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Review Article

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An introduction to “Second Order Subelliptic PDEs”: the scientific work of Ermanno Lanconelli

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Abstract: We present an overview of the scientific activity of Ermanno Lanconelli, to whom this volume is dedicated on the occasion of his birthday.

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1 Introduction

On the occasion of his 80th birthday, we want to recall the foundational ideas of our *maestro* Ermanno Lanconelli, which allowed him to open rich and still very fruitful research directions, attracting great interest from the scientific community.

Lanconelli started his professional career under the advice of Bruno Pini, with early contributions concerning Fourier transform multipliers and the theory of strongly weakly singular integrals.

The first breakthrough of Lanconelli came in the ’70s when he focused on the boundary behaviour of the Perron-Wiener solution to the Dirichlet problem for the heat equation. This perspective finds its roots in abstract potential theory and it manages to unify the study of elliptic and parabolic equations, which had classically been studied independently. Another milestone in this unification process was established in the early ’80s with a celebrated series of papers in collaboration with Bruno Franchi, which marks the birth of the subelliptic theory. In these works, they used for the first time a distance modelled after the lack of ellipticity of the operators in order to address the interior regularity of totally degenerate operators, with a technique borrowed from elliptic theory.

The main body of the work of Lanconelli can be considered the full realization of this ambitious unification project for elliptic, parabolic and subelliptic equations. He recognized the privileged role of the fundamental

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solution, which has the same informational content as the operator but is already expressed in integral form, and gives rise to a compact operator. In particular, he introduced the idea of interpreting the geometry of the operator directly on the level sets of the fundamental solution and reconstructing with these much of the classical results of potential theory. In this way, over the years, he has achieved important results regarding the asymptotic behaviour of fundamental solutions, the interior and boundary regularity of linear and nonlinear subelliptic equations, representation formulas of mean-value type, and inverse problems in potential theory.

His results find application not only in purely analytical problems but are of interest to problems arising from complex geometry, such as the Levi pseudocurvature equation, for which he obtained particularly valuable results. Additionally, they have applications in the kinetic theory of gases, probability, finance, and vision problems.

2 The first breakthroughs

2.1 Strongly weakly singular integrals

The first works of Lanconelli are devoted to the study of multipliers of the Fourier transform [1–3] and their application to the following problems posed by the PDEs theory:

1) characterization of traces of functional spaces generalizing the classical Sobolev spaces (spaces related to PDEs product of “quasi-elliptic” operators [4, 5]);
2) L^p and asymptotic estimates for the solutions of the Cauchy problem for the wave equation, the Schrödinger equation, and some of their generalizations [6];
3) characterization of the spectrum of constant coefficients ODEs [7].

The most important results from this group of works are those related to the study of Fourier transform multipliers, almost simultaneously studied by Fefferman and Stein within their general theory of strongly weakly singular integrals, a generalization of the classical Calderón-Zygmund theory. Particularly deep is the work [1], which introduces an original technique based on the use of Besov spaces. The applications contained in [6] are nowadays recognized as dispersive estimates, and they were the first to be optimal with respect to the index of the interpolation space.

2.2 Potential theory for elliptic-parabolic operators

An important part of the research by Lanconelli started in the 70’s and is devoted to the potential theory for uniformly parabolic operators, which was later developed by him into the study of elliptic-parabolic operators. Since the beginning of the 20th century, many mathematicians have been attracted by the Dirichlet problem for the heat equation in bounded open sets: the difficulty (and the main difference with the elliptic setting) is that, even for smooth sets, there might be no solutions for generic continuous Dirichlet data at the boundary. The first results for non-cylindrical domains are due to Levi [8], Gevrey [9], Petrowsky [10] and later by Kohn-Nirenberg [11], Effros-Kazdan [12]. A delicate issue is to understand at which boundary points the assigned Dirichlet datum is attained by the solution. For the Laplace operator such a question was settled by Wiener in his celebrated 1924 paper [13] and, since then, uniformly elliptic operators with various assumptions on the coefficients were addressed. Lanconelli’s interest in these problems was influenced by the work of Bruno Pini, who attacked the problem for the heat equation with ideas and techniques typical of the elliptic potential theory: in [14] Pini provided sufficient conditions of geometric nature on the boundary of some special planar domains (1-space dimension) in order to have boundary continuity for the Perron-Wiener solution. Lanconelli’s investigations in this subject initiated in [15] where he studied in a systematic (and yet very flexible) way the geometric characterization of the regularity of the boundary points for the Perron-Wiener solution to the heat equation in generic bounded open sets in $\mathbb{R}^{N+1}$. The starting point is the notion of parabolic capacity, which is a capacity defined with respect to the heat kernel, together with the understanding of the equilibrium measure and of the reflection of the parabolic geometry into the metric properties of the capacity. In [15], he formulated a Wiener-type condition in terms of the divergence of a series involving the capacity of the level set of the heat kernel, and he proved that such a condition is necessary for the regularity of the boundary point. Some years later, Evans and Gariepy in [16] were able to prove that the Lanconelli condition is also sufficient for the regularity.

