d}H)(f^*X_F)=f^*((\mathrm{d}H)(f_*f^*X_F))=f^*((\mathrm{d}H)(X_F))=f^*\{H,F\} \ .$$

Vice versa, it can be shown that a diffeomorphism $f : M \to M$ that preserves all the Poisson brackets, is a symplectomorphism.

Example 5.4.2. Let Q be any smooth manifold. Then T^*Q is naturally endowed with a symplecitc structure. First: there is a canonical 1-form, $\theta \in \mathcal{C}^\infty(T^*Q, T^*(T^*Q))$, called the *Liouville form*, defined as follows

$$\theta_p(v) := p((\mathrm{d}\pi)_p(v)) \ ,$$

where $p \in T^*M$, $v \in T_p(T^*M)$ and $\pi : T^*Q \to Q$ is the canonical projection. Then, $\omega := \mathrm{d}\theta$ defines a symplectic structure on T^*Q. Indeed, first notice that ω is automatically closed. For the non-degeneracy condition, it is useful to fix local coordinates for T^*Q. Let q^i, $i = 1, \dots, n = \dim(Q)$, be coordinate functions for Q on an open set U. Then $p \mapsto (q^1, \dots, q^n, p_i, \dots, p_n)$, where $q = \pi(p)$ and $p = p_i(\mathrm{d}q^i)_q$, are local coordinates for T^*M on $\pi^{-1}(U)$. Then, we can see that in these coordinates we have

$$\theta_p = p_i(\mathrm{d}q^i)_q \ .$$

Therefore, we obtain $\mathrm{d}\theta = \mathrm{d}p_i \wedge \mathrm{d}q^i$, i.e., at each $T_p(T^*Q)$, in the chosen coordinates, ω coincides with the standard symplectic form on $\mathbb{R}^{2n}$.

Local theory

The well known Darboux Theorem (see e.g. [9]), states that every symplectic manifold locally looks the same. More precisely if $\mathbb{R}^{2n}$ is given coordinates (q^i, p_i), then the two form $\omega_0 := \mathrm{d}q^i \wedge \mathrm{d}p_i$, $i = 1, \cdots, n$, is a symplectic form on $\mathbb{R}^{2n}$. Then the theorem states that given (M, ω) symplectic manifold of dimension $2n$, for all $x \in M$ there is an open neighborhood $U \subseteq M$ of x and a diffeomorphism $\phi : U \to \mathbb{R}^{2n}$ such that $\phi^*\omega_0 = \omega|_U$.

More generally, this theorem can be relativized to submanifolds. We have that given $S \subseteq M$ a submanifold, if $\omega, \tilde{\omega}$ are two symplectic forms on M that coincide on S, i.e. $\omega_x = \tilde{\omega}_x$ for all $x \in S$, there exist two open neighborhoods $U, \tilde{U}$ of S and a diffeomorphism $\psi : U \to \tilde{U}$, with $\psi|_S = \mathrm{id}_S$ and $\psi^*\tilde{\omega} = \omega$.

Submanifolds

We recall four types of submanifolds in the context of symplectic Geometry. If $S \subseteq M$ is a submanifold of (M, ω), symplectic manifold of dimension $2n$, we call S

i) *isotropic* if for all $x \in S$, $T_xS \subseteq (T_xS)^\S$;

ii) *coisotropic* if for all $x \in S$, $T_xS \supseteq (T_xS)^\S$;

iii) *Lagrangian* if for all $x \in S$, $T_xS = (T_xS)^\S$;

iv) *symplectic* if for all $x \in S$, $T_xS \cap (T_xS)^\S = \{0\}$,

where $(T_xS)^\S := \{v \in T_xM \ : \ \omega_x(v, w) = 0, \text{ for all } w \in T_xS\}$ denote the symplectic orthogonal. It is clear from the definition that $2n = \dim(T_xM) = \dim(T_xS) + \dim((T_xS)^\S)$. Therefore, the dimensions of isotropic, coisotropic and Lagrangian submanifolds are respectively less than n, greater than n, and equal to n. Examples of isotropic submanifolds are simple smooth curves for dimensional reasons. Examples of coisotropic submanifold are given by level sets of functions. Take $F : M \to \mathbb{R}$ then for generic $c \in \mathbb{R}$ the subset $S := F^{-1}(c)$ is an embedded submanifold. We have $T_xS = \ker((\mathrm{d}F)_x)$. Notice that $X_F(x) \in T_xS$ for all $x \in S$, in fact: $(\mathrm{d}F)_x(X_F(x)) = \omega_x(X_F(x), X_F(x)) = 0$. Therefore, given $w \in (T_xS)^\S$, we have $(\mathrm{d}F)_x(w) = \omega_x(X_F(x), w) = 0$, i.e., $w \in T_xS$.

