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University students' sense-making of the asymptotic behavior of functions

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Contents

Abstract

This study analyzes the sense-making of university students about the asymptotic behavior of functions focusing on solving processes. It examines how students engage with unfamiliar tasks, adopting Sfard's Commognitive framework as the main theoretical lens. Across its two phases, the project progressively refines the analysis of students' activity, moving from the identification of approaches, to the role of resistance and its connection with elements of commognition theory, and finally to the characterization of reasoning.

In the preliminary phase, the focus was on investigating the influence of the presence and use of different visual mediators on students' performance and solving processes in tasks requiring them to determine straight lines intersecting a given parabola. The participants were first-year engineering students who solved the tasks individually, and their written answers were collected for analysis. This phase led to the identification of two main categories of approaches enacted by students: approaches guided by rituals and approaches guided by the graph. This categorization made it possible to observe how specific visual mediators included in the task text shaped both students' performance and their solving processes, showing that visual mediators do not merely accompany mathematical activity but actively contribute to structuring it.

The second phase deepened the investigation by integrating the notion of resistance, in Wertsch's sense, with the commognitive constructs of routines, visual mediators, and aspects of mathematical reasoning. In this phase, engineering and mathematics students worked in pairs on unfamiliar tasks during task-based interviews, in which their discourse, gestures, and written work were collected and analyzed. The analysis focused on points at which the solving process was impeded and on how students eventually managed, or failed, to overcome such impediments, through which instances of resistance were identified. The results indicate that the characteristics

of routines are closely related to resistance: when resistance is not overcome, routines tend to display limited flexibility and restricted applicability, whereas overcoming resistance is associated with routines that are more adaptable and can be mobilized across a wider range of situations.

This phase also extended the perspective by focusing explicitly on visual mediators and their relation to resistance within students' solving processes. The results suggest a connection between the use of different visual mediators and the emergence of moments of resistance. In particular, resistance appears to arise when different realizations of the same mathematical object—such as algebraic expressions, graphs, or gestures—lead to outcomes that students struggle to reconcile. These moments tend to be overcome when students construct new narratives that connect such realizations, thus enabling a more coherent solving process.

Finally, the research explored students' approaches and substantiation when solving tasks involving the asymptotic behavior of functions. From a commognitive perspective, these approaches were characterized in terms of processual and structural aspects of reasoning. The analyses revealed considerable variation in students' activity: some approaches were predominantly processual or predominantly structural, whereas others integrated both dimensions without a clear predominance. Substantiation was frequently based on multiple approaches rather than on a single dominant mathematical aspect. At the same time, the visualization of asymptotic behavior itself at times became a source of impediment, highlighting the difficulty of applying a familiar tool effectively in specific solving contexts.

Overall, the results indicate that visual mediators, realizations, and resistance are deeply interconnected and play a decisive role in shaping students' solving processes when working with the asymptotic behavior of functions. These findings offer useful insights for the teaching of this mathematical topic, suggesting the importance of promoting stronger connections between different realizations during instruction.

Chapter 1

Introduction

The starting point of this research is the analysis of university students' sense-making regarding asymptotic behavior, a concept typically introduced in at least one calculus course. In this section, we summarize the notions required to solve the proposed tasks, which are generally addressed in secondary school and in introductory university calculus. We also outline the object of study in light of the existing literature, situating the research within the broader scholarly context.

1.1 Framing of the research problem

This study investigates university students' solving processes involving the asymptotic behavior of functions. As highlighted in the literature, the learning of asymptotic behavior has not been extensively examined, particularly with respect to growth at infinity. Existing research has primarily focused on asymptotes and on the difficulties students experience when working with this notion (e.g., Katalenić et al., 2020; Katalenić et al., 2023; Mpofu and Pournara, 2018; Mpofu and Mudaly, 2020). More generally, research in mathematics education addressing university students has explored mathematical discourse and the challenges linked to coordinating different realizations of the same concept, as well as the learning of the function concept (e.g., Güçler, 2016; Nachlieli and Tabach, 2012; Ryve et al., 2013).

Within this field, our study contributes to addressing the gap concerning students' sense-making of asymptotic behavior. Making sense of this notion requires coordinating several mathematical elements (functions, limits, different realizations and visual mediators). This complexity led us to focus on the solving process, examining how students engage with unfamiliar tasks by analyzing their verbal and non-verbal communication, the approaches they adopt, and the meanings they construct through discourse and interaction. To this end, the research is framed within Commognitive Theory (Sfard, 2008; Lavie et al., 2019), and is locally integrated (Prediger et al., 2008) with the notion of resistance (Wertsch, 1998) and with Jeannotte and Kieran's model of mathematical reasoning (Jeannotte and Kieran, 2017). Integrating resistance into the commognitive framework allows us to connect with the constraints described in the precedent-search-space (Lavie et al., 2019) and to gain deeper insight into students' approaches when facing unfamiliar mathematical tasks. It also enables us to explore how the use of routines relates to the overcoming—or persistence—of resistance. The study was conducted in two phases: a preliminary phase, based on a mixed-methods design (Johnson and Onwuegbuzie, 2004), and a main study consisting of task-based interviews following Goldin's methodological approach (Goldin, 2000). Participants were university students enrolled in engineering and mathematics programs. The tasks involved contexts familiar to students, such as the intersection of functions, but differed from those typically encountered in school or university courses, as they aimed to investigate the notion of asymptotic behavior without explicitly naming it. Results from the preliminary phase indicated that the use of different visual mediators significantly influenced students' responses and performance (see Chapter 5; Gambini et al., 2024). These results were confirmed in the main study, which further showed that the presence of multiple realizations often triggered the emergence of resistance (see Chapter 6, and Viola, Under review). Additionally, the results suggest that the characteristics of routines are strongly related to whether resistance is overcome: overcoming resistance appeared to be associated with greater flexibility and applicability (see Chapter 7, Viola and Gambini, Under review). Finally, by examining the different approaches adopted by students in relation to the aspects of mathematical reasoning, it was possible to analyze how they substantiated their arguments, highlighting connections among substantiation, the overcoming of resistance, and the absence of a single dominant aspect of mathematical reasoning (see Chapter 8, Viola et al., Submitted).

1.1.1 Mathematical overview on the asymptotic behavior of functions

The notion of an asymptote can be defined as a line whose distance from a curve tends to zero as a point on the curve tends to infinity along an unbounded branch of the curve. For functions of a single variable, asymptotes correspond to the asymptotic lines of their graphs in the Cartesian plane. The horizontal asymptote of a function f is defined as the line $y = l$ that satisfies the condition

$$\lim_{x \to +\infty} f(x) = l.$$

Similarly, the oblique asymptote of f is defined as the line $y = kx + l$ that satisfies the condition

$$\lim_{x \to +\infty} (f(x) - kx - l) = 0.$$

These definitions include cases where the function intersects its asymptotes finitely or even infinitely many times. The oblique asymptote can be understood as a linear approximation of the function as $\to +\infty$. Generalizing this idea, one can observe that a function may be approximated not only by a linear function—as in the case of the oblique asymptote—but also by other classes of functions, thus extending the notion of asymptote in this way

$$\lim_{x \to +\infty} (f(x) - g(x)) = 0$$

(Janaszak, 2013; Salas et al., 2021;Yerushalmy, 1997). This perspective highlights how functions can be approximated by simpler ones to facilitate calculations. Furthermore, Dobbs (2011) emphasized a geometric sense of the asymptotic concept: if f and g are two functions within a given family of functions, and $g(x) \neq 0$, then

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1.$$

This relation is not equivalent to the previously mentioned one and is referred to as the asymptotic equivalence of functions (De Bruijn, 2014).

Chapter 2

Theoretical framework

This research is framed within a sociocultural perspective, emphasizing a participatory vision of learning rather than viewing it as mere knowledge acquisition. Students are actively involved in the learning process and activities, with communication playing a key role, as it enables them to contribute to the activity and engage in social practices that are essential for cognitive development. From this perspective, mathematics is not conceived as a static body of knowledge to be transmitted and acquired individually, but rather as a historically and socially developed form of communication in which learners gradually learn to participate. This view is consistent with a Vygotskian (1978) foundation, according to which collectively implemented and historically constituted activities precede the emergence of specifically human cognitive abilities. Thinking can be understood as an individualized version of communication—an internalized discursive interaction in which a single person takes on the roles of all interlocutors. Given that Commognition is grounded in this socio-cultural account of learning (Vygotsky, 1978), it highlights the centrality of context and interaction: mathematical understanding develops through participation in shared discursive practices, and such discourse evolves as students engage with others while working on tasks. For this reason, our analysis adopts a socio-cultural lens, paying close attention to how students interact, how they make use of mathematical tools, and how their discourse transforms during the solving process.

Within this framework, we ground our research in Sfard's (2008) Theory of Commognition, placing significant emphasis on communication and the development of discourse. Additionally, we integrate locally (Prediger et al., 2008) the concept of resistance, as described by Wertsch (1998). Moreover, we focus on students' sense-making, which in the literature has been described as the search for meaning and internal coherence within mathematical expressions, as well as the coordination between symbolic relations and the phenomena they are intended to represent. Sense-making in, and with, mathematics is widely seen as a central goal of mathematics education (Arcavi, 1994). It entails that mathematical activity should be meaningful, that learners are empowered to interpret and work with situations, and that mathematical tools can be used to make progress both within and beyond mathematics (Arcavi, 1994; Gupta and Elby, 2011).

In our study, this perspective guides the analysis of students' discourse, with attention to how they interpret the mathematical tools they use, how they connect different realizations of the same object, and how meaning emerges—or fails to emerge—as they reason through the tasks. Thus, sense-making becomes a key lens for examining students' solving processes, particularly in moments when coordinating graphical and algebraic realizations requires them to articulate what the mathematics is intended to express. Within this perspective, we also focus on different aspects of mathematical reasoning highlighted in the model of Jeannotte and Kieran (2017), in order to analyze which forms of reasoning become prevalent during the solving process and whether this prevalence is connected to students' ability to deal with moments of resistance in problem solving. In the following paragraphs, these theoretical elements are described in detail, clarifying their connection to this study.

In this study, we use the term approach to describe the way students engage with a task, encompassing the specific actions, reasoning steps, and choices they implement to arrive at a solution. An approach reflects how students tackle a problem, including the selection and coordination of different realizations. The term tool, on the other hand, refers to a mathematical instrument or resource that students use within their approach to support their solving process. A tool can take many forms, such as an algebraic system, a graph, an analysis of asymptotic behavior, or any representation that aids in solving the task. It is important to note that a tool is distinct from the approach: the same tool may be employed in multiple approaches, and an approach may combine several tools to construct a solution.

2.1 Commognition Theory

The theoretical framework guiding this research is Sfard's (2008) theory of Commognition, which examines the evolution of discourse in the teaching and learning process. This discursive approach to mathematics education, initially introduced by Kieran and colleagues (2002), contrasts with the traditional view of learning as an individual act of knowledge acquisition. Instead, it emphasizes participation within a broader socio-cultural context, considering both the environment in which learning takes place and the role of mediating tools. Our study adopts this perspective, focusing on how discourse evolves as students engage with the mathematical solving process.

Sfard (2008) further developed the discursive approach by introducing the concept of *Commognition*, a term that merges "communication" and "cognition" to highlight their interconnection. She argues that many learning difficulties in mathematics stem from ambiguities within language and existing discourses. Within this framework, mathematical discourse is characterized by four key elements: *word use*, *narratives*, *visual mediators*, and *routines*.

Word use refers to specialized mathematical terminology and common words that acquire distinct meanings in mathematical contexts (e.g., "field" or "root"). According to Sfard (2008), the use of a word in discourse develops through various stages: passive use, routine-driven use, phrase-driven use, and object-driven use. In the passive stage, the student does not actively use the word, but responds to it through established routines when others use it. In the routine-driven stage, the word is used actively, though its use is limited to familiar routines and embedded within fixed discursive sequences. In the phrase-driven stage, the word is no longer tied to a specific routine but is used within particular phrases. Finally, in the object-driven stage, the word "*gets a life of its own* as a noun" (Sfard, 2008, p. 181), and the student can use it flexibly and appropriately across various contexts and sentence structures.

Narratives refer to verbal or written accounts concerning mathematical objects, their properties, and the relationships between them. These narratives are subject to validation, which depends on the discourse context and the interlocutors involved. The criteria and modes of validation may vary across different types of discourse. In the case of mathematical discourse, validation is grounded in deductive reasoning and logical consistency. Within this framework, the narratives recognized and endorsed by the mathematical community include theorems, axioms, definitions, proofs, and similar constructs.

Routines, as defined by Sfard, are composed of a set of metarules that guide repetitive patterns of discursive activity. These routines can be divided into two main components: the how and the when of the routine.

The *how* component refers to the metarules that govern the manner in which the discursive action is carried out—that is, the procedural unfolding of the routine. The *when* component consists of metarules that determine the contexts or situations in which it is considered appropriate to enact a particular routine. The when aspect can be further subdivided into two categories: applicability conditions and closure conditions.

Applicability conditions describe the types of situations in which an individual is likely to activate or evoke a given routine. Closure conditions define the circumstances under which a person interprets the execution of a routine as successfully completed. Prompts play a key role in the applicability of routines. These are cues or stimuli that can initiate a routine and may take various forms: they can be verbal, written, or environmental. Prompts may originate externally (e.g., from a teacher or a task) or internally (from the individual's own initiative or expectations).

Routines are categorized into three types: *rituals*, *deeds*, and *explorations*. *Rituals* are defined as sequences of discursive actions whose primary goal (closure condition) is to establish and maintain social connections with others, rather than to produce narratives or transform mathematical objects. *Deeds* refer to a set of rules aimed at constructing or manipulating mathematical objects, rather than producing narratives. For example, they may be characterized by the use of numerical terms as the only specific mathematical words involved. *Explorations* can be understood as a type of routine whose performance is considered complete when it results in the production or substantiation of endorsed narratives. The process of endorsement follows the specific rules and norms of mathematical discourse. Exploratory routines may involve three kinds of narrative activity.

- *Construction*: the creation of new narratives that have the potential to be endorsed within the mathematical community.
- *Recalling*: the act of bringing to mind narratives that have already been endorsed.
- *Substantiation*: the evaluation of previously constructed narratives to determine whether

they should be endorsed or not.

Learning progresses as students transition from ritualistic behaviors to deeds and explorations. This progression reflects a shift from a process-oriented approach, focused primarily on carrying out procedures regardless of the outcome, to a product-oriented approach, where the goal placed on the output produced by the implementation of the routine (Lavie et al., 2019). Rituals often represent the initial stage in the implementation of routines and are a natural, sometimes unavoidable, part of their development. When faced with new situations, individuals tend to rely on behaviors previously used in similar contexts. This tendency also applies to the learning process and the evolution of discourse: through imitation, students begin by engaging in ritualized actions. Over time, and through a process of rationalization, these rituals may gradually evolve into explorative routines. The development of routines involves a process of individualization, where students begin with socially shared routines. To analyze how students transition from rituals to explorations— a process known as *de-ritualization*— several factors must be considered (see Table 2.1), including *flexibility*, *applicability*, *agentivity*, and the *ability to objectify discourse* (Sfard, 2008; Lavie et al., 2019). Understanding this transition allows us to determine the types of routines students employ in solving processes. For instance, when analyzing the intersection of two functions, students may use different approaches, such as solving an algebraic system or reasoning about their asymptotic behavior. Examining solving processes through the lens of de-ritualization provides insight into students' evolving mathematical discourse.

Learning is understood as a transformation of discourse, and two distinct types of learning can be identified: object-level learning and meta-level learning. Object-level learning refers to the development of an existing discourse through the expansion of vocabulary, the creation of new narratives, and the introduction of new routines. Meta-level learning, on the other hand, involves changes in the underlying metarules that govern discourse. This means that students begin to solve both familiar and unfamiliar tasks in new ways and may alter how they use familiar terms. Meta-level learning does not occur in autonomy; rather, it happens when students are exposed to a new discourse characterized by different metarules. This encounter often leads to a commognitive conflict, a situation in which different speakers apply different metarules. Such conflict is often essential, as it becomes the driving force behind meta-level learning.

Element of de-ritualization process	Description	Adaptation to our context
Flexibility	*The rise in the flexibility of a routine means that there is now more than one way to perform the task … This happens when the child realizes that other, hitherto unrelated, procedures can be used to perform the same task.*(Lavie et al., 2019, p. 167). Refers to the possibility of choosing among different methods to approach a task, with the awareness of their availability. It reflects a shift in students' actions toward exploratory engagement.	To determine whether two straight lines intersect, one can either compare their slopes or solve a system of equations. A student displays flexibility when they recognize multiple possible paths and take them into account in their reasoning.
Bondedness	*we call routine "bonded" if the output of any given step in its procedure, if not yet the desired final product, feeds in (is used as an input) in latter steps* (Lavie et al., 2019, p. 168). Involves carrying out a series of connected steps, where each result is used as a basis for the next. This reflects a structured and coherent exploration.	This feature is difficult to observe in our study, as the tasks proposed are not organized into sequential phases that naturally build upon each other.
Applicability	*considering the range of task situations for which its performances so far are likely to constitute precedents.*, (Lavie et al., 2019, p. 169). Refers to the range of contexts in which a certain mathematical process or tool, previously encountered, can be recalled and reused meaningfully. Ritualstend to limit this dimension, as they are not product-oriented but rather based on the mechanical repetition of procedures.	When students are asked to determine if two functions intersect, the term "intersection" triggers previous experiences involving this concept, allowing them to draw upon familiar approaches developed in earlier tasks.
Performer's agentivity	It occurs when performer is free *to decide what should be done and when* (Lavie et al., 2019, p. 170) students make decisions during the resolution process independently, without relying on others. In ritual-based activity, students typically follow instructions without exercising such autonomy. The influence of a teacher or textbook, even if not physically present, shapes every aspect of the work.	Students show autonomy when, for example, they choose to solve a system of equations with full understanding of what it entails and how the outcomes relate to the task at hand.
Objectification	*As time goes by and one's experience with the given routine grows, the story becomes more and more abstract, that is, it starts being told as a narrative on mathematical rather than concrete objects.*(Lavie et al., 2019, p. 170). Refers to the ability to think in terms of mathematical structures or properties rather than relying solely on previous examples. It indicates a deeper abstraction from past experiences.	An example is when students consider the general behavior of a parabola, such as its growth toward infinity, and use this understanding to interpret a situation, rather than drawing from prior examples alone.
Substantiability	Occurs when students respond *to the request for a substantiation of what [they] did* (Lavie et al., 2019, p. 171) evaluating the validity and effectiveness of their approach without seeking confirmation from authority figures, relying instead on internalized	For instance, when students examine whether comparing slopes is a valid way to determine the intersection of lines, they do so by judging the reasoning themselves rather than seeking external approval.

Table 2.1: Elements of de-ritualization process linked to our context

When individuals face novel situations, they often rely on memories of past experiences to navigate the unfamiliar. This process unfolds within what is called a *precedent-search-space (PSS)*, where the selection of past experiences is guided primarily by perceived contextual similarities. In any given task situation—understood as a context that prompts individuals to act—they draw upon these precedents to interpret and respond to the current challenge. Innovation arises when individuals extend beyond the usual limits of their PSS, applying precedents in ways that are not typically associated with the given context (Lavie et al., 2019). In light of this perspective, we introduce the notion of *resistance* as the set of difficulties students encounter when engaging with unfamiliar tasks. Resistance, in this sense, marks the boundaries of their precedent-based reasoning and represents the hurdles they must surpass in order to arrive at innovative responses (see Section 2.2).

Visual mediators include diagrams, graphs, gestures, and algebraic expressions that support communication. Visual mediators, as described by Sfard (2008), facilitate discourse by making mathematical objects more tangible. They act as "providers of images" that help coordinate communication among learners. For example, gestures and diagrams serve as *realizations of mathematical signifiers*, aiding comprehension and interaction. The ability to manipulate these realizations is crucial for constructing endorsed narratives and developing a deeper understanding of mathematical concepts. A key aspect of this research involves analyzing realization trees, a concept introduced by Sfard (2008) to describe how different realizations of a mathematical object are connected hierarchically. In a realization tree, each node functions both as a realization of the preceding node and as a signifier for the subsequent one, demonstrating the recursive nature of realization. Different learners may perceive realization trees in distinct ways, influencing their approach to problem-solving. Sfard emphasizes that fluency in switching between realizations is fundamental for mathematical reasoning.

In this study, realization trees are examined from a socio-cultural perspective to investigate students' solving processes in pairs. For example, as demonstrated by Nachlieli and Tabach (2012), multiple realizations of a function—such as an algebraic expression, a graph, and a table of values—can coexist within a single discourse. For instance, in the context of functions, Nachlieli and Tabach (2012) describe several possible realizations, such as an algebraic formula, a graphical representation, or a table of values (see Figure 2.1, inspired by their work). Figure 2.2 illustrates

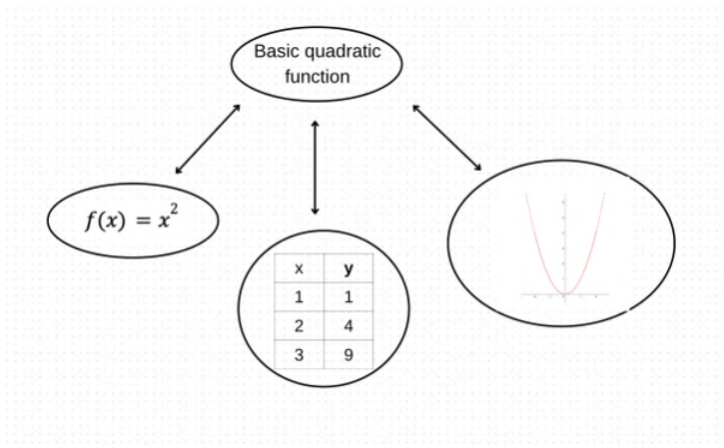


Figure 2.1: Realization tree of functions

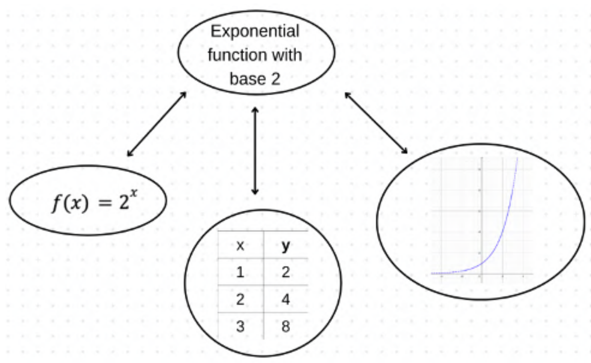


Figure 2.2: Realization tree of exponential function

the realization tree for the exponential function, relevant for this study. In addition, this study investigates how different visual mediators influence students' performance, and how difficulties in using these mediators can trigger resistance.

Through the lens of Commognition Theory, this study investigates how students' mathematical discourse evolves during the solving process, with particular attention to the interplay between visual mediators, realization trees, routines, and the process of de-ritualization. This framework allows us to explore how mathematical meaning is constructed and negotiated within a socio-cultural context. In particular, we examine how elements of Commognition Theory relate to the notion of resistance—understood here as the boundaries of the precedent-search space (PSS) that students must overcome to reach innovation. Through analysis of students' mathematical discourse, we observed how the use of different visual mediators influenced their task performance and interacted with moments of resistance. Furthermore, we identified a range of approaches adopted by students, linking elements of the de-ritualization process to instances of resistance and connecting them to various aspects of mathematical reasoning.

2.2 Appropriation process and resistance

This study is grounded in a socio-cultural perspective, which emphasizes that learning emerges through interaction with others (Vygotsky, 1978). Within this framework, the notion of *appropriation* is central and has been interpreted in various ways across the literature (Bakhtin, 1981; Harre, 1983; Moschkovich, 2004; Rogoff, 1995). Harre (1983) described appropriation as the process by which something external is adapted to meet the needs of the individual, while Bakhtin (1981) referred to it in linguistic terms—as making others' words one's own. Exposure to diverse perspectives within this collaborative context fosters deeper understanding and supports further development. Rogoff (1995), whose perspective is adopted in this study, defines participatory appropriation as the transformation that occurs through an individual's active engagement in an activity. This view stands in contrast to static models of learning, such as internalization, by emphasizing that understanding is shaped through participation in culturally and socially situated practices. In this perspective, the learner is not seen as separate from the activity but rather as embedded within it. Appropriation thus occurs dynamically as individuals interact with tools and meanings, shaping both their current thinking and their readiness for future participation.

According to Rogoff, understanding is transformed through participation, and this participation is the appropriation process. As students engage in a shared activity, they become increasingly prepared to navigate similar situations in the future. Communication plays a key role in this process (Moschkovich, 2004; Rogoff, 1995): the sharing of difficulties and negotiation of meaning between participants leads to mutual adjustments, the construction of shared understanding, and the pursuit of common goals. Engaging with different viewpoints in a collaborative setting enhances understanding and supports further development. The notion of appropriation as a process of transformation rather than mere imitation is also supported by the work of Moschkovich (2004). According to this research, appropriation occurs through joint activity and the co-construction of shared meaning. In particular, students appropriate new ways of interpreting tasks and develop the ability to apply these new interpretations and goals. This process positions students as active participants in appropriation process.

Unlike internalization that divides time into separate stages (past, present, and future), Rogoff's view sees development as a continuous and dynamic process. The past informs the present, which in turn shapes the future. Previous experiences are not simply left behind but are extended and transformed as they contribute to present activities and future trajectories. In this way, learning is seen as an evolving process in which new elements are added and existing understandings are refined.

In parallel, the concept of *resistance* plays a significant role in this process. In mathematics education, resistance has been conceptualized in different ways. Some scholars interpret it as students' reluctance to engage with mathematical learning due to motivational or belief-related barriers (Hannula, 2006; Eynde et al., 2006). Others understand it as the tendency to cling to familiar methods rather than adopting new strategies (Nunokawa and Kuwayama, 2004). In this research, however, we adopt Wertsch's (1998) interpretation, where resistance is described as a form of "friction" between an available mediational tool and its application in a specific context. From this perspective, resistance is not a mere obstacle but an integral part of the appropriation process. It manifests when learners encounter impediments in transferring and applying mathematical tools in unfamiliar situations. If such friction becomes too strong, the learner may abandon the use of the tool altogether, creating a discontinuity in the reasoning process. Thus, the challenge of overcoming resistance is seen as fundamental to learning and appropriation. Resistance can be understood as the tension that emerges in the space between the intended meaning of a mathematical tool and the way it is used in practice. It appears when students encounter impediments such as recalling formulas, carrying out calculations, or interpreting different realizations of the same mathematical object. In our study, resistance was particularly evident in the attempt to connect the meaning of asymptotic behavior with its application in specific contexts. For instance, some students could describe asymptotic behavior accurately in words, yet represented it graphically as an oblique growth of the function.

This dynamic has been observed in various studies, such as those by Carlsen (2010, 2018), which explored how students struggled to apply known mathematical concepts—such as geometric series or harmonic oscillation in new contexts. These studies highlighted the distinction between knowing a concept and being able to operationalize it appropriately, showing that resistance often emerges in the gap between familiarity and functional use.

In our study, resistance is identified within students' solving processes, especially when they are required to choose and coordinate between multiple mathematical tools. The tasks were deliberately designed to allow for different ways to solve, enabling the identification of moments where students encountered resistance. For example, when attempting to solve a task using algebraic decomposition despite the existence of a visible root in the graph, students often struggled—indicating resistance generated by their reliance on a preferred, but ineffective, tool. Similarly, another form of resistance was observed when students attempted to determine the number of straight lines intersecting a parabola at exactly one point. In such cases, students engaged both graphical and analytical strategies, but the inconsistency between the two led to confusion and tension. These situations illustrate how resistance can arise from the interaction between different realizations and tools, highlighting the complexity of appropriation in mathematical reasoning. In this research, a detailed analysis of the appropriation process is challenging, as the experiment was not conducted over an extended period but rather as a single episode. This limitation hinders a thorough observation of the dynamic appropriation process. Consequently, the study directs its attention to the notion of resistance within the solving process, exploring its connection to the Theory of Commognition and the mathematical reasoning.