Lanconelli faced the study of uniformly parabolic operators with varying and possibly non smooth coefficients in a series of papers [17–19], where he studied capacities with respect to different Gaussian kernels and he found both necessary and sufficient conditions which are sharp with respect to the Petrowsky counterexamples. Later, in the late ’80s, he extended the Lanconelli-Evans-Gariepy criterion to parabolic operators in divergence form with smooth coefficients with Garofalo in [20] and with $C^{1,\mathrm{dini}}$-coefficients together with Fabes and Garofalo [21].

2.3 The beginnings of subelliptic theory

In a group of works [22–25], Lanconelli, in collaboration with Franchi, introduced the idea of associating an appropriate metric to a second-order linear differential operator with a positive semi-definite characteristic form. Using a suitable Sobolev space he was able to obtain, in this degenerate setting, results analogous to the ones known in the Riemannian setting. The class of operators considered here is modelled on the following one:

$$L_\alpha u = \partial^2_{x_1x_1} u + |x_1|^{2\alpha} \partial^2_{x_2x_2} u,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $\alpha > 0$. It was already known that if α was an integer, the operator was hypoelliptic, thanks to the results of Hörmander [26] but nothing was known for general values of α. Franchi and Lanconelli introduce suitable vector fields associated with the operators and prove a connectivity property based on their integral curves. This generalizes Chow theorem (see [27, 28] or [29]) to a non smooth setting and allowed them to introduce a natural control distance associated with the operator L_α. Then they introduce Sobolev spaces associated with the vector fields and prove a Poincaré-type inequality. These tools enabled them to adapt to this context the Moser’s iterative method and obtain a Hölder-type theorem for weak solutions [22–24] and a Harnack inequality for non-negative weak solutions in [25].

These results can be considered the beginning of the sub-Riemannian theory since they were obtained simultaneously and independently to the extraordinary development of the theory of subelliptic operators expressed in terms of Hörmander vector fields due to Bony [30] and to the school of Stein. Let us recall the foundational results of Folland [31], Rothschild-Stein [32], Jerison and Sánchez-Calle [33] on the existence of the fundamental solutions for sums of squares of vector fields plus a drift associated with Hörmander-type fields. The Poincaré inequality [34] and the estimates of the distance in terms of the exponential map by Nagel, Stein and Wainger in [35] will be proved a few years later. Hence, the papers of Franchi and Lanconelli contain, in fact, the first proof of the Poincaré inequality in a non Euclidean setting.

In addition, it is important to note that the operators studied by Franchi and Lanconelli do not fall within the setting of Hörmander operators, for non integer values of α. The fundamental ideas contained in these works contributed to the opening of a broad research field now known as the sub-Riemannian Geometric Analysis.