Poisson Manifolds

One can also introduce a notion of Poisson bracket in the more general context of Poisson Geometry, where one drops the non degeneracy condition. We call a Poisson manifold a couple $(M, \{\cdot,\cdot\})$, where $\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathbb{R}$, called the Poisson bracket, is a bilinear map such that $(\mathcal{C}^\infty(M), \{\cdot,\cdot\})$ is a *Poisson algebra*, i.e. a Lie algebra and $\{\cdot,\cdot\}$ is a derivation. In particular, functions in the center a Poisson algebra, are the so-called *Casimir functions*.

The presence of a Poisson bracket on M still allows to define a homomorphism of Lie algebras $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M, TM)$ as

$$H \mapsto X_H := \{\cdot, H\} \ .$$

The vector field X_H is called the Hamiltonian vector associated with H. Notice that $F \in \mathcal{C}^\infty(M)$ is a Casimir function if and only if $X_F = 0$.

A Poisson structure is more flexible than a Symplectic structure, e.g. it does not need M to have even dimension. Moreover, every submanifold of a Poisson manifold is Poisson, simply by restricting the Poisson bracket.

Since the homomorphism defined by the Poisson brackets only depends on $\mathrm{d}H$, we can equivalently define a smooth section $P \in \mathcal{C}^\infty(M, \Lambda^2(TM))$ as

$$\{F, H\}(x) := P_x((\mathrm{d}F)_x, (\mathrm{d}H)_x) \ .$$

Such tensor is called the *Poisson tensor*. Then there is also a natural map $P^\# : T^*M \to TM$ defined as

$$P_x(\alpha, \beta) := \alpha(P^\#_x(\beta)) \ ,$$

for all $\alpha, \beta \in T^*_xM$ and $x \in M$. We define the *characteristic distribution* of P as the image $P^\#(T^*M) \subseteq TM$ and define the the *rank of* $(M, \{\cdot,\cdot\})$ *a the point* $x \in M$ as the dimension of $P^\#_x(T^*_xM) \subseteq T_xM$.

Example 5.4.3. We list some important examples of Poisson manifolds.

i) A symplectic manifold (M, ω) is clearly a Poisson manifold. In particular, due to the non-degeneracy condition, the Casimir function of the Poisson structure given by the symplectic structure are only the constant functions. The Poisson tensor is given by the bilinear map ω^{-1}. Moreover, the rank of such Poisson structure does not depend on the point ad is always equal to $\dim(M)$.

ii) Let G be a compact Lie group and let $\mathfrak{g}^*$ be the dual of its Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}^*$ is a Poisson manifold, where the Poisson bracket is given by the *Lie-Possion bracket*
$$\{F, H\}(\alpha) := \alpha([(\mathrm{d}F)_\alpha, (\mathrm{d}H)_\alpha]) \ ,$$
for any $\alpha \in \mathfrak{g}^*$, $F, H \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and where we used identification $T^*_\alpha \mathfrak{g}^* = (T_\alpha \mathfrak{g}^*)^* \simeq (\mathfrak{g}^*)^* \simeq \mathfrak{g}$, in which the last isomorphism is the canonical duality. We can easily see that such a Poisson bracket is non degenerate along the coadjoint orbits. We have that $T_\alpha G(\alpha) = \mathrm{ad}^*_\xi \alpha$ for $\xi \in \mathfrak{g}$. It is possible to show that the rank of this Poisson structure at α is given by the dimension of the coadjoint orbit through α, $\dim(G(\alpha))$. By the principal orbit theorem, there is an open and dense set where the rank is constant. Moreover, we recognize that $\{\cdot,\cdot\}|_{G(\alpha)}$ coincides with the Poisson bracket of the Lie-Poisson symplectic structure on $G(\alpha)$.