2.3 Mathematical Reasoning

As highlighted by Jeannotte and Kieran (2017), there is no common definition of mathematical reasoning (MR) that encompasses all the aspects described in the literature. Their model was developed to address this gap by providing a conceptual framework that integrates the diverse features of mathematical reasoning. Grounded in Commognitive Theory, the model views mathematical reasoning as a discursive activity. From their review of the literature, Jeannotte and Kieran identified four key features of mathematical reasoning: (i) the activity/product dichotomy, (ii) the inferential nature of MR, (iii) the goals and functions of MR, and (iv) its structural and process aspects. They argued that, in combination with Commognitive Theory, these elements make it possible to define mathematical reasoning as "a process of communication with others or with oneself that allows for inferring mathematical utterances from other mathematical utterances" (Jeannotte and Kieran, 2017, p. 7). The inferential nature of mathematical reasoning is embedded in this definition. In relation to the first feature, within the commognitive perspective discourse is conceived both as an activity and as a product: each communicational act involves an activity aspect and a product aspect, which are further reflected in the processual and structural dimensions. The other features are likewise integrated into these dimensions. For example, the goals and functions of mathematical reasoning are captured within both the processual and structural aspects.

The processual approach refers to meta-discursive processes based on narratives about mathematical objects and the relationships between them. It involves processes such as identifying similarities and differences, validation, and exemplification, which in turn support the other categories. This approach includes the formulation of mathematical expressions, their manipulation, and computational work.

The structural approach, by contrast, concerns the description of mathematical objects and their relationships within a system formed through the combination of discursive elements. It includes the recalling of theorems and mathematical properties and is characterized by deductive, inductive, and abductive reasoning.

This research adopts the above definition of mathematical reasoning within a commognitive perspective and integrates the concept of resistance to conduct an in-depth analysis of the impediments students encounter while solving tasks. It further examines the relationship between resistance—both when overcome and when persisting—and the specific aspects of mathematical reasoning that prevail in the solving process.

2.4 Networking and Research Questions

In this study, Commognition theory was locally integrated with the concept of resistance. As emphasized by Bikner and colleagues (Bikner-Ahsbahs et al., 2014; Prediger et al., 2008), local integration refers to the development of a new theoretical framework by connecting specific elements from two distinct theories. This integration is often asymmetrical, which means that the contributing theories do not influence the framework equally, an aspect evident in the present work. Here, the concept of resistance was incorporated into Commognition Theory by interpreting resistance as the constraints of the precedent-search-space (PSS) that students must overcome to achieve innovation. The term "locally" is essential, as it highlights that this is not a full unification of theories, but rather a local integration, where one element of one theory was incorporated into the other. In this study, the notion of resistance—originally formulated within a socio-cultural perspective (Wertsch, 1998) and not specifically in mathematics education—was embedded into Commognition Theory to deepen the analysis of students' discursive activity.

The purpose of this research is to investigate how students engage in making sense around specific mathematical concept, the asymptotic behavior of functions, by observing their solving processes, with particular attention to the approaches they adopt. At first, the analysis focused on the use of different visual mediators and their relationship with the routines employed (see chapter 5). In a second phase, the attention shifted to how the use of various visual mediators influences students' approaches to solving tasks, not only in terms of the routines applied but also with respect to the emergence of resistance (see chapter 6). Furthermore, the study explores the relationship between routine use and resistance, aiming to understand whether overcoming resistance is associated with specific aspects of the de-ritualization process (see chapter 7). In addition, it was focused on the analysis of students' approaches to problems involving the asymptotic behavior of functions, exploring how resistance emerges in relation to different aspects of mathematical thinking (see chapter 8).

The research questions that guided this research are the following:

RQ1: In an analytical geometry task concerning the parabola and straight lines intersecting it, are there differences (both in terms of performance and the solution procedures chosen) if the same task is administered with the aid of different visual mediators? Does the use of different visual mediators in solving processes affect performance?

RQ2: How does the use of distinct visual mediators leading students to conflicting solutions relate to the manifestation of resistance during the solving of Calculus tasks?

RQ3: In what ways do the relationships between resistance and routine use manifest themselves when university students solve Calculus tasks?

RQ4: What approaches do the pairs of students use with respect to the given tasks about the asymptotic behavior of functions? In what ways do the students substantiate their reasoning with respect to the approaches used?

Chapter 3

Literature review

In this chapter, we present the state of the art related to studies that adopt Commognition Theory (Sfard, 2008), in order to situate the present research within this theoretical field, as well as studies that examine resistance as defined by Wertsch (1998). Specifically, Section 3.1.1 reviews research that applies Sfard's framework to investigate students' difficulties with the concept of function—particularly with asymptotic behavior—which is closely connected to the focus of this work. Adopting a sociocultural and participatory perspective, and following Sfard's view of learning as the development of discourse, Section 3.1.2 discusses studies on university students' mathematical discourse (at both undergraduate and master's levels) to outline the state of the art in commognitive research at the tertiary level. Given that this research focuses on routines and visual mediators, Section 3.1.3 examines studies on routines and de-ritualization processes (Lavie et al., 2019) across different school levels. In addition, Section 3.1.4 reviews research addressing students' difficulties in managing different realizations, offering a broader context for the issues investigated in this study. Section 3.1.5 provides an overview of work on mathematical reasoning from a commognitive perspective. Finally, Section 3.2 presents research on resistance, an element that was integrated into the main study of this thesis.

3.1 Studies conducted from a commognitive perspective

In 2016, Güçler (2016) pointed out that, despite extensive research on functions, only a small fraction of studies employed a socio-cultural framework (e.g., Nachlieli and Tabach, 2012). In recent years, Sfard's Commognition Theory(2008) has been applied in various studies to investigate students' mathematical discourse across different scholastic grades and domains (Biza, 2017; Biza, 2021; Biza and Nardi, 2023; Dell'Agnello et al., 2022; Fernández-León et al., 2021; Nachlieli and Tabach, 2022; Viirman and Nardi, 2019; Viola, 2023). A smaller subset of these studies specifically focused on students' discourse related to functions (e.g., Güçler, 2016; Mpofu and Mudaly, 2020). Additionally, some research explored university-level teachers' discourses (Viirman, 2014; Viirman, 2015) and the differences and interactions between the discourses of students and teachers (Güçler, 2013; Nardi et al., 2014).

Specifically, certain studies have examined specific elements of mathematical discourse, such as narratives (Biza, 2017; Biza and Nardi, 2023), routines(Fernández-León et al., 2021; Nachlieli and Tabach, 2022; Viirman and Nardi, 2019), and visual mediators (Ryve et al., 2013; Lu et al., 2023; Khalloufi-Mouha, 2021) and the process of de-ritualization(Lipper and Karavi, 2023; Nachlieli and Tabach, 2022; Viirman and Jacobsson, 2023). This research project contributes to the growing body of work on university students' mathematical discourse, particularly in the domain of calculus.

3.1.1 Students' difficulties in learning the concept of functions and asymptote

The concept of function is widely recognized as a very difficult subject for students to understand. Numerous studies have investigated the barriers students encounter (Eisenberg, 1991; Trujillo et al., 2023), such as challenges in interpreting and linking various representations of functions (Monk, 1994; Sfard, 1992; Tall, 1996; Hitt, 1998; Alajmi and Al-Kandari, 2022; Wilkie, 2024), examining the relationships between two variables (Monk, 1994). Additionally, research underscores the ongoing difficulties students face in comprehending the covariational nature of functions (Carlson et al., 2002; Weber and Thompson, 2014). According to Sfard (1992), a significant challenge lies in helping students develop the dual perspective of understanding functions as both dynamic processes and fixed objects. Existing research on the concept of asymptote has drawn on different theoretical perspectives, most notably the Anthropological Theory of the Didactic (ATD) (Katalenić et al., 2020;Katalenić et al., 2023). Within this framework, analyses of students' praxeologies reveal that functions and asymptotic behavior are often treated as two separate objects. Several explanations have been proposed for this phenomenon: first, in some educational contexts—such as Croatian mathematics education—there is a strong emphasis on algebraic manipulation with the equations of lines and parabolas. Second, prior studies suggest that students tend not to operate with notions at the object level (Güçler, 2016; Nachlieli and Tabach, 2012). In line with this, Katalenić and colleagues (2020) found that students viewed the function and its asymptote as distinct entities, rather than recognizing the asymptote as a property of the function. A third explanation is that students tend to adopt a pointwise perspective—evaluating specific points on the graph of the asymptote—instead of a global perspective, where the function and its asymptote are seen as approaching each other at infinity.

Other studies have examined asymptotes through different frameworks, such as Tall and Vinner's notion of concept image and concept definition (Arnal-Palacián and Pérez-Montilla, 2025), or by using graphing technologies to investigate asymptotes (Yerushalmy, 1997) or focusing on the horizontal asymptote (Kidron, 2011). From a socio-cultural perspective, some research has also addressed students' understandings of asymptotes. However, the topic remains underexplored, and most contributions focus narrowly on the concept of asymptote itself rather than on the broader exploration of asymptotic behavior of functions. As noted by Güçler (2016), research on functions conducted from a socio-cultural perspective is limited. Below, some of these studies are presented in a commognitive perspective.

Nachlieli and Tabach (2012) analyzed learning-teaching processes in the classroom, focusing on discourse about objects that students are not fully aware of. To explore discourse on the concept of function, the researchers observed videos of grade 7 students in their initial engagement with the topic. They studied the objectification of discourse on functions, highlighting the principles on which teachers' actions should be based to support students' objectification of mathematical discourse. The results showed that the initial input for meta-level learning was provided by the teacher, who guided students in participating in discourse on functions. Meanwhile, students developed a lower-level discourse on different realizations of functions, such as algebraic expressions, graphs, and tables. Importantly, students were able to engage in discourse on functions even without having full awareness of the concept. By the end of the study, despite still being at an early stage of the objectification process, students demonstrated sufficient ability to participate in discussions on different realizations of functions and to switch between them. This provides a solid foundation for future discourses.

In the same line of research, the work of Güçler (2016) is situated, studying a didactic approach focused on specific metarules as a potential support for students' learning of the concept of function. This study is part of a broader investigation aimed at exploring whether the use of specific metarules can lead to a shift in students' discourse on calculus. The research was conducted with pre-service and in-service teachers attending a calculus course. The researcher video-recorded three classroom sessions and collected students' reflections and interviews. The study aimed to analyze how students realized functions and how they incorporated them into definitions, particularly in terms of the metarules they used. Another objective was to examine students' underlying assumptions behind their definitions, as different students may have different realizations and assumptions. The results of this study revealed that explicitly addressing the metarules of discourse can support students' learning, as it helped them change the way they talk about functions and increased their awareness. However, the presence of metarules in discourse did not eliminate difficulties; rather, students became more aware of these difficulties and their nature. The approach used in the research not only helped participants reflect on their own understanding of functions but also encouraged them to think about how to teach this concept to their students. These studies examined students' discourse on functions, their awareness of this concept, and how it supports learning. The following studies, instead, focus more specifically on asymptotes and are therefore more closely related to the object of our research.

Mpofu and Pournara (2018) investigated learners' discourse on the hyperbola through task-based interviews with five Grade 10 learners from a township school, adapting the discourse profile specifically for this function. Their analysis showed that, although learners' graphs were generally correct, their language was largely colloquial and their interpretations mainly visual, indicating that their routines remained strongly ritualized. Moreover, while the tables of values produced by learners did not display asymptotic behavior, their graphs included two asymptotes. Interestingly, four learners spoke as if the hyperbola had only one asymptote and struggled to recognize the axes as asymptotes. In a subsequent study, Mpofu and Mudaly (2020) extended this line of research to the discourse of 112 Grade 11 students on the graphical, algebraic, and verbal realizations of the asymptotes of hyperbolic and exponential functions. Using students' written responses to a test, they developed a discourse profile for these functions, adapted from the arithmetic discourse profile. The findings indicated that students' mathematical discourse was still largely grounded in ritualized routines, with the de-ritualization process not yet complete. The authors argued that, to enhance mathematics learning, it is necessary to guide students toward more explorative approaches, for instance by encouraging them to justify their answers. Although these studies focus on asymptotes, they do not address asymptotic behavior nor do they involve university students. For this reason, the following sections present an overview of research conducted within the commognitive framework at the university level, even if not directly centred on this specific mathematical content.

3.1.2 University students' discourse in a commognitive perspective

In this section, we will present some examples of studies on the discourses of university students (both bachelor's and master's students) that investigate different aspects highlighted in Commognitive Theory, such as commognitive conflict, the elements that characterize mathematical discourse, and learning at both the meta-level and object-level.

In Biza's research (2017) students' discourses about the tangent line were explored. The results showed that students used different types of discourses (geometrical, analytical and algebraic) and multiple types of discourses are involved in the same justification. In addition, this study suggests that the type of task proposed to students can trigger a specific kind of discourse. For example, a request to calculate a formula is in the algebraic field and can trigger an algebraic and analytical responses, while the presence of a graph can trigger a geometric discourse.

Biza (2023) explored the manifestation of what the researcher calls the "discursive footprint", i.e., the discursive characteristics of how a mathematical concept is encountered by students across different domains during their schooling. Biza analyzed 182 written responses from undergraduate students and identified five themes in the manifestation of the "discursive footprint": *apparent replication of word use in different narratives*, *geometry-local hybrid discourse*, *endorsement* *of conflicting narratives*, *enrichment of familiar narratives with new words*, and *mathematical analysis as a subsuming discourse*. The footprint, as defined by Biza, encompasses various characteristics of mathematical discourse regarding tangents: the word use related to tangents, narratives concerning them, visual mediators associated with tangents, and the routines used to verify whether a line is tangent to a given curve.

Biza and Nardi (2023) explored the role of the use of online digital resources in the mathematical problem-solving, analyzing how this type of resources could influence their problem-solving activity. They asked to students to solve a task and to evaluate the solutions proposed by others. Specifically, they analyzed the mathematical and pedagogical discourse of students enrolling bachelor in Mathematics and Education.

In several studies, object-level and meta-level learning have been the focus of research, with attention on students' difficulties in these types of learning or their characteristics. For example, Ioannou investigated students' learning of group theory (2016; 2017; 2018). This research was part of a larger project involving second-year mathematics students enrolled in a compulsory group theory course. The studies reported in these works (Ioannou, 2016; Ioannou, 2017; Ioannou, 2018) were based on interviews with 13 voluntary students and an analysis of their coursework. The objectives of these studies were to examine students' understanding of the concept of subgroups Ioannou, 2016; the difficulties students encountered in object-level learning, meta-level learning, and commognitive conflicts in their solutions (Ioannou, 2018); and the occurrence of commognitive conflicts arising from the incommensurability of other mathematical concepts (Ioannou, 2017). In 2016, Ioannou found that students struggled with proving non-emptiness, as they did not perceive the necessity of demonstrating it. Ioannou (2016) further emphasized that commognitive conflicts arose during the transition from the discourse of set theory in secondary school to the discourse of group theory at university. The 2018 study revealed that difficulties at the object-level learning stage could be characterized by three key elements: confusion between groups and sets, reflected in a commognitive conflict regarding algebraic structures; the use of visual mediators that led to difficulties in transitioning from previous discourses on complex analysis or set theory to new discourses on group theory; the influence of problem context on students' application of routines. This study also identified a connection between object-level learning and meta-level learning, as objectification derived from other discourses affected the endorsement of metarules guiding routines. Additionally, the research found evidence of commognitive conflicts: the first emerging from the introduction of new terms in group theory at the object-level learning stage, and the second concerning metarules, specifically the omission of a proof of non-emptiness. The latter appears to stem from prior discourses in secondary school, where sets are typically assumed to be non-empty as they are defined in relation to their elements.

Weingarden's (2024) research also investigated meta-level and object-level learning, focusing on their features. Specifically, Weingarden conducted a study on the discourses of 15 pre-service secondary teachers attending an advanced university course for secondary mathematics pre-service teachers, focused on mathematical connections. During the course, the lecturer introduced and utilized the realization tree mediator in an assignment. The study aims to investigate the types of mathematical connections pre-service teachers employed in constructing the realization tree mediator, using the "saming narratives", defined as "including the so-called translation between one realization and another"(Weingarden and Heyd-Metzuyanim, 2023), and the characteristics of meta-learning and object-learning in the saming narratives produced by students. The results of the study suggest that different mathematical connections are associated with the characteristics of object-learning and meta-learning. Other studies explored the commognitive conflict in the transition from high school to university. For example, Thoma and Nardi (2018) analyzed the examination scripts of first-year undergraduate students in a mathematics course. The researchers investigated unresolved commognitive conflict in mathematical discourse during the transition from school to university. Thoma and Nardi identified four manifestations of unresolved commognitive conflict in students' discourse: two related to differences between the discourse used in school and that used at university, and two associated with changes in discourse within the university setting. The results of this study suggest that, despite lecturers guiding students in switching from one discourse to another, final exam scripts revealed unresolved commognitive conflicts in transitions between discourses (either from school to university or within the university itself). In 2019, Thoma and Nardi presented a study on the responses of twenty-two students in the examination on Sets, Numbers, and Probability course, focusing on the optional questions about the surjectivity and injectivity of functions. The results highlighted students' difficulty in recalling the definition of injectivity, especially in their word use and visual mediators. The differences between the discourses and practices adopted in secondary school and university seem to influence students' actions, such as the lack of the closing condition in substantiation routines.

The transition from secondary school to university was also the focus of research conducted by Schuler-Mayer (2019). The researcher studied the transition between secondary and tertiary education. Specifically, they conducted research involving students in the penultimate year of secondary school. The study took place in a transition course attended by students from different classrooms who were interested in enrolling in STEM programs at university. The transition course incorporated discourses and practices typical of university mathematics, such as defining and proving, and introduced new algebraic structures, aiming to create a bridge between familiar school discourses and university-level discourses. Students encountered commognitive conflicts in the parallel use of these discourses, as they initially perceived them as part of the same discourse.

Some researchers have focused on the interactions between the lecturer and students, highlighting the difficulties arising from different discourses. In 2013, Güçler explored students' difficulties in learning the concept of a limit, with particular attention to the discourses of both students and the lecturer. The research was conducted in a classroom of undergraduate students at the introductory level of calculus. Researchers video-recorded lessons on the concept of a limit, and afterward, students were interviewed individually. The discourses of both students and the instructor were analyzed by identifying the elements of mathematical discourse according to Sfard: visual mediators, word use, routines, and narratives. The results revealed that the instructor's use of word choices, visual mediators, and metarules was coherent. However, although students used the same word choices, visual mediators, and metarules to endorse narratives, their usage was inconsistent. This issue also emerged from students' statements, as they believed their approach aligned with that of the lecturer. This suggests a miscommunication between students and the lecturer. The interaction between students and the lecturer was also presented in the research by Nardi and colleagues, in one of the studies explained in their paper (2014). Nardi and colleagues analyzed shifts in discourse during the initial study of Calculus across three different studies. In their paper, they presented three research cases exploring the discourses of university students and teachers, as well as the interactions between them. In the first case, they analyzed discourse in problem-solving activities within small groups. The results showed that word use and visual mediators contributed to the validation and approval of narratives, aligning with the definition of effective communication. The second study examined university lectures on the construction of the concept of function, focusing on explorations and “saming”. In the third case, the interaction between the lecturer and students was investigated, analyzing the presence of commognitive conflicts, with particular attention to rituals, explorations, and deeds. The key difference between the lecturer and students lay in the production of narratives and the approval process. Our research is situated within university-level studies that adopt the commognitive framework. At the same time, related work has concentrated on other components of commognition—such as routines and the process of de-ritualization. The following paragraphs therefore outline the key contributions of studies that have examined these particular aspects.

3.1.3 Routines and de-ritualization process

In this paragraph, we will summarize some studies that have investigated the presence of routines and the de-ritualization process in students' discourse (Viirman and Nardi, 2019; Fernández-León et al., 2021), and how task design can promote the de-ritualization process, also analyzing the presence of hybrids between rituals and explorations (Viirman and Nardi, 2019; Christiansen et al., 2023).

The research conducted by Fernández-León and colleagues (2021) fits into this line, as they studied the mathematical discourse of undergraduate students, focusing on the routines that characterize their discourse when defining and describing concepts in 2D and 3D geometry. The results showed that students used different discourses when defining in 2D and 3D geometry, revealing that discourses about 3D geometry are not a generalization of those about 2D geometry. In 2024 researchers (Toscano et al., 2024) conducted a study with pre-service secondary teachers, master students in Secondary Education, investigating how construct and select definitions in the task involving prototypical and non-prototypical solids. Findings releved differences between selection and construction of definition about prototypical and non-prototypical solids. In addition, the results revealed the occurrence of two commognitive conflicts involving non-prototypical solids.

In 2019, Nachlieli and Tabach introduced two new terms, developing a model to interpret teachers' actions in terms of rituals and explorations. This model framed other studies, such as Christiansen and colleagues' research (2023), which investigated the presence of hybrids between rituals and explorations. Nachlieli and Tabach conducted a research that introduced two new notions: “ritual-enabling Opportunities To Learn” (OTLs) and “exploration-requiring OTLs” to analyze teaching practices through the lenses of ritual and exploration. The definition of Opportunities To Learn used in their study was taken from the National Research Council: “circumstances that allow students to engage in and spend time on academic tasks. . . ” (Findell et al., 2001, p. 333). Their definitions focused on the teacher’s actions, which should provide tasks that can be solved using previously learned procedures but still serve as a learning opportunity for students (ritual-enabling OTLs). Additionally, they described teacher actions aimed at providing tasks that cannot be solved by applying prior procedures and instead require the production of a new narrative (exploration-requiring OTLs). In their study, the researchers observed 11 lessons from grade 8 classes in different parts of the world (Australia, USA and Hong Kong). The researchers stated that ritual-enabling OTLs could be seen as a necessary initial phase for developing an explorative approach and constructing a new discourse. In accordance with the de-ritualization process, teaching actions should begin with ritual-enabling OTLs and gradually progress toward exploration-requiring OTLs, creating a continuum of opportunities for students. Both ritual-enabling and exploration-requiring OTLs should be integrated into teaching practices.

In 2022, Nachlieli and Tabach conducted a study to analyze classroom learning, viewing it as routinization in accordance with the conceptualization of Lavie and colleagues (2019). The aim of the study was to conceptualize learning and explore how it can be operationalized. The study was conducted with 18 prospective mathematics teachers during a course at a college of education. The researchers analyzed both the lessons and the students’ work. They examined students’ learning using the characteristics of the de-ritualization process (flexibility, bondedness, substantiality, objectification, applicability, agentivity). This perspective on learning guides students toward meta-level and object-level learning. The former requires mediation by an expert, such as the teacher, while the latter can be achieved through group activities.

In accordance with this perspective, Christiansen and colleagues 2023 proposed a model that incorporates the notions of ritual-enabling and exploration-requiring OTLs (Nachlieli and Tabach, 2019), alongside the concept of de-ritualization (Lavie et al., 2019), in order to examine hybrid forms of exploratory and ritualized routines. They hypothesized that hybrids of ritual and exploratory routines possess characteristics from both categories. The researchers observed and analyzed lessons from different scholastic grades, identifying the presence of hybrid forms of rituals and explorations. The lessons in different settings shared a common starting point: encouraging exploratory activities among students. However, the guidance provided by teachers varied, influencing the students' agentivity and substantiation. The researchers analyzed the teachers' discourse to observe the existence of hybrids between rituals and explorations from the teachers' perspective. The results showed evidence of these hybrid forms in the three lessons analyzed. According to this model, hybrid forms depend on students' agentivity and substantiation, based on varying levels of engagement with these characteristics. This research aligns with the findings of Viirman and Nardi, where rituals and explorations are not perceived as two extremes but as part of a continuum. In fact, Viirman and Nardi examined the evolution of mathematical discourse in their studies, focusing specifically on the interplay between ritualized and exploratory approaches. In particular, they analyzed the discourse generated by 12 undergraduate biology students during modeling activities (2019). They investigated the presence of exploratory and ritualized approaches in the students' engagement with the proposed activities. The researchers' findings revealed evidence of commognitive conflict in the students' work, both within the mathematical domain (intra-mathematical) and in its interpretation within a biological context (extra-mathematical). Furthermore, the results revealed that, initially, during the first session, students adopted a more ritualistic approach, while in the final session they were more confident and employed an exploratory approach. The analysis highlighted a fluid transition between ritualized and exploratory approaches, rejecting a simplistic dichotomy. In the study conducted by Tetaj and Viirman (2019), the findings confirm the results obtained by Viirman and Nardi (2019). Undergraduate biology students were involved in solving a biology-related task. The results showed that, despite the students' exploratory approach, they applied the routines in a ritualized manner. In their 2021 study (Viirman and Nardi, 2021), the results highlighted meta-level learning guided by a de-ritualization process of graphing and modeling routines, influenced by narratives produced about the modeling and by prior experiences related to the interpretation and construction of graphs. This study emphasized the necessity of designing modeling activities to connect previous routines and enable their application in other contexts.

Lipper (2023) studied the teaching characteristics that support the de-ritualization process in students' understanding of proof. 40 teaching episodes on theorems and proofs were observed and analyzed. The features of teachers' mathematical discourse created a connection between the theorem and the initiation of formal proving. An exploratory approach by students was encouraged through teachers' emphasis at the beginning of the proof process. Teachers promoted an exploratory approach and supported the de-ritualization process, which is complex for students and requires time for full development, by making students aware of their actions. In a recent study on the de-ritualization process, Heyd-Metzuyanim and colleagues (2025) introduced the notion of regulative routines within commognitive theory to account for students' strategies when engaging with mathematical tasks. Incorporating this construct into the commognitive framework enables a more refined analysis of solving and de-ritualization processes. The study, conducted with a 7th-grade student working on an algebraic problem, employed a Discourse Mapping Diagram to examine regulative routines in relation to the task's structure. The results demonstrated that as students progressed from ritualized to more exploratory routines, the number of regulative routines increased, accompanied by greater flexibility, applicability, and boundedness.

This study is situated within the same line of research examining different types of routines and the process of de-ritualization, analysing how these elements appear in students' discourse and how routines evolve. Another central component of this work concerns the role of visual mediators, which will be reviewed in the following paragraph.

3.1.4 Students' difficulties in management of different realizations and visualization and studies on visual mediators

Numerous studies have examined the concept of functions, with a particular emphasis on their various representations (Hitt, 1998; Gagatsis and Shiakalli, 2004; Elia et al., 2007), highlighting the challenges students face in transitioning between them. Kop and colleagues (2015; 2017) investigated how mathematics students classify functions, revealing their difficulties in linking algebraic and graphical forms. Similarly, Montenegro and colleagues (2018) observed that although visual representations contain elements that could support students' understanding, they are not always interpreted effectively. Stylianou and Silver (2004) explored the role of visual representations in problem-solving, analyzing the differences and similarities between novices and experts. Lesh and colleagues underscore the significance of representing mathematical concepts for achieving a deeper comprehension. They argue that some representations can highlight specific properties while concealing others (Lesh et al., 1987). A substantial body of research has examined the notion of function, with particular attention to the role of its different representations (Hitt, 1998; Gagatsis and Shiakalli, 2004; Elia et al., 2007), emphasizing the difficulties students encounter when shifting between them. This idea is exemplified in the work of Zazkis and Liljedahl (2004), who illustrate how multiple representations are essential for understanding prime numbers.