3 A systematic theory of subelliptic equations

3.1 Parabolic and subelliptic mean value formulas

The first results extending the mean value formulas from the classical Laplace setting to the heat equation one are due to Pini [36] (see also Watson [37]): any harmonic function is represented as the mean on the level set of the fundamental solution with respect to a suitable kernel. One of the ideas proposed by Lanconelli is to express mean value formulas on the level sets of the fundamental solution for a much wider class of operators, obtaining very natural formulas with surprisingly simple proofs. In Fabes-Garofalo [38] and Garofalo-Lanconelli [20, 39], the authors recognized that the kernel is a function of the fundamental solution and its derivatives, clarifying its geometrical meaning. This key remark allows to extend mean value and representation formulas to a much larger class of operators for which the existence of the fundamental solution is known. In [40], Lanconelli and Garofalo used the explicit expression of the fundamental solution of the Kolmogorov equation provided in [41], to obtain mean value formulas and Harnack inequalities. For general sub-Laplacians, mean value theorems for sums of squares of vector fields were proved in Jacob [42], and by Citti, Garofalo and Lanconelli in [43]. The proofs are simple and elegant, only based on the divergence theorem, since the fundamental solution with the pole in the center of a ball coincides with the Green function up to a constant. The same formula was adapted to larger classes of Kolmogorov-Fokker-Planck equations in [44] after having obtained the explicit expression of the fundamental solution in [45] (see below).

The relation between “representation formulas” and the Poincaré inequality and the deep geometric meaning of this last inequality were studied in a series of papers by Saloff-Coste [46], Hajłasz and Koskela [47], Maheux and Saloff-Coste [48], Garofalo and Nhieu [49] and Franchi, Lu and Wheeden [50]. Then, Lanconelli tackled again the more fundamental problem of how to extend the Poincaré inequality to a more general sub-Riemannian setting analyzing geometric properties of the subunit (horizontal) curves associated with the vector fields. This was accomplished in collaboration with Morbidelli in [51] for very general families of vector fields. They introduced a weaker analogue of the notion of exponential map, called *almost exponential map*, and obtained a suitable Poincaré inequality for vector fields, further extending the approach of [23]. See also the papers [52] and [53].

3.2 Kolmogorov-Fokker-Planck operators

A particularly important class of operators for which it has been possible to determine an explicit fundamental solution is that of Kolmogorov-Fokker-Planck operators. The simplest example of such an operator was introduced by Kolmogorov in [41] and takes the form

$$\mathcal{L}u(x,y,t):=\frac{1}{2}\Delta_x u(x,y,t)+x\cdot\nabla_y u(x,y,t)+\partial_t u(x,y,t),$$

where $(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$. The differential equation $\mathcal{L}u=0$ appears in several applicative fields, which include the stochastic theory, the kinetic theory and its applications to the modeling of the financial markets, in some vision models, and in models of artificial vision. In particular, Kolmogorov considered in [41] the Langevin process $(X_t, Y_t)_{t\geq 0}$ in the phase space $\mathbb{R}^n\times\mathbb{R}^n$

$$\begin{cases} X_t = x_0 + W_t, \\ Y_t = y_0 + \displaystyle\int_0^t X_s\,ds, \end{cases}$$

(here $\left(W_t\right)_{t\geq 0}$ denotes an n-dimensional Wiener process) and proved that its density p exists and is the fundamental solution to the equation $\mathcal{L}u=0$. The operator $\mathcal{L}$ can be written in the form of *sum of squares of vector fields plus a drift* $\mathcal{L}=X_1^2+\cdots+X_n^2+Y$, where $X_j=\partial_{x_j}$, $j=1,\dots,n$ and $Y=x\cdot\nabla_y+\partial_t$, which therefore falls into the famous class of operators subsequently studied by Lars Hörmander in [26].

In the article [45], in collaboration with Polidoro, the above results have been extended to a general class of Kolmogorov operators satisfying the Hörmander’s condition. The main contribution of the paper is the explicit expression of the Lie group structure with respect to which the Kolmogorov operator is invariant. This invariance property paved the way for the use of Harmonic Analysis and Functional Analysis techniques on Lie groups in the study of the operator $\mathcal{L}$. We refer to the survey articles [54, 55] by Anceschi, Piccinini and Rebucci and to their bibliography for the recent developments of the theory in this field.

Among the most important developments of this field, we recall the classical regularity theory for Kolmogorov operators with Hölder continuous coefficients (see Manfredini [56], Polidoro [57]), the regularity theory for weak solutions (see Bramanti, Cerutti, Manfredini, Polidoro, Ragusa [58–61]), degenerate Ornstein-Uhlenbeck operators, which constitute the stationary counterpart of the operators studied in [45] (see Bramanti, Cupini, Lanconelli and Priola [62, 63]). The regularity properties of the solutions to the stationary equations have important consequences in the stochastic theory.