The item *ii*) of the previous example is a particular case of the fact that if $(M, \{\cdot,\cdot\})$ is a Poisson manifold and there is an open $U \subseteq M$ in which the Poisson structure has constant rank $\dim((P^\#)) = 2k$, then the characteristic distribution is integrable and there is a unique induced symplectic structure on each integral leaf $S \subseteq U$ given by $\omega_S^{-1} = P|_S$. Hence, we can say that constant rank Poisson manifolds are foliated by symplectic manifolds.

A *Poisson map* between two Poisson manifolds $(M_i, \{\cdot,\cdot\}_i)$, $i = 1, 2$, is a smooth map $f : M_1 \to M_2$ such that for all $F, H \in \mathcal{C}^\infty(M_2)$, one has
$$f^* \{F, H\}_2 = \{f^*F, f^*H\}_1 \ .$$
Unlike in the symplectic case, we are not limited to immersions. In fact, recall that a smooth map between symplectic manifolds (M_i, ω_i), $i = 1, 2$, such that $f^*\omega_2 = \omega_1$, forces $\dim(M_1) \leq \dim(M_2)$. We call such a map a *symplectic map*.

An important fact to note is that if we have a symplectic map, this need not to be a Poisson map between the respective Poisson structures induced by the symplectic ones, as the next example shows.

Example 5.4.4. Let $\mathbb{R}^4$ be endowed with the standard symplectic structure. In Darboux coordinates, we have $\omega = \mathrm{d}q^1 \wedge \mathrm{d}p^1 + \mathrm{d}q^2 \wedge \mathrm{d}p^2$. Let $V = \mathrm{span}(\partial_{q^1}, \partial_{p^1}) \subset \mathbb{R}^4$ be the (q^1, p^1)-plane. It is clear that V is a symplectic submanifold of $(\mathbb{R}^4, \omega)$. In particular, the inclusion $\iota : V \hookrightarrow \mathbb{R}^4$ is a symplectic map . Both V and $\mathbb{R}^4$ are Poisson manifolds with respect to the Poisson bracket $\{\cdot,\cdot\}_V$ and $\{\cdot,\cdot\}$ induced by $\omega|_V$ and ω, respectively. We observe that ι is not a Poisson map. In fact, let $H, K \in \mathcal{C}^\infty(\mathbb{R}^4)$ be constant on V, i.e. $\partial_{q^1} K = \partial_{q^1} H = 0 = \partial_{p^1} H = \partial_{p^1} K$. Then, in general, $\iota^*\{H, K\} \neq 0$ but $\{\iota^*H, \iota^*K\}_V \equiv 0$.

5.4.3 Almost-complex structures

Given a symplectic manifold (M, ω) and given $\mathbf{g}$ any Riemannian metric on M, we can consider the invertible skew-symmetric endomorphism associated to ω using $\mathbf{g}$, i.e.

$A \in \mathcal{C}^\infty(M, \mathrm{End}(TM))$, such that $\omega(X,Y) = \mathbf{g}(AX,Y)$ for all vector fields X, Y on M. We have that $\mathbf{g}(AX,Y) = -\mathbf{g}(X,AY)$, hence, $A^T = -A$. Therefore, $-A^2 = A^TA$ is a symmetric endomorphism. By polar decomposition there exist unique $R \in \mathcal{C}^\infty(M, O(TM,\mathbf{g}))$ and a $\mathbf{g}$-symmetric $\mathbf{g}$-positive definite $S \in \mathcal{C}^\infty(M, \mathrm{End}(TM))$ such that $A = RS$. Then $-A^2 = A^TA = SR^TRS = S^2$, hence $-A^2$ is positive definite and A commutes with S. We set $J := S^{-1}A$, and check that $J^2 = S^{-2}A^{-2} = -\mathrm{id}_{TM}$, i.e. J is a *almost complex structure* on M. Moreover, $\omega(X, JY) = \mathbf{g}(AX, JY) = \mathbf{g}(SX,Y)$ is positive definite, and ω and $\mathbf{g}$ are J-invariant, i.e. $\mathbf{g}(JX,JY) = -\mathbf{g}(J^2X,Y) = \mathbf{g}(X,Y)$ and, similarly, $\omega(JX,JY) = \omega(X,Y)$.