Within Sfard's (2008) Commognition Theory, visualization is recognized as a fundamental aspect of learning mathematical objects. This theory identifies visual mediators, such as graphical realizations, as essential tools for recognizing and conveying mathematical concepts in discourse. Unlike everyday communication, which relies on images of physical objects, mathematical discourse employs specialized symbolic artifacts designed for precise mathematical expression. Specific studies have investigated the role of visual mediators in group communication (Ryve et al., 2013) and their impact on students' performance (Gambini et al., 2024; Viola, 2023). Results suggest that the presence and application of different visual mediators influence students' success and their use of routines in problem-solving. Specifically, the 2023 study investigated the role of visual mediators in students' performance and the routines they employed. The participants were high school students in grades 11 and 13, who were asked to determine the number of tangents to a given circumference. The same mathematical task was presented in two different modalities: one with a graphical realization and the other with an algebraic realization. This design enabled the analysis of the strategies activated by students depending on the type of stimulus provided. The data were analyzed through the theoretical lenses of representations management and Sfard's commognitive theory. These frameworks made it possible to examine students' strategies in depth and to reflect on how stimulus realizations and teaching practices influenced their choice of strategies, with particular attention to the role of graphical realization and the routines activated.

Three distinct strategies for solving the task were identified; however, their distribution across the two cohorts was not homogeneous. The routines adopted by students also appeared to be shaped by teaching practices, as notable differences emerged between the two groups. When the task was presented in algebraic form, students tended to rely on familiar routines (in the sense of relying on approaches that students typically employ in comparable situations). Conversely, when a graph was included in the problem statement, they appeared more confident and more inclined to adopt alternative approaches to solving the task. Dell'Agnello and colleagues (2022) demonstrated that employing visual mediators during problem-solving aids in recognizing the roots of polynomials. Meanwhile, Nachlieli and Tabach (2012) explored how the teaching and learning of mathematical concepts unfold in the classroom, paying particular attention to different realizations of functions.

Some studies have explored group communication and the role of visual mediators in establishing effective communication. For example, Ryve and colleagues (2013) studied the role of visual mediators in student communication within groups across different settings. To this end, the researchers worked with students aged 12–13 and first-year undergraduate mathematics students. In the first case, the proposed task involved a game activity related to the concept of probability, while in the second case, the task focused on the concepts of function, derivative, and proof by induction. The study highlights how visual mediators and technical terms influence communication and contribute to making it effective. The essential role of visual mediators in effective communication appears to generate a conflict: on one hand, students attempt to create and sustain communication mediated by visual mediators; on the other hand, group dynamics reveal a motivation to employ different realizations. These results suggest that when communication is heavily mediated by a specific visual mediator, there may be a risk of neglecting or avoiding the use of other realizations.

Another example is provided by the research of Khalloufi-Mouha (2021), who analyzed the function of visual mediators and mathematical vocabulary in group discussions to establish effective communication. This concept was introduced by Sfard and Kieran (2001): "The different utterances of the speakers evoke responses that are in tune with the speakers' meta-discursive expectations" (p. 49). The goal of the researcher was to operationalize the concept of the focus project. To achieve this, the study investigated how the use of visual mediators and mathematical vocabulary helped students express their personal focus project and develop a shared focus project, thus characterizing effective communication. To analyze this, the researcher video-recorded six groups, each composed of three first-year biology students. The students were required to solve an applied mathematics problem that involved the use of a mathematical model. To operationalize the concept of the focal project, the researcher highlighted its observable properties.

The results of this study aligned with the findings of Ryve and colleagues (2013). In fact, it emerged that students encountered difficulties in establishing effective communication, despite the importance of group work. This may be due to the use of the same visual mediator in different ways. While using the same visual mediator helps connect projects, it does not guarantee effective communication. The researcher concludes that effective communication is possible in two situations: first, when students assess and adjust their personal focal project by comparing it with others while working in a common context; second, when one student proposes their personal focal project, and the others accept it.

Other studies have explored the different realizations of mathematical objects, their use by students or teachers, and their presence in textbooks. For example, Haghjoo and colleagues (2023) examined the written discourse in textbooks, with particular attention to "primary objects", concrete "discursive objects", and realizations related to the concept of the derivative. The aim of this study was to investigate how primary objects and concrete discursive objects are used in textbooks and what opportunities they provide for students to apply an explorative approach in mathematical discourse. The researchers analyzed 14 mathematics textbooks, identifying primary objects (images in the textbooks) and concrete discursive objects (images, texts, as well as examples and exercises). They also created a realization tree to identify the different realizations of the derivative used in the textbooks. The results of this study highlighted that in the latest versions of Iranian textbooks, greater importance was given to the use of different realizations. Additionally, the latest versions included more primary objects and concrete discursive objects.

3.1.5 Mathematical Reasoning in the Commognitive Framework

Another aspect analysed in this study is the substantiation of discourse, particularly in relation to different components of mathematical reasoning. The term mathematical reasoning has been widely used without a shared definition, and different aspects have been emphasized by various researchers (Yackel and Hanna, 2003; Jeannotte and Kieran, 2017). Some studies (Arsac, 1996; Cabassut, 2005) have highlighted its dual nature—structural and processual—others have stressed its role in producing knowledge (Arsac, 1996; Lithner, 2008), while still others have examined different forms of reasoning, such as deductive, inductive, or abductive, often privileging one aspect over the others (Duval, 1995; Reid, 2003; Rivera, 2008; Meyer, 2010). The structural aspect has been characterized through activities such as conjecturing (Stacey et al., 1982; Stylianides, 2008), generalizing (Stylianides, 2008), exemplifying (Stacey et al., 1982), proving (Duval, 1995; Stylianides, 2008), arguing (Pedemonte, 2002), and convincing (Cabassut, 2005). By contrast, the process aspect has not been deeply explored in the literature from an epistemological perspective.

In this area, Jeannotte and Kieran 2017 developed a model that conceptualizes mathematical reasoning by incorporating both structural and processual aspects. This model, described in detail in Section 2.3, was elaborated within the commognitive framework. Building on this line of research, other scholars have adopted the model proposed by Jeannotte and Kieran (2017). For example, Valenta and colleagues (Valenta et al., 2024) developed a framework for identifying object-level narratives and analyzing the processes involved in their construction and substantiation. Their framework distinguishes between constructions based on procedures and those based on properties and relations, while substantiation is divided into implicit (deriving from the construction itself) and explicit (relying on procedures or on procedures combined with relations).

This framework was later refined following its implementation and analysis in Norwegian mathematics classrooms, where new categories for both construction and substantiation were added. The data showed that the framework can be a valuable tool for recognizing typical teaching situations, offering insights for both teachers and researchers.

3.2 Studies addressing the appropriation process and resistance

The concepts of appropriation and resistance originate in a sociocultural perspective and were not initially developed within mathematics education. Several studies have explored the appropriation process (Gruver and Hawthorne, 2023; Lai and White, 2012; Lai and White, 2014; Lobato et al., 2023), but not many have studied resistance (Carlsen, 2010; Carlsen, 2018), especially at the university level, where there are no studies on resistance. In this section, we will present some studies that have analyzed the appropriation process as a process involving active participation and interaction with others (Gruver and Hawthorne, 2023; Lai and White, 2012; Lai and White, 2014; Lobato et al., 2023), with particular attention to Carlsen's studies (2010; 2018), which include the element of resistance.

For example, Gruver and Hawthorne (2023) analyzed the types of classroom interactions when students are involved in the mathematical practice of imposing structure. They video-recorded classroom interactions for three days, focusing on the actions of teachers and the work of small groups. Moreover, seven students were interviewed after the last day. They identified three types of classroom interactions: engaging students in the practice, providing different realizations of the practice, and reflecting on different instantiations of the practice. The results suggested that teachers should reflect on how to guide students toward appropriate mathematical practices. While, Lai and White, in their studies (2012;2014), explored the interactions of small groups that characterize the appropriation process in geometric reasoning. In 2012, they conducted research on the learning of quadrilaterals, investigating whether interaction in small groups can support learning about geometric shapes and their relationships, as well as engagement in manipulation within a dynamic geometry environment. In 2014, they focused on collaborative learning and how collaborative tasks can support students' learning. The results of these studies suggest that students have difficulty appropriating others' approaches when they do not share a common focus of attention (Lai and White, 2014). Furthermore, the appropriation of peers' ways of discussing shapes and sharing technological features is a critical element of the learning process (Lai and White, 2012).

The studies by Carlsen (2010; 2018) are particularly relevant to this work, as they examine the processes of appropriation and resistance.

In 2010, Carlsen studied the appropriation process regarding geometric series. The researcher observed classroom activities in an upper secondary school, focusing on a group of six students during 13 group sessions, where they tackled tasks related to geometric series. Specifically, the aim of this research was to analyze how students appropriated the concept of geometric series in collaborative small-group problem-solving. The results of this study showed that resistance in the appropriation process manifested in students' difficulties managing different elements of the formulas and in their insecurities. Moreover, resistance emerged through conflicting arguments in their reasoning. Over the course of the sessions, students' participation evolved, becoming more collaborative and leading to a shared understanding. Additionally, the construction of shared meanings for both the concept and its theoretical components suggests that students gained mathematical knowledge. Another example of research analyzing resistance is Carlsen's 2018 study. In this research, Carlsen explored students' mathematical reasoning regarding the concept of sinusoidal functions. He video-recorded a small group of four upper secondary school students engaged in collaborative problem-solving over five weeks. The results indicate that students' mathematical reasoning was characterized by resistance in estimating the parameters of the algebraic expression of a sinusoidal function. Resistance emerged when students struggled to recall terms or displayed hesitation and a lack of accuracy while carrying out the task. The findings of this research highlight how resistance shapes mathematical reasoning and influences both its structural and procedural aspects. In this study, the notion of resistance is integrated into the commognitive framework to address a gap in the literature: understanding what happens when students attempt to modify or shift their routines, how they respond to the resulting impediments, and what may cause these difficulties.

Chapter 4

Methodology

4.1 Preliminary study on the concept of asymptotic behavior

The preliminary phase of the research adopted a mixed-methods approach (Johnson and Onwuegbuzie, 2004), incorporating a task with closed questions to analyze both students' responses and the methods they employed. This segment of the study concentrated on the role of visual mediators and routines, as conceptualized within the framework of Commognition Theory (Sfard, 2008). To investigate the impact of visual mediators, we developed a task comprising three multiple-choice items, presented in two distinct formats (Type 1 and Type 2; see Figures 4.1 and 4.2). In the Type 1 version, students were given the algebraic equation of a parabola, whereas in Type 2, they were presented with a graphical realization of the same function. Although the input differed, the content and structure of the questions remained identical, requiring students to determine the number of straight lines passing through given points that intersect the parabola.

The task involved three specific situations: one in which the point was external to the parabola, where three distinct lines intersect it at exactly one point; another in which the point lay on the parabola, with two distinct lines intersecting it at exactly one point; and a third case, which we referred to as the internal point, where no tangent lines could be drawn, and only the vertical line passing through the point intersected the parabola at a single point. Each question provided

Type 1

Consider the parabola of equation $y = x^2 + 1$.

How many lines through $A(0,\ 0)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

Justify your answer.

How many lines through $B(0,\ 1)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

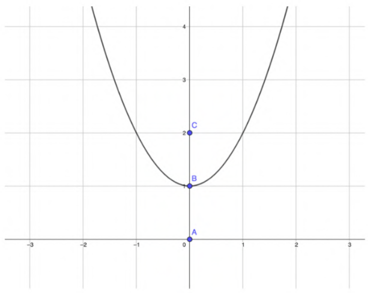
And through $C(0,\ 2)$*?*

- 0
- 1
- 2
- 3

Justify your answer.

Figure 4.1: First version of the task, Type 1

Type 2



Consider the parabola in the figure

How many lines through $A(0,\ 0)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

How many lines through $B(0,\ 1)$ *have a single intersection with the given parabola?*

- 0
- 1
- 2
- 3

And through $C(0,\ 2)$*?*

- 0
- 1
- 2
- 3

Justify your answer.

Figure 4.2: First version of the task, Type 2

four options—0, 1, 2, or 3—and students were asked to justify their chosen answers in writing. The participants included 48 first-year engineering students, and the study was conducted at the conclusion of their first-semester Calculus course, during which they had studied asymptotic behavior, functions, and relationships between functions. To compare the effects of different realizations on students' reasoning, participants were randomly assigned to either the Type 1 or Type 2 group. Their multiple-choice responses and written explanations were subsequently analyzed to explore whether and how the type of visual mediator influenced students' approaches during the solving process.

In this preliminary phase, the data consisted of students' written responses and accompanying justifications. The analysis focused on aspects of mathematical discourse, particularly the use of visual mediators and routines. Students' responses were examined through a qualitative content analysis approach (Mayring, 2014). We employed a concept-driven coding framework following Kuckartz (2019), organizing the students' solution paths into two primary categories: Guided by Graph (GG) and Guided by Rituals (GR), described in the next chapter 5.

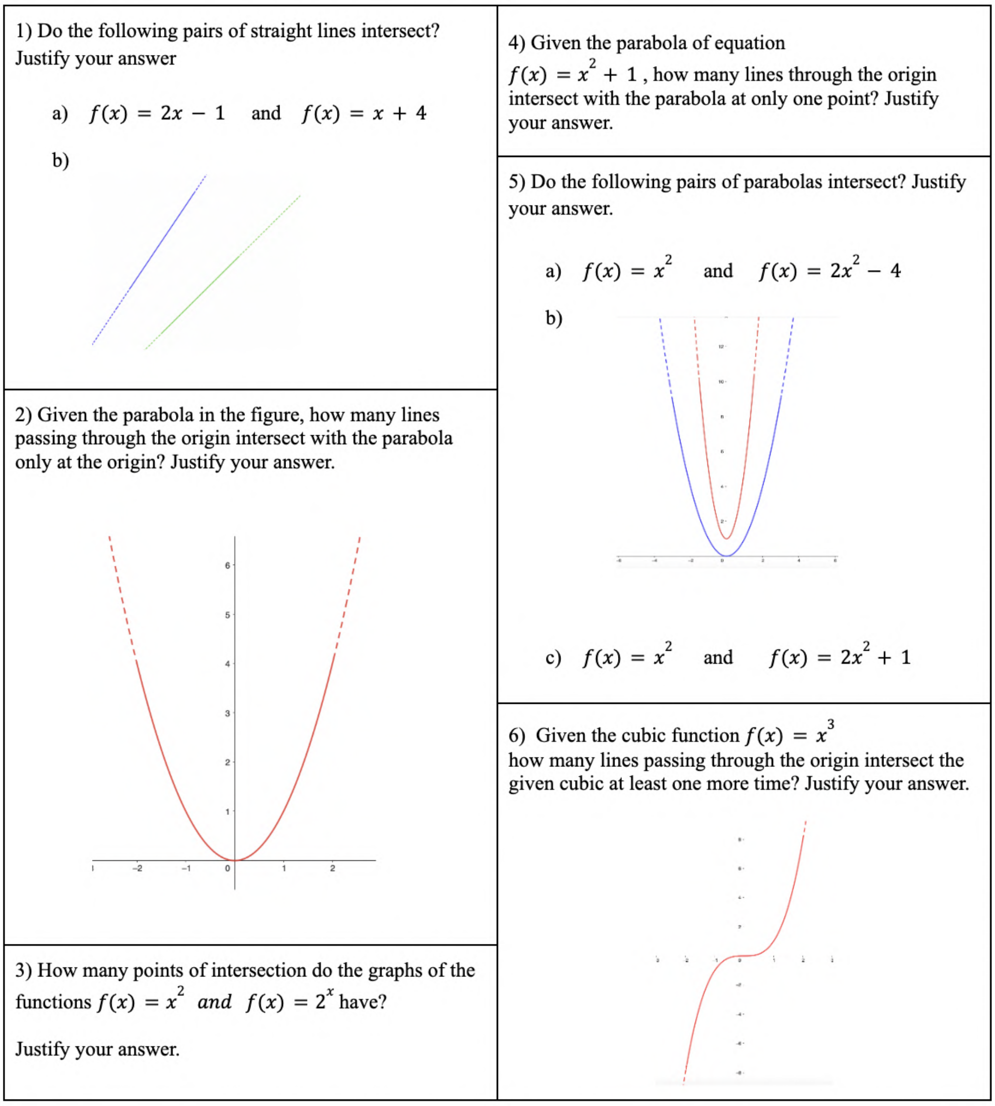
4.2 Main study on the concept of asymptotic behavior

In the main study of this research, a different methodology was adopted. Specifically, in this phase, we employed a qualitative approach rather than the mixed-method design used in the preliminary phase in order to examine students' solving processes more deeply, with particular attention to the mathematical discourse involved. Moreover, both the number of students involved and the type of data collected differed, reflecting the limitations and developments that emerged from the preliminary phase, which are explained in detail in section 5.3. To this end, task-based interviews were conducted, allowing us to follow the full development of each student's approach. The interviews were carried out in pairs, in accordance with a socio-cultural perspective that recognizes the importance of context and interaction in shaping how meaning is made during the activity. In this part of the research, the focus was on examining how students encounter and address resistance during mathematical activity, with particular attention to the role of visual mediators and the enactment of routines throughout the solving process. The analysis concentrated on how specific features of the de-ritualization process are managed and on the ways in which students' approaches are connected to different aspects of mathematical reasoning.

To support this investigation, a set of six unfamiliar tasks was developed, each involving different algebraic, graphical, or symbolic realizations of mathematical objects. These tasks diverged from typical exercises found in high school or university textbooks and were designed to elicit conditions in which familiar approaches might become ineffective—thereby allowing resistance to emerge and be observed.

To examine the solving processes of students, we used task-based interviews inspired by Goldin's approach (2000). The students worked in pairs to complete the tasks, encouraging interaction and shared reasoning. These sessions, along with follow-up interviews, were video recorded. During the interviews, students were asked to comment on what they had done while completing the tasks. This made it possible to observe resistance episodes and how students reacted to them. The approach is consistent with a socio-cultural perspective, where learning is seen as a participative and interactive activity.

Each task was designed to allow for multiple solution approaches and to encourage the emergence of resistance by proposing new contexts in which students had to apply familiar



1) Do the following pairs of straight lines intersect? Justify your answer

a) $f(x) = 2x - 1$ and $f(x) = x + 4$

b)

2) Given the parabola in the figure, how many lines passing through the origin intersect with the parabola only at the origin? Justify your answer.

3) How many points of intersection do the graphs of the functions $f(x) = x^2$ *and* $f(x) = 2^x$ have?

Justify your answer.

4) Given the parabola of equation $f(x) = x^2 + 1$, how many lines through the origin intersect with the parabola at only one point? Justify your answer.

5) Do the following pairs of parabolas intersect? Justify your answer.

a) $f(x) = x^2$ and $f(x) = 2x^2 - 4$

b)

c) $f(x) = x^2$ and $f(x) = 2x^2 + 1$

6) Given the cubic function $f(x) = x^3$ how many lines passing through the origin intersect the given cubic at least one more time? Justify your answer.

Figure 4.3: Tasks

tools. Specifically, the tasks focused on analyzing the asymptotic behavior of functions—whether between polynomials of the same or different degrees, or between polynomial and exponential functions—without explicitly referring to this concept, but rather by asking students whether the functions intersect and how many points of intersection they have.

Task 1 required students to determine whether two straight lines intersect, presented through two realizations: algebraic (1a) and graphical (1b). Students adopted different approaches, such as comparing slopes, solving a system of equations, or extending the graphical realizations. While reasoning about slopes was sufficient, the other approaches—though more elaborate—also led to correct conclusions, namely that the two lines intersect.

Tasks 2 and 4 asked students to establish how many straight lines intersect a parabola at exactly one point, again combining algebraic and graphical realizations. Students could either solve a system involving the parabola and a family of lines through the origin, or reason in terms of asymptotic behavior. However, the algebraic method alone identified only tangent lines, not vertical ones, requiring additional reasoning. These tasks represent an evolution of those designed for the preliminary phase. In Task 2, a graphical realization was provided with a point lying on the parabola, resulting in two straight lines intersecting it at exactly one point. In contrast, Task 4 presented only an algebraic realization with an external point, where three straight lines intersect the parabola at exactly one point.

Task 3 involved determining the number of intersection points between a parabola and an exponential function. Here, solving a system was not feasible, so students had to reason about the behavior of the functions at infinity or rely on graphical constructions. By using these tools, they could identify three points of intersection between the two graphs.

Task 5, which presented two parabolas in both algebraic (items a and c) and graphical form (item a), asked students to determine whether the curves intersect. This could be addressed either by solving a system of equations or by reasoning about their growth at infinity, with both approaches leading to correct solutions: the curves intersect in item a, while they do not intersect in items b and c.

Task 6 examined how many lines intersect a cubic function at more than one point. As in previous tasks, students could either set up a system with the cubic and a sheaf of lines, or reason graphically and in terms of asymptotic behavior. Both methods proved effective, showing that infinitely many lines intersect the cubic function at more than one point.

The tasks were open-ended and presented through different realizations of the same mathematical objects. This allowed us to identify when resistance occurred and how students handled it, particularly in relation to the approach used. The comparison between different visual mediators was central to this investigation, especially in observing how certain realizations facilitated the solving process, while others seemed to trigger hesitation or confusion.

By observing not only the written outcomes but also the verbal exchanges between students, the study aims to better understand how meaning is constructed during mathematical activity. The findings provide insight into how students move between familiar and unfamiliar contexts, and how they confront the challenges involved in adapting to new realizations and approaches.

Students were also asked whether they had previously encountered tasks of a similar nature; their responses indicated a general unfamiliarity with the proposed tasks.

In 2023 Viirman and Jacobsson presented strategies for designing tasks aimed at promoting the de-ritualization process. The researchers provided examples of tasks designed according to the characteristics of de-ritualization. Specifically, giving students tasks that allow for multiple solution methods supports flexibility. To foster bondedness, students can be asked to describe the role of each step in a procedure, analyzing its purpose. Applicability can be encouraged by presenting additional tasks that can be solved using the chosen procedure or by designing new problems that require its application. Agentivity is supported by engaging with other characteristics or asking students to justify their solutions. Objectification can be promoted by using tasks that trigger an objectification process, such as creating new narratives with different realizations or procedures. To support substantiability, students can be asked to analyze the applied procedure and provide justifications for each step.

The design of the tasks aimed to investigate specific aspects of the de-ritualization process, particularly concerning the nature of students' engagement with routines and whether their approaches were more ritualistic or more explorative. Each task was deliberately constructed to allow for multiple modes of resolution—for instance, through the manipulation of algebraic expressions involving functions or through the interpretation of asymptotic behavior—with the goal of fostering flexibility in mathematical activity, as suggested by Viirman and Jacobsson. In addition, the research sought to facilitate objectification by incorporating different realizations of the same mathematical object within the tasks or, during interviews, by encouraging students to reflect on alternative ways of engaging with the proposed mathematical activity, in line with the framework outlined by Viirman and Jacobsson (2023).

A central focus of the study was the identification and analysis of resistance, along with the ways in which students responded to it. Particular attention was devoted to whether resistance was successfully overcome and how these episodes were connected to the use of routines. The tasks were therefore designed to make visible potential points of tension or discontinuity in students' engagement and to examine how these were related to the process of de-ritualization and its possible transformation.

Students' diverse academic backgrounds—specifically, the distinction between those enrolled in engineering and those in mathematics programs—were taken into account during the analysis, recognizing that prior educational experiences may influence how they approach the proposed tasks. For this reason, participant selection intentionally included two distinct cohorts, enabling a comparative investigation of the approaches adopted by each group. A total of 19 interviews were conducted: 11 pairs of engineering students and 8 pairs of mathematics students, all from two Italian universities. All participants had successfully completed at least the first course in Calculus, allowing for meaningful comparisons among students with differing degrees of mathematical familiarity. The data corpus included written responses, video recordings, and transcriptions of the interviews.

The analytical framework was grounded in commognition theory (Sfard, 2008), with particular attention to the identification of narrative components, routine enactments, and the use of visual mediators. The notion of resistance was interpreted following Wertsch's (1998) conceptualization—as a moment of tension or disruption that learners must navigate and overcome to advance in their mathematical activity.

Specifically, to identify the commognitive elements, we analyzed students' discourse both during the solving process and the interviews, examining spoken and written words, gestures, pauses of silence, tone of voice, drawings, and algebraic expressions. To identify resistance, we focused on the obstacles students encountered when using a tool—for instance, difficulties in managing algebraic expressions, difficulties in performing calculations, or difficulties in linking different realizations of the same mathematical object. Signals that revealed the presence of resistance included hesitation in the solving process, manifested through expressions of confusion or uncertainty such as “I don’t know” or “I’m not sure”. Not overcoming resistance was characterized by a shift to a different tool, with students abandoning the one previously chosen. In contrast, overcoming resistance consisted in dealing with the constraints encountered and continuing to use the originally selected tool to complete the task.

In Table 4.1, we outline the procedure adopted for the coding and analysis of the data. The examination included both students’ verbal and non-verbal forms of communication, taking into account spoken language, facial expressions, gestures, and tone of voice.

In addition, the collected data were analyzed in two complementary ways. First, we compared the approaches used in the same task across different pairs of students, distinguishing also between the two groups (engineering and mathematics). Second, we examined the approaches adopted by the same pair throughout the entire solving process across the tasks, in order to investigate changes and analogies in their reasoning when tackling different tasks involving the same mathematical concept, namely the asymptotic behavior. These two modes of analysis provided a more comprehensive perspective on the data, allowing for both cross-pair comparisons and within-pair examinations.

In the following chapters, “S” refers to students (e.g., S1 = Student 1) and “I” to the interviewer. Lettered labels identify each example (A, B, C, D,... = Examples 1–4,...), while numerical indices refer to the line numbers within each example (e.g., [A1] = line 1 of Example A). Thus, a reference such as [A1]–[A3] indicates that the excerpt spans lines 1–3 of Example 1.

Analytical category	Operational definition	Observable indicators	Decision rule	Example
Resistance	A constraint emerging within mediated action that both restricts and shapes the course of activity	(i) hesitation or expressions of uncertainty (e.g., "I don't know," "I'm not sure"); (ii) difficulty in managing a routine/tool or seeking external confirmation; (iii) lack of coordination between realizations; (iv) challenges in interpreting results	Code an episode as resistance when at least one indicator appears in a mini-sequence. Code as overcome if students later coordinate realizations, interpret results appropriately, and use the tool effectively; otherwise, code as not overcome.	*Isn't this like the concept of parallel lines? I mean, to make parallel lines intersect, the slopes just need to be different. Here, it's similar. This would have solutions... No, well, if it had been a plus, no." [...] "No, but it's more complicated than lines."* Indicators (i) and (ii) co-occur
Flexibility	Awareness and consideration of more than one viable approach to solve the task	Mention or attempt of > 1 tool; switching between realizations (graphical/algebraic); explicit comparison of tools	Code when at least one alternative strategy is proposed and meaningfully evaluated	First evokes an alternative tool *"If the coefficient had been one, it could have been a translation of the other"* then switches to a different tool deemed easier to manage. Explicit reference to two different realizations: *"So from the drawing we can conclude this, and now we solve this, and it has no solution."*
Applicability	Recognition that a tool used in a previous situation can be employed in the new context	Reference to previous tasks or prompt words leading to tool activation (e.g., "intersection" prompt for solving a system)	Code when a connection to a previous situation is invoked as justification for tool selection	*"Isn't this like the concept of parallel lines? I mean, to make parallel lines intersect, the slopes just need to be different. Here, it's similar.";"Here, it's the same reasoning as before in item b."*
Objectification	Shift toward reasoning about mathematical objects and properties rather than specific instances	References to general properties (e.g., growth at infinity of a parabola); general or structural reasoning	Code when object-level properties are explicitly mobilized to justify a step	This is a quantity less than zero ($a - b$), this is a quantity greater than zero (x^2), this is a quantity greater than zero (c). A negative thing cannot be equal to a positive thing, that's it.