3.3 Asymptotic behavior of the fundamental solution

In the works referenced in the previous sections, the fundamental solution has been used to define and estimate capacitary terms as well as to obtain local regularity results. However, this tool became central in Lanconelli’s work, thanks to a long series of studies dedicated to the investigation of the asymptotic behaviour of solutions to degenerate parabolic equations in collaboration with Bonfiglioli, Bramanti, Brandolini, and Uguzzoni [64–66]. Specifically, they studied operators of the following type

$$\sum_{i,j=1}^{n} a_{ij}X_iX_j - \partial_t,$$

where the matrix $A = (a_{ij})_{ij}$ is positive definite, and the fields $(X_i)_{i=1}^{m}$ satisfy the Hörmander condition in an open subset of $\mathbb{R}^n$.

Local estimates for the fundamental solution were well-known due to Jerison-Sánchez-Calle [33], while Gaussian-type estimates had been proven by Varopoulos in the case of groups (see [67] and references therein), by Polidoro for degenerate Kolmogorov operators in [68], and by Kusuoka-Stroock in the absence of a group structure (see [69] and references therein). The asymptotic estimate has been first proved for constant coefficient operators (see [64]) using an idea first introduced in [70]. In this case, the homogeneity of the operator plays a key role: the estimates of the fundamental solution are obtained on lenticular bounded sets which invade the space, so that the fundamental solution is obtained by passing to the limit. In addition, uniform estimates of the fundamental solution are obtained, depending only on the least eigenvalue of the matrix A (see [64]). Once obtained the uniform estimates for the model operator with constant coefficients, one can apply the Levi parametrix method and establish existence and Gaussian bounds for the fundamental solution of the operators with variable Hölder continuous coefficients [65]. The method, well known in the elliptic case and already used by [33] in subelliptic settings, was deeply simplified here due to the asymptotic Gaussian estimates, which ensure convergence of all the considered integrals. Analogous estimates have been obtained for a more general class of non-divergence equations structured on Hörmander vector fields in [66], with non regular coefficients.

3.4 Potential theory for sub-Laplacians in stratified Lie groups

A good knowledge of the properties of the fundamental solution, together with the representation formulas, allows us to perform a complete potential theory that parallels the one of the classical Laplace operator, using the same ideas presented above for the elliptic-parabolic operators. This approach has been introduced by Bonfiglioli, Lanconelli and Uguzzoni in the book [71], which can be considered one of the main reference text for experts in subelliptic PDEs. The book contains an exhaustive potential theory for the sub-Laplacians. The lack of explicit Poisson integral formulas forces the authors to follow an abstract approach and the Perron–Wiener method for the Dirichlet problem, adapted to sub-Laplacians. Also using the properties of the fundamental solution, they prove: sub-mean characterizations of the subharmonic functions with respect to the considered operator, and applications to the notion of convexity in Carnot groups (see also [72]); representation theorems for subharmonic functions, maximum principles on unbounded domains (see also [73]); the proof that the distance associated with sub-Laplacians satisfies the Eikonal equation (see also [74]); a notion of capacity adapted to the geometry of the operator and polar sets, Wiener’s criterion, Poisson-Jensen type formulas (see also [75]).

The explicit expression of a solution given by the representation formula was later on used to provide a notion of average solution of the Dirichlet problem [76], of the Poisson problem [77], and to provide estimates for boundary datum in L^p [78].

In the book, they also provide direct proofs of a special case of the Campbell–Hausdorff formula, the Rothschild–Stein lifting theorem, the Chow connectivity theorem, Taylor’s formula in the setting of Carnot groups.

In a more general setting, Wiener-type results were obtained by Lanconelli in collaboration with Uguzzoni in [79], with Tralli and Uguzzoni in [80] and with Kogoj and Tralli in [81]. Precisely, a deep Potential Analysis and uniform boundary estimates for the Perron-Wiener solution to the Dirichlet problem has been obtained only assuming a two sided Gaussian-type estimate for Γ.

