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Research paper

Singularity analysis of spatial single-DOF mechanisms based on the locations of the instantaneous screw axes

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ABSTRACT

The instantaneous kinematics of a mechanism can be fully described through the instantaneous screw axes (ISAs) associated to the relative motions between mechanism's links. Characterizing the instantaneous kinematics of a mechanism includes its singularity analysis, which is the identification of mechanism configurations (singularities) where the instantaneous degrees-of-freedom (DOFs) of the mechanism "somehow" change. In single-DOF mechanisms with time-independent either holonomic or first-order non-holonomic constraints, the locations of the instantaneous screw axes only depend on the mechanism configuration. This property is fully exploited in planar single-DOF mechanisms where all the singularity conditions have been associated to the coincidence of particular instant centers (ICs), thus making the singularity identification possible with a simple analysis of the mechanism configuration. Differently, for single-DOF spatial mechanisms, the geometric conditions on ISAs that identify a singular configuration have not been enunciated for the general case, yet. This paper fills this lack by providing the explicit expressions of all the possible instantaneous input-output relationships, the exhaustive enumeration of the geometric conditions on the ISAs that identify singular configurations and a geometric and analytic technique for the singularity analysis of spatial single-DOF mechanisms, which is the natural extension of the IC-based ones conceived for planar single-DOF mechanisms.

1. Introduction

The vast majority of mechanisms have time-independent (scleronomic) constraints that are either holonomic constraints or non-holonomic constraints whose dependence on motion-variables' rates is linear (i.e, they are first-order non-holonomic constraints) [1]. These constraints model the mechanism's instantaneous kinematics through equations, which are linear and homogeneous in the motion-variable rates, with coefficients that uniquely depend on the mechanism configuration (i.e., on the mechanism's generalized coordinates, which usually coincide with the actuated-joint variables). Such an equation system can be regarded as a linear and homogeneous input-output model where the rates of the generalized coordinates are the inputs, whose number is equal to the mechanism's DOFs, and the rates of all the other (secondary) motion variables are the outputs, whose number is equal to the number of equations. Hereafter, this system will be referred to as instantaneous input-output relationship (IOR) and, for the sake of brevity, the term "mechanism" will refer to a mechanism with scleronomic constraints that are either holonomic or first-order non-holonomic.

At a given mechanism configuration, if, in the IOR, the two coefficient matrices (Jacobians) that multiply the tuple collecting all the inputs and the tuple collecting all the outputs, respectively, are not rank deficient, the IOR allows the computation of a unique solution

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Nomenclature

DOF	degree of freedom
Hel	helical motion
Tra	translational motion
IFK	instantaneous forward kinematic
IIK	instantaneous inverse kinematic
IOR	instantaneous input-output relationship
SISO	single-input-single-output
SIMO	single-input-multiple-output
VC	velocity coefficient
a	coefficient that multiplies the output variable in the IOR
b	coefficient that multiplies the input variable in the IOR
q	generalized coordinate of the single-DOF mechanism
s	a secondary variable of the single-DOF mechanism
ISA_{ji}	instantaneous screw axis (Fig. 1) of the relative motion between links j and i
A_{ji}	a point lying on ISA_{ji}
$\mathbf{u}_{ji}$	unit vector parallel to ISA_{ji}
p_{ji}	pitch of the relative motion between links j and i
$\omega_{ji}\mathbf{u}_{ji}$	angular velocity of the relative motion between links j and i; ω_{ji} is its signed magnitude
$\boldsymbol{\tau}_{jk}$	unit vector parallel to the translation direction of the relative motion between links j and k, which is an instantaneous pure translation
$\nu_{jk}\boldsymbol{\tau}_{jk}$	translation velocity of the relative motion between links j and k, which is an instantaneous pure translation; ν_{jk} is its signed magnitude
(P, **w**)	line passing through point P and parallel to unit vector **w**; according to these notations, (A_{ji}, $\mathbf{u}_{ji}$) is another way to denote ISA_{ji}
P	position vector locating the generic point P in a Cartesian reference
${}^{i}\mathbf{v}_{P\,\mid\,j}$	velocity of point P when the point is fixed to link j and the velocity is measured from link i
Q_{ji}, Q'_{ji}	points lying on ISA_{ji} that are intersections between ISA_{ji} and a line that is perpendicular to ISA_{ji} and to another ISA (see Figs. 2 and 3)
h_{ji}	$(= (Q_{ji} - Q'_{ji}) \cdot \mathbf{u}_{ji})$ signed distance between Q_{ji} and Q'_{ji}
$\mathbf{n}_{jir}$	unit vector of the common normal to ISA_{ji} and ISA_{ir}, if the relative motions *ji* and *ir* are both helical motions, or of a line perpendicular to ISA_{ji} and $\boldsymbol{\tau}_{ir}$, if the relative motions *ji* and *ir* are an helical motion and a translational motion, respectively (see Fig. 2)
$y_{ji,ir}$	$(= (Q_{ir} - Q_{ji}) \cdot \mathbf{n}_{jir})$ signed distance between Q_{ir} and Q_{ji} (see Fig. 2)
$\alpha_{ji,ir}$	angle between ISA_{ji} and ISA_{ir}, if the relative motions *ji* and *ir* are both helical motions, or between ISA_{ji} and $\boldsymbol{\tau}_{ir}$, if the relative motions *ji* and *ir* are an helical motion and a translational motion, respectively (see Fig. 2)

for the output tuple when the input tuple is assigned and vice versa. The determination of the input tuple (of the output tuple) for an assigned output tuple (input tuple) is named instantaneous inverse (forward) kinematic (IIK (IFK)) problem. The mechanism configurations where one or the other or both of the two above-mentioned Jacobians are rank deficient are singular configurations (singularities), which make the IIK problem or the IFK problem or both the problems undetermined [2–6]. Accordingly, in the literature [2], three types of singularities have been identified: (I) those where the IIK problem is indeterminate, also named serial singularities, (II) those where the IFK problem is indeterminate, also named parallel singularities, (III) those where both the IIK and the IFK problems are indeterminate.

The indeterminacy of the IIK problem kinematically means that the actuated-joint rates (i.e., the inputs) can be different from zero even though the links (output links) the secondary motion-variable rates refer to are locked, that is, at a serial singularity, the number of actuated-joints that affect the output-links' motion is reduced, which brings to the conclusion that the instantaneous DOFs of the output links are lower than the ones they have at a non-singular configuration. Such a condition can only occur at the boundaries of the variation range (workspace) of the secondary variables that make the configuration singular. Moreover, the duality [7] that the virtual work principle states between kinematics and statics brings one to conclude that, at a serial singularity, some external loads applied to the output links are directly equilibrated by the mechanism structure without any contribution of the generalized torques in the actuated joints.

The indeterminacy of the IFK problem kinematically means that the output links can have an elementary motion even though the actuated joints are locked, that is, at a parallel singularity, the number of actuated-joints necessary to fully control the output-links' motion increases, which brings to the conclusion that the instantaneous DOFs of the output links are more than the ones they have at a non-singular configuration. Such a condition may occur [2] only when the mechanism contains closed kinematic chains and, usually, inside the workspace boundaries of the secondary variables that make the configuration singular. Moreover, the duality [7] between kinematics and statics brings one to conclude that, at a parallel singularity, some external loads applied to the output links need infinite generalized torques in the actuated joints to be equilibrated, that is, the mechanism breaks down when it tries to reach a parallel singularity.

Hunt [8] introduced the concept of "stationary configuration" and "uncertainty configuration" to classify singularities according to instantaneous DOFs (instantaneous mobility) of the studied mechanism. In particular, stationary configurations are mechanism configurations where at least one joint variable of a non-actuated joint is inactive (i.e., has a null rate) even though all the actuated-joint rates are different from zero (i.e., the inactive joint variable has reached an absolute or local extreme of its variation range, which implies that the instantaneous mobility of the mechanism is decreased). Differently, uncertainty configurations are mechanism configurations where at least one joint variable of a non-actuated joint can have a rate different from zero even though all the actuated joints are locked (i.e., the actuated joints are not sufficient to fully control that joint variable, which implies that the instantaneous mobility of the mechanism is increased). Roughly speaking (see [9] pp. 302-303), stationary and uncertainty configurations correspond to the above-defined serial and parallel singularities, respectively.

Whatever be the adopted classification of mechanism singularities, their identification, during the mechanism design, and their avoidance, during the mechanism life, is a primary requirement to satisfy through suitable mechanism sizing and motion control [10–12]. That is the reason why the singularity analysis has become a mandatory step during the design of a novel mechanism and there is an ample literature on singularities (see, for instance, [13–21].

In single-DOF mechanisms, the IOR is a linear single-input-multiple-output (SIMO) model. Any linear SIMO model can be seen as a number of independent linear single-input-single-output (SISO) models that work in parallel. Accordingly, the IOR analysis for a single-DOF mechanism can be conducted by separately analyzing the independent SISO systems into which it can be broken down. The two IOR Jacobians of a SISO system reduce themselves to two scalar coefficients, whose values depend on the mechanism configuration: one that multiplies the actuated-joint rate and the other that multiplies the rate of the output joint (secondary) variable. As a consequence, the ratios between the rates of any secondary motion variable and of the actuated joint only depend on the mechanism configuration (i.e., on the generalized coordinate of the mechanism), which also brings the conclusion that, in single-DOF mechanisms, the ratios between motion-variables rates always depend only on the mechanism configuration. Such ratios are named [22,23] velocity (or influence) coefficients (VCs).

In a SISO system, the analysis of the VC relating the rate of the output variable and the rate of the input variable can fully replace the analysis of its IOR to identify all its singularities. In single-DOF planar mechanisms, such a VC can always be analytically expressed as a function of the locations of three or, at most, four instant centers (ICs) [24]; this feature made it possible to devise general geometric and analytic techniques for their singularity analysis based on the IC locations [8,25–28]. Recently, in [29], this author has addressed the determination of the instantaneous screw axes (ISAs) in single-DOF spatial mechanisms by taking advantage from the fact that, in single-DOF mechanisms, the ISA locations depend only on the mechanism configuration. The focus and the main result of Ref. [29] has been the deduction of a technique that extends the techniques, used in single-DOF planar mechanisms for the IC determination, to the ISA determination in single-DOF spatial mechanisms. Incidentally, in [29], this author, while devising this technique, has also deduced the analytic and geometric expressions of the above-mentioned VC as a function of the locations of three or, at most, four ISAs, which generalize the ones based on the IC locations that hold for planar mechanisms. In this paper, such expressions, firstly, are integrated with other general expressions to cover all the possible cases; then, are analyzed to provide the exhaustive enumeration of all the geometric conditions on ISAs that identify singular configurations; eventually, the deduced exhaustive enumeration of singularity conditions is used to devise a general analytic and geometric technique for the singularity analysis of single-DOF spatial mechanisms that is based on the locations of the ISAs. The proposed technique is the natural extension of the ones proposed for planar single-DOF mechanisms that are based on the IC locations and, as those techniques, it has the merit of making singularity configurations identifiable through a simple analysis of its 3D-CAD representation. As far as this author is aware this approach is novel and the exhaustive enumeration of all the geometric conditions on ISAs that identify singular configurations of single-DOF spatial mechanisms is presented for the first time in the literature.

The paper is organized as follows. Section 2 provides the necessary background materials, deduces the additional general expressions of the VCs, and illustrates the method. Section 3 presents the application of the method to a relevant case study. Then, Section 4 discusses the results and, eventually, Section 5 draws the conclusions.

2. Materials and methods

Here, firstly the adopted notations will be defined and some necessary background concepts will be reminded. Then, the general expressions of the IOR for a SISO mechanism will be deduced and, finally, the proposed singularity-analysis method will be presented.

In a mechanism with m links, the relative motions between link pairs[1] are m × (m–1)/2 and the instantaneous screw axes, ISA_{ji} ($\equiv ISA_{ij}$) with $j,i \in \{1,\ldots,m \mid (j \neq i) \& (j > i)\}$, that identify their instantaneous status are as many. Nevertheless, if the mechanism has f DOFs, the number of independent relative motions is the minimum value between (m–1) and f since, if (m–1) relative motions are known, the remaining ones can be determined by means of the relative motion theorems [30], that is, at most (m–1) relative motions are independent in the worst case, which is the one where the m links are not connected through kinematic pairs.

[1] Here, for a generic link pair constituted by links *i* and *j*, the motion of link *i* with respect to link *j* and its inverse motion (i.e., the motion of link *j* with respect to link *i*) are considered the same relative motion since the velocity fields associated to the two motions are immediately obtained from one another through a change of sign.

Hereafter, with reference to the motion of link j with respect to link i (Fig. 1), the $\mathrm{ISA_{ji}}$ will be geometrically identified by the ordered couple (A_{ji}, $\mathbf{u}_{ji}$) where A_{ji} is a point belonging to $\mathrm{ISA_{ji}}$ and $\mathbf{u}_{ji}$ is a unit vector parallel to $\mathrm{ISA_{ji}}$; the angular velocity, $\boldsymbol{\omega}_{ji}$, will be put equal to $\omega_{ji}\mathbf{u}_{ji}$ and will be assigned through its signed magnitude ω_{ji}. In addition, if $\omega_{ji}\neq 0$, that is, the relative motion is a helical motion (Hel), the velocity[2], ${}^{i}\mathbf{v}_{\mathrm{A_{ji}}|j}$, of point A_{ji} ($\in\mathrm{ISA_{ji}}$) will be put equal to $\omega_{ji}p_{ji}\mathbf{u}_{ji}$ and assigned through the pitch p_{ji} of the Hel; otherwise (i.e., if $\omega_{ji}=0$), the relative motion will be a translational motion (Tra) and its translation velocity will be put equal to $\nu_{ji}\boldsymbol{\tau}_{ji}$ where ν_{ji} is the signed magnitude of the translation velocity and $\boldsymbol{\tau}_{ji}$ is the unit vector that gives the positive direction of translation[3].