Table 4.1: Descriptions of the approaches and the number of student pairs adopting each

Chapter 5

Routines and visual mediators in university students' solving processes

In this chapter, we focus on answering the first research question (RQ1), regarding differences in terms of performance and solution strategies in relation to the realizations provided to students, and whether the use of specific visual mediators can affect their performance. Part of the results from this preliminary phase are included in the paper presented at the Twenty-Sixth ICMI Study: Advances in Geometry Education (Gambini et al., 2024). In this preliminary phase, we collected and analyzed only the written answers.

5.1 Categorization of the answers and analysis

In the analysis of students' written protocols, two main approaches emerged in relation to the use of visual mediators and symbolic-algebraic routines. These were categorized as **Guided by Graph (GG)** and **Guided by Rituals (GR)**.

The GG category includes all protocols in which the graphical realization—either provided in the task or introduced independently by the student—played a central role in the resolution process. This encompasses both responses to Type 2 tasks, where the graph was explicitly part of the problem statement, and Type 1 tasks, in which students voluntarily created a graphical realization despite its absence in the graphical input. For instance, one protocol classified under the GG category (Figure5.1) illustrates how a student used sketches of a parabola to support and justify their answers.

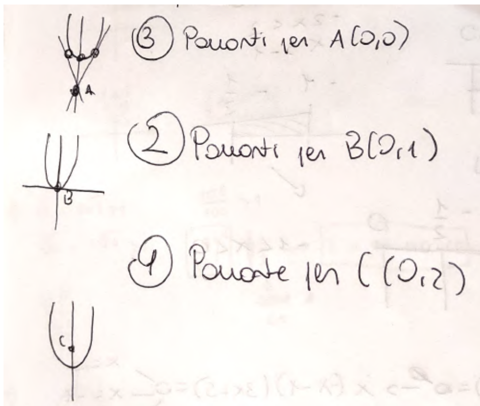


Figure 5.1: Example of a GG protocol of Type 2

The student represented the intersection of lines with the curve through three specific points, labeling them: *"3 lines passing through A(0;0)," "2 lines passing through B(0;1),"* and *"1 line passing through C(0;2)."* Notably, the graphic was not only instrumental in arriving at the response but also in articulating a clear justification. It is worth noting that the student did not use any algebraic calculation; instead, they relied on the graph provided in the task and, to make the explanation clearer, replicated the drawing for each case.

Another example (Figure 5.2) shows a Type 1 protocol, where no visual aid was initially provided. Here, the student reconstructed the parabola graphically, plotted the given points, and drew the intersecting lines, stating: *"As we can see in the figure, each point lies on the y-axis."* This demonstrates the autonomous use of visual reasoning in the absence of explicit visual input in the task. Here too, despite the presence of an algebraic realization, the student used the graph to solve the task, managing two different realizations and switching from an algebraic to a graphical one.

In both examples, the use of the graph led to correct answers. However, differences can be

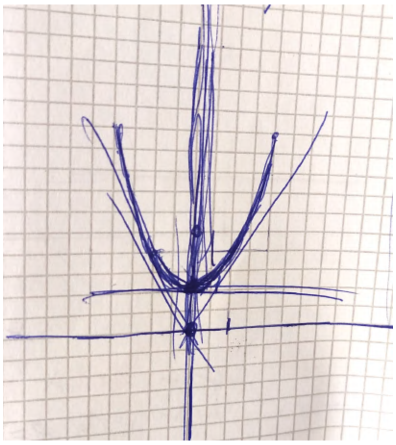


Figure 5.2: Example of a GG protocol of Type 1

observed in how the solutions were communicated: while the first student divided the three cases into separate graphs, the second represented all cases within a single graph, drawing the lines multiple times and, for instance, showing that the vertical line belongs to all three cases.

By contrast, the GR category includes those protocols in which students relied primarily on algebraic manipulation, with or without the support of graphical tools. These responses typically reflected an approach commonly taught in Italian secondary schools, which involves setting up a system between the parabola and a generic straight line passing through a given point. The solution method consists of substituting the line's equation into the parabola's equation, reducing the system to a second-degree equation in the slope parameter and solving it by imposing a zero discriminant to find the tangent lines. We referred to this category as Guided by Rituals (GR) because the implementation of this method appeared to be more process-oriented than product-oriented. In fact, while the method can be effective, its rigid application often leads to omissions or misinterpretations—for example, it typically neglects vertical lines, making it necessary to rely on a visual realization, even if only imagined, to properly account for them. An illustrative example is shown in Figure 5.3, where a student applied the system between the sheaf of lines and the parabola to determine the slope of the lines that intersect the curve at a single point.

The approach appeared to be process-oriented: the student applied the same method even for point C, internal point of the parabola, and due to calculation errors, identified two values of the

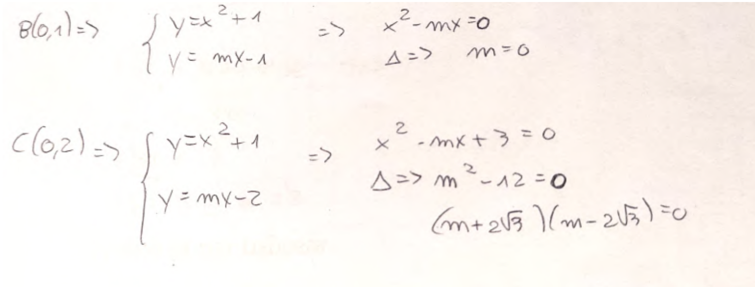


Figure 5.3: Example of a GR protocol of Type 1

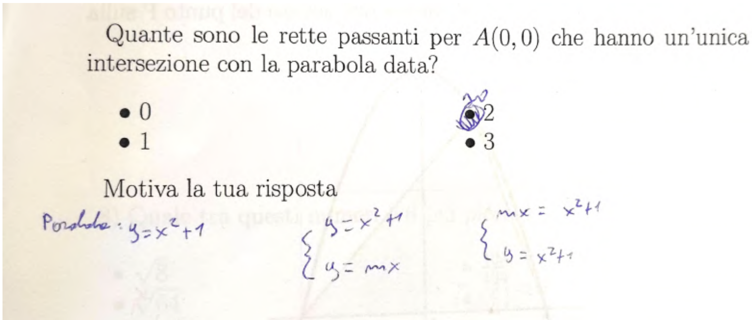


Figure 5.4: Example of a GR protocol of Type 2

slope, concluding that there were two lines tangent to the parabola. The student seemed unaware both of the suitability of the chosen method and of the correctness of the obtained results. For these reasons, this protocol was classified as Guided by Rituals, reflecting a routine driven by the need to reproduce familiar procedures rather than to reason toward the mathematical goal.

Even when a graphical realization was included in the task (Type 2), as illustrated in Figure 5.4, some students ignored it entirely, instead attempting to solve the task using the aforementioned algebraic procedure.

In one such case, the student began formulating the equations but did not complete the process and appeared uncertain about the correctness of the response, initially selecting “2” but then deleting it and writing “no” without providing an alternative answer. This suggests that, despite having access to visual information, the student defaulted to familiar routines, likely internalized through repeated classroom exposure. The implementation of this approach was categorized as GR, as the orientation was again more process-oriented than product-oriented. This categorization reflects not only the inefficiency of relying solely on this method but also the fact that, even in the presence of a graph—which offered an easier path to the solution—the student chose to derive the algebraic expression of the parabola and use it to solve the task. Furthermore, the presence of the graph may have contributed to the student's uncertainty, possibly due to a conflict between what was observed visually and the algebraic manipulations being applied.

The comparison between the GG and GR categories provides further insight into the impact of visual mediators and routines on students' performance. The presence of a graph within the task statement had a clear influence on outcomes: almost half of the students working on Type 2 tasks (46%) produced fully correct responses, compared to only 25% for Type 1 tasks. Responses were considered fully correct when students provided accurate answers for all items—specifically, identifying 3, 2, and 1 lines in the respective cases. Overall, 35% of students answered correctly across all questions; yet, the difference between task types remains significant. Among those who responded correctly, the vast majority (88%) explicitly relied on graphical realizations in their solutions. The remaining two students provided correct answers without offering any explanations, preventing further analysis of their solving processes. Conversely, students categorized as GR relied exclusively on algebraic routines—most notably, the technique for determining the slope of tangent lines—which inherently excludes vertical lines, since these are not representable as functions. As a result, none of the students in the GR category produced a fully correct response in either task type. Table 5.1 presents the percentages of correct answers by task type and by the use of GG or GR approaches.

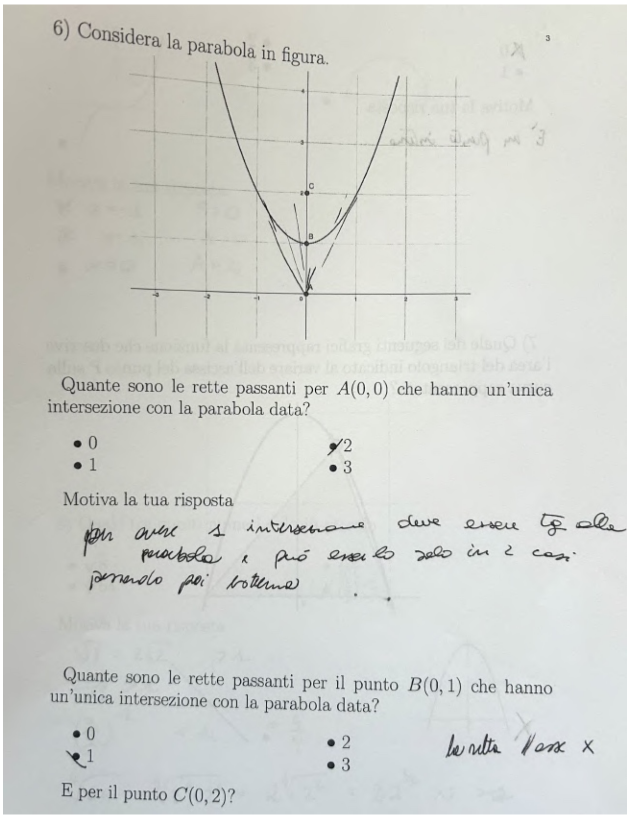
	%Correct Answers	% GG	% GR
Type 1	25%	54%	21%
Type 2	45.8%	58%	12,5%

Table 5.1: Percentages of correct answers by task type and by use of GG or GR approaches

This contrast was also evident in the way students approached the task and interpreted the questions. The goal was not to find the equations of the lines, but rather to determine how many lines intersect the parabola at exactly one point passing through a given location. Students who relied on graphical realizations were able to identify the correct number of lines more easily and could qualitatively represent them in their sketches. In contrast, students classified in the GR category encountered greater difficulty in visualizing the situation, and the algebraic method they applied did not enable them to identify all the relevant lines. The procedure observed in the GR protocols reflects a widespread practice in Italian mathematics education and was likely applied in a ritualized manner by students, reproducing techniques familiar from their high school experience with intersections between parabolas and lines. This approach was interpreted as ritualized because students appeared more focused on mechanically following a known procedure than on constructing meaning from the task. Consequently, their reasoning was process-oriented rather than product-oriented. This pattern is clearly visible in several protocols where students uncritically applied the same algebraic procedure even to the internal point of the parabola, without recognizing its inappropriateness or interpreting the resulting outputs.

Some responses could not be classified as either GG or GR. In these cases, students reasoned using properties of tangents and the relationships between points and the parabola. Specifically, they stated that for an external point there are two tangent lines, for a point on the parabola there is one tangent line, and for an internal point there are no tangent lines. Such statements appeared in both task types. Examples of these responses are shown in Figure 5.5. The use of these statements indicates that these students were neither guided by rituals nor by the graphical realization; indeed, some students with access to the graph did not identify the presence of the vertical line.

These responses cannot be interpreted as ritualized, since the students did not display a purely process-oriented performance, yet they also cannot be described as explorative. Their routines may therefore lie somewhere between ritual and exploration. However, hybrid routines were not analyzed in depth at this stage, as they were examined more extensively in the main study (see Chapter 7). Moreover, the use of the graph did not always lead to correct answers. As shown in the following example (see Figure 5.6), one student identified only the vertical line while overlooking the tangent line, revealing a lack of consideration for the parabola's properties and a reliance solely on visual cues. This perceptual aspect was further investigated in the main study (see Chapter 6 and Chapter 8).



6) Considera la parabola in figura.

Quante sono le rette passanti per $A(0,0)$ che hanno un'unica intersezione con la parabola data?

- 0
- 1
- 2
- 3

Motiva la tua risposta

per avere 1 intersezione deve essere tg alla parabola e può esserlo solo in 2 casi passando poi esterna

Quante sono le rette passanti per il punto $B(0,1)$ che hanno un'unica intersezione con la parabola data?

- 0
- 1
- 2
- 3

la retta l'asse x

E per il punto $C(0,2)$?

Figure 5.5: Student protocol identifying only the tangent lines
Transcription: "To have only one intersection, it has to be tangent to the parabola, and this can occur only in two cases, passing externally," and "the line x-axis."

5.2 Discussion

To respond to the research question, we observed that the inclusion of graphical realizations in the tasks had a clear impact on students' performance, leading to a higher percentage of correct responses compared to tasks presented without graphical realizations. Specifically, students performing Type 2 tasks achieved a higher rate of accurate answers than those working on Type 1 tasks. It became evident that the choice of visual mediators significantly influenced students' ability to provide correct solutions. Graphical realizations helped students visually discern the number of lines intersecting the parabola, whereas the algebraic approach—focused on calculating the slope—often failed to reveal all the necessary lines. This result underscores the effectiveness of incorporating graphical tools when addressing such tasks. Students who

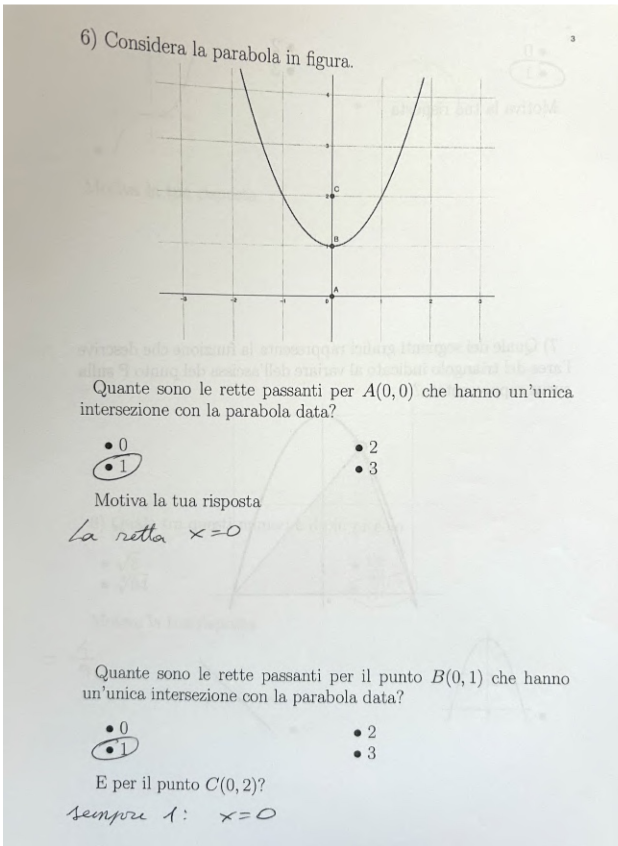


Figure 5.6: student protocol identifying only the vertical line.
Transcription: "The line $x = 0$," and "always 1, $x = 0$."

engaged in GG approaches demonstrated flexibility between algebraic and graphical realizations, particularly in Type 1 tasks where no graph was provided. This result reflects difficulties already documented in other studies regarding the connection between different realizations (Hitt, 1998; Kop et al., 2017) and confirms that graphs can provide students with valuable means to solve tasks when used appropriately (Montenegro et al., 2018). Nevertheless, despite the potential usefulness of graphical representations and the opportunities they offer for solving the task, difficulties in managing these representations were evident—especially when students failed to extract meaningful information from them, as also noted by Montenegro and colleagues (2018). On one hand, some students did not identify the tangent lines and focused only on the vertical line, emphasizing the strong influence of perception; on the other hand, others recognized the tangent lines but failed to consider the vertical one, despite its correspondence with the easily visible y-axis. Conversely, when students implemented ritualized approaches, a lack of flexibility was observed, and their solving processes became process-oriented rather than product-oriented. This contrast is particularly clear in cases where students applied the algebraic system even to the internal point of the parabola, demonstrating a lack of awareness of the meaning of the tools they employed. The concept of flexibility and its limitations was addressed in greater depth in the main study and is discussed in the following chapters. More broadly, these preliminary results illustrate how social and institutional factors shape mathematical activity. The persistence of ritualized procedures within the GR category suggests that students tended to rely on familiar techniques to conform to perceived expectations of teachers, rather than engaging in reasoning aimed at solving the task itself. Students who adopted a GR-type approach sometimes combined it with the use of a graph, but in an ineffective way: their answers were primarily guided by the implementation of the algebraic system rather than by the information provided by the graphical representation. This was particularly evident in their lack of consideration of the vertical line, which indicates a limited integration between algebraic and graphical realizations. In this sense, the distinction between GG and GR strategies influenced not only students' success rates but also the orientation of their reasoning—toward solution and justification in the former case, and toward procedure and compliance in the latter. As highlighted in the analysis, not all students' answers could be classified within these two categories, as some responses lacked sufficient explanations or information about their solving processes, making their categorization difficult.

5.3 Refining research: limits of the preliminary phase and developments in the Main Study

In the preliminary stage of the study, several limitations were identified, particularly the lack of an in-depth analysis of students' discourses and their problem-solving processes. The written responses alone were insufficient for such an analysis, as students often provided only brief answers without elaboration. The use of closed-ended questions further limited the possibility of understanding their reasoning. In many cases, students gave answers without offering any explanation, relying instead on graphical representations to illustrate or obtain their solutions—this was evident when they used graphs to identify straight lines with only one intersection with the parabola—or by employing an algebraic approach, which became apparent when they explicitly set up a system between the sheaf of straight lines and the equation of the parabola. To address these issues in the main study, two major changes were introduced. First, the tasks were reformulated as open-ended questions, to encourage students to justify their reasoning rather than simply select an answer. Second, a different methodological approach was adopted: task-based interviews. During these sessions, students were video-recorded throughout the entire solving process without researcher intervention, and they were subsequently interviewed to gain deeper insights into their reasoning. Additionally, during the preliminary phase, some students reported to the researcher that the correct answer was not included among the multiple-choice options—for example, stating that "there are infinitely many lines," which was not among the available answers. This feedback further motivated the decision to convert the tasks into open-ended questions and to design additional items focusing on the asymptotic behavior of functions. These new tasks aimed to explore students' sense-making processes concerning this concept across different functional contexts, including cases involving functions of the same order (e.g., two parabolas) and of different orders (e.g., a parabola and an exponential function). These methodological changes inevitably reduced the number of participants, leading to a shift from a mixed-methods approach to a qualitative design. The emphasis thus moved from the percentage of correct answers to a detailed analysis of students' solving processes. Because a comprehensive examination of protocols, interviews, and transcripts required close attention to individual reasoning, it was not feasible to include a large sample. Furthermore, the task-based interviews were conducted in pairs to encourage interaction and collaboration, with a specific focus on social dynamics during solving process. This deeper, qualitative analysis made it possible to focus on the students' discourses, the routines they enacted and their characteristics, and the ways in which they used visual mediators. It also allowed for the identification of resistance and an examination of how resistance related to commognitive elements. In addition, aspects of mathematical reasoning were analyzed in relation to the approaches adopted by the students and to how they substantiated their discourse.

Chapter 6

The role of visual mediators in the manifestation of resistance

In this chapter, we address the second research question (RQ2), which concerns the role of visual mediators in the manifestation of resistance during the process of solving calculus tasks. Part of the results discussed here are drawn from the article currently under review (Viola, Under review). In the main study, data were collected and analysed from students' written responses, video recordings, and interview transcripts. Specifically, to answer this research question, these data sources were examined in depth, with particular attention to the use of different visual mediators and their relationship to the manifestation of resistance during solving process.

6.1 Results

In this section, we focus on the solving processes observed in Tasks 2, 3, 4, and 5 (see Figure 4.3), analysing the approaches adopted by students with particular attention to the visual mediators they employed. The analysis of the data revealed several recurring patterns. Table 6.1 summarizes how pairs used different visual mediators across tasks, while Table 6.2 presents the approaches observed in Task 5, organized by item. In this table, we distinguish between students who relied primarily on graphical realizations, those who worked mainly with algebraic realizations, and those who integrated both. It is important to note that gestures were not coded as a separate category since they often accompanied and complemented each of these types of realizations. In fact, gestures frequently played a mediating role, enriching both graphical and algebraic reasoning throughout the solving process.

	Task 2	Task 3	Task 4
Graphical visual mediator	8	10	6
Algebraic visual mediator	4	0	8
Both	7	9	3

Table 6.1: Pairs' use of graphical and algebraic visual mediators in Task 2, Task 3 and Task 4

	Item a	Item b	Item c
Graphical visual mediator	2	13	2
Algebraic visual mediator	11	1	11
Both	6	5	6

Table 6.2: Pairs' use of graphical and algebraic visual mediators in Task 5

We observed a general tendency among students to employ, during the solving process, the same type of visual mediator that was presented in the task statement - particularly in Tasks 2, 4, and 5 - confirming the results from the preliminary phase (Chapter 5). However, in Task 3, it was not possible to solve the problem relying exclusively on an algebraic approach. Despite this constraint, it is noteworthy that nearly half of the students spontaneously integrated both types of visual mediators in their reasoning, highlighting that the algebraic visual mediator was nonetheless employed in this context. In this section, however, the focus shifts toward a deeper examination of the solving processes, with specific attention to the emergence of resistance and the ways in which students addressed it. The table indicates that both groups employed all visual mediators. Nevertheless, mathematics students most frequently relied on algebraic mediators, either alone or together with graphical ones, while engineering students mainly favoured graphical mediators. This contrast can be understood considering the different academic backgrounds: mathematics students are typically accustomed to express ideas in formally structured ways and therefore hesitate to base their conclusions exclusively on graphical evidence. In contrast, engineering students seem more comfortable supporting their arguments through graphical realizations without necessarily integrating additional algebraic work.

In the following paragraphs, representative excerpts from each task are presented, selected to illustrate typical reasoning patterns, the interaction between different visual mediators, and moments where resistance either emerged.

6.1.1 Examples of overcoming of resistance

In this section, we present examples from Task 2 and Task 4 illustrating two pairs who successfully overcame resistance during their solving processes.

Example from Task 2

In the Task 2, two mathematics students began by examining the graph provided in the task. Their initial interpretation led them to claim that infinitely many lines intersect the parabola at exactly one point. To verify this intuition analytically, they constructed what they referred to as a "system," equating the sheaf of lines passing through the origin with the algebraic expression of the parabola. A segment of their dialogue is reproduced below:

A1_S1: I think there are infinitely many lines.

A2_S2: Yes, because if you take this straight line, for example, it does not intersect the parabola again.

A3_S1: We can solve this analytically.

[Students equalize equation of the sheaf of straight lines and the equation of parabola, as it is possible to see in Figure 6.1]

A4_S2: This system always has solution [They referred to the equalization of the two equations as a system]

A5_S1: Yes, it's absurd!

A6_S2: Yes, there should be infinite.

A7_S1: The parabola grows to infinity faster than the straight line…

A8_S2: Yes

A9_S1: Ah, of course! The straight line is slower than parabola, so at the beginning it stays above and after the straight line will be under, so there are two points of intersection!

The students appeared surprised by the outcome of the system, as they initially expected an infinite number of straight lines intersecting the parabola at exactly one point. Although they correctly interpreted the meaning of the algebraic procedure they had applied, the result

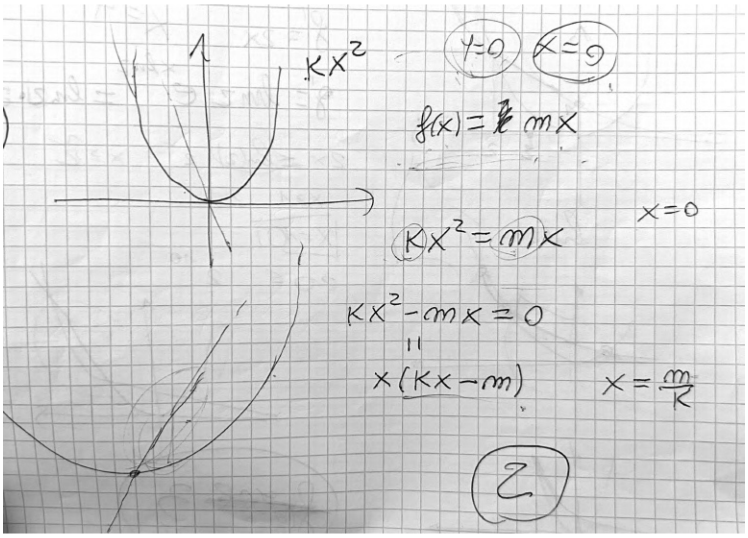


Figure 6.1: Mathematics students' protocol about Task 2

contradicted their graphical expectation. This mismatch led to a moment of resistance after setting up the system involving the sheaf of lines and the parabola [A4]–[A8], particularly when the outcomes produced by different visual mediators appeared inconsistent. Their comments - *"Yes, it's absurd!"* and *"Yes, there should be infinite"* - reveal uncertainty and difficulty in making sense of the contradiction, prompting them to perceive the algebraic tool as ineffective in this context. The resistance emerged specifically during the transition from the graphical realization to the algebraic one. However the students began to reflect on the growth of both the straight line and the parabola at infinity, gradually connecting the graphical realization with the algebraic result they had obtained. This process enabled them to construct a new narrative [A9] that reconciled the two mediators. The algebraic realization ultimately served as a resource for reinterpreting their results and restoring coherence between the conflicting visual mediators.

This episode illustrates how discrepancies between mediators can trigger resistance and how such resistance can be resolved when students successfully manage the transition between realizations. In this case, the pair's realization tree was primarily composed of graphical and algebraic realizations, which they progressively integrated throughout their solving process.

Example from Task 4

In this example, a pair of engineering students solved Task 4 by drawing the parabola and focusing on tangent lines, without initially invoking the parabola's algebraic formula. During the interview the researcher prompted them to consider a line close to the y-axis, which led to a sustained reflective exchange:

B1_S3: Even a line very close to the y-axis will intersect the parabola at some point.

B2_I: Okay, can you tell me why?

B3_S3: Because the angular coefficient is inclined.

B4_S4: Oh, wait, but then the parabola opens up. What if the line is really close to the y-axis?

B5_S3: Is never reaches it?

B6_S4: The line opens more slowly, so it doesn't reach the parabola.

(students indicate with gestures that the growths of the parabola at the infinity is oblique, such as have an obliques asymptote)

B7_S3: So, infinitely close?

B8_S4: But if you tilt the line a bit more, they'll eventually intersect.

B9_S3: I think there's a limit, beyond which they don't intersect anymore.

(...) Students discussed about the "limit"

B10_S4: It's not possible. These lines should be parallel to parabola at the infinity

B11_S3: Maybe only the vertical line is parallel to parabola at the infinity.