3.5 Liouville-type theorems

The doubling property of the balls and the Poincaré inequality on the same balls, mean value or Poisson–Jensen-type representation formulas and global Harnack inequalities are essentially the main tools used by Lanconelli to prove Liouville properties for large classes of linear second-order PDEs with underlying sub-Riemannian structures.

The work [82] is an abstract formalization of the ideas contained in the works of Franchi and Lanconelli of the early 1980s [22–24]. Lanconelli and Kogoj introduced X-elliptic operators, a class of “uniformly subelliptic” operators, and they gave a condition in terms of the properties of the Carnot-Charathéodory metric structure sufficient to guarantee the geometric framework and the functional analysis setting that allow Moser’s method to apply, thus obtaining a scale-invariant Harnack inequality for weak solutions. *One-side Liouville-type properties* - i.e., the non-negative global solutions have to be constant - for classes of X-elliptic operators were proved by Lanconelli with Gutiérrez [83] and with Kogoj [84]. Additionally, in [84], by exploiting the doubling property and the Poincaré inequality, a polynomial Liouville-type theorem for X-elliptic operators with non-smooth coefficients was proved via the extension to this new setting of a celebrated result by Colding and Minicozzi [85] related to the Laplace-Beltrami operator on Riemannian manifolds with non-negative Ricci curvature.

A further wide class of degenerate operators for which Lanconelli studied Liouville properties are the hypoelliptic and invariant ones on Lie groups. In 1983 the works of Geller [86] and Rothschild [87] underlined how hypoellipticity has a fundamental importance in Liouville theorems of polynomial type. Polynomial Liouville theorems were proved by Lanconelli in collaboration with Bonfiglioli for sub-Laplacians on Carnot groups [88] and by Lanconelli in collaboration with Kogoj for invariant hypoelliptic operators [89, 90]. Kogoj and Lanconelli continued studying various Liouville properties for evolution operators and their *time-stationary* counterparts. As a consequence of a global parabolic Harnack inequality, in [91], they proved a *Liouville theorem at time $t = -\infty$* for the evolution operators. From this theorem, they deduced the one-side Liouville property for the relative stationary counterparts. Furthermore, without requiring homogeneity but only invariance with respect to a group law and proving representation formulas of Poisson-Jensen-type, they obtain many uniqueness results for both solutions and subsolutions in L^p spaces [92]. All the results hold as very particular cases for the heat operators on stratified Lie groups and some of them are new even for the classical heat operator. In the case of Kolmogorov-Fokker-Planck operators and their time-stationary counterparts, the Ornstein-Uhlenbeck operators, are collected in the survey [93]. Nowadays, Liouville theorems for these classes of operators are of great interest both from an analytic and a stochastic point of view [94–97].

A noteworthy theorem obtained by Priola and Zabczyk in 2004 [96] with probabilistic methods states that the Ornstein-Uhlenbeck operator $\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla\rangle$, where A and B are $N \times N$ real matrices so that $\mathcal{L}$ is hypoelliptic and with B having spectrum in the left-half of the complex plane, has the L^∞-Liouville property, i.e. the global bounded solutions have to be constant. It is an open problem to know if the solutions to the equation $\mathcal{L}v = 0$ *bounded just from one side have to be constant* and if this can be proved with purely analytic methods. In the case where the operators considered are invariant with respect to groups of dilations, this result was proved by Kogoj and Lanconelli in [89]. For non-homogeneous operators Lanconelli with Kogoj and Priola have obtained in [98] a first result under the assumption that the matrix B is diagonalizable over the complex field with all its eigenvalues on the imaginary axis. Very recent developments have been appearing in this direction as this volume is being published, see [99, 100].

3.6 Inverse mean value formulas and stability results

Symmetry results and uniqueness of radial solutions were considered by Lanconelli in different settings at various stages of his career. We would like to highlight in particular the 1996 paper [101] coauthored with Franchi and Serrin who had a strong impact on the scientific community working on nonlinear equations. In the most recent years, Lanconelli has been focusing on rigidity and stability results related to inverse mean value formulas, and we briefly describe here this line of research.