With these notations, the following relationships hold:

(a) if $\omega_{ji}\neq 0$ (Hel), then $A_{ji}=A_{ij}$, $\mathbf{u}_{ji}=\mathbf{u}_{ij}$, $p_{ji}=p_{ij}$ and $\omega_{ji}=-\omega_{ij}$;
(b) if $\omega_{ji}=0$ (Tra), then $\boldsymbol{\tau}_{ji}=\boldsymbol{\tau}_{ij}$ and $\nu_{ji}=-\nu_{ij}$.

Therefore, the instantaneous relative motion between links j and i is fully defined when, in the case $\omega_{ji}\neq 0$ (Hel), $\mathrm{ISA_{ji}}$ (i.e., the ordered couple (A_{ji}, $\mathbf{u}_{ji}$)), p_{ji} and ω_{ji} or, in the case $\omega_{ji}=0$ (Tra), ν_{ji} and $\boldsymbol{\tau}_{ji}$ are known. Hereafter, the pure rotation, which occurs when $\omega_{ji}\neq 0$ and $p_{ji}=0$, will be considered only a special case of Hel (i.e., the one with $p_{ji}=0$) and dealt with the formulas deduced for the Hel case. Accordingly, the links of a mechanism can perform only two types of instantaneous spatial motions (i.e., Hel and Tra) and, consequently, if only the instantaneous motions of the input and the output links are considered, only four types of SISO models can be identified for single-DOF spatial mechanisms:

- The Hel-Hel where both the input link and the output link perform an helical motion;
- The Hel-Tra where the input link performs an helical motion and the output link performs a translational motion;
- The Tra-Hel where the input link performs a translational motion and the output link performs an helical motion;
- The Tra-Tra where both the input link and the output link perform a translational motion.

The spatial version of the Aronhold-Kennedy theorem [29,31–36] states that "*In a system of three bodies the three ISAs of the three possible relative motions share a common normal.*" With the above notations, this author, in [29], deduced the following formulas for a three-body system (i, o and f denote the indices of the input link, of the output link and of the remaining third link (frame), respectively):

(i) for the Hel-Hel case (here (A_{fi}, $\mathbf{u}_{fi}$), p_{fi} and ω_{if} ((A_{of}, $\mathbf{u}_{of}$), p_{of} and ω_{of}) are the input-motion (output-motion) data (see Fig. 2a and remind the relative motion theorems [30]))

$$\frac{\omega_{of}}{\omega_{if}}=\frac{\mathbf{n}_{ofi}\cdot(\mathbf{u}_{if}\times\mathbf{u}_{oi})}{\mathbf{n}_{ofi}\cdot(\mathbf{u}_{of}\times\mathbf{u}_{oi})}\equiv\frac{(\mathbf{u}_{of}\cdot\mathbf{u}_{oi})-(\mathbf{u}_{if}\cdot\mathbf{u}_{oi})(\mathbf{u}_{if}\cdot\mathbf{u}_{of})}{(\mathbf{u}_{of}\cdot\mathbf{u}_{oi})(\mathbf{u}_{of}\cdot\mathbf{u}_{if})-(\mathbf{u}_{if}\cdot\mathbf{u}_{oi})}\Rightarrow\frac{\omega_{of}}{\omega_{if}}=\frac{\sin\alpha_{oi,if}}{\sin\alpha_{oi,of}}\equiv\frac{\cos\alpha_{oi,of}-\cos\alpha_{oi,if}\cos\alpha_{of,if}}{\cos\alpha_{oi,of}\cos\alpha_{of,if}-\cos\alpha_{oi,if}} \tag{1a}$$

$$p_{of}=p_{fi}+\frac{y_{of,oi}\left[\left(\frac{\omega_{of}}{\omega_{if}}\right)-(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})\right]+(y_{of,fi}-y_{of,oi})\left[(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})-\left(\frac{\omega_{of}}{\omega_{if}}\right)^{-1}\right]}{\|\mathbf{u}_{of}\times\mathbf{u}_{fi}\|} \tag{1b}$$

$$p_{oi}=p_{fi}+\frac{y_{of,oi}\left[\left(\frac{\omega_{of}}{\omega_{if}}\right)-(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})\right]+y_{of,fi}(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})}{\|\mathbf{u}_{of}\times\mathbf{u}_{fi}\|}\Rightarrow\frac{\omega_{of}}{\omega_{if}}=\frac{(p_{oi}-p_{fi})\|\mathbf{u}_{of}\times\mathbf{u}_{fi}\|+y_{if,oi}(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})}{y_{of,oi}} \tag{1c}$$

$$\left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi}=\omega_{of}\mathbf{u}_{of}-\omega_{if}\mathbf{u}_{if}\\ \omega_{oi}\mathbf{u}_{oi}=\omega_{og}\mathbf{u}_{og}-\omega_{ig}\mathbf{u}_{ig}\end{array}\right\}\Rightarrow\overset{\text{iff }\mathbf{u}_{og}\neq\pm\mathbf{u}_{ig}}{\downarrow}\quad\omega_{of}\mathbf{u}_{of}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})=\omega_{if}\mathbf{u}_{if}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})\Rightarrow\frac{\omega_{of}}{\omega_{if}}=\frac{\mathbf{u}_{if}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})}{\mathbf{u}_{of}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})} \tag{1d}$$

where, the angle $\alpha_{jk,ln}$ (see Fig. 2a) with $jk,ln\in\{oi, of, if\mid jk\neq ln\}$ is the angle counterclockwise with respect to $\mathbf{n}_{ofi}$ that goes from $\mathbf{u}_{jk}$ to $\mathbf{u}_{ln}$; Eq. (1b) and (1c) have been obtained from those reported in [29] through simple algebraic manipulations; also, the following definitions have been introduced (see Fig. 2a)

$$\mathbf{n}_{ofi}=\frac{\mathbf{u}_{of}\times\mathbf{u}_{fi}}{\|\mathbf{u}_{of}\times\mathbf{u}_{fi}\|};\ y_{of,fi}=(Q_{fi}-Q_{of})\cdot\mathbf{n}_{ofi}=(A_{fi}-A_{of})\cdot\mathbf{n}_{ofi};\ y_{of,oi}=(Q_{oi}-Q_{of})\cdot\mathbf{n}_{ofi}=(A_{oi}-A_{of})\cdot\mathbf{n}_{ofi} \tag{2a}$$

$$y_{if,oi}=(Q_{oi}-Q_{if})\cdot\mathbf{n}_{ofi}=(A_{oi}-A_{if})\cdot\mathbf{n}_{ofi}=y_{of,oi}-y_{of,fi} \tag{2b}$$

[2] Hereafter, ${}^{i}\mathbf{v}_{\mathrm{P}|j}$ will denote the velocity of point P when the point is fixed to link j and the velocity is measured from link i.

[3] In the case of a translation, the $\mathrm{ISA_{ji}}$ is the line at infinity (ideal line) that is the geometric locus of the points at infinity of the planes perpendicular to the translation direction.

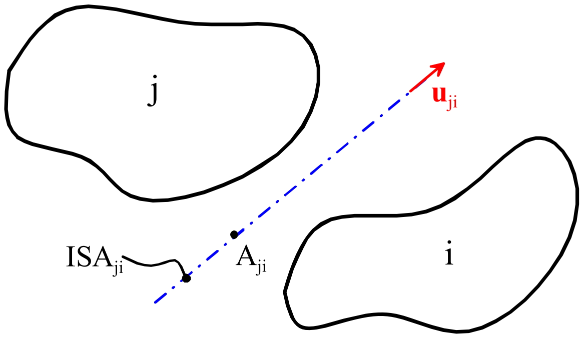

Fig. 1. Two links, named *i* and *j*, whose relative motion is a helical motion: notations.

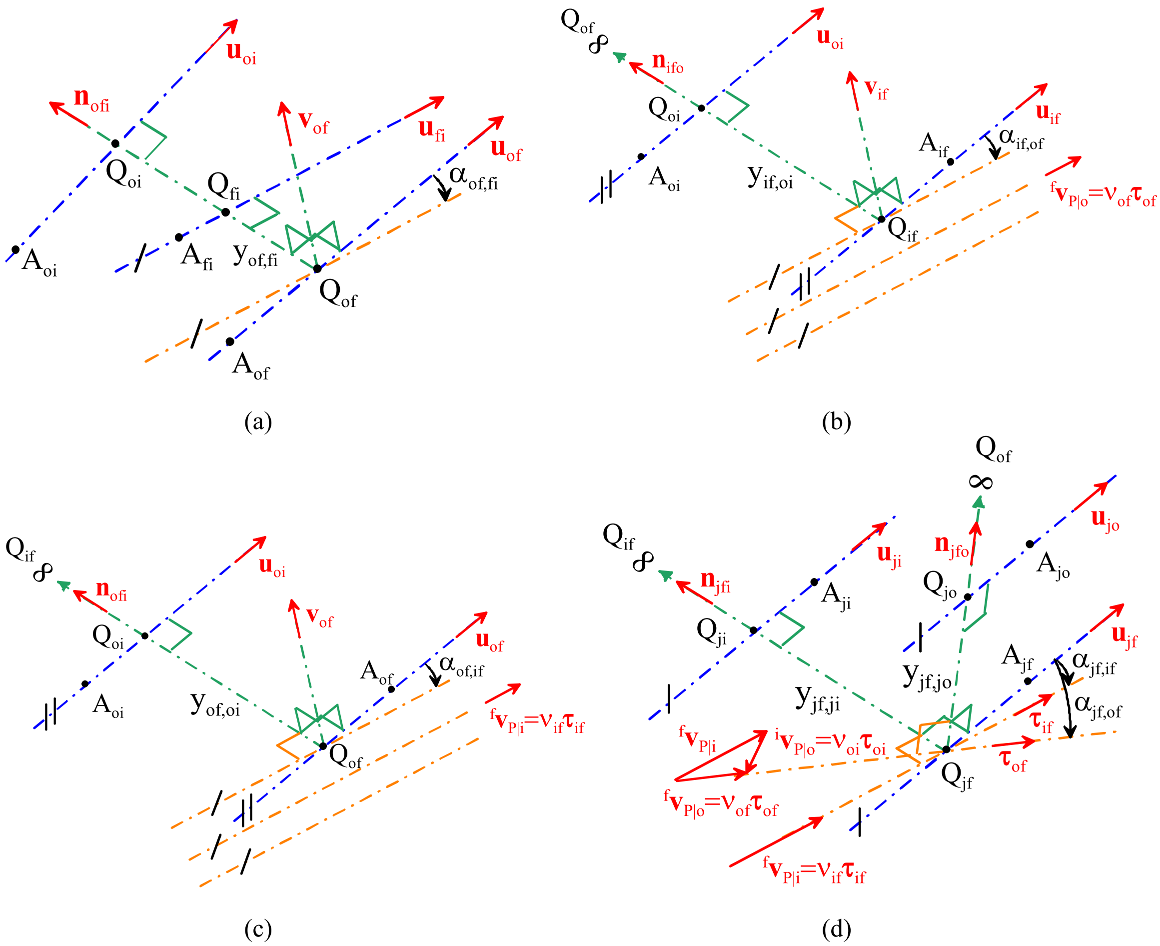

Fig. 2. Three-body system: (a) both the *of* and *fi* motions are helical motions, (b) the *of* motion is a translational motion and the *if* motion is a helical motion, (c) the *of* motion is a helical motion and the *if* motion is a translational motion, (d) both the *of* and *if* motions are translational motions.

and link g of Eq. (1d), which is an expression of the VC alternative to Eq. (1a), is any fourth link chosen so that $\mathbf{u}_{og} \neq \pm\mathbf{u}_{ig}$.