B12_S4: I think they'll intersect in the end. I think the only lines that intersect the parabola just once are the tangents and the vertical one. Even if the slope is very steep, I still think the line eventually intersects the parabola again.

In this example, the students displayed a distorted perception of asymptotic behavior, interpreting the growth of the parabola at infinity as a widening of its arms (*"the parabola opens up"*). This perception appeared to trigger a moment of resistance, as they changed their answer on the basis of this misinterpretation [B4]–[B6]. During the discussion, they reflected at length on whether the line and the parabola might intersect again, introducing the idea of a "limit" [B7]–[B9] beyond which infinitely many lines would intersect the parabola only once. Eventually, they returned to their initial view, adding the vertical line and concluding that no such limit exists [B10]–[B12]; for them, only the vertical line and tangents have a single intersection with the parabola. Throughout the discussion, the students used gestures to represent the growth of both the parabola and the line. This shift from a static to a dynamic image seemed to contribute to the emergence of resistance, as suggested by their words [B4]–[B6], when the graph-intended as the primary tool-appeared not to provide a coherent answer (hesitation and uncertainty). Considering a line closer to the y-axis and using gestures to convey its growth, they implicitly introduced a dynamic realization that contrasted with the static graph. The interplay between these two visual mediators (gestures and graph) produced conflicting interpretations, triggering resistance.

After reflecting through the connection between the mediators, the students eventually overcame this resistance by constructing a new narrative. Their approach was exploratory: although their explanation was not mathematically correct, it enabled them to move forward [B10]–[B12]. They grounded this new narrative primarily in the graphical mediator. Following an initial difficulty, they reorganized their approach around a graphical interpretation and formulated the idea of a "parallelism" between the parabola and the lines-a mathematically inaccurate but internally coherent narrative that they accepted as valid. This construction allowed them to reconcile the conflicting results produced by different mediators and to overcome the resistance. In this episode, the realization tree appears largely grounded in graphical realizations. The connections students attempted to build involved graphs and gestures, juxtaposing static and dynamic images of the same object. Algebraic realizations and tables of values did not play a role in this case.

6.1.2 Examples of not overcoming resistance

Example from Task 4

This extract presents another pair of engineering students who experienced a moment of resistance that they did not resolve. These students first relied on the graph and subsequently tried to confirm their visual hypothesis by setting up the algebraic "system":

C1_S5: We reasoned on the behavior of the straight lines, so straight lines with a fairly high coefficient intersect the parabola at one point. After we decided to implement the system between a sheaf of straight lines and parabola, but I do not know what it does not work.

C2_S6: We expected infinite lines, but we found only 3

C3_S5: Something does not work, because the value of angular coefficient equal to 0 is not ok,

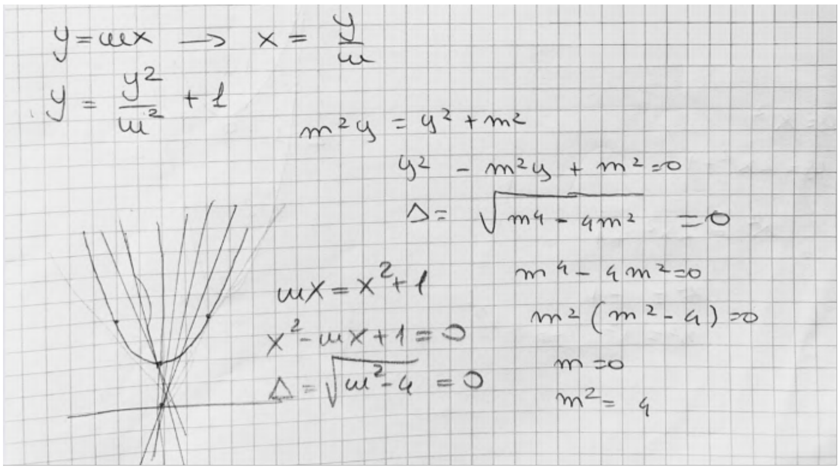


Figure 6.2: Engeenering students' protocol about Task 4

because the horizontal line passing through the origin does not intersect the parabola

C4_I: How do you interpret this?

C5_S6: Maybe we made a mistake in implementing the system

C6_S5: The lines are infinite, as we see in the figure, but I do not know why the system does not work.

In Figure (6.2) the students' protocol is shown. As described during the interview, they began by sketching the parabola and attempting to address the question through its visual realization. Only afterwards did they choose to set up the system, whose solutions, however, did not correspond to what they anticipated. The way they handled the system suggests an use closer to a ritualized use of the procedure: the students did not articulate its underlying purpose, nor did they comment on why three different slopes emerged. Although they stated that the value $m = 0$ should be excluded, they were unable to justify this claim [C5]–[C6].

Their dialogue reveals a distinct moment of resistance, visible in their struggle to apply mathematical tools appropriately to this situation [C1]–[C3]. Their hesitation and difficulty in interpreting the outcomes indicate that the coexistence of multiple visual mediators played a role in generating this resistance. In particular, the students were unable to integrate the results derived from the graphical and algebraic approaches, which led to a loss of coherence between the two realizations. The shift from the graph to the algebraic procedure proved especially troublesome: the algebraic method did not appear reliable to them, as it did not confirm what the graph initially suggested [C6]. In this episode, the different visual mediators contributed more to confusion than to support. Although both graphical and algebraic realizations were present in their realization tree, the students did not manage to establish meaningful links between them, nor did they recognize their equivalence [C6]. Rather than constructing new narratives to reconcile the different outcomes, they appeared to reproduce familiar procedures without demonstrating a solid grasp of their significance. This reading is reinforced by their difficulty in explaining why $m = 0$ is indeed a solution [C5]–[C6]. Their attention appeared focused on carrying out the algebraic steps rather than making sense of their output. As a result, when confronted with outcomes that did not align, they tended to assign greater credibility to the graphical realization.

Example from Task 3

This example features a pair of master's students in mathematics who struggled to apply analytical procedures appropriately. Initially they attempted a symbolic "system" (evident in the crossed-out section of their work, see Figure 5), then constructed graphs of the two functions and debated their mutual behavior. An excerpt follows:

D1_S7: Surely an intersection has it, because 2^x *goes to 0 on the left*

D2_S8: Yes, because it definitely passes through 1

D3_S7: The exponential goes a little bit, but the problem is whether the parabola also widens

D4_S8: The exponential widens, even if the parabola does not widen

D5_S7: But does the exponential have a vertical asymptote?

(They reason about the existence of a vertical asymptote and conclude no).

D6_S7: No, when x goes to plus infinity 2^x *goes to plus infinity, so it does not have an asymptote, so it has two points of intersection*

D7_S8: I imagine the exponential that as it grows it moves further and further away from the y-axis, so it depends on the parabola

D8_S7: If the parabola grows a little to the right and the exponential grows a little, they do not intersect

After, they go back to the system and reason about the logarithm, concluding that the two functions have only one point of intersection, as is possible to see in Figure 6.3.

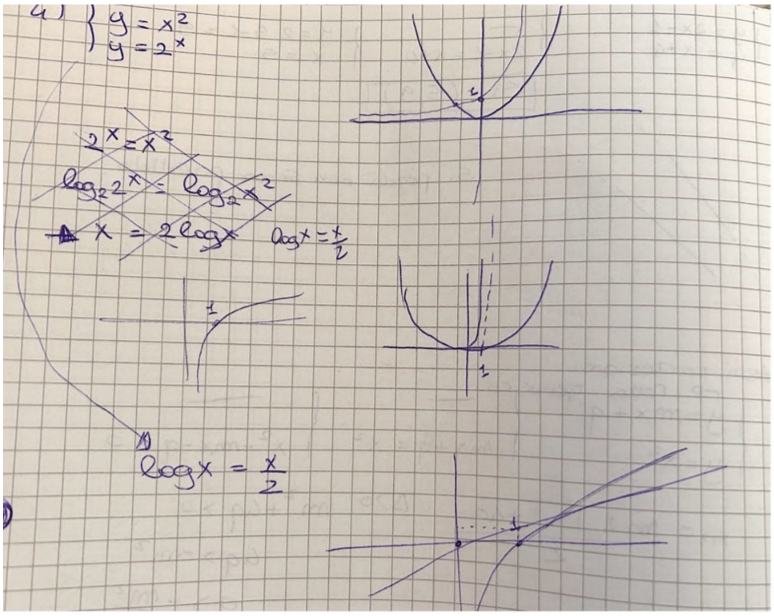


Figure 6.3: Students' protocol about Task 3

The students' discourse reveals that they interpreted the behavior of the two functions at infinity as if they were tending toward an oblique asymptote [D7]–[D8]. Starting from this idea of how the curves grow, they concluded that the graphs meet only once-an indication of a distorted view of asymptotic behavior. A moment of resistance emerged when they wavered between the possibility of one or two intersection points [D6]–[D8]. Although their work was initially grounded in the graph, their hesitation suggests difficulties both in interpreting the visual information and in relating it to the algebraic realization.

They later revisited the algebraic equation and the associated system. The algebraic outcome led them to sketch another figure (shown in the lower right of the protocol of Figure 6.3), once again attempting to connect the two mediators. Even though they seemed to attribute greater reliability to the algebraic result, they eventually sought confirmation in this new graphical realization, which also indicated a single intersection. The resistance is visible in their first attempt to use the algebraic tool: they abandoned it when they could not make sense of the outcome, turning back to it only after using the graph as support. A similar hesitation appeared in their reading of the first graph [D6]–[D8], where interpreting the image and linking it to the asymptotic behavior proved challenging.

Their attempt to coordinate algebraic and graphical realizations did not succeed, as they continued to identify only one intersection. The presence of multiple visual mediators seems to have prompted resistance: the students were unable to establish a connection between the two realizations yet, they believed they had reached a correct conclusion based on their final discussion involving logarithms. While solving the system, they showed limited awareness that composing functions can alter the domain of the original equation. By applying logarithms to both sides, they unintentionally restricted the domain and eliminated a solution-an issue that went unnoticed. In this episode, the students' realization tree relied on both algebraic and graphical realizations, but they struggled to manage the different realizations of the two functions. Their discourse did not involve the construction of new narratives; rather, they attempted to recall endorsed narratives concerning asymptotic behavior [D6]–[D8] and the algebraic tool, though in an imprecise manner.

Example from Task 2

The students initially argued that exactly two straight lines intersect the parabola at only one point, identifying them as the Cartesian axes. They reasoned that a line slightly rotated from the x-axis would intersect the parabola twice, whereas a line near the y-axis would not intersect it. They also discussed limits, claiming that x^2 grows faster than x, and used this idea to justify that the parabola "moves away" more quickly at infinity. Later, they considered a bundle of lines close to the y-axis and concluded that infinitely many such lines meet the parabola in a single point. At another moment, they used an argument based on the area between the line and the parabola: if the area is infinite, the curves intersect only once; if it is finite, they intersect twice. Overall, their reasoning reveals hesitation, a distorted perception of asymptotic behavior, and difficulty connecting algebraic and graphical realizations. An excerpt of the discussion is shown below:

E1_S9: Sure, Cartesian axes.

E2_S10: Yes.

E3_S9: If we take into account a line very close to the y-axis, the parabola will expand much more and will not intersect the straight line.

E4_S9: Even if you compute the limit as x to infinity of x^2 and the limit as x to infinity of the equation of a general line passing through the origin, something like $m \cdot x$, x^2 is always of higher order than x. So it goes to infinity faster than x.

E5_S10: I don't remember if this is only in theory. (hesitation)

E6_S9: I mean, this one reaches infinity earlier, right? So theoretically... (hesitation)

E7_S10: ...faster. (hesitation)

E8_S9: Yes, it goes faster. So if I consider a line like this...

E9_S10: ...the line is always further ahead.

E10_S9: Unless maybe it intersects everything.

E11_S10: Actually, it doesn't depend.

E12_S9: Certainly two lines that coincide with the Cartesian axes and then...

E13_S9: But I'm not sure if this reasoning is correct. I mean, it seems correct. If you go to infinity, it will definitely...

E14_S10: Let's compute the ratio.

E15_S9: Ah yes, you're right.

E16_S10: The smaller divided by the greater should go to zero.

E17_S10: $y = m \cdot x$*, where* m *is a constant.*

E18_S9: It depends, if $m = 0$ *then...*

E19_S9: Right, but if $m = 0$ *what does it mean?*

E20_S10: It means it's horizontal.

E21_S9: Horizontal, $y = 0$*.*

E22_S10: Yes, so then it wouldn't... but this analysis also focuses on the Cartesian axes.

E23_S9: Certainly, the two Cartesian axes intersect the parabola in exactly one point, I would say.

E24_S10: I would answer this way: the family of lines that differ from the vertical axis (the y-axis) so that the slope is always smaller than the slope of the tangent line to the parabola at infinity - meaning the slope must always be lower than the slope of that tangent line at infinity.

(...)

E25_S9: Listen, this reasoning is actually similar: if the function describing the distance between a line with slope m and the parabola is a function that, instead of decreasing so that they eventually meet, increases... yes, exactly, that's another reasoning - a function that theoretically describes all lines.

(...)

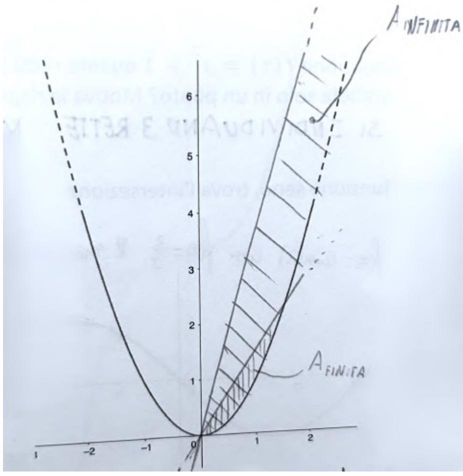


Figure 6.4: Students' protocol.

E26_S9: But how many of these lines are there? I mean, we must say how many.

E27_S10: I don't know. Well, even with a minimum slope... (hesitation)

E28_S9: That's the idea because they would be all those lines like this (gesture), and how many are there?

E29_S10: Infinitely many.

E30_S9: Infinitely many, I think so. Because even if you have up to here, okay, this is the last line...

E31_S10: Even with an infinitesimal minimum slope.

(...) (they try to understand how to write it)

E32_S10: By intuition, by analyzing the area enclosed between the line and the parabola - which describes the delta between the two - because if it is too steep, you intersect twice very quickly. So there must be a point where, as you see, what remains inside is decreasing, while in this other case it is increasing. (...). From the students' discussion, it is possible to observe clear difficulties in coordinating different realizations of the same mathematical object, as well as signs of hesitation, both of which indicate the presence of resistance. The students attempted to use asymptotic behavior to solve the task, but their gestures and verbal explanations suggested a distorted perception; they appeared to interpret growth at infinity as an oblique form of divergence. This distorted perception seems to influence their reasoning and trigger resistance, as they claimed that the parabola grows faster than the straight line, although this was not consistently reflected in their realizations, revealing challenges in connecting realizations.

The students also shifted their approach to justify their answer by referring to the area between the two functions. However, they did not compute this area formally; instead, they relied on perceptual arguments to support their claim. Although they attempted to relate realizations by calculating limits and discussing asymptotic behavior, they did not appear fully convinced by this line of reasoning and eventually changed tools (to the area-based argument), relying primarily on perceptual considerations rather than algebraic reasoning. This indicates that resistance was not overcome, as the students abandoned the previous tool instead of reinterpreting the results and coordinating the different realizations. The students' realization tree included both algebraic and graphical realizations, yet no meaningful connection emerged between them. They seemed to struggle to make sense of the mathematical concept, and the different realizations appeared to carry different, uncoordinated meanings. The students' discourse suggests that they were not yet aware of these differences, which further contributed to their difficulty in resolving the task.

6.2 Discussion

The analysis brought to light several points in which students encountered resistance during their solving processes. Across Tasks 2, 3, and 4, a recurring issue concerned students' distorted view of asymptotic behavior. Many students appeared to interpret the growth of functions at infinity as if the curves were tending toward an oblique asymptote, a misinterpretation that led some to claim the existence of infinitely many tangents based solely on what the drawing seemed to suggest rather than on analytical considerations. A comparable misunderstanding emerged in Task 3, where most pairs identified only one or two intersection points, overlooking the correct outcome. These tendencies indicate that students often perceived the graph through the lens of oblique growth instead of recognizing that the function approaches the vertical axis when moving toward infinity.

Although many students showed familiarity with algebraic tools, they often struggled to interpret the outputs and to connect them with graphical realizations. This difficulty echoes the distinction highlighted by Carlsen (2010; 2018) between knowing a mathematical tool and knowing how to use it appropriately in a given context. The results also underline a broader challenge in establishing links between algebraic procedures and graphs. As Montenegro and colleagues (Montenegro et al., 2018) point out, graphs can offer an accessible entry point for solving; however, students do not always perceive these potential resources, as illustrated in tasks involving parabolas and lines or exponential and parabolic functions. The comparison between engineering and mathematics students revealed contrasting preferences in the choice of visual mediators while showing similar struggles when facing unfamiliar tasks. Both groups displayed resistance when attempting to coordinate different realizations of the same mathematical object and exhibited comparable distorted perceptions about asymptotic behavior (see Figure 6.1.2). The fact that two cohorts with markedly different academic backgrounds encountered the same obstacles suggests that these difficulties may be widespread. This observation indicates the need for future studies involving additional student populations to determine how general these patterns are. Taken together, the results confirm and extend the preliminary results (Chapter 5), strengthening the claim that students' performance and approaches are strongly shaped by their use of visual mediators. The present work adds further insight by identifying resistance as a key indicator of difficulties in shifting between mediators. In many excerpts, the students' attempts to bridge algebraic and graphical realizations triggered moments of resistance. Moreover, gestures (Example B, 6.1.1) surfaced as an additional visual mediator and occasionally conflicted with the static drawings, revealing a tension between dynamic and static realizations of the same mathematical object.

The results suggest that resistance tends to emerge when students are unable to coordinate different realizations. This aligns with Sfard's (2008, p. 166) claim that the capacity to manage multiple realizations is central to mathematical understanding. When resistance was overcome, however, students shifted toward a more exploratory mode, developing new or hybrid narratives that integrated several realizations. In those moments, they showed an awareness of the meaning underlying the procedures they used and succeeded in weaving together algebraic, graphical, and gestural mediators. The type of visual mediator influenced the nature of the narrative: graphical realizations favored perceptual narratives, whereas algebraic ones encouraged rule-based or theorem-oriented accounts. Resistance persisted whenever these narratives remained disconnected, and students neither built new links nor retrieved endorsed ones.

When such coordination fails, students frequently privilege one realization-typically the one they view as more trustworthy. For instance, in Example C, the students leaned on graphical reasoning and set the algebraic result aside, whereas in another section (Example D) they discounted the graphical information and treated the algebraic tool as more authoritative. In both cases, the abandonment of one tool marks a moment of resistance: the students recognized the tool but were unable to adapt it to the specific solving context, illustrating Carlsen's (2018) distinction between possessing a tool and knowing how to incorporate it meaningfully.

Finally, the results shed light on an overlooked issue in the literature. Previous studies have recognized the benefits of working with different realizations as well as students' difficulties in coordinating them (Ryve et al., 2013; Haghjoo et al., 2023; Weingarden and Heyd-Metzuyanim, 2023), but none have examined how resistance emerges precisely when familiar tools are used in unfamiliar ways. Our analysis shows that resistance frequently appears when students attempt to mobilize algebraic, graphical, or gestural tools they already know but cannot adjust them to the demands of the task. This highlights the importance of considering not only which mediators students use, but also how they manage them when they need to engage in solving in new or atypical situations.

Chapter 7

Connection between resistance and routines: focus on the de-ritualization process

In this chapter, we address the third research question (RQ3), which examines how students' use of routines develops and interacts with the emergence of resistance during the solution of Calculus tasks. Some of the results presented here build on results discussed in a manuscript currently under review (Viola and Gambini, Under review). For the main study, we analyzed students' written solutions, video recordings of their work sessions, and interview transcripts. To answer this specific question, these data sources were examined in depth, with particular attention to the routines that students employed and to how these routines were connected to episodes in which resistance became apparent during solving process. The analysis focused in particular on routine features associated with the process of de-ritualization-specifically, their flexibility and their applicability across different contexts.

7.1 Results

In this section, we illustrate how resistance emerges during students' solving processes by presenting several episodes in which resistance was either overcome or remained unresolved. Our focus is on examining students' discourse and exploring how the use of routines interacts with the appearance of resistance, paying particular attention to the ways in which routines shift or adapt when students face such obstacles. For clarity, the analysis is organized according to whether resistance is surpassed or persists, allowing us to show-through different examples-how the relationship between routines and resistance unfolds across the tasks.

The discussion concentrates on Tasks 2, 3, 4, 5, and 6 (see Figure 4.3), where the presence of resistance was most pronounced. Task 1 was treated differently, as it did not present a fully unfamiliar situation. Despite being formulated in a new way, it essentially recalled a well-known setting: the intersection of two lines. Students were not required to compute the intersection numerically, but simply to determine whether the two lines met. All pairs answered both parts correctly. In item (a), 11 pairs approached the question by solving the linear system, while in item (b) every group relied on graphical reasoning-either by comparing slopes, extending the lines visually to foresee their meeting point, or by adding a third auxiliary line as justification. For this reason, Task 1 did not offer substantial evidence of resistance and was not examined in depth.

In the tasks where resistance was more evident or more frequently left unsolved, we observed the students' solving processes with a particular emphasis on how they used and modified routines in response to the constraints they encountered. Episodes in which resistance persisted were especially insightful, as they provided a window into how students managed (or failed to manage) the evolution of their routines when confronted with difficulties.

7.1.1 Use of routines in relation to the overcoming of resistance

Example from Task 5

In this case, the pair of mathematics students initially tried to extend the line of reasoning used in Task 1, focusing on the comparison of coefficients. They soon judged this approach to be too challenging in the new context and shifted to a different approach, formulating a system of equations instead. While addressing item (b), they began discussing the "width" of the parabolas and gradually connected this visual impression to the coefficients of x^2. This line of thought was then naturally taken up again in item (c). The excerpt we report concerns the episode in which the pair worked through Task 5:

A1_S1: Isn't this like the concept of parallel lines? I mean, to make parallel lines intersect, the slopes just need to be different. Here, it's similar. This would have solutions... No, well, if it had been a plus, no.

A2_S2: If the coefficient had been one, it could have been a translation of the other.

A3_S1: No, but it's more complicated than lines. If it had been +4, they wouldn't intersect.

(They solve the system and move on to b.)(...)

A4_S2: I'd say no, the inner one seems to be steeper. (observing the graph)(...)

A5_S1: We have $y =$, these are like ax^2, and the other one is another coefficient $x^2 + c$ (writes $bx^2 + c$).

A6_S2: From the drawing we assume that a is smaller than b.

A7_S1: But... you're saying $a < b$ yes, yes, you're right.

A8_S2: So from the drawing we can conclude this (they point to what they wrote, the first lines of part b), *and now we solve this, and it has no solution.*

A9_S1: Hopefully.

A10_S2: No, no, it really doesn't, c is positive.

A11_S1: Yes, it's positive.

A12_S2: It's important that c is positive in order to have no solutions.

A13_S1: It's similar to the one above, but here there was a "-4". In fact, if it had been "$+4$" we would have had problems, because then there would have been no solutions in the reals.

(...)(they solve it)

A14_S1: This is a quantity less than zero ($a - b$), this is a quantity greater than zero (x^2), this is a quantity greater than zero (c). A negative thing cannot be equal to a positive thing, that's it.

(...)(They outline the situation algebraically but don't use the system, reasoning instead about the coefficient of x^2. They move on to c.)

A15_S1: Here, it's the same reasoning as before in item b.

As the solving of the task unfolded, the students gradually reshaped their approach. At the outset, their work rested entirely on an algebraic treatment of the situation through a system

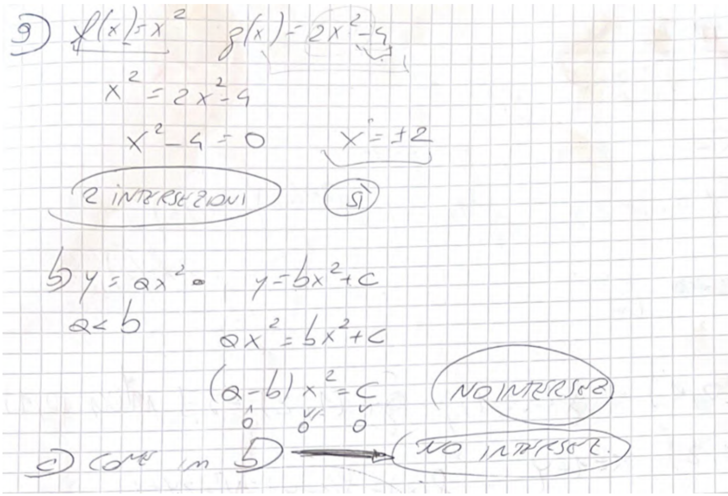


Figure 7.1: An example of a shift in the approach to solving the task. Protocol of the students' solution to Task 5
a) "two intersections, yes"; b) "no intersections"; c) "no intersections".

of equations. Moving to the later items, however, they began to weave together algebraic and geometric tools, formulating the expressions of the parabolas while, at the same time, relying on graphical interpretations to support their thinking (see Figure 7.1). Task 5 offers a clear example of how students shifted among different tools. In the earliest exchanges ([A1]–[A3]), a moment of resistance surfaced, expressed both in their hesitation [A1] and in their explicit difficulty in mobilizing a familiar tool within an unfamiliar setting [A3]. This difficulty aligns with the constraints of their PSS: although they recognized structural analogies with previous experiences, they were unsure how to adapt the tool to this non-linear context, noting that it felt *"more complicated than lines."*

After engaging with the graphic aspect of the task, they returned to algebra and organized their reasoning around the coefficients of x^2, as illustrated in Figure 7.1. The formulation of new narratives - together with the reactivation of endorsed ones - supported them in moving past the initial resistance, enabling them to employ a tool that had initially posed an obstacle. Solving item (b) thus played a pivotal role in helping them reconnect the situation to familiar routines within their PSS and to extend a previously known comparison (slopes) to a structurally analogous one (coefficients of x^2). Their approach [A8]–[A14] is product-oriented: they demonstrated understanding of the procedures they were enacting and showed confidence in switching among multiple tools while solving the task. Their movement between graphical and algebraic realizations was smooth, and their discourse exhibited coherent narratives that justified their choices.

This example illustrates their capacity to select and coordinate tools according to the needs of the context (see Figure 7.1), as well as their awareness of what each tool contributes [A12]–[A13]. A broader sense of applicability also emerged, since they increasingly acknowledged the possibility of transferring tools across contexts, visible both in their early exchanges [A1]–[A3] and in the resolution of item (c) [A15]. Their discourse also shows signs of objectification: they progressively detached from the specific drawing and reasoned about the mathematical objects and their properties [A8]–[A14], coordinating multiple realizations and moving flexibly among different ways of solving the task.

7.1.2 Use of routines in relation to the not overcoming of resistance

Example from Task 2

Here, we examine a pair of mathematics students engaged in Task 2. They first argued that the result is determined by the angular coefficients of the straight lines. To substantiate this idea, they attempted to translate their reasoning into algebraic terms by constructing a system of equations that combined a sheaf of straight lines with a parabola:

B1_S3: If you take a line very, very close like this, almost vertical, it will eventually open slowly, and so will the parabola, and they won't intersect. It depends on the angular coefficient.

B2_S4: Okay, got it. If you take a line like this (takes a line very close to the y-axis)...