In particular, we say that ω and $\mathbf{g}$ are *compatible* if $\omega(X,JY) = \mathbf{g}(X,Y)$. In that case, we have that $\mathbf{g}(SX,Y) = \mathbf{g}(X,Y)$, i.e., $S = \mathrm{id}_{TM}$ and $A = J$.

In summary, we have proved that given a symplectic manifold (M,ω), there exists an endomorphism $J \in \mathcal{C}^\infty(M, \mathrm{End}(TM))$ such that $J^2 = -\mathrm{id}_{TM}$ and such that $\omega(JX,JY) = \omega(X,Y)$. Then,

$$\mathbf{h}(X,Y) := \omega(X,JY) ,$$

is a compatible Riemannian metric such that $\mathbf{h}(JX,JY) = \mathbf{h}(X,Y)$. The collection $(M,\omega,J,\mathbf{h})$ is called an *almost-Kähler* manifold.

Notice that in an almost-Kähler manifold, we can describe the symplectic orthogonal in terms of J and $\mathbf{h}$. In fact, for $x \in M$, given $V \subseteq T_xM$, we have

$$V^\S = J_x(V^\perp) ,$$

where the orthogonal is taken with respect to $\mathbf{h}$. Indeed, given $w \in J_x(V^\perp)$, we have $w = J_x(u)$ for some $u \in V^\perp$. Then, for all $v \in V$, $\omega_x(v,w) = \omega_x(v,J_xu) = \mathbf{h}_x(v,u) = 0$, hence $w \in V^\S$. Vice versa, given $w \in V^\S$, for all $v \in V$, $0 = \omega_x(v,w) = \mathbf{h}_x(J_xv,w)$, and since J_x is an isomorphism, $w \in (J_x(V))^\perp = J_x(V^\perp)$ because $\mathbf{h}$ is J-invariant.

Let (M,ω) be symplectic, and let J be an almost complex structure. We call J a *complex structure* on M if J is induced by a *holomorphic structure* on M, i.e., an atlas for which the transition functions are given by biholomorphic maps. More precisely, if $\phi : U \to \mathbb{R}^{2n} = \mathbb{C}^n$ is a chart, we have that $\phi_*\circ J = \phi_*\circ i$, where i is the imaginary unit of $\mathbb{C}^n$. A celebrated theorem by Newlander and Nirenberg states that an almost-complex structure is complex if and only if the Nijenhuis tensor $N_J = 0$, where

$$N_J(X,Y) := [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY] .$$

We call an almost-Kähler manifold a *Kähler* manifold if J is integrable. In this case, ω is called the *Kähler metric.*

Kähler manifolds

In the compact case, a Kähler manifold $(M,\omega,J,\mathbf{h})$ is subject to strong topological restrictions. Using Hodge Theory, one can show that the Betti numbers of M must satisfy the following conditions

$$\begin{cases} b_i(M,\mathbb{R}) > 0 & \text{if } i = 0 \mod(2) \\ b_i(M,\mathbb{R}) = 0 \mod(2) & \text{if } i = 1 \mod(2) . \end{cases}$$

These conditions clearly cannot hold in the non compact case. Consider, for instance, the trivial example given by the Kälher manifold $(\mathbb{C}^n, \omega_0, J_0, \langle\cdot,\cdot\rangle)$, where ω_0 is the standard symplectic structure, J_0 is the complex structure induced by multiplication by the complex unit i and $\langle\cdot,\cdot\rangle$ is the standard scalar product.

We note that the condition on the even Betti numbers is not forced by the Kähler condition, but is also present in the symplectic case. In fact, on a compact symplectic manifold (M,ω), each k-multiple of the symplectic form $\omega^k := \omega \wedge \overset{k\text{-times}}{\cdots} \wedge \omega$, always defines a nonzero element $[\omega^k] \in H^{2k}(M,\mathbb{R})$. To see this we simply observe that if $\omega^k = \mathrm{d}\eta$ for some $(2k-1)$-form η, we have that $\omega^n = \omega^k \wedge \omega^{n-k} = \mathrm{d}\eta \wedge \omega^{n-k} = \mathrm{d}(\eta \wedge \omega^{n-k})$. If $\mathrm{Vol}(M) > 0$ denotes the symplectic volume of M, by Stokes Theorem we get a contradiction,

$$0 < \mathrm{Vol}(M) := \int_M \omega^n = \int_M \mathrm{d}(\eta \wedge \omega^{n-k}) = \int_{\partial M} \eta \wedge \omega^{n-k} = 0 \ .$$