(ii) for the Hel-Tra case (here (A_{if}, $\mathbf{u}_{if}$), p_{if} and ω_{if} ($\nu_{of}=-\nu_{fo}$ and $\boldsymbol{\tau}_{of}=\boldsymbol{\tau}_{fo}$) are the input-motion (output-motion) data (see Fig. 2b))

$$\frac{\nu_{of}}{\omega_{if}}=\frac{\left(A_{oi}-A_{if}\right)\cdot\left(\mathbf{u}_{if}\times\boldsymbol{\tau}_{of}\right)}{\boldsymbol{\tau}_{of}\cdot\left[\mathbf{u}_{if}\times\left(\mathbf{u}_{if}\times\boldsymbol{\tau}_{of}\right)\right]}\equiv\frac{y_{if,oi}\,\|\,\mathbf{u}_{if}\times\boldsymbol{\tau}_{of}\,\|}{\left(\mathbf{u}_{if}\cdot\boldsymbol{\tau}_{of}\right)^{2}-1}\equiv-\frac{y_{if,oi}}{\|\,\mathbf{u}_{if}\times\boldsymbol{\tau}_{of}\,\|}\equiv-\frac{y_{if,oi}}{\sin\alpha_{if,of}} \tag{3a}$$

$$p_{oi} = p_{if} - \frac{v_{of}}{\omega_{if}}\left(\boldsymbol{\tau}_{of}\cdot\mathbf{u}_{if}\right) \equiv p_{if} + y_{if,oi}\frac{\left(\boldsymbol{\tau}_{of}\cdot\mathbf{u}_{if}\right)}{\|\mathbf{u}_{if}\times\boldsymbol{\tau}_{of}\|} \equiv p_{if} + \frac{y_{if,oi}}{\tan\alpha_{if,of}} \tag{3b}$$

$$\mathbf{n}_{ifo} = \frac{\mathbf{u}_{if}\times\boldsymbol{\tau}_{fo}}{\|\mathbf{u}_{if}\times\boldsymbol{\tau}_{fo}\|};\; y_{if,oi} = \left(Q_{oi}-Q_{if}\right)\cdot\mathbf{n}_{ifo} = \left(A_{oi}-A_{if}\right)\cdot\mathbf{n}_{ifo} \tag{3c}$$

$$\left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi} = \omega_{of}\mathbf{u}_{of} - \omega_{if}\mathbf{u}_{if}\\ \omega_{of} = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega_{oi} = -\omega_{if} = \omega_{fi}\\ \mathbf{u}_{oi} = \mathbf{u}_{if}\end{array}\right. \tag{3d}$$

(iii) for the Tra-Hel case (here $\nu_{if}=-\nu_{fi}$ and $\boldsymbol{\tau}_{if}=\boldsymbol{\tau}_{fi}$ ((A_{of}, $\mathbf{u}_{of}$), p_{of} and ω_{of}) are the input-motion (output-motion) data (see Fig. 2c))

$$\frac{\omega_{of}}{v_{if}} = \frac{\boldsymbol{\tau}_{if}\cdot\left[\mathbf{u}_{of}\times\left(\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\right)\right]}{\left(A_{oi}-A_{of}\right)\cdot\left(\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\right)} \equiv \frac{\left(\mathbf{u}_{of}\cdot\boldsymbol{\tau}_{if}\right)^2-1}{y_{of,oi}\|\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\|} \equiv -\frac{\|\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\|}{y_{of,oi}} \equiv -\frac{\sin\alpha_{of,if}}{y_{of,oi}} \tag{4a}$$

$$p_{of} = p_{oi} + \left(\frac{\omega_{of}}{v_{if}}\right)^{-1}\left(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{of}\right) \equiv p_{oi} - y_{of,oi}\frac{\left(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{of}\right)}{\|\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\|} \equiv p_{oi} - \frac{y_{of,oi}}{\tan\alpha_{of,if}} \tag{4b}$$

$$\mathbf{n}_{ofi} = \frac{\mathbf{u}_{of}\times\boldsymbol{\tau}_{fi}}{\|\mathbf{u}_{of}\times\boldsymbol{\tau}_{fi}\|};\; y_{of,oi} = \left(Q_{oi}-Q_{of}\right)\cdot\mathbf{n}_{ofi} = \left(A_{oi}-A_{of}\right)\cdot\mathbf{n}_{ofi} \tag{4c}$$

$$\left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi} = \omega_{of}\mathbf{u}_{of} - \omega_{if}\mathbf{u}_{if}\\ \omega_{if} = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega_{oi} = \omega_{of}\\ \mathbf{u}_{oi} = \mathbf{u}_{of}\end{array}\right. \tag{4d}$$

(iv) for the Tra-Tra case (here $\nu_{if}=-\nu_{fi}$ and $\boldsymbol{\tau}_{if}=\boldsymbol{\tau}_{fi}$ ($\nu_{of}=-\nu_{fo}$ and $\boldsymbol{\tau}_{of}=\boldsymbol{\tau}_{fo}$) are the input-motion (output-motion) data (see Fig. 2d))

$$\frac{v_{of}}{v_{if}} = \frac{\boldsymbol{\tau}_{of}\cdot\left[\left(A_{jo}-A_{jf}\right)\times\mathbf{u}_{jf}\right]\left[\left(\mathbf{u}_{jf}\cdot\boldsymbol{\tau}_{if}\right)^2-1\right]}{\boldsymbol{\tau}_{if}\cdot\left[\left(A_{ji}-A_{jf}\right)\times\mathbf{u}_{jf}\right]\left[\left(\mathbf{u}_{jf}\cdot\boldsymbol{\tau}_{of}\right)^2-1\right]} \equiv \frac{y_{jf,jo}\|\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fi}\|}{y_{jf,ji}\|\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fo}\|} \equiv \frac{y_{jf,jo}\sin\alpha_{jf,if}}{y_{jf,ji}\sin\alpha_{jf,of}} \tag{5a}$$

$$\mathbf{n}_{jfi} = \frac{\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fi}}{\|\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fi}\|};\; y_{jf,ji} = \left(Q_{ji}-Q_{jf}\right)\cdot\mathbf{n}_{jfi} = \left(A_{ji}-A_{jf}\right)\cdot\mathbf{n}_{jfi} \tag{5b}$$

$$\mathbf{n}_{jfo} = \frac{\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fo}}{\|\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fo}\|};\; y_{jf,jo} = \left(Q_{jo}-Q_{jf}\right)\cdot\mathbf{n}_{jfo} = \left(A_{jo}-A_{jf}\right)\cdot\mathbf{n}_{jfo} \tag{5c}$$

$${}^{i}\mathbf{v}_{P|o} = {}^{f}\mathbf{v}_{P|o} - {}^{f}\mathbf{v}_{P|i} = \nu_{of}\boldsymbol{\tau}_{of} - \nu_{if}\boldsymbol{\tau}_{if} \Rightarrow \nu_{oi}\boldsymbol{\tau}_{oi} = \nu_{of}\boldsymbol{\tau}_{of} - \nu_{if}\boldsymbol{\tau}_{if} \Rightarrow \left\{\begin{array}{l}\boldsymbol{\tau}_{of} = \dfrac{\nu_{oi}\boldsymbol{\tau}_{oi}+\nu_{if}\boldsymbol{\tau}_{if}}{\|\nu_{oi}\boldsymbol{\tau}_{oi}+\nu_{if}\boldsymbol{\tau}_{if}\|}\\ \nu_{of} = \nu_{oi}\left(\boldsymbol{\tau}_{oi}\cdot\boldsymbol{\tau}_{of}\right)+\nu_{if}\left(\boldsymbol{\tau}_{if}\cdot\boldsymbol{\tau}_{of}\right)\end{array}\right. \tag{5d}$$

$$\left.\begin{array}{l}\left.\begin{array}{l}\omega_{ji}\mathbf{u}_{ji} = \omega_{jf}\mathbf{u}_{jf} - \omega_{if}\mathbf{u}_{if}\\ \omega_{if} = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega_{ji} = \omega_{jf}\\ \mathbf{u}_{ji} = \mathbf{u}_{jf}\end{array}\right.\\ \left.\begin{array}{l}\omega_{oj}\mathbf{u}_{oj} = \omega_{of}\mathbf{u}_{of} - \omega_{jf}\mathbf{u}_{jf}\\ \omega_{of} = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega_{oj} = -\omega_{jf} = \omega_{fj}\\ \mathbf{u}_{oj} = \mathbf{u}_{jf}\end{array}\right.\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega_{ji} = \omega_{jf} = -\omega_{oj}\\ \mathbf{u}_{ji} = \mathbf{u}_{jf} = \mathbf{u}_{oj}\end{array}\right. \tag{5e}$$

where P is a generic point and link j is any fourth link chosen so that it performs an helical motion with respect to link f with an ISA_{jf} (i.e., (A_{jf}, $\mathbf{u}_{jf}$) with $\mathbf{u}_{jf}=\mathbf{u}_{ji}=\mathbf{u}_{jo}$ since links i and o translate with respect to link f) that is neither parallel to $\boldsymbol{\tau}_{fi}$ nor parallel to $\boldsymbol{\tau}_{fo}$ (it is worth noting that such a choice is always possible since link j can be any link).

2.1. The general input-output instantaneous relationship

The general expression of the input-output instantaneous relationship (IOR) of a SISO mechanism is

$$a(q)\,\dot{q} = b(q)\,\dot{s} \tag{6}$$

where q is the chosen generalized coordinate of the single-DOF mechanism, $\dot{q}$ and $\dot{s}$ are respectively the input rate and the output rate of the IOR, and a and b are two scalar functions of the mechanism configuration (i.e., of q), which, here, play the role of 1×1 Jacobians of the IOR.

If the motion of both the input, i, and the output, o, links are referred to the same reference link, say link f, the analytic and geometric expressions of a and b can be directly deduced from the VC explicit expressions given by Eqs. (1), (3), (4), and (5) for all the possible cases (i.e., Hel-Hel, Hel-Tra, Tra-Hel, Tra-Tra). In general, the motion of the input link could be referred to a link different from the one used as reference for the motion of the output link. Accordingly, the most general expressions of a and b must be deduced for a system of four links, hereafter named i (input link), o (output link), f (reference link for the motion of link i), and r (reference link for the motion of link o). The relative motions of a four-link system are 6 (=4(4−1)/2), that is, the *if, of, ir, or, oi,* and *rf* relative motions. In the following part of this subsection the general analytic and geometric expressions of a and b for the four possible cases will be deduced.

2.1.1. Hel-Hel case

In this case, $\dot{q}$ and $\dot{s}$ are ω_{if} and ω_{or}, respectively. Fig. 3 shows the general spatial disposition of the six ISAs associated to the six possible relative motions for the Hel-Hel case. In Fig. 3, the six blue lines (A_{jk}, $\mathbf{u}_{jk}$) with $jk \in \{if, of, ir, or, oi, rf\}$ are the six ISA_{jk} and the following definitions have been introduced

$$h_{jk} = \left(Q_{jk} - Q'_{jk}\right) \cdot \mathbf{u}_{jk} \qquad jk \in \{if,\ of,\ ir,\ or,\ oi,\ rf\} \tag{7a}$$

$$\mathbf{n}_{jkl} = \frac{\mathbf{u}_{jk} \times \mathbf{u}_{kl}}{\| \mathbf{u}_{jk} \times \mathbf{u}_{kl} \|} \qquad j,k,l \in \{i,\ o,\ f,\ r | j \neq k \neq l \neq j\} \tag{7b}$$

Also, with reference to Fig. 3, the following further definitions can be introduced:

$$y_{kj,kl} = \left(Q_{kl} - Q_{kj}\right) \cdot \mathbf{n}_{jkl} = \left(A_{kl} - A_{kj}\right) \cdot \mathbf{n}_{jkl} \qquad j,k,l \in \{i,\ o,\ f,\ r | j \neq k \neq l \neq j\} \tag{8}$$

Eqs. (1) and (2) hold for the subset of three bodies {*o, i, f*}. The same formulas can be written for the three-body subset {*o, i, r*} by replacing index f with index r in Eqs. (1) and (2) as follows:

$$\frac{\omega_{or}}{\omega_{ir}} = \frac{\mathbf{n}_{ori} \cdot (\mathbf{u}_{ir} \times \mathbf{u}_{oi})}{\mathbf{n}_{ori} \cdot (\mathbf{u}_{or} \times \mathbf{u}_{oi})} \equiv \frac{(\mathbf{u}_{or} \cdot \mathbf{u}_{oi}) - (\mathbf{u}_{ir} \cdot \mathbf{u}_{oi})(\mathbf{u}_{ir} \cdot \mathbf{u}_{or})}{(\mathbf{u}_{or} \cdot \mathbf{u}_{oi})(\mathbf{u}_{or} \cdot \mathbf{u}_{ir}) - (\mathbf{u}_{ir} \cdot \mathbf{u}_{oi})} \Rightarrow \frac{\omega_{or}}{\omega_{ir}} = \frac{\sin\alpha_{oi,ir}}{\sin\alpha_{oi,or}} \equiv \frac{\cos\alpha_{oi,or} - \cos\alpha_{oi,ir}\cos\alpha_{or,ir}}{\cos\alpha_{oi,or}\cos\alpha_{or,ir} - \cos\alpha_{oi,ir}} \tag{9a}$$

$$p_{or} = p_{ri} + \frac{y_{or,oi}\left[\left(\frac{\omega_{or}}{\omega_{ir}}\right) - (\mathbf{u}_{or} \cdot \mathbf{u}_{ri})\right] + \left(y_{or,ri} - y_{or,oi}\right)\left[(\mathbf{u}_{or} \cdot \mathbf{u}_{ri}) - \left(\frac{\omega_{or}}{\omega_{ir}}\right)^{-1}\right]}{\| \mathbf{u}_{or} \times \mathbf{u}_{ri} \|} \tag{9b}$$

$$p_{oi} = p_{ri} + \frac{y_{or,oi}\left[\left(\frac{\omega_{or}}{\omega_{ir}}\right) - (\mathbf{u}_{or} \cdot \mathbf{u}_{ir})\right] + y_{or,ir}(\mathbf{u}_{or} \cdot \mathbf{u}_{ir})}{\| \mathbf{u}_{or} \times \mathbf{u}_{ir} \|} \Rightarrow \frac{\omega_{or}}{\omega_{ir}} = \frac{(p_{oi} - p_{ri}) \| \mathbf{u}_{or} \times \mathbf{u}_{ir} \| + y_{ir,oi}(\mathbf{u}_{or} \cdot \mathbf{u}_{ir})}{y_{or,oi}} \tag{9c}$$

$$\mathbf{n}_{ori} = \frac{\mathbf{u}_{or} \times \mathbf{u}_{ri}}{\| \mathbf{u}_{or} \times \mathbf{u}_{ri} \|};\ y_{or,ri} = (Q_{ri} - Q_{or}) \cdot \mathbf{n}_{ori} = (A_{ri} - A_{or}) \cdot \mathbf{n}_{ori};\ y_{or,oi} = (Q_{oi} - Q_{or}) \cdot \mathbf{n}_{ori} = (A_{oi} - A_{or}) \cdot \mathbf{n}_{ori} \tag{9d}$$