B3_S3: In this case [draws a straight line], *there is another intersection, because the straight line grows faster than the parabola. It depends on the angular coefficient. There could be an angular coefficient whereby the parabola grows faster than the straight line.*(perception of asymptotic behavior in an oblique way shown with the gestures of the hands). *You take a generic straight line, and this is the parabola expression*

B4_S4: Do you intend to make a system between a sheaf of straight lines and parabolas?

B5_S3: Yes.

(...)(They proceed to implement the system using a general sheaf of straight lines and the expression of a parabola (see Figure 7.2).)

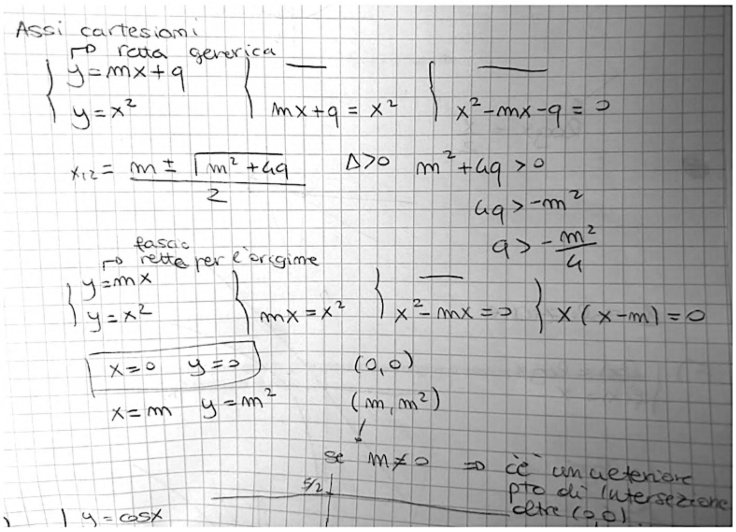


Figure 7.2: Students' protocol about Task 2 1) Cartesian axes 2) Generic straight line 3) Sheaf of straight lines passing through the origin 4) There is another point of intersection

B6_S3: So, this is the sheaf of straight lines passing through the origin, and this is the parabola... So one point of intersection is the origin, and the other is at $x = m$ and $y = m^2$.

B7_S4: This point (the origin) *is okay, but the other is not.*

B8_S3: Exactly, we have only the Cartesian axes.

B9_S4: We've shown that if you draw a straight line from the sheaf passing through the origin, we get another point of intersection.

B10_S3: If you take an m different from 0, it creates a point different from the origin. So, not any straight line in the sheaf is valid-only the Cartesian axes.

This pair made use of several tools while working through the task. They began with a graphical approach, then attempted to set up a system involving a generic sheaf of straight lines and the algebraic realization of the parabola, later narrowing this to a sheaf through the origin. A clear moment of resistance surfaced when they tried to engage with asymptotic behavior, particularly in [B1]–[B5]. Their hesitation and uncertainty are visible both in the transcript [B1]–[B3] and in the follow-up interview, where they admitted: *"For a line closer to the y-axis we can't say if it intersects the parabola or not"* and *"We did not think about the growing faster."* These comments point to difficulties in managing this tool and highlight the limitations imposed by their PSS when attempting to adapt it to this unfamiliar setting. Although they eventually reached a correct solution through the system, the resistance linked to asymptotic behavior was not resolved. The students did attempt to reason about growth at infinity - this is evident in [B1]–[B3] - yet they were unable to handle this realization productively. Their difficulty in deciding which function grows faster suggests a weak connection between the algebraic expressions and their graphical realizations. While their discourse related to the system was objectified - they interpreted the results correctly and demonstrated awareness of the procedure's purpose - their discourse on asymptotic behavior remained unobjectified. In [B1]–[B3], their explanations and gestures reveal a distorted perception, as they tended to interpret "growing faster" as following a more oblique trajectory, thus reinforcing their misunderstanding. Their approach can be described as product-oriented: once they opted for the system, they solved it with clarity and purpose, as shown in [B6]–[B10]. However, their flexibility was limited. They confidently applied the system but did not rely on asymptotic behavior as a meaningful tool. Even though their remarks in [B3] suggest an attempt to draw on asymptotic reasoning, they did not recognize its relevance for the solving process. In [B3], for instance, one student insisted that the outcome *"depends on the angular coefficient,"* while simultaneously using hand gestures that indicated growth toward infinity in an oblique fashion and the interview comments reinforce this lack of confidence. These observations show that the asymptotic tool had limited applicability for them in this context. Their struggle to use it effectively [B3] is a key factor behind the resistance that remained unresolved throughout the solving process.

Example from Task 2

This pair of engineering students initially identified the Cartesian axes visually, but they did not feel confident relying solely on this argument. They therefore attempted to justify their answer algebraically by solving a system, and, once they obtained the solutions, they seemed convinced of the correctness of their reasoning (see Figure 7.3). During the interview, they explained that they had first recognized the straight lines graphically and subsequently used the system of equations to determine their equations. However, the protocol shows that they actually found the intersection points, not the equations of the straight lines—yet they appeared unaware of this distinction. An excerpt of their discussion is reported below:

C1_I: So, did you find the intersection point that actually works?

C2_S5: Yes, I saw these two as the Cartesian axes.

(...)(The student initially looked confused then later concluded that he had misinterpreted)

C3_I: Okay, and how did you see them geometrically? (...)

C4_S5: Yes, because I imagined the other possibilities and I saw that they were all secant to the parabola, there were...(hesitation)

C5_I: Let's take a very steep line, $y = 100,000x$, extremely steep. Does it intersect the parabola?

C6_S6: At some point, yes.

C7_I: Why?

C8_S6: Yes, at some point, because it is almost like the y-axis, since the y-axis goes... okay.

C9_I: Alright, but if I shift just a tiny bit, a very tiny angle, so I take a line with an extremely large coefficient, $y = 10^{15}x$. Does it intersect the parabola? We are interested in the motivation.

C10_S6: Yes, because it is constantly greater - I mean, constantly greater... because it intersects it... let's see.

C11_I: You say it intersects it because even if it is extremely vertical, where does it intersect it? For large values of x, for small values of x, or closer to the origin?

C12_S6: For very small values of x, close to the origin, yes, or not...

C13_S5: I think... no, actually I don't think so. Because this very steep line grows much more rapidly than the parabola, so $y = mx$ is sometimes faster than the parabola, depending on m, but sometimes not. If m is small, no.

C14_I: So, from your reasoning, if m exceeds a certain value, you said that the line becomes too steep and the parabola cannot catch up anymore. Is that correct? Did I understand well?

C15_S5: Yes.

When the interviewer pointed out that they had not identified the Cartesian axes, the students showed signs of hesitation and uncertainty. At that moment, a form of resistance emerged: the tool they had relied on no longer appeared effective for justifying their claim, see [C1]-[C2]. This resistance persisted throughout the discussion. Indeed, the students did not return to the algebraic tool to validate their conjecture; instead, they relied exclusively on perceptual arguments and proposed conflicting answers.

One student claimed that the straight line and the parabola would intersect again, but only for small values of the angular coefficient. She later expressed doubts about her own argument,

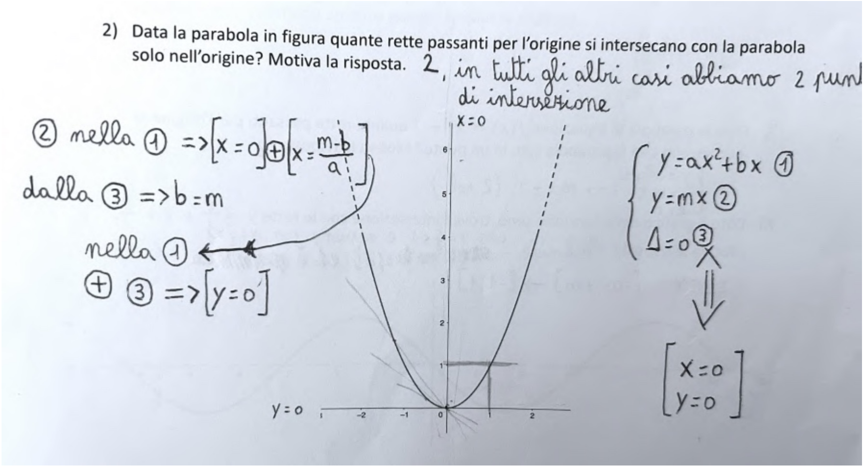


Figure 7.3: Students' protocol

revealing a lack of full awareness of the mathematical objects involved and an unobjectified discourse, in which objects are not treated as mathematical entities but rather as perceptual shapes [C13]. The other student argued that the intersection depends on the slope m, suggesting that for certain values of m the line "grows faster" toward infinity than the parabola [C13]-[C15]. This reveals difficulties related to asymptotic behavior, as it reflects the belief that a straight line may eventually exceed a parabola of the form x^2 in growth rate.

The students' flexibility also appeared limited. They did not use different ways to approach the problem and seemed to rely mainly on perception. Although they were aware of the tools available to solve the task, they did not shift efficiently between different realizations [C4]-[C10], which is another indication of resistance. Similarly, the applicability of previously used tools seemed restricted. They struggled to use a tool in a different situation (asymptotic behavior), and they did not correctly interpret the outcomes of the algebraic system, nor did they make productive use of the meanings derived from it. Their reasoning shows that they did not fully grasp the asymptotic behavior of functions, as they continued to treat the straight line as growing faster than the parabola. This also reflects an unobjectified discourse [C13].

The resistance remained unresolved because the students eventually abandoned the algebraic approach without interpreting the results and without establishing a correct connection between the different realizations involved in the task. The students' performance appears to be more process-oriented than product-oriented, as they do not seem fully aware of the meaning or implications of the results obtained. Moreover, their PSS seems limited, and they experience difficulties in overcoming the constraints of the task. In particular, they do not connect the different realizations involved, nor do they use the available algebraic tool in an efficient way. For instance, one student claimed that a generic straight line and the parabola would intersect again, but only for small values of m, while another student disagreed, relying on a distorted perception of the asymptotic behavior of the functions. The difficulty in dealing with a non-familiar task also emerges from their attempt to set up a system between a sheaf of straight lines and the equation of the parabola, which they eventually abandoned without correctly interpreting the results produced.

Example from Task 3

In this example, the pair of mathematics students began by attempting an algebraic approach: they set up the system (Figure 7.4) and tried to determine the degree of the resulting equation. Shortly afterwards, however, they turned to a graphical sketch of the functions, encountering difficulties similar to those experienced in Task 2.

D1_S7: Can't we set up a system of equations?

D2_S8: Yes, so let's write $x^2 = 2^x$*. Should we apply logarithms?*

D3_S7: Yes, I think that's the best approach. (...)(They apply logarithms to both sides but then get stuck)

D4_S7: If it's asking for points of intersection, we need to consider the degree of the equation, how many zeros it has...

D5_S8: When we find the intersection, the resulting equation would be $2^x - x^2 = 0$ *.*

D6_S7: Right, but how can you determine the degree of that equation?

(Moment of silence)

D7_S7: We can also visualize them graphically.

(...)(They draw the graphs (Figure 7.4). The drawing, partially erased, was nevertheless similar to the one shown in Figure 7.5.)

D8_S8: There will be two points of intersection.

D9_S7: Yes, because 2^x*... or it behaves like this* (redraws 2^x as narrower, closer to the y-axis) *same as before: if this grows like that and the other one opens...it's the same issue as before.*

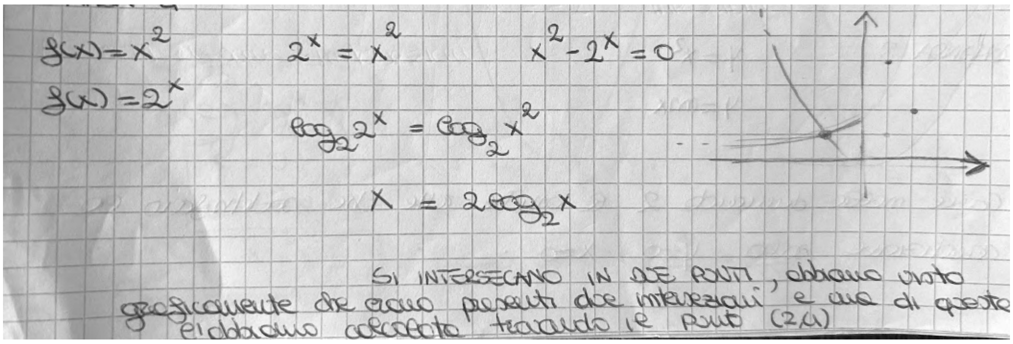


Figure 7.4: The actual drawing produced by the students in this example, which was partially erased "They intersect at two points; we saw graphically that there were two intersections, and we calculated one of them, finding the point (2, 4)."

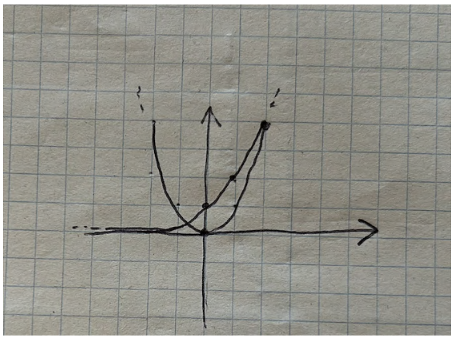


Figure 7.5: Representative example of a similar drawing as it appeared before being altered.

(...)(They plot several points to accurately sketch the parabola and the exponential function)

D10_S8: Ok, so in two points.

(...)(To justify this, they say they saw it graphically, and calculated one point of intersection: (2, 4)).

In the interview, the students explained that they interpreted the "faster-growing" function as the one whose graph appeared more slanted. This illustrates that, from the outset, they struggled to engage effectively with the task. Their first approach involved writing the algebraic system, but uncertainty quickly emerged - captured in the question, *"How can you determine the degree of that equation?"* - which prompted them to abandon the algebraic route and move to a graphical realization. The shift reveals a difficulty in managing the algebraic tool in this context.

Yet drawing the functions brought its own challenges. The students hesitated particularly over the long-term behavior of the exponential function, signaling a moment of resistance ([D6]–[D10]). While discussing their sketches, they explicitly referenced Task 2, identifying a parallel between the two situations (*"same issue as before"* [D9]).

This resistance was not resolved. The students were unable to establish a coherent link between the algebraic expressions and their graphical realizations, a difficulty tied to their distorted perception of asymptotic behavior. During the interview, they described exponential growth as continuing diagonally rather than tending toward the y-axis, reinforcing this misconception. Although they recognized that tools previously used might also apply here ([D8]–[D9]), they struggled to adapt them, revealing limited applicability of these tools within their PSS [D6].

Their attempt to coordinate different realizations was equally problematic. When sketching the functions, they relied heavily on plotting individual points, suggesting difficulty in connecting the algebraic form to the graphical behavior at infinity, and signaling reduced flexibility [D9]. Despite these obstacles, the pair maintained a product-oriented stance, as shown by their ongoing attempts to interpret the results arising from their solving process ([D4]–[D5]). Their discourse also showed signs of objectification: they moved beyond the immediate situation to articulate properties of mathematical objects, using terminology in a deliberate and reflective way.

Example from Task 5

In this example, a couple of mathematics students to determine whether the parabolas in item a) intersect decide to calculate the zeros with the x-axis. This approach seems to be inconclusive, so they decide to implement the system between the equations of the parabolas. In the item b) they reasoned on the widths of parabolas, formalizing this reasoning trying to express the link between the widths and the coefficient of x^2. In item c) tried to give an answer using the comparison of the coefficient of x^2, but then stated, *"maybe it is better to implement an algebraic system"*, the students seem to be unsure about how on proceed in their solving of the task and choose to use the system between the expressions of parabolas. We show the excerpt of their discussion during the solving process of the Task 5:

E1_S9: One is contained in the other.

E2_S10: Yes, it means that the first one only touches here, while this one touches here and here, and both are upwards because a is greater than zero. (S9 indicates the intersection of parabolas with the x-axis)

E3_S10: We need to be careful; it could be that one is narrower (inner parabola), *and one is wider* (outer parabola), *but sooner or later they will intersect for sure.*

E4_S9: Yes, sooner or later they will intersect for sure.

E5_S10: Yes.

E6_S9: Sorry, but if I want to find where they intersect, isn't it enough to just set up the system? It's even simpler and mathematically much more correct.

(They solve the system and move on to item b.)

E7_S9: This is the case I had in my head before. These (parabolas in the picture of the item b) *do not intersect; this one will keep opening, and this one will keep opening in a narrower way.*

E8_S10: One is contained within the other.

(S5 draws on the paper, which they later erased).

E9_S9: This one will continue to open (blue parabola) *and this one will also continue to open* (red parabola) *but more narrowly and should make a sort of diversions to meet the other one.*

E10_S10: They chase each other forever.

E11_S9: Because this one will also keep opening-it's not like it's going to close.

E12_S10: Earlier we said they could intersect or not, and we set up the system; here, since we don't have the equation, we only have the drawing...

E13_S9: We have the exact drawing, and it feels more automatic to think...

E14_S10: Parabolas do not intersect because the width of red parabola is smaller. How do we justify it?

E15_S9: The one with the bigger thing is narrower. (they don't recall the proper term for the coefficient, referring to it as thing)

(They try examples with $y = 4x^2$ and $y = x^2$, and draw some graphs.)

E16_S10: Don't try with the ones that only have x^2, because it's hard to tell at the vertex, how can you really see how wide it is or not? You'd need to test it by plugging in some points.

(S9 tries plugging in a few points and sees that the graph is narrower. They conclude that the larger the coefficient of x^2, the narrower the parabola is.)

E17_S10 (goes back to the earlier point): *The ones before intersected because this one here was wider* (points to the inner parabola), *and this one* (points to the outer one) *is narrower.*

E18_S9: The parabola with the largest coefficient of x^2 is narrower than the parabola with the

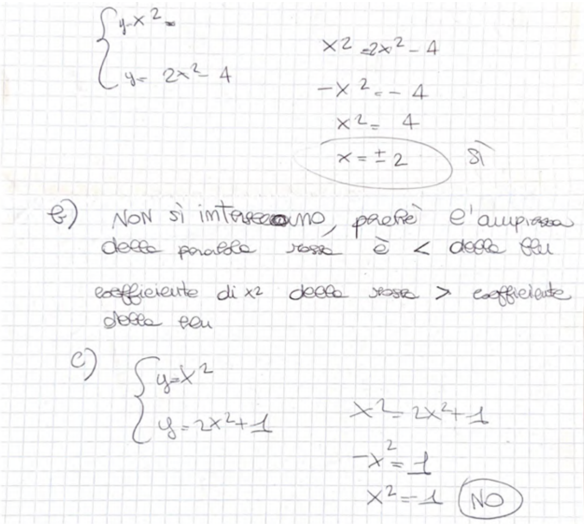


Figure 7.6: Example of resistance not overcome. Protocol of the students' solution to Task 5. a) "yes"; b) "do not intersect, because the width of the red parabola is smaller than the blue one. The coefficient of x^2 of the red parabola is greater than the coefficient of x^2 of the blue one"; c) "no"

smallest coefficient of x^2.

(They move on to item c and attempt to use coefficient comparison)

E19_S10: Then there are these (parabolas in item c), *which are like the previous ones* (referred to parabolas in item a), *but this one* (the second parabola in item c) *is shifted upwards and not downwards. So before, when it was shifted downwards, they intersected...*

E20_S9: But I think they will still intersect because we said the second one is narrower. Oh no, wait, it could be inside like the previous one, we'd better redo the system.

From students' responses in items [E1]-[E6], it is evident that their approach integrates both graphical and algebraic reasoning (Figure 7.6). However, uncertainty emerges clearly in their graphical interpretations, particularly in [E3]-[E4], where their reasoning is incorrect; they mistakenly assume that if the inner parabola appears narrower, intersections must exist. Additionally, students demonstrate uncertainty regarding the relationship between parabola width and the algebraic expression, as they hypothesize visually without explicitly considering the actual coefficient of x^2 in the equations. Later, in [E15]-[E18], one student explicitly remarks, *"Don't try with the ones that only have x^2, because it's hard to tell at the vertex, how can you* *really see how wide it is or not? You'd need to test it by plugging in some points".* This highlights their uncertainty regarding the coefficient-width relationship, prompting them to verify visually using specific points. The hesitation expressed in the question *"how can you really see how it is or not?"* represents a manifestation of resistance, marked by uncertainty and difficulty in recalling the connection between the coefficient of x^2 and the parabola's width (different realizations). This suggests that students encountered constraints in retrieving previous situations within their PSS.

The difficulty students experience in establishing and recalling connections between different realizations (graphical and algebraic) signals resistance to employing coefficient comparison as a tool. This resistance recurs clearly in the final stage, where students attempt to reuse a previously obtained result but, lacking confidence, ultimately revert to solving a system of equations ([E19]-[E20]). Here again, students struggled to recall previous situations within their PSS, despite the presence of items (a) and (b), which could have enriched their PSS and facilitated the overcoming of resistance. Students successfully applied coefficient comparison only when explicitly supported by graphical realizations ([E7]-[E18]). Although students recognize multiple ways to solve the task, their confidence in applying them independently is limited. Managing different realizations proves problematic, prompting students to seek algebraic verification (using a system of equations), rather than relying solely on geometric reasoning. For instance, in item c, although they initially considered comparing x^2 coefficients, they ultimately opted for solving a system algebraically ([E19]-[E20]), illustrating greater confidence in algebraic tools and thus limited flexibility ([E20]).

Their approach remains product-oriented; students clearly understand the meaning of solving the system and select this tool due to perceived simplicity. Nevertheless, managing alternative tools - such as coefficient comparison - poses challenges, revealing limited applicability and difficulty integrating different realizations. During interviews, students explicitly mentioned that comparing parabolas did not feel "spontaneous", admitting unfamiliarity and discomfort with such comparisons-even with linear functions. Thus, the applicability of tools used in previous experiences was restricted: despite familiarity with similar tools in linear contexts, as it was possible to observe in Task 1 where students compared angular coefficients of straight lines, students struggled to transfer these routines effectively to the unfamiliar setting involving parabolas. Furthermore, the presence of items (a) and (b) did not support this transfer: even when students recognized the situations as similar, the applicability of the tool remained restricted, as they did not succeed in using it in the new context [E19]–[E20]. The discourse was not fully objectified: at times, students referred to mathematical objects using vague terms such as "thing" (e.g., when discussing the coefficient [E15]). Moreover, the properties of mathematical objects were not always clearly articulated. For instance, the relationship between the parabola and the coefficient of x^2 appeared to be a point of struggle, and students did not seem fully aware of how to explore this connection, as suggested in [E16].

7.2 Discussion

The analyses presented in the previous section show that students experience various forms of difficulty when confronted with mathematical situations that differ from those they are used to. These difficulties primarily concern the management of the mathematical objects involved, and they echo results by Carlsen (2010,2018), who reports that learners often meet resistance when attempting to transfer a familiar tool into an unfamiliar environment. Our data indicate that this phenomenon remains visible even at the university level, recurring across all the examples discussed. Moreover, consistent with earlier studies, the specific realization through which a task is presented plays a role in shaping students' routine use (Gambini et al., 2024; Viola, 2023) and may itself contribute to generating resistance (see Chapter 6).

Resistance emerged particularly clearly in relation to asymptotic behavior. Students frequently interpreted growth at infinity as a diagonal tendency, even for parabolic or exponential functions. This distorted perception complicated their solving process in tasks and often led to prolonged uncertainty. In several instances, students attempted to rely on familiar tools but found themselves unable to adapt them effectively, sometimes reverting to algebraic systems as a way to control the general case. This illustrates Carlsen's (2018) argument that possessing a tool does not necessarily imply being able to use it in a flexible and effective way.

With respect to our research question (RQ3), when resistance is overcome, students' routines appear to evolve. They gain the ability to make sense of the chosen tool, interpret the outcomes it produces, and connect different realizations. In these cases, flexibility increases, and the tool becomes applicable across a wider range of situations. Collaborative discussion seems to play a decisive role in this development: interaction supports the articulation of mathematical ideas, helps dissolve initial obstacles, and strengthens both flexibility and applicability. Section 7.1.1 provides an example of this progression, showing how students moved from initial uncertainty to a coordinated use of algebraic and graphical reasoning.

Resistance generally surfaces when familiar tools seem inadequate for the new situation. Students who succeed in resolving it display greater freedom in choosing and combining methods ([A12]–[A13]) and a broader ability to recognize when previously used routines can be transferred ([A1]–[A3], [A15]). In this sense, resistance offers valuable insight into the constraints of the PSS that Commognitive Theory has not extensively addressed. Introducing the notion of resistance into a commognitive framework allows for a more fine-grained examination of why and how students modify their routines during the solving process. Our results suggest a clear pattern: overcoming resistance correlates with wide applicability and strong flexibility, whereas unresolved resistance corresponds to a restricted and rigid use of routines.

In contrast, the results also indicate an association between the persistence of resistance and the characteristics of students' routines. When resistance remains unresolved, students tend to exhibit limited flexibility and a narrow range of applicability. This is apparent (Section 7.1.2) where students rely confidently on a single routine but struggle to coordinate it with different realizations of the same mathematical object. Although they ultimately complete the tasks, their accounts during the interviews reveal uncertainty about how and when different tools should be activated. Episodes such as [B5], [D9] and [C13] show explicit difficulty in evaluating the appropriateness of a tool-particularly those related to asymptotic behavior-which in turn reduces students' confidence. Even when they perceive similarities to earlier tasks, their use of routines remains confined to familiar methods, revealing restricted applicability.

Across all examples, most of students showed product-oriented approaches and objectified discourse when working with tools that did not generate resistance. Conversely, tool-specific resistance often coincided with unobjectified discourse. Although most of pairs displayed exploratory intentions, the nature of this exploration varied depending on their degree of flexibility and applicability. Some students struggled to identify alternative approaches or to surpass the boundaries imposed by their PSS. We interpret these struggles as manifestations of resistance arising from the encounter with unfamiliar or poorly understood mathematical aspects. When resistance is resolved, flexibility and applicability expand, reinforcing Lavie and colleagues' (2019) claim that *"overcoming the constraints of the PSS is necessary for innovation"* (Lavie et al., 2019, p. 160).

Finally, incorporating resistance into a commognitive analysis provides a means to examine how exploratory and ritualized routines interweave. As noted by Viirman and Nardi (2019) and by Christiansen and colleagues (2023), these forms of acting should not be seen as mutually exclusive; rather, they coexist and interact dynamically. Resistance sheds light on this interaction by highlighting how routines transform when obstacles are encountered. Although some indications regarding substantiability and agentivity emerged, these dimensions were not explored sufficiently to draw firm conclusions. Future work could investigate these aspects more thoroughly, as well as bondedness-hard to observe in the current tasks. We intend to address this by designing multi-step tasks that allow us to examine how students use outputs as inputs, especially in relation to resistance.

Chapter 8

Mathematical reasoning and resistance

In this chapter, we turn to the fourth research question (RQ4), which investigates students' approaches in relation to aspects of mathematical reasoning, as well as the ways in which they substantiate their reasoning when resistance arises during the solution of Calculus tasks. Some of the insights presented here draw on results reported in Viola et al., Submitted. Within the broader project, students' written work, video-recorded sessions, and interview transcripts were analyzed. For this specific question, these materials were revisited in detail, focusing on the approaches students developed and on how these approaches intersected with aspects of mathematical reasoning and the substantiation of their arguments in moments where resistance emerged throughout the solving process.