Example 5.4.5. We list some famous examples of compact Kähler manifolds.

i) $\mathbb{C}P^n$ with the so called Fubini-Study metric;

ii) Complex tori $\mathbb{T}^{2n}$ with the standard symplectic strcuture and the standard flat metric;

iii) Every complex analytic subvariety of $\mathbb{C}P^n$, where the Kähler form is provided by the restriction of the Fubini-Study metric.

Item *iii)* is a consequence of the general fact that any complex submanifold of a Kähler manifold is Kähler, simply by restricting the Kähler metric.

5.5 Morse Theory

Given a topological space X, we say that X admits a *cell decomposition* if there exists a sequence of closed subspaces $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n := X$ such that $X_{i+1} = X_i \cup_{g_i} e^{k_i}$, where $g_i : \mathbb{S}^{k_i-1} \to X_i$, is the gluing map attaching the boundary of a k_i-cell $e^{k_i} \subseteq X$ to X_i. We denote such a decomposition with the couple (X, X_0). In the following, ν_i will denote the number of cells of dimension i in the decomposition (X, X_0).
A well know consequence of Morse Theory is that if $f : M \to \mathbb{R}$ is a Morse function on a complete Riemannian manifold M, and if $M^a \subseteq M^b$ are compact sub-level sets of f, relative to regular values $a \le b \in \mathbb{R}$, then (M^b, M^a) is a cell decomposition, where ν_i is equal to the number $\mu_i(f,a,b)$ of critical points of f of index i, comprised in $M^a \cap M^b$. Moreover, if $b_i(M^b, M^a) := \dim(H_i(M^b, M^a; F))$ is the i-th Betti number over some field F, we have the so called *weak Morse inequalities*:

$$b_i(M^b, M^a) \le \mu_i(f,a,b) \ .$$

Definition 5.5.1. Let $f : M \to \mathbb{R}$ be a Morse function over a complete Riemannian manifold. We call f a *F-perfect Morse function* if for all $a \le b \in \mathbb{R}$ regular values with compact sub-level sets, and $i = 0, \cdots, \dim(M)$, all Morse inequalities become equalities:

$$b_i(M^b, M^a) = \mu_i(f,a,b) \ .$$

5.5.1 Linking Cycles

Let $\overline{B}^k$ denote the unit closed ball of $\mathbb{R}^k$ and B^k its interior; hence $\partial\overline{B}^k = \mathbb{S}^{k-1}$ and $\overline{B}^k \setminus \partial\overline{B}^k = B^k$. We will consider a topological space X and $Y \subseteq X$ a closed subspace. Let $\mathcal{G} : (\overline{B}^k, \mathbb{S}^{k-1}) \to (X, Y)$, where $X = Y \cup_g e^k$, be the map so defined:

i) $g = \mathcal{G}|_{\mathbb{S}^k} : \mathbb{S}^{k-1} \to Y$ is the attaching map;

ii) $\mathcal{G}|_{B^k}$ is a homeomorphism of the open ball B^k onto the cell e^k.

Using the relative homology exact sequence for the couples $(B^k, \mathbb{S}^{k-1})$ and (X, Y), since $\mathcal{G}_* : H_*(\overline{B}^k, \mathbb{S}^{k-1}) \to H_*(X, Y)$ is an isomorphism and $H_*(\overline{B}^k, \mathbb{S}^{k-1}) \simeq H_*(\mathbb{S}^{k-1})$, we obtain the following exact sequence.

$$\cdots \to H_i(\mathbb{S}^{k-1}) \xrightarrow{g_*} H_i(Y) \to H_i(X) \to H_{i-1}(\mathbb{S}^{k-1}) \xrightarrow{g_*} H_{i-1}(Y) \to \cdots$$

If $i \neq k, k-1$, we get $H_i(X) \simeq H_i(Y)$. For $i = k, k-1$ we are left with two exact sequences

$$0 \to H_k(Y) \to H_k(X) \to \ker(g_*) \to 0 \ ;$$

$$0 \to \mathrm{Im}(g_*) \to H_{k-1}(Y) \to H_{k-1}(X) \to 0 \ .$$

This means that if we know g_*, we can compute $H_{k-1}(X)$ and $H_k(X)$. The following definition describes a case when this computation becomes trivial.