The relative motion theorems [30] bring one to write the following relationships (see Fig. 3 and Eqs. (7) and (8)):

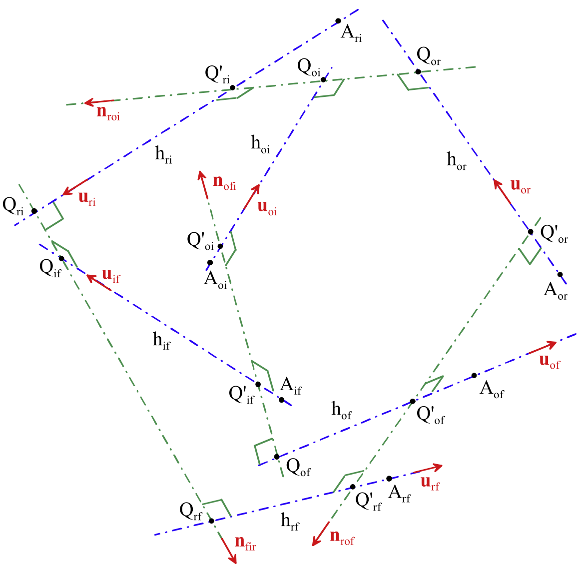

Fig. 3. General spatial disposition of the six ISAs associated to the six possible relative motions of a four-body system for the Hel-Hel case: the six blue lines (A_{jk}, $\mathbf{u}_{jk}$) with $jk \in \{if, of, ir, or, oi, rf\}$ are the six ISA_{jk}.

$$\left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi}=\omega_{of}\mathbf{u}_{of}-\omega_{if}\mathbf{u}_{if}\\ \omega_{oi}\mathbf{u}_{oi}=\omega_{or}\mathbf{u}_{or}-\omega_{ir}\mathbf{u}_{ir}\end{array}\right\}\Rightarrow\underset{\downarrow}{\text{iff } \mathbf{u}_{of}\neq\pm\mathbf{u}_{ir}}\quad \omega_{or}\mathbf{u}_{or}\cdot\left(\mathbf{u}_{of}\times\mathbf{u}_{ir}\right)=-\omega_{if}\mathbf{u}_{if}\cdot\left(\mathbf{u}_{of}\times\mathbf{u}_{ir}\right)\Rightarrow\frac{\omega_{or}}{\omega_{if}}=\frac{\mathbf{u}_{of}\cdot\left(\mathbf{u}_{if}\times\mathbf{u}_{ir}\right)}{\mathbf{u}_{or}\cdot\left(\mathbf{u}_{of}\times\mathbf{u}_{ir}\right)} \tag{10a}$$

$$\left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi}=\omega_{of}\mathbf{u}_{of}-\omega_{if}\mathbf{u}_{if}\\ \omega_{oi}\mathbf{u}_{oi}=\omega_{or}\mathbf{u}_{or}-\omega_{ir}\mathbf{u}_{ir}\end{array}\right\}\Rightarrow\omega_{of}\left(\mathbf{u}_{of}\cdot\mathbf{u}_{oi}\right)-\omega_{if}\left(\mathbf{u}_{if}\cdot\mathbf{u}_{oi}\right)=\omega_{or}\left(\mathbf{u}_{or}\cdot\mathbf{u}_{oi}\right)-\omega_{ir}\left(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi}\right)\Rightarrow\frac{\omega_{or}}{\omega_{if}}=\frac{\frac{\omega_{of}}{\omega_{if}}\left(\mathbf{u}_{of}\cdot\mathbf{u}_{oi}\right)-\left(\mathbf{u}_{if}\cdot\mathbf{u}_{oi}\right)}{\left(\mathbf{u}_{or}\cdot\mathbf{u}_{oi}\right)-\frac{\omega_{ir}}{\omega_{or}}\left(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi}\right)} \tag{10b}$$

$$\left.\begin{array}{l}p_{oi}\omega_{oi}\mathbf{u}_{oi}={}^{i}\mathbf{v}_{Q_{oi}|o}\equiv{}^{i}\mathbf{v}_{Q'_{oi}|o}\\ {}^{i}\mathbf{v}_{Q_{oi}|o}={}^{r}\mathbf{v}_{Q_{oi}|o}-{}^{r}\mathbf{v}_{Q_{oi}|i}\\ {}^{r}\mathbf{v}_{Q_{oi}|o}=\omega_{or}\left[p_{or}\mathbf{u}_{or}+\mathbf{u}_{or}\times\left(Q_{oi}-Q_{or}\right)\right]\\ {}^{r}\mathbf{v}_{Q_{oi}|i}=\omega_{ir}\left[p_{ir}\mathbf{u}_{ir}+\mathbf{u}_{ir}\times\left(Q_{oi}-Q'_{ir}\right)\right]\\ \left(Q_{oi}-Q_{or}\right)=-y_{oi,or}\mathbf{n}_{ior}\\ \left(Q_{oi}-Q'_{ir}\right)=-y_{oi,ir}\mathbf{n}_{oir}\end{array}\right\}\Rightarrow p_{oi}\omega_{oi}=\omega_{or}\left\{\left[p_{or}\left(\mathbf{u}_{oi}\cdot\mathbf{u}_{or}\right)-y_{oi,or}\parallel\mathbf{u}_{oi}\times\mathbf{u}_{or}\parallel\right]-\frac{\omega_{ir}}{\omega_{or}}\left[p_{ir}\left(\mathbf{u}_{oi}\cdot\mathbf{u}_{ir}\right)-y_{oi,ir}\parallel\mathbf{u}_{oi}\times\mathbf{u}_{ir}\parallel\right]\right\} \tag{10c}$$

$$\left.\begin{array}{l}p_{oi}\omega_{oi}\mathbf{u}_{oi}={}^{i}\mathbf{v}_{Q_{oi}|o}\equiv{}^{i}\mathbf{v}_{Q'_{oi}|o}\\ {}^{i}\mathbf{v}_{Q'_{oi}|o}={}^{f}\mathbf{v}_{Q'_{oi}|o}-{}^{f}\mathbf{v}_{Q'_{oi}|i}\\ {}^{f}\mathbf{v}_{Q'_{oi}|o}=\omega_{of}\left[p_{of}\mathbf{u}_{of}+\mathbf{u}_{of}\times\left(Q'_{oi}-Q_{of}\right)\right]\\ {}^{f}\mathbf{v}_{Q'_{oi}|i}=\omega_{if}\left[p_{if}\mathbf{u}_{if}+\mathbf{u}_{if}\times\left(Q'_{oi}-Q'_{if}\right)\right]\\ \left(Q'_{oi}-Q_{of}\right)=-y_{oi,of}\mathbf{n}_{iof}\\ \left(Q'_{oi}-Q'_{if}\right)=-y_{oi,if}\mathbf{n}_{oif}\end{array}\right\}\Rightarrow p_{oi}\omega_{oi}=\omega_{if}\left\{\frac{\omega_{of}}{\omega_{if}}\left[p_{of}\left(\mathbf{u}_{oi}\cdot\mathbf{u}_{of}\right)-y_{oi,of}\parallel\mathbf{u}_{oi}\times\mathbf{u}_{of}\parallel\right]-p_{if}\left(\mathbf{u}_{oi}\cdot\mathbf{u}_{if}\right)+y_{oi,if}\parallel\mathbf{u}_{oi}\times\mathbf{u}_{if}\parallel\right\} \tag{10d}$$

Eq. (10a) and (10b), where expressions (1c) and (9c) replace $\left(\frac{\omega_{of}}{\omega_{if}}\right)$ and $\left(\frac{\omega_{or}}{\omega_{ir}}\right)$, respectively, yield the following two alternative expressions for the IOR of the Hel-Hel case

$$\omega_{or}\left[\mathbf{u}_{or}\cdot\left(\mathbf{u}_{of}\times\mathbf{u}_{ir}\right)\right]=\omega_{if}\left[\mathbf{u}_{of}\cdot\left(\mathbf{u}_{if}\times\mathbf{u}_{ir}\right)\right] \tag{11a}$$

$$\frac{\omega_{or}}{\omega_{if}}=\frac{\left\{\left[\left(p_{oi}-p_{fi}\right)\parallel\mathbf{u}_{of}\times\mathbf{u}_{fi}\parallel+y_{if,oi}\left(\mathbf{u}_{of}\cdot\mathbf{u}_{fi}\right)\right]\left(\mathbf{u}_{of}\cdot\mathbf{u}_{oi}\right)-y_{of,oi}\left(\mathbf{u}_{if}\cdot\mathbf{u}_{oi}\right)\right\}\left[\left(p_{oi}-p_{ri}\right)\parallel\mathbf{u}_{or}\times\mathbf{u}_{ir}\parallel+y_{ir,oi}\left(\mathbf{u}_{or}\cdot\mathbf{u}_{ir}\right)\right]}{y_{of,oi}\left\{\left(\mathbf{u}_{or}\cdot\mathbf{u}_{oi}\right)\left[\left(p_{oi}-p_{ri}\right)\parallel\mathbf{u}_{or}\times\mathbf{u}_{ir}\parallel+y_{ir,oi}\left(\mathbf{u}_{or}\cdot\mathbf{u}_{ir}\right)\right]-y_{or,oi}\left(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi}\right)\right\}} \tag{11b}$$

which come to have the canonical form of Eq. (6) with the following positions

$$\dot{q}=\omega_{if};\ \dot{s}=\omega_{or}; \tag{12a}$$

$$\text{Eq. (11a)}\Rightarrow a(q)=\mathbf{u}_{of}\cdot\left(\mathbf{u}_{if}\times\mathbf{u}_{ir}\right);\ b(q)=\mathbf{u}_{or}\cdot\left(\mathbf{u}_{of}\times\mathbf{u}_{ir}\right); \tag{12b}$$

$$\text{Eq.(11b)}\Rightarrow\left\{\begin{array}{l}a(q)=\left\{\left[\left(p_{oi}-p_{fi}\right)\parallel\mathbf{u}_{of}\times\mathbf{u}_{fi}\parallel+y_{if,oi}\left(\mathbf{u}_{of}\cdot\mathbf{u}_{fi}\right)\right]\left(\mathbf{u}_{of}\cdot\mathbf{u}_{oi}\right)-y_{of,oi}\left(\mathbf{u}_{if}\cdot\mathbf{u}_{oi}\right)\right\}\left[\left(p_{oi}-p_{ri}\right)\parallel\mathbf{u}_{or}\times\mathbf{u}_{ir}\parallel+y_{ir,oi}\left(\mathbf{u}_{or}\cdot\mathbf{u}_{ir}\right)\right]\\ b(q)=y_{of,oi}\left\{\left(\mathbf{u}_{or}\cdot\mathbf{u}_{oi}\right)\left[\left(p_{oi}-p_{ri}\right)\parallel\mathbf{u}_{or}\times\mathbf{u}_{ir}\parallel+y_{ir,oi}\left(\mathbf{u}_{or}\cdot\mathbf{u}_{ir}\right)\right]-y_{or,oi}\left(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi}\right)\right\}\end{array}\right. \tag{12c}$$

In addition, the introduction of the expression of p_{ri} obtained from Eq. (9c) into Eq. (9b) gives

$$p_{or}=p_{oi}+\frac{\left(y_{or,oi}-y_{or,ri}\right)\left(\frac{\omega_{or}}{\omega_{ir}}\right)^{-1}-y_{or,oi}\left(\mathbf{u}_{or}\cdot\mathbf{u}_{ri}\right)}{\parallel\mathbf{u}_{or}\times\mathbf{u}_{ri}\parallel} \tag{13}$$

and, finally, the introduction of the explicit expression of p_{oi} provided by Eq. (1c) into Eq. (13) yields

$$p_{or}=p_{fi}+\frac{y_{of,oi}\left[\left(\frac{\omega_{of}}{\omega_{if}}\right)-\left(\mathbf{u}_{of}\cdot\mathbf{u}_{fi}\right)\right]+y_{of,fi}\left(\mathbf{u}_{of}\cdot\mathbf{u}_{fi}\right)}{\parallel\mathbf{u}_{of}\times\mathbf{u}_{fi}\parallel}+\frac{\left(y_{or,oi}-y_{or,ri}\right)\left(\frac{\omega_{or}}{\omega_{ir}}\right)^{-1}-y_{or,oi}\left(\mathbf{u}_{or}\cdot\mathbf{u}_{ri}\right)}{\parallel\mathbf{u}_{or}\times\mathbf{u}_{ri}\parallel} \tag{14}$$

where the explicit expressions of $\left(\frac{\omega_{of}}{\omega_{if}}\right)$ and $\left(\frac{\omega_{or}}{\omega_{ir}}\right)$ are given by Eqs. (1a) and (9a), respectively.

Eq. (14) completes the IOR of the Hel-Hel case (i.e., Eq. (11)) by providing the pitch, p_{or}, of the output helical motion as a function of the pitch, p_{fi}, of the input helical motion and of the spatial disposition of the six ISAs (i.e., of the mechanism configuration) shown in Fig. 3.