Among the various rigidity properties of the Euclidean balls one of the best known examples is related to the mean value formula for harmonic functions

> If D is an open set with finite measure and $x_0 \in D$ is a point such that $u(x_0)$ is the average of u on D for every integrable harmonic functions u in D, is it true that D is a ball centered at x_0?

The answer is yes and it has been given by Kuran [102], with a short and elegant proof. A similar question is about the Newtonian potential of a homogeneous body D, that is proportional, outside D, to the Newtonian potential of a mass concentrated at a point x_0 in D if and only if D is a Euclidean ball centered at x_0. The *if part* simply follows from Gauss Mean Value Theorem for harmonic functions. The *only if part* is a Theorem by Aharonov, Schiffer and Zalcman [103]. A version of this last result for sub-Laplacians has been proved by Lanconelli in [104], see Cupini and Lanconelli [105–108] for related results. The rigidity result for the heat operator has been established by Suzuki and Watson in [109], later generalized to the heat operator on stratified Lie groups and to Kolmogorov-Fokker-Planck-type operators by Lanconelli, in collaboration with Kogoj and Tralli, see [110, 111]. We refer to Netuka and Vesely [112] for a survey on problems related to the mean value property of harmonic and caloric functions.

This harmonic characterization of balls raises the question of its stability; i.e.:

> If D is an open set with finite measure and x_0 is a point of D such that $u(x_0)$ is close to the average of u on D for every integrable harmonic functions u in D, is it true that D is close to a ball centered at x_0?

A positive answer has been given in Cupini, Lanconelli, Fusco and Zhong [113]: there exists a constant $C(n)$ such that if $D \subset \mathbb{R}^n$ is an open set of finite measure and $x_0 \in D$, then

$$C(n)\frac{|D\backslash B(x_0, r_{x_0})|}{|D|} \leq \sup_{u\in\mathcal{H}(D)\cap L^1(D), u\not\equiv 0} \frac{\left|u(x_0) - \fint_D u(x)\,dx\right|}{\fint_D |u(x)|\,dx},$$

where $r_{x_0} = \operatorname{dist}(x_0, \partial D)$. Note also that this estimate implies Kuran’s Theorem as a corollary.

Similar questions can be posed for the surface averages: Let D be a bounded open set of $\mathbb{R}^n$ and let x_0 be a point of D. Assume that $u(x_0)$ equals the average of u on ∂D for every harmonic function u in D continuous up to the boundary. In this case one says that D is a harmonic pseudosphere centered at x_0. In general, harmonic pseudospheres are not spheres see [114]. As a consequence, the following problem naturally arises: when a pseudosphere is a sphere? The most general result in this direction was obtained by Lewis and Vogel [115] in 2002: a harmonic pseudosphere ∂D is a sphere if D is Dirichlet regular and the surface measure on ∂D has at most an Euclidean growth. Preiss and Toro [116], in 2007, proved the stability of Lewis and Vogel’s result: a bounded domain D satisfying the Lewis and Vogel’s regularity assumptions has the boundary geometrically close to a sphere centered at x_0 if the Poisson kernel of D with pole at x_0 is close to a constant. By using a family of harmonic functions introduced by Kuran in [102], in [117] Cupini and Lanconelli define a new harmonic invariant that measures the gap between the perimeter of a domain D and the perimeter of the biggest ball contained in D and centered at a fixed point x_0 of D. From the properties of this harmonic invariant they get new proofs, generalizations and partial improvements of several rigidity and stability theorems by [103, 115, 116, 118, 119] and by Fichera [120].

4 Applications

4.1 Cauchy–Riemann-settings: Levi curvature equations

Most of Ermanno Lanconelli’s scientific production focuses on linear PDEs, but he also made significant applications to non linear equations coming from CR geometry. As an example, Lanconelli obtained non-existence results for nonnegative solutions to semilinear equations in unbounded domains in the works [121, 122] with Uguzzoni, see also [123], for equations related to the CR Yamabe problem. However, his main contribution on this topic is a long series of foundational papers dedicated to the Levi equation.