8.1 Results

8.1.1 Approaches used by the students

In what follows, we address the first part of research question by outlining the various approaches that students employed while working on Tasks 1, 3, and 6 (see Figure 4.3). For each identified approach, we characterized its main features and examined the processual and structural aspects of mathematical reasoning involved in its development and substantiation, drawing on the dominant forms of reasoning displayed by the students. In reporting the results, engineering and mathematics students are labelled as E and M, respectively, with numbers (1–8 for mathematics and 1–11 for engineering) used to distinguish the student pairs. The approaches are presented in summary tables, which include both their descriptions and the number of student pairs who adopted them. Because a single pair could engage with more than one approach during the solving process, the total number of occurrences exceeds the number of participating pairs. The tables are followed by a detailed reconstruction of the students' solving trajectories, specifying which approach they initially adopted, how their approach evolved, and whether they later returned to an earlier line of reasoning.

In the first task, the approaches that students developed fell into four distinct types for Task 1a and into five further types for Task 1b. For Task 1a—where students were presented with an algebraic realization of two linear functions—several different approaches emerged.

Approaches	Description	M	E
System	This approach involves determining the intersection point of two straight lines by setting up and solving a system of equations. For instance, given the equations $y = 2x - 1$ and $y = x + 4$, students solved the system algebraically and, upon finding a unique solution, concluded that the lines intersect.	4	8
Different angular coefficients different slopes	In this approach, students relied on the observation that the lines have different angular coefficients (i.e., slopes). Recognizing this difference served as a sufficient criterion to assert that the lines are not parallel and therefore must intersect.	2	2
Different angular coefficients and algebraic proof	Here, students considered the difference in slopes but also sought additional validation by solving the system of equations or providing another algebraic check, such as confirming that the x-values do not cancel. This hybrid approach was categorized separately to reflect the combination of slope comparison and algebraic proof required to justify the intersection.	2	1
Drawing the intersection point	In this approach, students used a graphical method, drawing the two lines and identifying the intersection point visually.	1	1

Table 8.1: Descriptions of the approaches and the number of student pairs adopting each for Task 1a.

As summarized in Table 8.1, some students drew on more than one approach while working on the task. This occurred for one pair of mathematics students (eight pairs in total, but nine recorded uses of approaches) and for one pair of engineering students (eleven pairs, but twelve recorded uses). In both cases, the students initially solved the system of equations and only later turned to the graphical realization of the two lines. The mathematics pair eventually returned to the system after detecting a discrepancy between the two methods; they repeated the procedure and identified an error in their original algebraic work.

Overall, solving the system of equations between the two straight lines was the approach used most frequently, especially among engineering students, even though it was not the most efficient approach. When asked during the interviews whether they had considered relying on angular coefficients, many students replied with comments such as: *"It seemed faster to me"* (E4), *"It's the first thing that came to mind"* (E10), or *"It's easier for me"* (E3). For these students, setting up the system operated as an intuitive, almost automatic choice—despite involving more computation.

In several instances, students reasoned by implicitly appealing to a contrapositive argument, basing their explanation on the property that two lines are parallel only when their angular coefficients coincide. As one pair stated: *"Two lines are parallel only if their angular coefficients are the same. In this case, they aren't, so they will intersect"* (M4). They then supported this observation algebraically by equating the expressions of the two lines (i.e. $2x - 1 = x + 4$) to verify that *"the x-values don't disappear."* This behavior exemplifies the "Different angular coefficients + algebraic proof" approach: although students invoked a relevant mathematical property—and appeared to substantiate their reasoning through what could be interpreted as a contrapositive argument—they still felt the need to supplement it with an algebraic verification, albeit without solving a full system.

Despite the disciplinary differences between the two cohorts, both mathematics and engineering students often adopted similar approaches. For example, in Task 1a, several pairs from each group relied solely on the system of equations without considering angular coefficients, explaining choices such as *"I didn't think of it; I always start with the calculations, not by observing"* (M6) or *"It's immediate, it's the first thing"* (E9). When two pairs selected the same approach for a task, their behavior tended to remain comparable across later tasks as well; however, discrepancies frequently emerged in how they interpreted the outcomes, which sometimes led them toward different conclusions.

Drawing on Jeannotte and Kieran's (2017) distinction between processual and structural aspects of mathematical reasoning, the approaches observed in Task 1a can be interpreted as follows:

- **System and Drawing the intersection point:** These approaches are chiefly processual, relying on algebraic manipulation, computation, and validation through procedures rather than on explicit reference to mathematical properties.

- **Different angular coefficients / different slopes:** These approaches are primarily structural, as students justified their conclusions by comparing slopes or appealing to graphical realizations, without performing calculations.

- **Different angular coefficients + algebraic proof:** This approach blends both processual and structural elements. Students invoked general properties while also performing algebraic manipulations, moving back and forth between the two modes of reasoning.

In contrast, Task 1b provided only a graphical realization, and—unsurprisingly—the system-solving approach did not appear. The approaches that surfaced in this case are listed in Table 8.2.

Across both Task 1a and Task 1b, all approaches ultimately enabled students to arrive at a correct determination of the lines' intersection. In Task 1a, the system approach emerged as the most frequently used, particularly among engineering students. In contrast, in Task 1b the not parallel approach became the predominant choice. The comparison between the two tasks highlights how the nature of the given input influences the strategies selected by students, echoing results reported in preliminary phase (see Chapter 5). When the prompt included algebraic realizations, students tended to resort to solving a system of equations; when such expressions were absent, they relied more heavily on visual inspection of the graph and on the not parallel property to substantiate their reasoning.

An additional element worth noting is that half of the mathematics students in Task 1b sought some form of confirmation of the not parallel argument, whereas engineering students generally appeared less inclined to verify or formalize their conclusions. This tendency is also reflected in the Graphically approach, which was adopted exclusively by engineering students. Expressions such as *"Can we explain it mathematically?"* (E2) and *"It is mathematically more correct"* (M1)

Approaches	Description	M	E
Different angular coefficients/ different slopes	In continuity with the reasoning observed in Task 1a, some students inferred that the two lines had different slopes by inspecting their graphical realizations. In this case, the identification of the slope difference arose primarily from what they saw on the graph rather than from explicit reference to the underlying algebraic concept.	1	3
Drawing the intersection point	Other students chose to prolong the drawn lines, using this extension to check visually whether the lines met or to reinforce their conclusion about the presence or absence of an intersection.	3	3
Not parallel	Some pairs visually recognized that the lines were not parallel, yet did not articulate this observation in terms of angular coefficients or slopes. Unlike the Different slopes approach, it remains uncertain whether they explicitly associated the graphical non-parallelism with the corresponding mathematical property.	6	6
Proof with angles	A further approach involved noticing that the two lines had distinct angular coefficients and substantiating this by drawing an additional line. By comparing the angles formed between each given line and the newly introduced one, students concluded that the original pair of lines could not be parallel.	1	0
Graphically	Finally, some students relied purely on the graphical realization to locate the intersection point. They neither invoked notions such as parallelism or slope nor extended the lines; instead, their conclusion was drawn solely from visual perception of the provided graph.	0	2

Table 8.2: Descriptions of the approaches and the number of student pairs adopting each for Task 1b.

appeared frequently in both groups, indicating that algebraic reasoning is often perceived as mathematically more legitimate or rigorous.

Drawing on Jeannotte and Kieran's (2017) model of mathematical reasoning—which distinguishes between processual and structural aspects—it is possible to characterize the different approaches more precisely.

- **Drawing the intersection point and Proof with angles:** these approaches display a predominance of the processual aspect, as students focused on verifying, illustrating, or providing a form of validation either through a drawing or through a constructed proof.

- **Different angular coefficients / different slopes and Not parallel:** as in Task 1a, these approaches reflect a structural orientation, since students' arguments rested mainly on mathematical properties rather than on computations or formal proofs.

- **Mixed approaches (e.g., Not parallel combined with Drawing):** in these cases no single aspect prevails. Students intertwined structural reasoning (e.g., comparing slopes) with processual reasoning (e.g., visually confirming their inference).

In Tasks 3 and 6, the analysis becomes more intricate: students rarely relied on a single approach but frequently blended two or more approaches, sometimes shifting between them at different moments in their work. In total, five approaches were identified (see Table 8.3), though several pairs drew on more than one, as the table indicates. A detailed account of these combinations and their use is presented in the following section.

In Task 3, two mathematics pairs began by examining the graph and later turned to the system as a means of verification. In addition, two mathematics pairs and three engineering pairs drew on all three approaches, though they navigated them in distinct ways. More precisely, the two mathematics pairs initiated their reasoning with the AG approach and subsequently shifted to the S approach when they found it challenging to handle asymptotic behavior directly from the graphical realization. For two engineering pairs, the progression unfolded differently: one pair first relied on the G approach, then proceeded to set up the system, and finally—during the interview—introduced considerations related to asymptotic behavior. The other pair also started graphically, attempted to construct the system without completing its solution, and later moved toward an argument grounded in asymptotic behavior.

In Task 6, two engineering pairs began with graphical reasoning and later used the system to either reinforce or validate their conclusions. Another engineering pair, after working within the AG approach, also constructed the system for confirmation, noting during the interview that *"The calculation is less intuitive, longer, but also more reliable" (E8).*

Drawing on Jeannotte and Kieran's (2017) framework, the approaches can be distinguished according to their emphasis on processual or structural dimensions of mathematical reasoning:

- **G and AG:** These approaches primarily exhibit a structural character, as students relied on the behavior of the functions at infinity or on their graphical realization rather than producing explicit algebraic realizations. Their reasoning was grounded mainly in functional properties and asymptotic behavior.

Approaches	**Description**	**Task 3**		**Task 6**	
		M	E	M	E
Graphically (G)	Students relied primarily on the graphical realization, focusing on tangents or on the sheaf of lines without considering asymptotic behavior. In this line of reasoning, they often sketched specific lines from the sheaf or drew tangents to the curve, basing their conclusions on particular visual examples or on the perceptual impression of lines positioned closer to the y-axis.	6	3	2	6
Asymptotic behavior and use of the graph (AG)	Students examined the graph while explicitly invoking the asymptotic behavior of the functions. Their reasoning took place directly on the graphical realization—for instance, by referring to tangents or commenting on the differing growth rates of the parabola and the straight lines—and they used these observations to determine whether a line located nearer to the y-axis would intersect the parabola. This category includes all pairs who, either during the task or in the interview, mentioned asymptotic behavior alongside other graphical considerations.	1	8	1	5
System (S)	Students formulated a system of equations involving either the entire sheaf of lines or specific lines together with the equation of the function (parabola or cubic).	3	2	0	1
System + Graph (SG)	Students blended algebraic and graphical reasoning: they solved a system of equations (e.g., between a straight line and the parabola or cubic function) while simultaneously using the graph to support or validate their conclusions.	0	1	5	2
Asymptotic behavior using the graph + System + Graph (ASG)	Students brought together all previously described strategies in a coherent manner: they reasoned from the graph by referring to asymptotic behavior, applied algebraic methods through a system of equations, and used the graphical realization to confirm or reinforce their arguments. What distinguishes this group from other pairs is the coordinated integration of these elements rather than a simple shift from one approach to another.	2	1	0	0

Table 8.3: Descriptions of the approaches and the number of student pairs adopting each for Task 3 and Task 6

- **SG and ASG:** These reflect a more balanced interplay between processual and structural aspects. Students combined algebraic work with reference to mathematical properties, while simultaneously drawing on the graph to guide or justify their conclusions.

- **S:** This approach is predominantly processual, since students focused mainly on algebraic manipulation, making only limited use of general properties or theorems.

Across Tasks 3 and 6, no clear dominance of processual reasoning emerged, as the exclusively processual S approach did not appear on its own. In these tasks, the approaches employed by students did not always lead to correct conclusions. In several cases, relying exclusively on the graph—or on reasoning about asymptotic behavior derived from the graphical realization—was not sufficient. Students often found it challenging to work effectively with these realizations, and such difficulties also emerged in how they interpreted the outcomes obtained. In total, eight pairs did not reach correct results in either task, and these were largely the same pairs across both tasks. Moreover, even when students adopted the same approach, their interpretations sometimes differed, which resulted in inconsistent answers despite the use of an identical approach.

The distribution of approaches differed between Task 3 and Task 6, even though the same set of approaches appeared in both. Once again, the presence of a graphical realization influenced the students' choices: the system-based approach never appeared in isolation but was always combined with some form of graphical reasoning. In Task 3, the AG approach was observed only among engineering students. In Task 6, however, it was employed by pairs from both groups, remaining slightly more frequent among engineering students and appearing also in one mathematics pair. Additionally, in Task 6, the G approach was adopted more often, whereas in Task 3 it was less common, with students tending instead to combine multiple approaches. The full integration of all three approaches was less frequent in Task 6, a tendency that may reflect students' increased confidence after having already engaged with Task 3.

8.1.2 Juxtaposing mathematics students' and engineering students' approaches

In this section, we provide our response to the second part of the research question. Clear contrasts emerged between the two cohorts. In Task 3, engineering students tended to cluster around the AG approach, whereas mathematics students showed a more varied distribution across the remaining approaches. This pattern shifted in Task 6: most engineering pairs relied on G and AG, while mathematics students predominantly used G and SG. Only one mathematics pair employed AG, and none adopted a full combination of all three approaches, while engineering students were represented in every identified category.

Determining whether an approach should be regarded as correct or incorrect was not always straightforward, since much depended on how students interpreted the task and on the way they coordinated multiple approaches. Reasoning directly from the graphical realization is not mathematically inappropriate per se, yet it may confine the analysis to the visible portion of the graph, encouraging conclusions driven mainly by perception. For instance, some pairs stated: *"There are infinitely many lines passing through a point and therefore intersecting the parabola only once"* (E9), or *"It's not just that tangent there ... for example, that one has only a single intersection at the origin"* (M2). Nonetheless, if students implicitly consider the behavior of the function toward infinity—even without naming it explicitly—the correct conclusion may still be reached. One pair remarked, *"By not focusing only on the graph, yes"*; *"We excluded the other lines because we imagined it"* (E4). These students did not explicitly refer to asymptotic behavior during their discussion, yet when asked whether a line close to the y-axis intersects the parabola, they answered: *"No, because the parabola tends to infinity faster"* (E4).

Graphical reasoning alone can sometimes be sufficient, especially when students recognize the Cartesian axes in the realization and use them coherently. Conversely, relying exclusively on the system approach prevents students from identifying vertical lines, since the system only yields lines of the form $y = ax + b$. In such cases, an additional approach must be combined with the system to produce a correct response. By contrast, asymptotic behavior alone may already allow students to reach the correct conclusion, provided that such reasoning is applied correctly and can be meaningfully related to the graphical realization. However, as illustrated by the following case, students using the AG approach (e.g., E10) encountered difficulties with asymptotic reasoning in both tasks.

In Task 3, during the interview, the pair (E10) reconsidered their earlier reasoning, suggesting that a line very close to the y-axis might not intersect the parabola again because it *"opens more slowly,"* whereas the parabola *"opens faster"*. Their explanation revealed a perceptual interpretation of *"speed,"* linked to the obliqueness of the graph: *"The parabola opens a bit more as it goes to infinity...while the line opens but always a bit less."* In Task 6, similar issues emerged. They observed: *"Maybe it's a bit different from the parabola... because it's narrower,"* and *"...this one grows straighter."* When asked which one grows faster, one student named the parabola, the other the cubic. One explained: *"This one stays here, it doesn't go that way,"* accompanied by a gesture comparing a steep ascent with a diagonal trend, concluding that the parabola grows faster toward infinity.

When students combine several approaches, they possess all the conceptual resources needed to obtain the correct solution; however, the final outcome still depends on how they interpret and coordinate what each approach yields. This is evident in comments such as: *"Ah, of course! The straight line is slower than the parabola... so at the beginning it stays above and later it goes below—so there are two intersections!"* (M4), or *"It's strange... from the graph it seemed different... but yes, it makes sense; the parabola increases more steeply, so even if the line has a slope tending to infinity, at some point the parabola will meet it."* (E8). These cases illustrate an application of the ASG approach, where students connected diverse approaches, interpreted the outcomes of the algebraic system, related them to the graphical realization, and integrated reasoning about asymptotic behavior.

Another instance of the ASG approach occurred in Task 3 (see Figure 8.1). A pair of mathematics students (M3) set up a system between $y = ax^2$ and a family of lines through the origin. Their reasoning was characterized by hesitation and uncertainty, as they attempted to prove the existence of infinitely many lines —perceived from the graph— by looking for a condition on the angular coefficient of the straight lines. Yet moments later, doubts resurfaced. They retracted, acknowledging that regardless of m and a, the line and the parabola still intersect: *"There are still two intersections... you can always find them."* After further hesitation, they settled on the statement that only two intersections exist. Reflecting on their work, they recognized the influence of perceptual reasoning: *"We were stubborn, trying to prove something that couldn't be proven."* Asked about behavior at infinity, one concluded: *"Going to infinity, I see it as always spreading more."* This episode will be analyzed in detail in the following section (8.1.4).

8.1.3 Manifestations of resistance associated with the approaches

In this section, we provide an overview of how resistance manifested in relation to the approaches employed by students and the dominant aspects of mathematical reasoning observed in their solving processes (Wertsch, 1998). We focus particularly on whether students were able to overcome resistance, or not overcoming, or remained in an intermediate state, in which they neither resolved the difficulty nor shifted to a different approach, instead expressing uncertainty

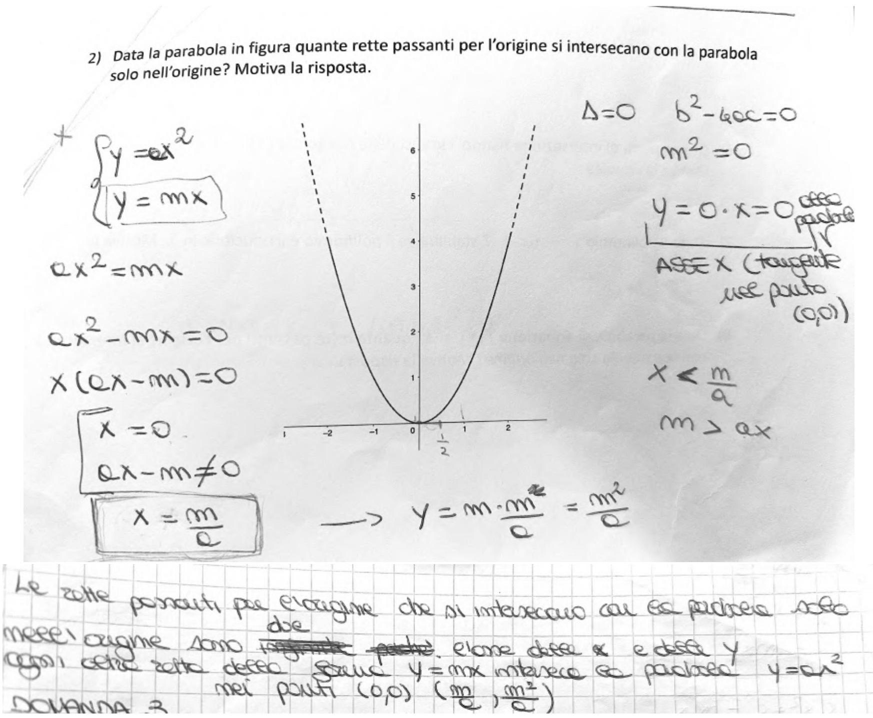


Figure 8.1: Figure 1a and 1b: Pair's protocols In b: "The lines through the origin that intersect the parabola only at the origin are two: the x-axis and the y-axis. Every other line of the form $y = mx$ intersects the parabola $y = ax^2$at the points $(0, 0)$ and $(m/a, m^2/a)$. Deleted part: infinite, because"

about their task solution. This intermediate state differs from the case of not overcoming resistance because students did not abandon the tool; however, they remained unsure about their answers. The explicit manifestation of this uncertainty is taken as evidence of an intermediate state of resistance. Instances where no clear signs of resistance were evident were classified as "no resistance." The analysis of students' interactions concentrates on Task 3 and Task 6, as these tasks were less familiar to the students and therefore revealed more pronounced examples of resistance, unlike Task 1, which, being relatively familiar, elicited minimal resistance. The distribution of resistance, in relation to both mathematical reasoning and the approaches used by students, is summarized in Table 8.4 and Table 8.5 for Tasks 3 and 6, respectively.

A clear contrast emerges in the category of not overcoming resistance: in Task 3 this occurred 16 times, whereas in Task 6 it appeared in 11 instances. The categories overcoming resistance and no resistance, instead, display only modest differences. In Task 3, these two categories together include 9 pairs (8 plus 1), while in Task 6 they involve 11 pairs (6 plus 5). This shift indicates

Task 3	Structural		Processual	Processual & Structural		Total
	G	AG	S	SG	ASG	
Resistance overcome	2	1	3	0	2	8
Resistance not overcome	6	7	2	1	0	16
In the middle	0	1	0	0	1	2
No resistance	1	0	0	0	0	1

Table 8.4: Number of student pairs by manifestation of resistance, approach and reasoning aspects used.

Task 6	Structural		Processual	Processual & Structural		Total
	G	AG	S	SG	ASG	
Resistance overcome	1	1	2	2	0	6
Resistance not overcome	5	4	1	1	0	11
In the middle	0	1	0	0	0	1
No resistance	3	0	0	2	0	5

Table 8.5: Number of student pairs by manifestation of resistance, approach and reasoning aspects used.

that, although resistance persisted more strongly in Task 3, a larger number of students in Task 6 either succeeded in overcoming it or did not encounter it at all. This tendency may reflect a form of progression across the solving process, whereby students began to identify structural similarities between the two tasks and applied the tool without experiencing the same obstacles. The intermediate condition shows only a minimal variation: two pairs in Task 3 and one in Task 6.

Considering the approaches adopted, resistance that remained unresolved is most frequently linked to the G approach and the AG approach. In the case of the G approach, students encountered difficulties in working solely with the graph without taking into account asymptotic behavior or growth at infinity. Often they switched to solving the system and obtained a result, yet they did not use the outcome of the system to make sense of the graph. As a result, resistance associated with this tool remained unresolved. Within the AG approach, students repeatedly showed uncertainty regarding the asymptotic behavior of the function, frequently interpreting growth at infinity in an oblique fashion, as though the function possessed an oblique asymptote. This incorrect interpretation created substantial obstacles, and only exceptionally (one pair in Task 3 and one in Task 6) did students manage to overcome resistance while continuing to use the same approach. Overall, students who relied on graphical reasoning (AG or G) experienced more pronounced difficulty in both tasks, with this mathematical idea becoming a central expression of resistance.

For instance, in Task 3 one student remarked during the interview: *"I'm quite convinced there is a limit; a limit line such that some lines fall inside it and others do not"* (E7), later adding: *"There are infinitely many within a certain interval"* (E7). Similarly, a pair of mathematics students argued: *"It depends on the slope. There might be a slope for which the parabola grows faster than the line"* (M6). These statements reflect the notion of a "limit line" that would bound a region containing infinitely many lines intersecting the parabola at a single point. Analogous reasoning appeared in Task 6. For example, one engineering student stated: *"If we approximate the slope of the curve with that of a straight line, then there will surely be a line parallel to the trend of the graph, and therefore below that line ..."* (E3). Another pair commented: *"We have to rule out a range of lines located in the first and third quadrants"* (E8). These excerpts show how a distorted understanding of asymptotic behavior heavily shaped students' reasoning and the way they substantiated their conclusions.

8.1.4 Examples of protocols of students

In this section, we present examples from two pairs of students (one from mathematics and one from engineering). These two pairs were chosen because they make us able to highlight the differences in the approaches they employed and the prevailing aspects of mathematical reasoning revealed. Moreover, differences between the two pairs can also be observed in terms of the issue of resistance across the two tasks.

Example of mathematics students A pair of mathematics students in Task 3 set up a system with $y = ax^2$ and the bundle of lines passing through the origin. They reason about the discriminant and state that it must be set equal to 0: *"As if we were finding the tangent by setting the discriminant equal to zero."* From this, they identify the x-axis as the tangent. Next, they consider the vertical line, with one student noting that all lines very close to the y-axis also seem to work. Reflecting on the results of the system, they remark: *"all lines that satisfy this condition."* They then recognize a contradiction: although they initially thought they found only one intersection, they reason again and conclude *"we need a condition on m,* *not on x."* One suggests: *"If I take one with a very small slope…This parabola goes to infinity faster than the line, so they don't intersect anywhere else."* At this point, they argue that if $m > ax$, the functions do not intersect. Believing they had identified a condition, they conclude there must be infinitely many lines. They attempt to substantiate this with the remark: *"It's always possible to find a line with a slope smaller than the parabola's width."* Nevertheless, doubts quickly emerge: they revisit their reasoning and say that regardless of the values of m and a, the line and parabola will still intersect at a point. This leads them to question their earlier claim, concluding instead that intersections must eventually occur. They then shift their attention to the condition $x = m/a$, but confusion persists: *"There are still two intersections.""You can always find them."* Eventually, they change their minds again and settle on the conclusion that there are only two intersections. During the interview, they acknowledge having relied too much on visual perception and admit uncertainty about their answer. One of them reflects: *"We were stubborn, trying to prove something that couldn't be proven."* When asked whether they considered the functions' behavior at infinity, one responds: *"Going to infinity, I see it as always spreading more."* The following excerpt illustrates a part of this discussion. To better clarify the reasoning, we have added comments and references to the figure, and explicitly indicated manifestation of resistance in the students' discourse. In the transcript, S1 and S2 denote Student 1 and Student 2, respectively (a notation that will also be used for Task 6).

They read the text. (...)

A1_S2: In fact, it comes out as $m = 0$, *which makes sense.*

A2_S1: Of course, that's the tangent.

A3_S2: I mean, if I substitute back into the generic line equation, it would be $y = 0 \cdot x$, *which is just 0, and that's the x-axis.*

A4_S1: And that's the tangent.

(They write it down.) *A5_S2: But also this one* (pointing to the vertical line).

A6_S1: Yes, but also all these ones here (pointing to lines close to the y-axis).

A7_S2: Right…

A8_S2: So starting from this, if we substitute the two solutions: with $x = 0$, *we get* $y = 0$, *which is this (points on the graph). And if we plug in* $x = m/a$, *then* $y = m \cdot m/a = m^2/a$. *So in theory all the lines satisfy this condition.*

A9_S1: It's not guaranteed that they only intersect the parabola once.

A10_S2: Ah, right, only one intersection with the parabola, exactly, because this one passes through the origin... (points at a line with two intersections as an example).

A11_S1: So that's the second intersection, basically (referring to what they found algebraically, see 8.1). *Now we should reason it out.* (sign of resistance)

A12_S2: About when it intersects only once...

A13_S1: When this second intersection doesn't exist. Because $x = 0$ *is always there, since we imposed that the line is* $y = mx$*. But we want that to be the only intersection. Ah wait,* $ax - m$ *must not be equal to zero, so it shouldn't be a solution.*

A14_S2: Ah, right.

A15_S1: So x *must be different from* m/a*. No wait, sorry, we need a condition on* m*, not on* x*.*

A16_S2: Because from the graph, in theory, the lines intersect the parabola only at the origin.

A17_S1: Why 2? If I take one with a very small slope...

A18_S2: Ah, right.

A19_S1: ...This parabola goes to infinity faster than the line, so they don't intersect anywhere else. (distorted perception of the asymptotic behavior)

A20_S2: We should find the slope where the line meets the parabola, so a condition on m*, the slope of the line, so that from that point onward...*

A21_S1: I think this is the "cutoff point," when $x = m/a$*.*

A22_S2: Ah, when $x = m/a$*, they intersect.*

A23_S1: So when m *is... if* $x < m/a$*, then* $m > ax$*.*

A24_S2: Yes.

A25_S1: Then they don't intersect.

A26_S2: They don't intersect.

A27_S1: Because we're here (points to the paper, giving examples). *Once it passes this point, it starts intersecting.*

(...)