2.1.2. Hel-Tra case

In this case, $\dot{q}$ and $\dot{s}$ are ω_{if} and ν_{or}, respectively. Fig. 4 shows the general spatial disposition of the six ISAs associated to the six possible relative motions for the Hel-Tra case, which, now, are one ideal line plus five finite lines.

In this case, the translation condition for the output link, that is, ω_{or}=0, and Eq. (3) when written for the set of three bodies {*i, o, r*} together with the relative motion theorems [30] yield (Fig. 4)

$$\frac{v_{or}}{\omega_{ir}} = \frac{(A_{oi}-A_{ir})\cdot(\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or})}{\boldsymbol{\tau}_{or}\cdot[\mathbf{u}_{ir}\times(\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or})]} = \frac{y_{ir,oi}\,\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\|}{(\mathbf{u}_{ir}\cdot\boldsymbol{\tau}_{or})^2-1} \equiv -\frac{y_{ir,oi}}{\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\|} \equiv -\frac{y_{ir,oi}}{\sin\alpha_{ir,or}} \tag{15a}$$

$$p_{oi} = p_{ir} - \frac{v_{or}}{\omega_{ir}}(\boldsymbol{\tau}_{or}\cdot\mathbf{u}_{ir}) \equiv p_{ir} + y_{ir,oi}\frac{(\boldsymbol{\tau}_{or}\cdot\mathbf{u}_{ir})}{\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\|} \equiv p_{ir} + \frac{y_{ir,oi}}{\tan\alpha_{ir,of}} \tag{15b}$$

$$\mathbf{n}_{iro} = \frac{\mathbf{u}_{ir}\times\boldsymbol{\tau}_{ro}}{\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{ro}\|};\ y_{ir,oi} = (Q_{oi}-Q_{ir})\cdot\mathbf{n}_{iro} = (A_{oi}-A_{ir})\cdot\mathbf{n}_{iro}; \tag{15c}$$

$$\left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi} = \omega_{or}\mathbf{u}_{or} + \omega_{ri}\mathbf{u}_{ri}\\ \omega_{or} = 0\\ \omega_{of}\mathbf{u}_{of} = \omega_{or}\mathbf{u}_{or} + \omega_{rf}\mathbf{u}_{rf}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\omega_{oi} = \omega_{ri} = -\omega_{ir}\\ \mathbf{u}_{oi} = \mathbf{u}_{ri} = \mathbf{u}_{ir}\\ \omega_{of} = \omega_{rf} = -\omega_{fr}\\ \mathbf{u}_{of} = \mathbf{u}_{rf} = \mathbf{u}_{fr}\end{array}\right\} \Rightarrow -\omega_{ir}\mathbf{u}_{oi} = \omega_{of}\mathbf{u}_{of} - \omega_{if}\mathbf{u}_{if} \Rightarrow \omega_{ir} = \omega_{if}\left[\mathbf{u}_{if} - \left(\frac{\omega_{of}}{\omega_{if}}\right)\mathbf{u}_{of}\right]\cdot\mathbf{u}_{oi} \tag{15d}$$

$$\omega_{oi}\mathbf{u}_{oi} = \omega_{of}\mathbf{u}_{of} - \omega_{if}\mathbf{u}_{if}$$

$$\mathbf{n}_{fir} = \frac{\mathbf{u}_{if}\times\mathbf{u}_{ir}}{\|\mathbf{u}_{if}\times\mathbf{u}_{ir}\|};\ \mathbf{n}_{ofi} = \frac{\mathbf{u}_{of}\times\mathbf{u}_{if}}{\|\mathbf{u}_{of}\times\mathbf{u}_{if}\|};\ \mathbf{n}_{fro} = \frac{\mathbf{u}_{fr}\times\boldsymbol{\tau}_{ro}}{\|\mathbf{u}_{fr}\times\boldsymbol{\tau}_{ro}\|};\ y_{rf,of} = (Q_{of}-Q_{rf})\cdot\mathbf{n}_{fro} = (A_{of}-A_{rf})\cdot\mathbf{n}_{fro}; \tag{15e}$$

The introduction of the expressions provided by Eqs. (1c) or (1d) for the VC $\left(\frac{\omega_{of}}{\omega_{if}}\right)$ into Eq. (15d) gives

$$\text{Eq.(1c)} \Rightarrow \omega_{ir} = \omega_{if}\frac{(p_{if}-p_{oi})\,\|\mathbf{u}_{of}\times\mathbf{u}_{if}\|\,(\mathbf{u}_{of}\cdot\mathbf{u}_{oi}) + (\mathbf{u}_{if}\cdot\mathbf{u}_{oi})y_{of,oi} - y_{if,oi}(\mathbf{u}_{of}\cdot\mathbf{u}_{if})(\mathbf{u}_{of}\cdot\mathbf{u}_{oi})}{y_{of,oi}} \tag{16a}$$

$$\text{Eq.(1d)} \underset{\downarrow}{\overset{\text{iff } \mathbf{u}_{og}\neq\pm\mathbf{u}_{ig}}{\Rightarrow}} \omega_{ir} = \omega_{if}\frac{(\mathbf{u}_{og}\times\mathbf{u}_{ig})\cdot[(\mathbf{u}_{if}\times\mathbf{u}_{of})\times\mathbf{u}_{oi}]}{\mathbf{u}_{of}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})} \tag{16b}$$

whose introduction into Eq. (15a) respectively yields the following two alternative IORs for the Hel-Tra case

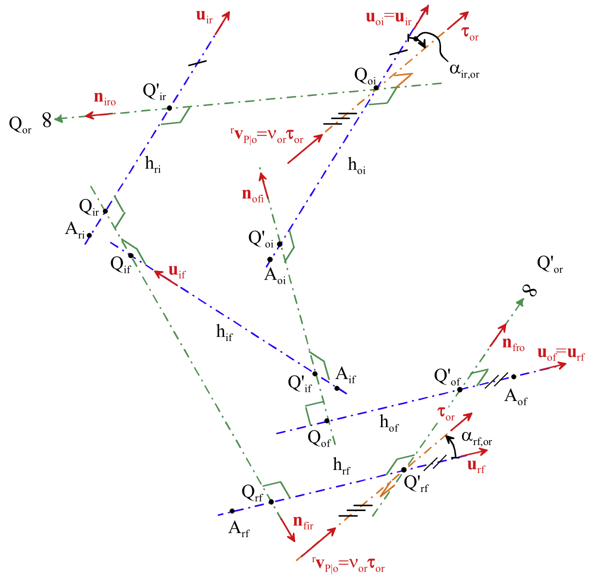

Fig. 4. General spatial disposition of the six ISAs (five finite lines and one ideal line) associated to the six possible relative motions of a four-body system for the Hel-Tra case: the five blue lines (A_{jk}, $\mathbf{u}_{jk}$) with $jk\in${*if, of, ir, oi, rf*} are the five finite ISA_{jk}; whereas, the ideal line, ISA_{or}, associated to the *or* translation is the ideal line shared by all the parallel planes that are all perpendicular to $\boldsymbol{\tau}_{or}$.

$$-v_{or}\frac{\| \mathbf{u}_{ir} \times \boldsymbol{\tau}_{or} \|}{y_{ir,oi}} = \omega_{if}\frac{(p_{if}-p_{oi}) \| \mathbf{u}_{of} \times \mathbf{u}_{if} \| (\mathbf{u}_{of}\cdot\mathbf{u}_{oi}) + (\mathbf{u}_{if}\cdot\mathbf{u}_{oi})y_{of,oi} - y_{if,oi}(\mathbf{u}_{of}\cdot\mathbf{u}_{if})(\mathbf{u}_{of}\cdot\mathbf{u}_{oi})}{y_{of,oi}} \tag{17a}$$

$$-v_{or}\frac{\| \mathbf{u}_{ir} \times \boldsymbol{\tau}_{or} \|}{y_{ir,oi}} = \omega_{if}\frac{(\mathbf{u}_{og}\times\mathbf{u}_{ig})\cdot[(\mathbf{u}_{if}\times\mathbf{u}_{of})\times\mathbf{u}_{oi}]}{\mathbf{u}_{of}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})} \tag{17b}$$

which come to have the canonical form of Eq. (6) with the following positions

$$\dot{q} = \omega_{if};\ \dot{s} = v_{or};\ b(q) = -\frac{\| \mathbf{u}_{ir} \times \boldsymbol{\tau}_{or} \|}{y_{ir,oi}} \tag{18a}$$

$$\text{Eq. (17a)} \Rightarrow a(q) = \frac{(p_{if}-p_{oi}) \| \mathbf{u}_{of} \times \mathbf{u}_{if} \| (\mathbf{u}_{of}\cdot\mathbf{u}_{oi}) + (\mathbf{u}_{if}\cdot\mathbf{u}_{oi})y_{of,oi} - y_{if,oi}(\mathbf{u}_{of}\cdot\mathbf{u}_{if})(\mathbf{u}_{of}\cdot\mathbf{u}_{oi})}{y_{of,oi}}; \tag{18b}$$

$$\text{Eq. (17b)} \Rightarrow a(q) = \frac{(\mathbf{u}_{og}\times\mathbf{u}_{ig})\cdot[(\mathbf{u}_{if}\times\mathbf{u}_{of})\times\mathbf{u}_{oi}]}{\mathbf{u}_{of}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})} \tag{18c}$$

2.1.3. Tra-Hel case

In this case, $\dot{q}$ and $\dot{s}$ are v_{if} and ω_{or}, respectively. The formulas and the spatial disposition of the ISAs of this case (Fig. 5) can be immediately deduced from those of the Hel-Tra case (i.e., those reported in Section 2.1.2 and in Fig. 4) by exchanging the link indices *i* and *o* and the link indices *f* and *r*. So doing, Eqs. (15a), (15b) and (17) yield the following relationships:

$$-v_{if}\frac{\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \|}{y_{of,oi}} = \omega_{or}\frac{(p_{or}-p_{oi}) \| \mathbf{u}_{ir} \times \mathbf{u}_{or} \| (\mathbf{u}_{ir}\cdot\mathbf{u}_{oi}) + (\mathbf{u}_{or}\cdot\mathbf{u}_{oi})y_{ir,oi} - y_{or,oi}(\mathbf{u}_{ir}\cdot\mathbf{u}_{or})(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi})}{y_{ir,oi}} \tag{19a}$$

$$-v_{if}\frac{\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \|}{y_{of,oi}} = \omega_{or}\frac{(\mathbf{u}_{og}\times\mathbf{u}_{ig})\cdot[(\mathbf{u}_{or}\times\mathbf{u}_{ir})\times\mathbf{u}_{oi}]}{\mathbf{u}_{ir}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})} \tag{19b}$$

$$\left.\begin{array}{l} p_{of} = p_{oi} + \dfrac{v_{if}}{\omega_{of}}(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{of}) \\ \dfrac{v_{if}}{\omega_{of}} = -\dfrac{y_{of,oi}}{\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \|} \end{array}\right\} \Rightarrow p_{oi} = p_{of} + y_{of,oi}\frac{(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{of})}{\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \|} \equiv p_{of} + \frac{y_{of,oi}}{\tan\alpha_{of,if}} \tag{19c}$$

with

$$\left.\begin{array}{l} \omega_{oi}\mathbf{u}_{oi} = \omega_{of}\mathbf{u}_{of} - \omega_{if}\mathbf{u}_{if} \\ \omega_{if} = 0 \\ \omega_{ir}\mathbf{u}_{ir} = \omega_{if}\mathbf{u}_{if} + \omega_{fr}\mathbf{u}_{fr} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \omega_{oi} = \omega_{of} \\ \mathbf{u}_{oi} = \mathbf{u}_{of} \\ \omega_{ir} = \omega_{fr} = -\omega_{rf} \\ \mathbf{u}_{ir} = \mathbf{u}_{fr} = \mathbf{u}_{rf} \end{array}\right. \tag{20a}$$

$$\mathbf{n}_{rof} = \frac{\mathbf{u}_{or}\times\mathbf{u}_{of}}{\| \mathbf{u}_{or}\times\mathbf{u}_{of} \|};\ \mathbf{n}_{iro} = \frac{\mathbf{u}_{ir}\times\mathbf{u}_{or}}{\| \mathbf{u}_{ir}\times\mathbf{u}_{or} \|};\ \mathbf{n}_{rfi} = \frac{\mathbf{u}_{rf}\times\boldsymbol{\tau}_{if}}{\| \mathbf{u}_{rf}\times\boldsymbol{\tau}_{if} \|};\ y_{fr,ir} = (Q_{ir}-Q_{fr})\cdot\mathbf{n}_{rfi} = (A_{ir}-A_{fr})\cdot\mathbf{n}_{rfi}; \tag{20b}$$

Eq. (19a) and (19b) are the two alternative IORs of the Tra-Hel case and come to have the canonical form of Eq. (6) with the following positions

$$\dot{q} = v_{if};\ \dot{s} = \omega_{or};\ a(q) = -\frac{\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \|}{y_{of,oi}}; \tag{21a}$$

$$\text{Eq. (19a)} \Rightarrow b(q) = \frac{(p_{or}-p_{oi}) \| \mathbf{u}_{ir} \times \mathbf{u}_{or} \| (\mathbf{u}_{ir}\cdot\mathbf{u}_{oi}) + (\mathbf{u}_{or}\cdot\mathbf{u}_{oi})y_{ir,oi} - y_{or,oi}(\mathbf{u}_{ir}\cdot\mathbf{u}_{or})(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi})}{y_{ir,oi}}; \tag{21b}$$

$$\text{Eq. (19b)} \Rightarrow b(q) = \frac{(\mathbf{u}_{og}\times\mathbf{u}_{ig})\cdot[(\mathbf{u}_{or}\times\mathbf{u}_{ir})\times\mathbf{u}_{oi}]}{\mathbf{u}_{ir}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})} \tag{21c}$$