Definition 5.5.2. We say that the pair (X, Y) is of *linking type* if the fundamental class $[\mathbb{S}^{k-1}]$ lies in the kernel of $g_* : H_{k-1}(\mathbb{S}^{k-1}) \to H_{k-1}(Y)$. Equivalently, there is $\mu \in H_k(X)$ which is mapped to $[\mathbb{S}^{k-1}]$. We call μ a *linking cycle*. We also say that a cell decomposition (X, X_0) is of *linking type* if all pairs (X_i, X_{i-1}) are of linking type.

Observe that at the level of the exact homology sequence of (X, Y), the cell decomposition (X, Y) is of linking type if and only if the map $H_k(X) \to H_k(X, Y)$ is surjective.

Note that the inclusion maps $\iota_i : X_i \to X$ induce injections in homology. Then, if μ_i is a linking cycle for the pair (X_i, X_{i-1}), we can identify it with an element of $H_*(X)$.

Theorem 5.5.3. *If (X, X_0) is a cell decomposition of linking type, given any ring R, we have*

$$H_*(X, R) = H_*(X_0, R) \oplus \left(\bigoplus_{i=1}^{n} R\mu_i \right) \ ,$$

where $\mu_i \in H_{k_i}(X)$ corresponds to the linking cycle of (X_i, X_{i-1}).

We use a specialization of definition 5.5.2 to the case of Morse Theory.

Definition 5.5.4. Let M be a complete Riemannnian manifold and $f : M \to \mathbb{R}$ be a Morse function bounded from below. Let $p_1, \cdots, p_n$ be critical points of f such that $f(p_i) \leq f(p_{i+1})$. Let also k_i denote the index of p_i. If $a > f(p_n)$ is a regular value, then M^a admits a cell decomposition $(X = M^a, X_0 = \emptyset)$ such that $X_i \cup_{g_i} e^{k_i}$ is a deformation retract of X_{i+1} and $p_i \in X_{i+1} \cap X_i$. We say that p_i is of *linking type* over a ring R if (X_{i+1}, X_i) is of linking type. We also say that f is *of linking type* if all its critical points are of linking type.

Theorem 5.5.5. *Let $f : M \to \mathbb{R}$ be a Morse function bounded from below, over a complete Riemannian manifold M. Then if f is of linking type and $a \in \mathbb{R}$ is a regular value,*

$$H_*(M^a, R) = \bigoplus_{i=1}^{n} R\mu_i \ ,$$

where $\mu_i \in H_{k_i}(M^a)$ are the linking cycles of the cell decomposition of Definition 5.5.4. In particular, f is a perfect Morse function.

Example 5.5.6. A very famous example of a Morse function of linking type is given by the height function defined on a torus $T^2 \subseteq \mathbb{R}^3$ with axis orthogonal to the z-axis in $\mathbb{R}^3$. An easy non-example is given by the height function defined on a J-shaped surface (homeomorphic to a 2-sphere). The homology created by the saddle is canceled by the first local maximum.

5.5.2 Bott-Samelson type

Definition 5.5.7. Let $f : M \to \mathbb{R}$ be a Morse function and let $x \in \mathrm{Crit}_k(M, f)$. Let Y be a compact, connected manifold of dimension k, oriented over some coefficient ring R and let $\psi : Y \to M$ be a smooth map. We call the couple (Y, ψ) a cycle of *Bott-Samelson type* for f at x, over R, if $(f \circ \psi) : Y \to \mathbb{R}$ has a unique, nondegenerate, maximum at $y \in \psi^{-1}(x)$.

We will use the following result (for a proof, see [93]).