A28_S1: I'm just not fully convinced about...

(hesitation sign of resistance)

A29_S2: More than anything, we can't determine an exact number. We can only determine *conditions for which...*

A30_S1: But wait, the question is: how many lines pass through the origin? If we've found a condition on m, then there are infinitely many lines. Like, even if it's just from 0 to $\frac{1}{2}$, since x works, multiplied by... you find the right m. Anyway, in that interval there are infinitely many numbers, so there are infinitely many lines. If the question is "how many", it's infinite, because m can take infinitely many values, even if the interval is small.

A31_S2: Are we sure?

A32_S1: Yes.

(...)(They start writing the solution and wonder how to phrase it and as they write, doubts arise)

A33_S2: The lines through the origin that intersect the parabola only at the origin are infinite... hmm, why?

A34_S1: Because...

A35_S2: Let's write: because it's always possible to find a line with a slope smaller than the parabola's width, I don't know if it's called width.

A36_S1: I don't remember.

(Silence)

A37_S1: Now I'm starting to have some doubts, I mean, we found this intersection, m/a...(uncertainty, sign of resistance)

A38_S2: Or we could try an example, let's take a line...

A39_S1: Because it feels like the way we found it, it's as if this point always exists, whatever m and a are, I always get this point where they intersect.

A40_S2: Yeah, I'm thinking...

A41_S1: Maybe what we said isn't actually true.

A42_S2: Either way, sooner or later they intersect.

A43_S1: Because the intersection point is this m/a. So whatever m and a are, the intersections are at $x = 0$ and at $x = m/a$. So even if we take an m...

A44_S2: I don't know, let's try a numerical example, like setting aa to something. Let's move on for now and come back to this later?

A45_S1: Yeah, I agree, I'm stuck.

(They come back to it after a while.)

A46_S2: There are at least two. And the y-axis comes from $x = 0$ as a solution, so to speak.

A47_S1: I mean, this $x = 0$ actually happens only when $m = 0$. So I'm convincing myself that there can't be more.

A48_S2: There are just two.

A49_S1: Like, in the limiting case of very small m and large a... in the limit m is basically 0, and then it would just be $x = 0$, the y-axis.

A50_S2: So let's change it.

They modify their answer, writing that there are two lines, the coordinate axes, and that every other line of the form $y = mx$ intersects the parabola at the points $(0, 0)$ and $(\frac{m}{a}, \frac{m^2}{a})$ (see Figure 8.1).

In Task 3, this pair adopted ASG approach, as they simultaneously employed algebraic and graphical reasoning, attempting to integrate the results of the system with their graphical interpretations. Resistance emerged when they began reasoning about the behavior of the functions at infinity. At that point, the algebraic results obtained from the system appeared obscure to them, leading the students to try to substantiate the existence of infinitely many lines by interpreting the system's solution as a kind of "cutoff point", that is, the last line intersecting the parabola at two points. As the students themselves confirmed, their reasoning was driven by an attempt to prove what they initially perceived, which was likely influenced by a distorted perception of the asymptotic behavior of the function. Their discourse reveals a clear need to formally substantiate the impressions derived from perception. However, the lack of connection between these perceptual impressions and the results of the algebraic realizations appears to be closely related to the manifestation of resistance. This approach does not reveal a clear predominance of one single aspect of mathematical reasoning (Jeannotte and Kieran, 2017); rather, the students attempted to keep algebraic and graphical reasoning together, simultaneously working with mathematical properties and algebraic realizations. However, the effort to coordinate all these aspects seems to have contributed to the emergence of resistance. As Carlsen' results (2018) highlight, there is a significant difference between knowing a tool and using it within a specific context (Wertsch, 1998). In this case, the students knew the tools they were employing, evidenced by the correct statement *"This parabola goes to infinity faster than the line"*, yet their graphical interpretation was flawed. In particular, they perceived the growth at infinity in an oblique manner, as confirmed in the interview: *"Going to infinity, I see it as always spreading more."* Resistance here appears not to have been fully overcome. The students themselves admitted in the interview that they were unsure about their conclusion, revealing lingering confusion and hesitation. Thus, this case can be interpreted as an intermediate state of resistance, where the obstacle was neither completely resolved nor completely abandoned. In Task 6, the pair constructed the system between the cubic function and a generic line passing through the origin. From their reasoning, they concluded that additional intersections exist only when the slope m is positive. They connected the algebraic results to the graphical realization, arguing that in this case infinitely many lines are possible, since all lines with $m > 0$ satisfy the condition. The following excerpt illustrates the students' discussion.

(They read the text.)

B1_S2: Let's take $f(x) = x^3$ *and then a generic line passing through the origin.*

B2_S1: Yes, always $y = mx$.

B3_S2: $y = mx$*, and let's find the intersection.*

B4_S1: Yes.

B5_S2: $mx = x^3$ (they solve the equation).

(...)

B6_S2: So one solution is $x = 0$*, and another is* $x = m$.

B7_S1: So basically...

B8_S2: If $x = 0$*, then* $y = 0$.

B9_S1: Right, and that's the first line through the origin, so it's the first intersection with the cubic.

B10_S2: But...

B11_S1: Then I get other intersections only if m is positive (points both to the figure and the algebraic equation, linking the two realizations). *If m is positive, I have these two additional intersections. If m is negative, my line is over here* (uses the pen to simulate a line in the second and fourth quadrants, then moves it as if simulating a sheaf of lines) *and I don't get other intersections.*

B12_S2: If m is negative, we'd have a square root of...

B13_S1: Right, it doesn't exist, so I have no intersections. But if m is positive, I always find two more intersections.

B14_S2: At least two.

B15_S1: Yes, besides zero, of course.

B16_S2: And so they are...

B17_S1: (re-reads the text) *Well, infinite this time* (reference to the Task 3), *maybe.*

B18_S2: Infinite.

B19_S1: Because all positive m values work.

(They write the answer: "The lines are infinite, I can take lines of the type $y = mx$ with $m > 0$")

B20_S2: Or even equal, greater or equal, after all, if $m = 0$*...*

B21_S1: No, if $m = 0$*, then there's only one intersection.*

B22_S2: Right, ok.

In Task 6, the students adopted an SG approach, simultaneously using both the system and the graph and connecting the two realizations, though without explicitly referring to asymptotic behavior. Unlike in Task 3, where resistance emerged and was only partially overcome, here the students interpreted the results of the system without encountering major obstacles. No clear manifestation of resistance was observed, and it can be hypothesized that they recognized similarities with the earlier situation, an interpretation supported by their remark, *"Well, infinite this time,"* referring back to Task 3. The absence of resistance may thus be interpreted as a sign of learning, suggesting that students did not perceive the context as entirely new, and the management of the system and graph was less challenging than it had been in Task 3. As in Task 3, no single aspect of mathematical reasoning prevailed; structural and processual elements were present equally (Jeannotte and Kieran, 2017).

A pair of engineering students approached Task 3 primarily through graphical reasoning. One of them identified the tangent, while the other suggested considering additional lines. However, the first pointed out that these lines would intersect the parabola again. They then discussed the idea of constructing a family of lines and checking which ones intersect with the parabola, but without carrying out any calculations they first stated "two" intersections. After reconsidering, they reduced this to only one, the tangent, explicitly excluding the vertical line. During the interview, they explained that they had not considered the vertical line because they were uncertain whether it could be regarded as a valid line. When asked about a line very close to the y-axis, they replied that it would still intersect again. Pressed to substantiate why, they referred to the angular coefficient, suggesting that *"the slope is very steep."* When reminded that in the case of a parabola one cannot reason in terms of angular coefficients, they reconsidered and proposed that perhaps a line very close to the y-axis would never intersect again, because it *"opens a bit more slowly,"* while the parabola *"opens a bit faster."* This reasoning reveals a perceptual interpretation of "speed" in terms of obliqueness. At the end, when asked once more how many intersections exist, they appeared uncertain, asking themselves whether the line might meet the parabola again: *"I don't know."* Eventually, they concluded that there are infinitely many lines very close to the y-axis. An excerpt from the interview illustrates this reasoning in greater detail. To make the reasoning clearer, we provided explanatory comments and we explicitly highlighted where resistance emerged in the students' discourse. In the transcript, the abbreviations S3 and S4 are used for Student 3 and Student 4, respectively; this convention is also maintained in Task 6.

I: Did you also consider lines very close to the y-axis?

C1_S3: But very close...

C2_S4: But at infinity it will intersect the parabola.

C3_I: Why?

C4_S3: Because the slope is very steep.

C5_I: But what does that have to do with the parabola? Aren't we talking about two lines?

C6_S4: But if it's not perpendicular...

C7_S3: Ah no! But this (meaning the parabola) *opens up, and if just a little, little bit...*

C8_S4: Ah, so then it never reaches it.

C9_S3: I suppose it opens a bit more slowly.

C10_S4: And this one is a bit faster.

C11_I: Which one is slower?

C12_S3: The line.

C13_I: How can you tell graphically which one grows faster?

C14_S3: The parabola opens up a bit more as it goes toward infinity (gestures with the hand to indicate an oblique direction), *while the line opens up but always a bit less.* (distorted perception of asymptotic behavior)

(...)(The interviewer asks for confirmation of their perception of the two behaviors, and they confirm)

C15_I: So how many lines are there?

C16_S3: Eh...

C17_S4: Definitely more than two... we need to find a way to understand when the line intersects the parabola... uhmm (hesitation, sign of resistance)

C18_S3: And what if they intersect after all?

C19_S4: If one grows much faster than the other, at infinity... there isn't really a limit.

C20_S3: Exactly, I don't know... infinite...

C21_S4: I'm still a bit unsure... (uncertainty, sign of resistance)

(We move on in the interview and return to it later)

In Task 3, students relied solely on the AG approach, reasoning about asymptotic behavior using the graph. While they correctly recognized that the parabola grows faster than the line (*"I: Which one is slower? S3: The line"*), their perception of growth toward infinity was distorted. They interpreted it in an oblique manner, as reflected both in statements such as *"The parabola opens up a bit more as it goes toward infinity"* and in gestures indicating a slanted direction. This misperception of growth at infinity appears to have caused the manifested resistance, as the asymptotic behavior tool appears to have acted as an obstacle in their reasoning and as a limitation for the effective use of the AG approach. During their reasoning, resistance manifested through doubts and confusion, and the chosen tool seemed insufficient for solving the task. In this case, students did not completely overcome the resistance; in our interpretation, they remained in an intermediate state, as they did not abandon the tool but struggled to manage it effectively. Here, a predominance of the structural aspect of mathematical reasoning is evident (Jeannotte and Kieran, 2017), as students relied more on understanding mathematical properties than on performing calculations or manipulating algebraic realizations. Unlike the other pair, this pair did not attempt to substantiate their reasoning algebraically, and their approach was primarily structural, based on reasoning about the growth of functions toward infinity. Regarding Task 6, the students reasoned graphically, stating that all lines in the first and third quadrants passing through the origin are valid. They wrote that all lines through the origin are included, except for

$y = 0$ and those in the second and fourth quadrants. During the interview, interviewer asked about the y-axis ($x = 0$), and they replied that they had not considered it and that it should be excluded. Interviewer also asked about a line very close to the y-axis: *"Maybe it's a bit different from the parabola... because it's a bit narrower." "...this one grows straighter"* interviewer asked which one grows faster, and one answered the parabola, while the other said the cubic. *"This one stays here, it doesn't go that way,"* (accompanied by a gesture showing one going up and the other going diagonally). The other agreed and added that she was confused. They then concluded that there is a greater chance of intersection. When talking about the parabola, one said that a line very close to the y-axis does not intersect the parabola again, while the other was unsure and did not know how to determine this limit. Interviewer then proposed a specific line, $y = 1,000,000x$, and they tested it, but said it does not intersect. At the end, one concluded that it will not intersect, while the other maintained that it will eventually intersect, extending the same reasoning to the cubic as well. The following excerpt is taken from the interview.

(...)

D1_I: If we take a line very close? (and points to a line very close on the sheet)

D2_S3: To the y-axis?

D3_I: Yes.

D4_S3: Maybe it's a bit different from the parabola...

D5_I: Why?

D6_S3: Because it's a bit narrower, I mean it's more... but still, it grows, it grows...

D7_S4: I mean this one grows straighter, it doesn't go like that (gestures obliquely), *it goes more like this* (shows with hands that it grows closer to the y-axis).

D8_I: And which one grows faster, the parabola or the cubic?

D9_S4: The parabola grows faster, yes, I think it's the parabola that grows faster.

D10_S3: This one, at infinity this one, right? (pointing to the cubic on the paper)

(...) S4 shows S3 that the parabola grows faster to infinity, taking numbers on the y-axis and showing that the x-values are still small, and S3 is convinced.

D11_S4: So maybe this one has more chances of meeting it.

(We go back to Task 3 and reason about this)

D12_S3: I think sooner or later they intersect.

D13_I: But you told me that the one that grows faster goes obliquely, right? (gestures with hands)

D14_S4: I still believe that in an infinitely small interval they don't meet.

D15_S3: I couldn't say, what limits this, what number tells me "ok"? (uncertainty, sign of resistance)

D16_S4: Something infinitely small.

D17_S3: Yes, but something concrete.

D18_S4: Maybe 0.0001*, I don't know.*

D19_S3: I don't know, I'll stick to my idea.

(...) (We move on to Task 6, and I ask whether they still think the same about lines very close to the y-axis. Afterward, we reasoned about Task 3 and Task 6 together.)

D20_S4: Yes.

D21_S3: Eh, yes, it makes sense for a slope that goes up to a certain point...

D22_S4: But there must be a way to figure it out, but we don't know it.

D23_I: If we take a line like $y = 1,000,000x$*, do you think it will never meet the parabola and the cubic?*

D24_S4: Eh yes, because it's so close to the y-axis... or maybe since it's slightly shifted and not perfectly parallel, at infinity it will intersect.

D25_S3: If $y = 1,000,000x$ *and* $y = x^2$*, for the line* $y = 1,000,000x$ *if* $x = 1$*, then* $y = 1,000,000$*, and for the parabola it's 1,* $y = 1$*... eh, yes, exactly.*

D26_S4: So?

D27_S3: So they don't intersect at that point.

D28_I: So for the line $y = 1,000,000x$ *it doesn't intersect this parabola here?*

D29_S4: No, for me in an infinitesimal neighborhood they don't intersect.

D30_S3: I think sooner or later they do intersect, somehow they must meet.

(...) (Interviewer asks if it's the same for the cubic, and they both confirm.)

In Task 6, students appeared confused and, like the other pair, frequently referred back to Task 3, sometimes reflecting on both tasks simultaneously. Their uncertainty and hesitation, expressed through statements such as *"I don't know"* or *"I couldn't say"*, reflected impediments encountered during the task and may indicate moments of resistance, manifested as tensions between the tool and its user. They relied solely on the AG approach, highlighting a predominance of the structural aspect of mathematical reasoning (Jeannotte and Kieran, 2017). Even when considering numerical examples, they did not write them down or engage directly with the algebraic realizations of the objects. Based on their perception of growth toward infinity, students concluded that the cubic function grows slower than the parabola, as observed from the graph. In this case, they did not overcome the resistance, remaining in an intermediate state: they retained the tool but struggled with its application. At the end of the task, the pair disagreed: one student believed there is a limit and an infinitesimal interval where infinitely many lines do not intersect the function, while the other maintained that the straight line and function (parabola or cubic) intersect. Notably, the straight line $y = 1,000,000x$ did not intersect either function according to their reasoning. Unlike the previous pair, Task 6 was recognized as a situation similar to Task 3 during the interview; however, no learning or improvement in tool use was evident. The management of the approach remained challenging, leading students to make incorrect theoretical conclusions based on their perception of the graph.

8.2 Discussion

Regarding the first part of RQ4, the analysis shows that students drew on a range of approaches that varied according to whether the task was framed through graphical, algebraic, or mixed realizations. The distribution of these approaches across the tasks is reported in Tables 8.1, 8.2, and 8.3. When asymptotic behavior was not explicitly referenced in the task statement, students did not consistently activate this tool on their own. Nevertheless, when reasoning about lines situated near the y-axis, many students spontaneously appealed to growth at infinity and invoked asymptotic behavior. Overall, engineering students displayed a more frequent use of asymptotic reasoning, whereas mathematics students tended to rely on graphical and algebraic approaches, often combining them.

Turning to second part of research question, the results indicate that many students encountered substantial challenges in substantiating their reasoning related to asymptotic behavior. We interpret these difficulties as manifestations of resistance that arise when familiar tools must be mobilized in settings that differ from those in which they were originally learned. Resistance can be observed in: a distorted perception of growth at infinity and problems in coordinating graphical realizations with results obtained through algebraic reasoning. Students generally achieved more coherent substantiation when both structural and processual aspects were present and mutually supportive. By contrast, when the structural dimension dominated, substantiation tended to become more fragile.

Clear contrasts also emerge between the two cohorts. Mathematics students more often combined different approaches, or shifted among approaches, suggesting the absence of a dominant line of reasoning and instead a relatively balanced interplay between structural and processual aspects. Engineering students, in contrast, tended to depend more on structurally oriented approaches, most notably those grounded in the graph, sometimes accompanied by explicit reference to asymptotic behavior, sometimes not. These tendencies plausibly reflect differences in disciplinary trajectories: mathematics students, having followed several calculus courses, appeared more used to contexts in which formal substantiation plays a central role, whereas engineering students seemed less oriented toward formal substantiation. Despite these distinctions, the two cohorts also exhibited shared difficulties. Even though asymptotic behavior is addressed repeatedly across calculus courses, both groups experienced comparable challenges in coordinating different realizations. In particular, articulating verbal or algebraic realizations with graphical ones proved demanding. Representing growth at infinity was especially problematic: several students displayed an oblique interpretation of growth, as if functions such as parabolas or exponentials approached oblique asymptotes.

Consistent with Carlsen (2018), resistance appeared in connection with both processual and structural aspects of reasoning. Our results nevertheless suggest that when structural aspects dominate, they may impede students in overcoming resistance related to asymptotic behavior. A recurrent source of difficulty lay in students' conceptualization of growth at infinity, frequently expressed through oblique descriptors. This misinterpretation created tension between previously learned meanings and task-specific observations, complicating judgments about whether a line intersects a parabola once or more than once and giving rise to protracted debate and uncertainty. In such situations, resistance (Wertsch, 1998) became particularly visible as students attempted to transfer familiar discursive tools to a new context. Frequently, they eventually turned to algebraic reasoning as a means of handling the general case, illustrating the distinction highlighted by Carlsen (2018) between possessing a tool and being able to deploy it effectively. These challenges may be interpreted as taking place at the level of object-level learning, where established discourses must be adapted to new contexts. From this perspective, resistance reflects the obstacles that emerge when known discursive resources are applied in unfamiliar situations—an essential difficulty of object-level learning.

In line with prior research, our results corroborate that students often struggle to coordinate different realizations of a mathematical object, a difficulty also documented in studies on asymptotes (Mpofu and Pournara, 2018; Mpofu and Mudaly, 2020) and, more generally, in research on the notion of function (Hitt, 1998; Kop et al., 2017). Building coherent links across realizations remains demanding, suggesting a need for instructional approaches that deliberately emphasize parallel work with multiple realizations and promote explicit connections among them.

Chapter 9

Final reflections and further developments

The analyses developed in this study paint a detailed picture of how students make sense of functions and their behavior at infinity, revealing the central role played by asymptotic behavior in shaping both their successes and their difficulties. Across the tasks, students frequently relied on graphical realizations to orient themselves, yet their interpretations of these graphs were often partial or distorted—particularly when dealing with growth at infinity. A recurrent result is that many students perceive asymptotic behavior obliquely, interpreting the graph as if the function possessed an oblique asymptote even when this is not mathematically justified. This misperception, observed in both mathematics and engineering students, suggests that the distortion is not tied to a specific academic background but reflects a broader difficulty in coordinating intuitive visual impressions with formal mathematical objects. In the analytical chapters, each research question was addressed in detail and supported with examples drawn from students' work. Below, we offer a concise synthesis of the responses to the research questions.

The analyses conducted across this study provide a comprehensive picture of how students engage with the proposed tasks, highlighting the decisive influence of visual mediators and the central role of resistance in the coordination of different realizations. Concerning the first research question (RQ1), the results show that the inclusion of graphical realizations has a clear positive impact on students' performance: tasks accompanied by a graph yielded higher percentages of correct answers than those presented without one. In Type 2 tasks, students were particularly successful, suggesting that the choice of visual mediators directly shapes the possibility of identifying correct solutions. Graphical realizations enabled students to immediately see how many lines intersect the parabola, whereas an exclusively algebraic approach—focused on slopes—often obscured important intersections. These results resonate with earlier research showing that students often struggle to connect different realizations in a coherent way (Hitt, 1998; Kop et al., 2017). At the same time, they reaffirm that graphs, when interpreted effectively, can serve as powerful supports for solving mathematical tasks (Montenegro et al., 2018). Yet, even though graphical realizations offer valuable cues for understanding the problem, many students were unable to take advantage of them: several participants had trouble identifying relevant features or drawing conclusions from what was displayed—a difficulty also highlighted by Montenegro and colleagues (2018).

However, this flexibility was neither uniform nor widespread. Many students struggled to extract meaningful information from graphs: some overlooked tangent lines and focused solely on the vertical line, while others noticed the tangents but ignored the vertical one, despite its immediate perceptual salience as the y-axis. Conversely, when students relied on ritualized (GR) approaches, their reasoning became rigid, strongly process-oriented, and disconnected from the meaning of the tools employed, as seen when they applied algebraic systems even to internal points of the parabola. Social and institutional expectations also appeared to shape these behaviors: several students seemed to activate familiar routines to meet perceived teacher expectations rather than to respond to the mathematical demands of the task.

Addressing RQ2, the analysis revealed multiple points of resistance, particularly in relation to asymptotic behavior. Across several tasks, students interpreted growth at infinity as a diagonal tendency, attributing oblique asymptotes to parabolic or exponential functions. These distortions, consistent with prior studies, were exacerbated by difficulties in coordinating algebraic results with graphical realizations—an issue aligned with Carlsen's distinction between possessing a tool and knowing how to use it meaningfully. Differences emerged between mathematics and engineering students in their choice of visual mediators, yet both groups displayed similar struggles when confronting unfamiliar tasks and exhibited comparable misconceptions about asymptotic behavior. Resistance often surfaced as tension between realizations—algebraic, graphical, and gestural—that students struggled to integrate.

While previous studies acknowledge both the advantages of engaging with multiple realizations and the challenges students face in coordinating them (Ryve et al., 2013; Haghjoo et al., 2023; Weingarden and Heyd-Metzuyanim, 2023), none have focused on how resistance arises when familiar tools must be used in ways that depart from standard or expected practices. Our analysis shows that resistance frequently appears precisely at these moments: students attempt to draw on algebraic, graphical, or gestural resources they already possess but struggle to adapt them to the new demands of the task. When students manage to overcome this resistance, however, their explanations become richer and more integrated, revealing a deeper grasp of the purposes and meanings of the mediators they employ. This underlines the importance of considering not only which mediators students use but also how they manage them when working in unfamiliar contexts.

Regarding RQ3, the results highlight a clear relationship between resistance and the characteristics of students' routines. When resistance remained unresolved, routines were marked by limited flexibility and narrow applicability: students tended to rely on a single familiar approach and struggled to coordinate alternative realizations. Although they usually completed the tasks, their interview accounts revealed uncertainty about when and why to activate particular tools. In contrast, when resistance was resolved, routines evolved: students made sense of their chosen tools, interpreted their outputs, and linked different realizations more coherently. Collaborative discussion played a central role in this process, supporting exploratory behavior and easing the constraints imposed by the PSS. Introducing the notion of resistance into a commognitive framework thus offers a more refined lens for examining how and why routines change when students encounter unfamiliar mathematical features. Finally, bringing resistance into a commognitive perspective also makes it possible to analyse more precisely how exploratory and ritualized routines interact with one another. As observed by Viirman and Nardi (2019) and by Christiansen and colleagues (2023), these two modes of acting are not opposites; rather, they develop side by side and influence each other in complex and dynamic ways.

Turning to RQ4, students activated different approaches depending on whether the task was presented through graphical, algebraic, or mixed realizations. When asymptotic behavior was not explicitly mentioned, students did not consistently activate it; however, when reasoning about lines near the y-axis, many students spontaneously appealed to growth at infinity. Yet substantial challenges emerged in substantiating their reasoning about asymptotic behavior. Coherent substantiation appeared more frequently when structural and processual aspects supported one another, whereas overreliance on structural features led to fragile or contradictory arguments. Mathematics students tended to combine or shift between approaches more fluidly, while engineering students more frequently adopted structurally oriented reasoning grounded in the graph. Despite these contrasts, both groups struggled to coordinate algebraic, graphical, and verbal realizations, particularly when interpreting growth at infinity, which was often perceived obliquely. Consistent with Carlsen, resistance emerged in relation to both structural and processual aspects. Our results further indicate that dominant structural reasoning sometimes hindered students in overcoming resistance related to asymptotic behavior.

This research also highlights an issue insufficiently explored in the literature: the sense-making of asymptotic behavior. Students' distorted perception of growth at infinity appears widespread across backgrounds and academic trajectories. This suggests that university instruction—especially in early calculus courses—should devote greater attention to the relationship between algebraic and graphical realizations rather than treating the topic predominantly algebraically. A more robust construction of the concept could help prevent students from treating different realizations as distinct objects and support a more flexible and meaningful use of available tools. Strengthening flexibility and applicability is therefore crucial: choosing a tool should be motivated by convenience, not by difficulties in using alternatives. Increasing these "desirable characteristics of routines" promotes exploratory approaches and enables students to move beyond the constraints of the PSS. As observed in our results, substantiation also plays an important role. At the level of object-level learning, students must adapt established discourses to new contexts, and it is precisely in this shift that these challenges arise. The difficulty of using a tool in an unfamiliar context highlights Carlsen's (2018) point that there is a difference between knowing a tool and being able to apply it appropriately in context. Seen this way, resistance emerges when familiar discursive resources are brought into situations that are not yet familiar—one of the core difficulties of object-level learning. In our interpretation, overcoming resistance corresponds to overcoming these constraints, paving the way toward innovation, as suggested by Lavie and colleagues (2019).

Finally, the study shows that resistance appears consistently and can be identified through at least two markers in all examined excerpts. This observation calls for further research to refine analytical tools for studying resistance and to examine its interaction with students' routines across broader and more diverse populations.

This study presents some limitations that should be addressed in future research. First, the relatively small number of participants does not allow for broad generalization of the results. Expanding both the sample size and the diversity of academic backgrounds would make it possible to determine how widespread the distorted perception of growth at infinity truly is and to assess the extent to which it varies across different cohorts of students. Second, the design of the tasks did not allow for a systematic investigation of boundedness, leaving this aspect underexplored. Moreover, examining only a single episode from each interview made it possible to analyze students' reasoning at specific moments, but not to observe how their routines and interpretations developed over time.

Future studies should therefore investigate how students transition between realizations across extended periods and how object-level learning unfolds when familiar discursive tools must be adapted to new or unfamiliar contexts. Studies involving different groups of students could reveal not only the prevalence of these distortions but also how routines evolve in relation to resistance, offering insights into how learning processes occur and how instructional design might promote the development of routines that help students overcome such resistance. Longitudinal designs and multi-step tasks—in which the output of one step becomes the input for the next—appear particularly promising for deepening our understanding of how students' routines transform, how they coordinate multiple realizations, and how resistance shapes mathematical sense-making and solving processes. Observing learning as it unfolds across sessions would provide richer information for a deeper analysis of the objectification of discourse and of how students appropriate and reconceptualize mathematical tools.

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