Moreover, Eq. (19c) combined with Eqs. (9a), (9b), (9c) and (20a) yields

$$p_{or} = p_{of} + y_{of,oi}\frac{(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{oi})}{\| \mathbf{u}_{oi} \times \boldsymbol{\tau}_{if} \|} - \frac{(y_{or,ri}-y_{or,oi})\frac{\mathbf{n}_{iro}\cdot(\mathbf{u}_{or}\times\mathbf{u}_{oi})}{\mathbf{n}_{iro}\cdot(\mathbf{u}_{ir}\times\mathbf{u}_{oi})} + y_{or,oi}(\mathbf{u}_{or}\cdot\mathbf{u}_{ri})}{\| \mathbf{u}_{or} \times \mathbf{u}_{ri} \|} \tag{22}$$

2.1.4. Tra-Tra case

In this case, $\dot{q}$ and $\dot{s}$ are v_{if} and v_{or}, respectively, and the relative motion theorems [30] bring one to write:

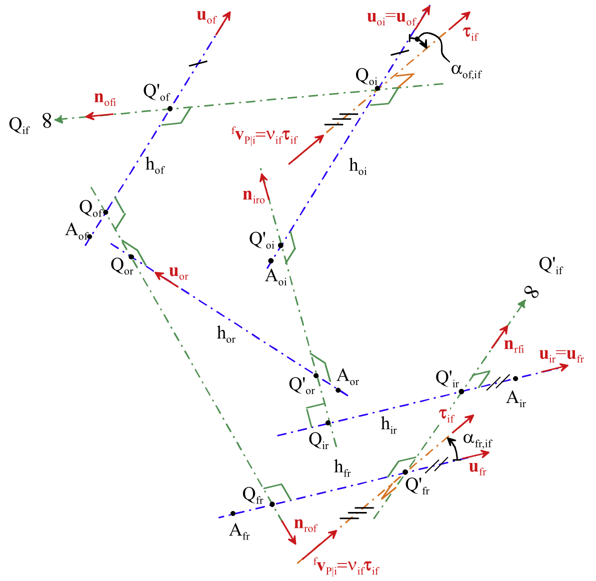

Fig. 5. General spatial disposition of the six ISAs (five finite lines and one ideal line) associated to the six possible relative motions of a four-body system for the Tra-Hel case: the five blue lines (A_{jk}, $\mathbf{u}_{jk}$) with $jk\in\{of, ir, or, oi, rf\}$ are the five finite ISA_{jk}; whereas, the ideal line, ISA_{if}, associated to the *if* translation is the ideal line shared by all the parallel planes that are all perpendicular to $\boldsymbol{\tau}_{if}$.

$$\left.\begin{array}{l}\left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi}=\omega_{of}\mathbf{u}_{of}-\omega_{if}\mathbf{u}_{if}\\ \omega_{if}=0\\ \omega_{ir}\mathbf{u}_{ir}=\omega_{if}\mathbf{u}_{if}+\omega_{fr}\mathbf{u}_{fr}\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\omega_{oi}=\omega_{of}\\ \mathbf{u}_{oi}=\mathbf{u}_{of}\\ \omega_{ir}=\omega_{fr}=-\omega_{rf}\\ \mathbf{u}_{ir}=\mathbf{u}_{fr}=\mathbf{u}_{rf}\end{array}\right.\\ \left.\begin{array}{l}\omega_{oi}\mathbf{u}_{oi}=\omega_{or}\mathbf{u}_{or}+\omega_{ri}\mathbf{u}_{ri}\\ \omega_{or}=0\\ \omega_{of}\mathbf{u}_{of}=\omega_{or}\mathbf{u}_{or}+\omega_{rf}\mathbf{u}_{rf}\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\omega_{oi}=\omega_{ri}=-\omega_{ir}\\ \mathbf{u}_{oi}=\mathbf{u}_{ri}=\mathbf{u}_{ir}\\ \omega_{of}=\omega_{rf}=-\omega_{fr}\\ \mathbf{u}_{of}=\mathbf{u}_{rf}=\mathbf{u}_{fr}\end{array}\right.\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\omega_{if}=\omega_{or}=0\\ \\ \omega_{oi}=\omega_{of}=\omega_{ri}=\omega_{rf}\\ \\ \mathbf{u}_{oi}=\mathbf{u}_{of}=\mathbf{u}_{ir}=\mathbf{u}_{rf}\Rightarrow\left\{\begin{array}{l}\mathbf{n}_{fro}=\dfrac{\mathbf{u}_{rf}\times\boldsymbol{\tau}_{or}}{\|\mathbf{u}_{rf}\times\boldsymbol{\tau}_{or}\|}=\dfrac{\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}}{\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\|}=\mathbf{n}_{iro}\\ \mathbf{n}_{rif}=\dfrac{\mathbf{u}_{ir}\times\boldsymbol{\tau}_{if}}{\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{if}\|}=\dfrac{\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}}{\|\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\|}=\mathbf{n}_{ofi}\end{array}\right.\end{array}\right. \tag{23}$$

Therefore, the general spatial disposition of the ISAs, which now are four parallel finite lines and two ideal lines, is the one shown in Fig. 6.

In this case, Eqs. (4a) and (4b) separately written for the four sets of three links {*i, f, o*}, {*o, r, i*}, {*r, f, i*}, and {*r, f, o*} yield

$$\omega_{of}=-v_{if}\frac{\|\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\|}{y_{of,oi}};\ \omega_{ir}=-v_{or}\frac{\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\|}{y_{ir,oi}}; \tag{24a}$$

$$\omega_{rf}=-v_{if}\frac{\|\mathbf{u}_{rf}\times\boldsymbol{\tau}_{if}\|}{y_{rf,ri}};\ \omega_{fr}=-v_{or}\frac{\|\mathbf{u}_{fr}\times\boldsymbol{\tau}_{or}\|}{y_{fr,of}}; \tag{24b}$$

$$p_{of}=p_{oi}-y_{of,oi}\frac{(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{of})}{\|\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\|};\ p_{ir}=p_{oi}-y_{ir,oi}\frac{(\boldsymbol{\tau}_{or}\cdot\mathbf{u}_{ir})}{\|\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\|}; \tag{24c}$$

$$p_{rf}=p_{ir}-y_{rf,ri}\frac{(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{rf})}{\|\mathbf{u}_{rf}\times\boldsymbol{\tau}_{if}\|};\ p_{fr}=p_{of}-y_{fr,of}\frac{(\boldsymbol{\tau}_{or}\cdot\mathbf{u}_{fr})}{\|\mathbf{u}_{fr}\times\boldsymbol{\tau}_{or}\|}. \tag{24d}$$

By considering that $\omega_{of}=\omega_{rf}=-\omega_{fr}=\omega_{ri}=-\omega_{ir}$ and that $\mathbf{u}_{oi}=\mathbf{u}_{of}=\mathbf{u}_{ir}=\mathbf{u}_{rf}$ (see Eq. (23)), formulas (24) provide the following relationships

$$v_{if}\|\mathbf{u}_{oi}\times\boldsymbol{\tau}_{if}\|y_{rf,of}=-v_{or}\|\mathbf{u}_{oi}\times\boldsymbol{\tau}_{or}\|y_{rf,ri} \tag{25a}$$

$$p_{of}=p_{oi}-y_{of,oi}\frac{(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{oi})}{\|\mathbf{u}_{oi}\times\boldsymbol{\tau}_{if}\|};\ p_{ir}=p_{oi}-y_{ir,oi}\frac{(\boldsymbol{\tau}_{or}\cdot\mathbf{u}_{oi})}{\|\mathbf{u}_{oi}\times\boldsymbol{\tau}_{or}\|}; \tag{25b}$$

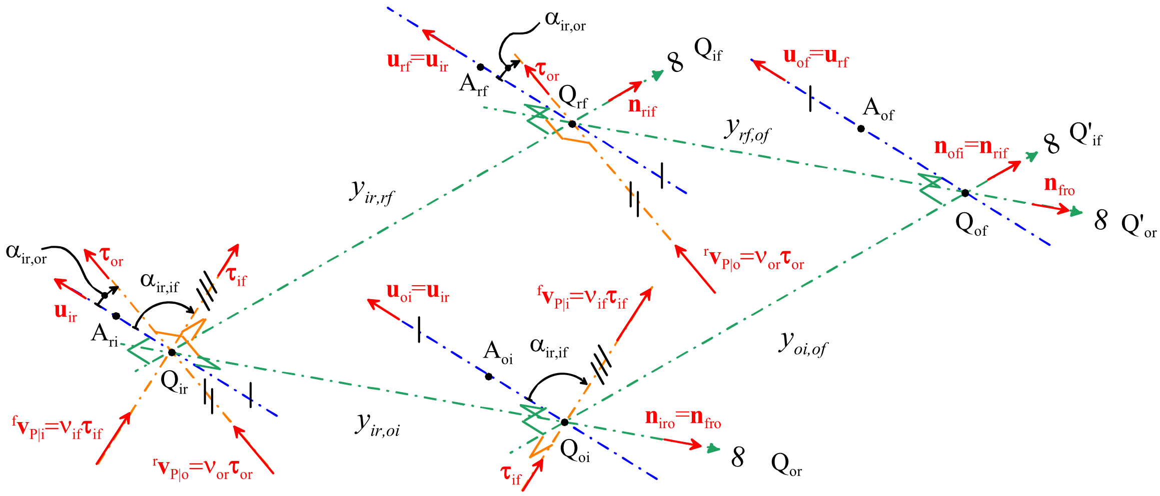

Fig. 6. General spatial disposition of the six ISAs (four finite parallel lines and two ideal lines) associated to the six possible relative motions of a four-body system for the Tra-Tra case: the four parallel blue lines (A_{jk}, $\mathbf{u}_{jk}$) with $jk\in\{of, ir, oi, rf\}$ are the four finite ISA_{jk}; whereas, the ideal line ISA_{if} (ISA_{or}) associated to the *if* (*or*) translation is the ideal line shared by all the parallel planes that are all perpendicular to $\boldsymbol{\tau}_{if}$ (to $\boldsymbol{\tau}_{or}$).

$$p_{rf} = p_{ir} - y_{rf,ri}\frac{\left(\boldsymbol{\tau}_{if}\cdot\mathbf{u}_{oi}\right)}{\|\ \mathbf{u}_{oi}\times\boldsymbol{\tau}_{if}\ \|};\ p_{fr} = p_{of} - y_{fr,of}\frac{\left(\boldsymbol{\tau}_{or}\cdot\mathbf{u}_{oi}\right)}{\|\ \mathbf{u}_{oi}\times\boldsymbol{\tau}_{or}\ \|}. \tag{25c}$$

$$\left(A_{oi}-A_{of}\right)\cdot\left(\mathbf{u}_{oi}\times\boldsymbol{\tau}_{if}\right) = \left(A_{ri}-A_{rf}\right)\cdot\left(\mathbf{u}_{rf}\times\boldsymbol{\tau}_{if}\right) \Rightarrow y_{rf,ri} = y_{of,oi} \tag{25d}$$

$$\left(A_{of}-A_{fr}\right)\cdot\left(\mathbf{u}_{rf}\times\boldsymbol{\tau}_{or}\right) = \left(A_{oi}-A_{ir}\right)\cdot\left(\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\right) \Rightarrow y_{fr,of} = y_{ir,oi} \tag{25e}$$

Eq. (25a) is the sought-after IOR of the Tra-Tra case and comes to have the canonical form of Eq. (6) with the following positions:

$$\dot{q} = \nu_{if};\ \dot{s} = \nu_{or};\ a(q) = \|\ \mathbf{u}_{oi}\times\boldsymbol{\tau}_{if}\ \|\ y_{rf,of};\ b(q) = -\ \|\ \mathbf{u}_{oi}\times\boldsymbol{\tau}_{or}\ \|\ y_{rf,ri} \tag{26}$$

Eventually, by taking into account formulas (25d) and (25e), the subtraction of Eqs. (25c) from Eqs. (25b) yields

$$p_{of} - p_{rf} - p_{oi} + p_{ir} = 0 \tag{27}$$

2.2. Singularity conditions

Eq. (6) is the general form of the IOR of a single-DOF mechanism with only one output variable (SISO system). After having simplified the expressions of coefficients a and b in Eq. (6) by eliminating all their common factors, a type-I (serial) singularity (i.e., the IIK is indeterminate) occurs for each q value that makes either coefficient a equal to zero or coefficient b equal to infinity; a type-II (parallel) singularity (i.e., the IFK is indeterminate) occurs for each q value that makes either coefficient b equal to zero or coefficient a equal to infinity; finally, a type-III singularity (i.e., both the IIK and the IFK are indeterminate) occurs for each q value that makes both the coefficients a and b equal to zero. In the previous part of this section, the explicit expressions that relate the coefficients a and b to the mechanism configuration through the spatial disposition of the involved ISAs have been deduced for all the possible cases (i.e., Hel-Hel, Hel-Tra, Tra-Hel, and Tra-Tra). Such expressions are summarized in Tables 1 and 2. In the following part of this section, they will be analyzed to identify all the geometric conditions on the ISAs that correspond to mechanism singularities.