Levi curvatures can be defined as the elementary symmetric functions of the eigenvalues of the Levi form of real hypersurfaces in $\mathbb{C}^{n+1}$. Understanding the role of the Levi form in pseudoconvexity problems is a classical topic in several complex variables and in the study of domains of holomorphy. The prescribed Levi curvature problem can be addressed as the problem of finding a real hypersurface with an assigned $2n$-dimensional boundary with vanishing total Levi curvature (Levi-flat in case $n = 1$). This problem is closely related to the construction of the envelope of holomorphy, and the first existence results were obtained by Bedford and Gaveau [124], by Bedford and Klingenberg [125], and Tomassini [126] with purely geometrical methods. Slodkowski and Tomassini in [127, 128] developed a PDE approach: for graphs the problem can, in fact, be translated into studying a Dirichlet-type problem for a nonlinear degenerate-elliptic equation. Under (strict) pseudoconvexity assumptions, the sub-bundle of the tangent bundle which is invariant under the complex structure is non involutive, and the underlying degenerate-elliptic equation inherits Hörmander-type properties. In [129], Citti expressed the equation of prescribed mean Levi curvature as sum of squares of non linear vector fields, allowing to apply the theory of sub-Riemannian operators. Subsequently, in a series of works, Citti and Montanari have progressively clarified many important properties of such Levi equation, all based on its sub-Riemannian nature [130]. These studies culminated, in 2000, in a paper [131] by Citti, Lanconelli and Montanari published in Acta Mathematica, in which it is shown that in case $n = 1$ the weak viscous solutions of the Levi curvature equation for $C^{2,\alpha}$ domains are actually C^∞ if the Levi curvature is strictly positive. The proof of the regularity result in [131] is particularly challenging since it is the first work where non linear vector fields have been considered. It is made in two steps. First, the authors apply a linearization procedure, reducing the operator to a linear one, expressed as sum of squares of $C^{1,\alpha}$ Hölder continuous vector fields. Because of the lack of good estimates of the fundamental solution for this type of operators, a freezing method is applied to reduce the problem to a Hörmander type ones, for which a fundamental solution is available and allows to conclude the result. We stress how this type of problem asks for good *a priori* estimates for fundamental solutions of operators represented in terms of non-smooth vector fields, as discussed in the previous sections.

The regularizing properties of these nonlinear equations in case $n > 1$ present many difficulties, and there are still open problems in this direction. Concerning the analysis of higher-order curvatures of pseudoconvex graphs, Lanconelli and Montanari [132] proved a general strong maximum principle and, together with Gutiérrez, they showed in [133] the existence of non-smooth solutions to the complex Monge Ampère type equations arising from the prescription of the total Levi curvature.

Another challenging direction of research related to the prescribed Levi curvature is related to the lack of symmetries with respect to rigid motions, which do not preserve the underlying complex structure. In particular, it is presently not known whether spheres are the only closed hypersurfaces with constant Levi curvature. Lanconelli addressed this problem in collaboration with Jorge Hounie in [134], where they showed an Alexandrov-type result in $\mathbb{C}^2$ in the class of Reinhardt domains. Further partial results and different approaches in this direction were pursued in [135, 136].

4.2 Developments for applied sciences

The work of Lanconelli contributed not only to the development of the Partial Differential Equations theory but also to some research fields where these equations occur: problems described in terms of variables related by differential constraints. These constraints reduce the spaces of allowed directions for diffusion or propagation, hence introducing a horizontal tangent space. Indeed these equations have been initially introduced by Kolmogorov in the study of Kinetic Theory of gases, and it is well known that the Heisenberg group arises in the description of the phase space.

More recently mathematical models of brain functionality or visual perception have been expressed through subelliptic PDEs by Petitot and Tondut [137], Duits in [138, 139] and Citti and Sarti [140]. A study of the Kuramoto model for synchronization phenomena based on degenerate Kolmogorov-Fokker-Planck equations is performed on the article [141] by Pecorella, Polidoro and Vernia. Several applications to the modeling of financial markets are described in Pascucci's monograph [142]. Recent applications to Kinetic Theory have been developed by several authors. We refer to the article by Golse, Imbert, Mouhot and Vasseur [143], and to the recent article [55] by Anceschi, Piccinini and Rebucci. Research activity in these fields is currently very active and under a rapid development.

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