Theorem 5.5.8. *If (Y, ψ) is a Bott-Samelson cycle for $f : M \to \mathbb{R}$ at $x \in X$, over R, and if $[Y] \in H_k(Y, R)$ is the orientation class of Y, then $\psi_*([Y]) \in H_k(M, R)$ is a linking cycle for f.*

Remark 5.5.9. The height function on the torus T of Example 5.5.6 is, in fact, of Bott-Samelson type. More precisely, let $p_0 \in T$ be the minimum. Then $Y = p_0$ and ψ is the inclusion. For the first saddle point $p_1 \in T$, $Y = \mathbb{S}^1$, and ψ maps Y to the lowest vertical circle inside T. If $p_2 \in T$ is the second saddle point, then $Y = \mathbb{S}^1$, and ψ maps Y into the internal equator of T. Lastly, if $p_3 \in T$ is the maximum, $Y = T$, and ψ is the identity.

5.5.3 Energy functional

Let $(M, \mathbf{g})$ be a complete Riemannian manifold and L be an embedded submanifold. Let

$$\Omega_{p,L} := \{\gamma : [0,1] \to M \ : \ \gamma(0) = p, \ \gamma(1) \in L\} \ .$$

By the formula for the first variation of the energy, the critical points of the energy functional

$$E : \Omega_{p,L} \to \mathbb{R} \ , \qquad \gamma \mapsto E(\gamma) := \int_0^1 \|\dot\gamma(t)\|^2_{\mathbf{g}} \, \mathrm{d}t \ ,$$

are geodesics joining p and L, and that such geodesics are orthogonal to L. We will denote the set of all critical points of E on $\Omega_{p,L}$ as $\mathrm{Crit}(E, \Omega_{p,L})$. A classical result of Morse Theory tells us that the null space of the Hessian of E at $\gamma \in \mathrm{Crit}(E, \Omega_{p,L})$, is given by the finite dimensional vector space of the L-Jacobi fields that vanish at the endpoints of γ, Λ^L_γ. If $\dim\left(\Lambda^L_{\gamma|_{[0,t]}}\right) = k > 0$, $\gamma(t)$ is called a *focal point* of L along γ of index k. It turns out that $\gamma(t)$ is a focal point of L of index k of along a geodesic γ starting orthogonally from L if and only if $\gamma(t)$ is a critical value of the normal exponential map $\exp^\perp_{\gamma(0)} : \nu_{\gamma(0)} L \to M$ and k is equal to the dimension of the kernel of $\left(\mathrm{d}\exp_{\gamma(0)}\right)_{t\dot\gamma(0)}$, i.e. if and only if $\gamma(t)$ is a focal point in the sense of Subsection 5.3.3. When p is not a focal point of L, the energy functional E is a Morse function on $\Omega_{p,L}$ and the index of a critical point $\gamma \in \mathrm{Crit}(E, \Omega_{p,L})$ is given by the finite sum $\sum_{t\in[0,1]} \dim\left(\Lambda^L_{\gamma|_{[0,t]}}\right)$.

Another fundamental fact is that for all values of the energy $r \in \mathbb{R}$, the full space of paths of energy less than or equal to r, $\Omega^r_{p,L}$, admits as a deformation retract the internal part of the space of (sufficiently) broken geodesics of energy less then or equal to r, starting orthogonally to L, that we denote with $\mathcal{B}^r(L,p)$. The space $\mathcal{B}^r(L,p)$ can be given the structure of a finite dimensional smooth manifold (see [78]). Therefore, we can apply to $\mathcal{B}^r := \mathcal{B}^r(L,p)$ the methods of Morse Theory. In particular, suppose that p is not focal for L. Then $\mathcal{B}^r$ has the structure of a finite CW-complex with one cell of dimension k for each critical point in $\mathrm{Crit}(E,\Omega^r_{p,L})$ of index k. By a direct limit procedure, it is then possible to show that the full path space $\Omega_{p,L}$ has the homotopy type of a countable CW-complex that contains one cell of dimension k for each critical point in $\mathrm{Crit}(E,\Omega_{p,L})$ of index k. The direct limit is obtained from successive inclusions

$$\emptyset = \Omega^{r_0}_{p,L} \subset \Omega^{r_1} \subset \Omega^{r_2} \subset \cdots ,$$

where $0 = r_0 < r_1 < r_2 < \ldots$ are increasing, non critical values of the energy such that each interval (r_i, r_{i+1}) contains only one critical point. This last condition is not restrictive, since we can always perturb a Morse function around a critical point, lowering or increasing the critical value, still obtaining a Morse function (see [78]).

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The Thesis is based on the following papers

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