2.2.1. Hel-Hel case

In this case, if link f coincides with link r, then $\dot{q}$ and $\dot{s}$ are ω_{if} and ω_{of}, respectively, and formulas (1a), (1c) and (1d) must be considered according to the particular disposition of the ISAs (see Table 1 and Fig. 2a); otherwise (i.e., if $r\neq f$), $\dot{q}$ and $\dot{s}$ are ω_{if} and ω_{or}, respectively, and formulas (10a) and (10b) together with the accompanying relationships (1c) and (9c) must be considered according to the ISA disposition (see Table 2 and Fig. 3).

If link r coincides with link f, there are only three links, that is, links i, o, and f; therefore, the three possible relative motions (i.e., *if*, *of* and *io* motions) among these links are associated to three ISAs that must share a common normal [29] with the direction of unit vector $\mathbf{n}_{ofi}$ (Fig. 2a). Moreover, the relative-motion theorems [30] impose that, if two out of these ISAs are parallel to each other, also the remaining third must be parallel to them; accordingly, either the three ISAs are all parallel or none of them is parallel to any other. As a consequence, the expressions of coefficients a and b deduced from Eq. (1a) either are both different from zero when there is no parallelism among the ISAs or are both equal to zero when the ISAs are all parallel; accordingly, Eq. (1a) can be used only when the ISAs are not parallel at a non-singular configuration. Indeed, it always provides an indeterminate expression ($\equiv$0/0) when the ISAs are

Table 1
Coefficients a and b of Eq. (6) and geometric/kinematic conditions on ISAs that are valid when the motion of both the input, i, and the output, o, links are referred to the same reference link, f.

Case	$\dot{q}$	$\dot{s}$	a	b	Geometric&Kinematic Conditions on ISAs
Hel-Hel(*)	ω_{if}	ω_{of}	$\mathbf{n}_{ofi}\cdot(\mathbf{u}_{fi}\times\mathbf{u}_{oi})$ or $(p_{oi}-p_{fi})\Vert\mathbf{u}_{of}\times\mathbf{u}_{fi}\Vert+y_{if,oi}(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})$ or $\mathbf{u}_{if}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})$	$\mathbf{n}_{ofi}\cdot(\mathbf{u}_{of}\times\mathbf{u}_{oi})$ or $y_{of,oi}$ or $\mathbf{u}_{of}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})$	$\frac{\omega_{of}}{\omega_{if}}=\frac{(p_{oi}-p_{fi})\Vert\mathbf{u}_{of}\times\mathbf{u}_{fi}\Vert+y_{if,oi}(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})}{y_{of,oi}}$
Hel-Tra	ω_{if}	v_{of}	$-y_{if,oi}$	$\Vert\mathbf{u}_{if}\times\boldsymbol{\tau}_{of}\Vert$	$\begin{cases}\omega_{oi}=-\omega_{if}=\omega_{fi}\\ \mathbf{u}_{oi}=\mathbf{u}_{if}\end{cases}$
Tra-Hel	v_{if}	ω_{of}	$\Vert\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\Vert$	$-y_{of,oi}$	$\begin{cases}\omega_{oi}=\omega_{of}\\ \mathbf{u}_{oi}=\mathbf{u}_{of}\end{cases}$
Tra-Tra(**)	v_{if}	v_{of}	$y_{jf,jo}\Vert\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fi}\Vert$	$y_{jf,ji}\Vert\mathbf{u}_{jf}\times\boldsymbol{\tau}_{fo}\Vert$	$\begin{cases}\omega_{ji}=\omega_{jf}=-\omega_{oj}\\ \mathbf{u}_{ji}=\mathbf{u}_{jf}=\mathbf{u}_{oj}\end{cases}$

(*) Link g is any fourth link chosen so that $\mathbf{u}_{og}\neq\pm\mathbf{u}_{ig}$
(**) Link j is any fourth link chosen so that it performs a helical motion with respect to link f with an ISA_{jf} that is neither parallel to $\boldsymbol{\tau}_{if}$ nor parallel to $\boldsymbol{\tau}_{of}$.

Table 2
Coefficients a and b of Eq. (6) and geometric/kinematic conditions on ISAs that are valid when the motion of the input link, i, is referred to link f and the motion of the output link, o, is referred to link r.

Case	$\dot{q}$	$\dot{s}$	a	b	Geometric&Kinematic Conditions on ISAs
Hel-Hel	ω_{if}	ω_{or}	$\mathbf{u}_{of}\cdot(\mathbf{u}_{if}\times\mathbf{u}_{ir})$(*) or 1st formula of Eq. (12c)	$\mathbf{u}_{or}\cdot(\mathbf{u}_{of}\times\mathbf{u}_{ir})$(*) or 2nd formula of Eq. (12c)	$\begin{cases}\frac{\omega_{or}}{\omega_{if}}=\frac{\frac{\omega_{of}}{\omega_{if}}(\mathbf{u}_{of}\cdot\mathbf{u}_{oi})-(\mathbf{u}_{if}\cdot\mathbf{u}_{oi})}{(\mathbf{u}_{or}\cdot\mathbf{u}_{oi})-\frac{\omega_{ir}}{\omega_{or}}(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi})}\\ \textit{with}\\ \frac{\omega_{of}}{\omega_{if}}=\frac{(p_{oi}-p_{fi})\Vert\mathbf{u}_{of}\times\mathbf{u}_{fi}\Vert+y_{if,oi}(\mathbf{u}_{of}\cdot\mathbf{u}_{fi})}{y_{of,oi}}\\ \frac{\omega_{or}}{\omega_{ir}}=\frac{(p_{oi}-p_{ri})\Vert\mathbf{u}_{or}\times\mathbf{u}_{ir}\Vert+y_{ir,oi}(\mathbf{u}_{or}\cdot\mathbf{u}_{ir})}{y_{or,oi}}\end{cases}$
Hel-Tra	ω_{if}	v_{or}	$\frac{(\mathbf{u}_{og}\times\mathbf{u}_{ig})\cdot[(\mathbf{u}_{if}\times\mathbf{u}_{of})\times\mathbf{u}_{oi}]}{\mathbf{u}_{of}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})}$(**) or Eq. (18b)	$-\frac{\Vert\mathbf{u}_{ir}\times\boldsymbol{\tau}_{or}\Vert}{y_{ir,oi}}$	$\begin{cases}\omega_{oi}=\omega_{ri}=-\omega_{ir}\\ \mathbf{u}_{oi}=\mathbf{u}_{ri}=\mathbf{u}_{ir}\\ \omega_{of}=\omega_{rf}=-\omega_{fr}\\ \mathbf{u}_{of}=\mathbf{u}_{rf}=\mathbf{u}_{fr}\end{cases}$
Tra-Hel	v_{if}	ω_{or}	$-\frac{\Vert\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\Vert}{y_{of,oi}}$	Eq. (21b) or $\frac{(\mathbf{u}_{og}\times\mathbf{u}_{ig})\cdot[(\mathbf{u}_{or}\times\mathbf{u}_{ir})\times\mathbf{u}_{oi}]}{\mathbf{u}_{ir}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig})}$(**)	$\begin{cases}\omega_{oi}=\omega_{of}\\ \mathbf{u}_{oi}=\mathbf{u}_{of}\\ \omega_{ir}=\omega_{fr}=-\omega_{rf}\\ \mathbf{u}_{ir}=\mathbf{u}_{fr}=\mathbf{u}_{rf}\end{cases}$
Tra-Tra	v_{if}	v_{or}	$\Vert\mathbf{u}_{oi}\times\boldsymbol{\tau}_{if}\Vert y_{rf,of}$	$-\Vert\mathbf{u}_{oi}\times\boldsymbol{\tau}_{or}\Vert y_{rf,ri}$	$\begin{cases}\omega_{if}=\omega_{or}=0\\ \omega_{oi}=\omega_{of}=\omega_{ri}=\omega_{rf}\\ \mathbf{u}_{oi}=\mathbf{u}_{of}=\mathbf{u}_{ir}=\mathbf{u}_{rf}\end{cases}$

(*) This expressions hold if and only if $\mathbf{u}_{of}\neq\pm\mathbf{u}_{ir}$
(**) Link g is any fifth link chosen so that $\mathbf{u}_{og}\neq\pm\mathbf{u}_{ig}$

all parallel, and, when the ISAs are not parallel, it always provides values of coefficients a ($=\sin\alpha_{oi,if}$) and b ($=\sin\alpha_{oi,of}$) that are different from zero[4]. Differently, Eq. (1c) can take into account both the cases provided that the three ISAs have not a common intersection and the two involved pitches are not equal to one another; such a condition always is not satisfied, for instance, in spherical mechanisms. The geometric conditions not-covered by Eq. (1c) are covered by Eq. (1d), which, unfortunately, needs a fourth

[4] This is due to the particular way Eq. (1a) was deduced in [29] (i.e., the dot product of the relationship $\omega_{oi}\mathbf{u}_{oi}=\omega_{of}\mathbf{u}_{of}+\omega_{fi}\mathbf{u}_{fi}$ by $(\mathbf{n}_{ofi}\times\mathbf{u}_{oi})$), which, in practice, excluded singular configurations (i.e., where $\mathbf{n}_{ofi}$ is indeterminate) and ISAs' parallelism (i.e., where $\mathbf{u}_{oi}\Vert\,\mathbf{u}_{of}\,\Vert\mathbf{u}_{fi}$).

link, link g, with $\mathbf{u}_{og} \neq \pm\mathbf{u}_{ig}$ to be used. The analysis of Eq. (1c) reveals that, if $\mathbf{u}_{of} \neq \pm\mathbf{u}_{fi}$ (i.e., the ISAs are not parallel), coefficient a is equal to zero (i.e., a serial singularity occurs) at mechanism configurations that satisfy the following condition

$$\tan\alpha_{of,if} \equiv \frac{\| \mathbf{u}_{of} \times \mathbf{u}_{fi} \|}{\mathbf{u}_{of} \cdot \mathbf{u}_{fi}} = \frac{y_{if,oi}}{p_{if} - p_{oi}}; \tag{28}$$

whereas, coefficient b is equal to zero (i.e., a parallel singularity occurs) when $y_{of,oi} = 0$, that is, at mechanism configurations where $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{oi}}$ intersect one another. Moreover, the same analysis reveals that, if $\mathbf{u}_{of} = \pm\mathbf{u}_{fi}$ (i.e., the ISAs are all parallel), Eq. (1c) becomes

$$\frac{\omega_{of}}{\omega_{if}} = \frac{y_{if,oi}}{y_{of,oi}}; \tag{29}$$

thus, coefficient a is equal to zero (i.e., a serial singularity occurs) when $y_{if,oi} = 0$, that is, at mechanism configurations where $\mathrm{ISA_{if}}$ coincides with $\mathrm{ISA_{oi}}$; whereas, coefficient b is equal to zero (i.e., a parallel singularity occurs) when $y_{of,oi} = 0$, that is, at mechanism configurations where $\mathrm{ISA_{of}}$ coincides with $\mathrm{ISA_{oi}}$. It is worth noting that Eq. (29) generalizes the one that holds for planar single-DOF mechanisms, where the signed distances $y_{if,oi}$ and $y_{of,oi}$ are computed through the positions of the instant centers (see Eq. (5a) of Ref. [27]), since it only needs equal pitches (i.e., non-necessarily, the pitches must be all equal to zero as it happens in planar mechanisms) and parallel ISAs to be valid. Eventually, the analysis of Eq. (1d), which holds even in the cases where Eq. (1c) is not valid (i.e., $\mathrm{ISA_{if}}$, $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{oi}}$ share a common intersection (i.e., $Q_{if} \equiv Q_{of} \equiv Q_{oi}$) and, at the same time, $p_{if} = p_{oi}$), reveals that coefficient a is equal to zero (i.e., a serial singularity occurs) when $\mathbf{u}_{if} \cdot (\mathbf{u}_{og} \times \mathbf{u}_{ig}) = 0$, that is, at mechanism configurations where $\mathrm{ISA_{if}}$, $\mathrm{ISA_{og}}$ and $\mathrm{ISA_{ig}}$ are all coplanar; whereas, coefficient b is equal to zero (i.e., a parallel singularity occurs) when $\mathbf{u}_{of} \cdot (\mathbf{u}_{og} \times \mathbf{u}_{ig}) = 0$, that is, at mechanism configurations where $\mathrm{ISA_{of}}$, $\mathrm{ISA_{og}}$ and $\mathrm{ISA_{ig}}$ are all coplanar. It is worth noting that Eq. (1c) generalizes the one that holds for spherical single-DOF mechanisms (see Eq. (6) of Ref. [37]).

If link r is different from link f, there are four links, which yield six relative motions and as many ISAs. Therefore, there are four triplets of links, that is, {i, o, f}, {i, o, r}, {i, r, f}, and {o, r, f}. Each of these triplets of links identifies a triplet of ISAs, associated to their relative motions, that must share a common normal [29] and, as a consequence, can be all either parallel or not parallel. Accordingly, the condition of being "all parallel" or "all not parallel" holds only for each of the following four sets of ISAs: {$\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$, $\mathrm{ISA_{if}}$}, {$\mathrm{ISA_{oi}}$, $\mathrm{ISA_{or}}$, $\mathrm{ISA_{ir}}$}, {$\mathrm{ISA_{ir}}$, $\mathrm{ISA_{if}}$, $\mathrm{ISA_{rf}}$}, and {$\mathrm{ISA_{or}}$, $\mathrm{ISA_{of}}$, $\mathrm{ISA_{rf}}$}. Since these ISA sets always share one ISA that is different for each couple of ISA triplets, the parallelism of all the six ISAs is a particular geometric condition that can occur if and only if, simultaneously, two of these ISA triplets have parallel ISAs.

In this case, Eq. (11a), which comes from Eq. (10a), holds provided that $\mathbf{u}_{of} \neq \pm\mathbf{u}_{ir}$, which, in particular, excludes the parallelism of all the six ISAs (e.g., it does not hold for single-DOF planar mechanisms); whereas, Eq. (11b), which comes from Eqs. (10b), (1c) and (9c), holds for the geometric conditions where Eq. (11a) is not valid, but it gives an indeterminate expression (i.e., it is not valid) when the condition ($p_{oi} = p_{ri} = p_{fi}$)&($\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$, and $\mathrm{ISA_{if}}$ share a common intersection)&($\mathrm{ISA_{oi}}$, $\mathrm{ISA_{or}}$, and $\mathrm{ISA_{ir}}$ share a common intersection) is true (e.g., it does not hold for single-DOF spherical mechanisms). The analysis of Eq. (11a), reveals that coefficient a is equal to zero (i.e., a serial singularity occurs) when $\mathbf{u}_{of} \cdot (\mathbf{u}_{if} \times \mathbf{u}_{ir}) = 0$, that is, at mechanism configurations where $\mathrm{ISA_{if}}$, $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{ir}}$ are all parallel to one plane; whereas, coefficient b is equal to zero (i.e., a parallel singularity occurs) when $\mathbf{u}_{or} \cdot (\mathbf{u}_{of} \times \mathbf{u}_{ir}) = 0$, that is, at mechanism configurations where $\mathrm{ISA_{of}}$, $\mathrm{ISA_{or}}$ and $\mathrm{ISA_{ir}}$ are all parallel to one plane. It is worth noting that Eq. (11a) generalizes the one that holds for spherical single-DOF mechanisms (see Eq. (5) of Ref. [37] and its equivalent expressions). The analysis of Eq. (11b) reveals that, if $\mathbf{u}_{of} \neq \pm\mathbf{u}_{ir}$ (i.e., the ISAs are not all parallel), coefficient a is equal to zero (i.e., a serial singularity occurs) at mechanism configurations that satisfy one or the other of the following two conditions

$$\left[(p_{oi} - p_{fi}) \| \mathbf{u}_{of} \times \mathbf{u}_{fi} \| + y_{if,oi}(\mathbf{u}_{of} \cdot \mathbf{u}_{fi})\right](\mathbf{u}_{of} \cdot \mathbf{u}_{oi}) = y_{of,oi}(\mathbf{u}_{if} \cdot \mathbf{u}_{oi}) \Rightarrow \left[(p_{oi} - p_{fi})\tan\alpha_{of,if} + y_{if,oi}\right]\cos\alpha_{of,if}\cos\alpha_{of,oi} = y_{of,oi}\cos\alpha_{if,oi} \tag{30a}$$

$$(p_{oi} - p_{ri}) \| \mathbf{u}_{or} \times \mathbf{u}_{ir} \| + y_{ir,oi}(\mathbf{u}_{or} \cdot \mathbf{u}_{ir}) = 0 \Rightarrow \tan\alpha_{or,ir} \equiv \frac{\| \mathbf{u}_{or} \times \mathbf{u}_{ir} \|}{\mathbf{u}_{or} \cdot \mathbf{u}_{ir}} = \frac{y_{ir,oi}}{p_{ri} - p_{oi}}; \tag{30b}$$

whereas, coefficient b is equal to zero (i.e., a parallel singularity occurs) at mechanism configurations that satisfy one or the other of the following two conditions

$$y_{of,oi} = 0 \tag{31a}$$

$$(\mathbf{u}_{or} \cdot \mathbf{u}_{oi})\left[(p_{oi} - p_{ri}) \| \mathbf{u}_{or} \times \mathbf{u}_{ir} \| + y_{ir,oi}(\mathbf{u}_{or} \cdot \mathbf{u}_{ir})\right] = y_{or,oi}(\mathbf{u}_{ir} \cdot \mathbf{u}_{oi}) \Rightarrow \left[(p_{oi} - p_{ri})\tan\alpha_{or,ir} + y_{ir,oi}\right]\cos\alpha_{or,ir}\cos\alpha_{or,oi} = y_{or,oi}\cos\alpha_{ir,oi} \tag{31b}$$

Moreover, the same analysis reveals that, if the six ISAs are all parallel, Eq. (11b) becomes

$$\frac{\omega_{or}}{\omega_{if}} = \frac{y_{ir,oi}(y_{if,oi} - y_{of,oi})}{y_{of,oi}(y_{ir,oi} - y_{or,oi})} \equiv \frac{y_{ir,oi}y_{if,of}}{y_{of,oi}y_{ir,or}} \tag{32}$$

thus, coefficient a is equal to zero (i.e., a serial singularity occurs) when either $y_{ir,oi} = 0$ (i.e., when $\mathrm{ISA_{ir}}$ coincides with $\mathrm{ISA_{oi}}$) or $y_{if,of} = 0$ (i.e., when $\mathrm{ISA_{if}}$ coincides with $\mathrm{ISA_{of}}$); whereas, coefficient b is equal to zero (i.e., a parallel singularity occurs) when either $y_{of,oi}$

$= 0$ (i.e., when $\mathrm{ISA_{of}}$ coincides with $\mathrm{ISA_{oi}}$) or $y_{ir,or} = 0$ (i.e., when $\mathrm{ISA_{ir}}$ coincides with $\mathrm{ISA_{or}}$). It is worth noting that Eq. (32) generalizes the one that holds for planar single-DOF mechanisms, where the parameters $y_{ir,oi}$, $y_{if,of}$, $y_{of,oi}$ and $y_{ir,or}$ are computed through the positions of the instant centers on the motion plane (see Eq. (6) of Ref. [27]).

2.2.2. Hel-Tra case

In this case, if link r coincides with link f, then $\dot{q}$ and $\dot{s}$ are ω_{if} and v_{of}, respectively, and Eq. (3a) must be considered together with the particular disposition of the ISAs (see Table 1 and Fig. 2b); otherwise (i.e., if $r \neq f$), $\dot{q}$ and $\dot{s}$ are ω_{if} and v_{or}, respectively, and Eqs. (17a) and (17b) must be considered according to the ISA disposition (see Table 2 and Fig. 4).

If link r coincides with link f, the analysis of Eq. (3a) reveals that coefficient a is equal to zero (i.e., a serial singularity occurs) when $y_{if,oi} = 0$, that is, at mechanism configurations (see Eq. 3c) where $\mathrm{ISA_{if}}$ coincides with $\mathrm{ISA_{oi}}$; whereas, coefficient b of Eq. (3a) is equal to zero (i.e., a parallel singularity occurs) when $\| \mathbf{u}_{if} \times \boldsymbol{\tau}_{of} \| = 0$, that is, at mechanism configurations where $\mathrm{ISA_{if}}$ is parallel to the translation direction of the output link (i.e., where $\mathbf{u}_{if} = \pm\boldsymbol{\tau}_{of}$).

If link r is different from link f, the six relative motions among the four links are associated to the ideal line $\mathrm{ISA_{or}}$ and to the five finite lines $\mathrm{ISA_{ir}}$, $\mathrm{ISA_{fr}}$, $\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$, and $\mathrm{ISA_{if}}$ that must satisfy the following geometric conditions (see Fig. 4 and formulas (15d)):

- $\mathrm{ISA_{of}}$ must be parallel to $\mathrm{ISA_{rf}}$;
- $\mathrm{ISA_{oi}}$ must be parallel to $\mathrm{ISA_{ir}}$;
- the three lines $\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{if}}$ must share a common normal and can be either all parallel or all not parallel since they are associated to the relative motion among the triplet of links {*i, o, f*} [29].

Eq. (17a) covers all the possible ISAs' dispositions, but the one where $\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{if}}$ share a common intersection and, at the same time, $p_{if} = p_{oi}$, since coefficient a of Eq. (17a) becomes indeterminate ($\equiv 0/0$) when this condition occurs. As a consequence, Eq. (17b), which has the same coefficient b as Eq. (17a), must be analyzed only with reference to this special case and uniquely to determine further singularities related to coefficient a. The analysis of Eq. (17a) reveals that a parallel singularity occurs (i.e., coefficient b is equal to zero or coefficient a goes to infinity) either if $\| \mathbf{u}_{ir} \times \boldsymbol{\tau}_{or} \| = 0$ (i.e., at mechanism configurations where $\mathbf{u}_{ir} = \pm\boldsymbol{\tau}_{or}$) or if $y_{of,oi} = 0$ (i.e., at mechanism configurations where $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{oi}}$ intersect one another); whereas, the same analysis brings one to the conclusion that a serial singularity occurs (i.e., coefficient b goes to infinity or coefficient a is equal to zero) either if $y_{ir,oi} = 0$ (i.e., at mechanism configurations where $\mathrm{ISA_{oi}}$ coincides with $\mathrm{ISA_{ir}}$) or if

$$\frac{(\mathbf{u}_{if} \cdot \mathbf{u}_{oi})}{(\mathbf{u}_{of} \cdot \mathbf{u}_{oi})} y_{of,oi} - y_{if,oi}(\mathbf{u}_{of} \cdot \mathbf{u}_{if}) = (p_{oi} - p_{if}) \| \mathbf{u}_{of} \times \mathbf{u}_{if} \| \quad \Rightarrow \quad \frac{\cos\alpha_{oi,if}}{\cos\alpha_{oi,of}} y_{of,oi} - y_{if,oi}\cos\alpha_{of,if} = (p_{oi} - p_{if})\sin\alpha_{of,if} \tag{33}$$

It is worth stressing that, if the five finite ISAs are all parallel[5] (e.g., as it happens in planar single-DOF mechanisms), Eq. (17a) becomes

$$-v_{or}\frac{\| \mathbf{u}_{ir} \times \boldsymbol{\tau}_{or} \|}{y_{ir,oi}} = \omega_{if}\frac{y_{of,oi} - y_{if,oi}}{y_{of,oi}} \equiv \omega_{if}\frac{y_{of,if}}{y_{of,oi}} \quad \Rightarrow \quad -v_{or} \| \mathbf{u}_{ir} \times \boldsymbol{\tau}_{or} \| y_{of,oi} = \omega_{if} y_{ir,oi} y_{of,if} \tag{34}$$

which generalizes the IOR of single-DOF planar mechanisms (see Eq. (11') and (27) of Ref. [27]) where any translation direction (i.e., any $\boldsymbol{\tau}_{or}$) lies on the motion plane and the ISAs are all perpendicular to the same plane, that is, $\| \mathbf{u}_{ir} \times \boldsymbol{\tau}_{or} \|$ is always equal to 1.

Eventually, if $\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{if}}$ share a common intersection and, at the same time, $p_{if} = p_{oi}$, the analysis of coefficient a of Eq. (17b) reveals that the conditions $\mathbf{u}_{og} \neq \pm\mathbf{u}_{ig}$ and $\mathbf{u}_{of} \cdot (\mathbf{u}_{og} \times \mathbf{u}_{ig}) \neq 0$ can be always satisfied by suitably choosing link g. Therefore, coefficient a of Eq. (17b) never goes to infinity and is equal to zero (i.e., a further serial singularity occurs) if $\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{if}}$ are all parallel, which, in this case, implies that they lie on the same finite line.

2.2.3. Tra-Hel case

In this case, if link r coincides with link f, then $\dot{q}$ and $\dot{s}$ are v_{if} and ω_{of}, respectively, and Eq. (4a) must be considered together with the particular disposition of the ISAs (see Table 1 and Fig. 2c); otherwise (i.e., if $r \neq f$), $\dot{q}$ and $\dot{s}$ are v_{if} and ω_{or}, respectively, and Eqs. (19a) and (19b) must be considered according to the ISA disposition (see Table 2 and Fig. 5).

If link r coincides with link f, the analysis of Eq. (4a) reveals that coefficient a is equal to zero (i.e., a serial singularity occurs) when $\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \| = 0$, that is, at mechanism configurations where $\mathrm{ISA_{of}}$ is parallel to the translation direction of the input link (i.e., where $\mathbf{u}_{of} = \pm\boldsymbol{\tau}_{if}$); whereas, coefficient b of Eq. (4a) is equal to zero (i.e., a parallel singularity occurs) when $y_{of,oi} = 0$, that is, at mechanism configurations (see Eq. 4c) where $\mathrm{ISA_{of}}$ coincides with $\mathrm{ISA_{oi}}$.

If link r is different from link f, the six relative motions among the four links are associated to the ideal line $\mathrm{ISA_{if}}$ and to the five finite lines $\mathrm{ISA_{ir}}$, $\mathrm{ISA_{or}}$, $\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$, and $\mathrm{ISA_{fr}}$ that must satisfy the following geometric conditions (see Fig. 5 and formulas (20a)):

- $\mathrm{ISA_{oi}}$ must be parallel to $\mathrm{ISA_{of}}$;

[5] It is worth stressing that, since $\mathrm{ISA_{of}}$ is parallel to $\mathrm{ISA_{rf}}$ and $\mathrm{ISA_{oi}}$ is parallel $\mathrm{ISA_{ir}}$, if $\mathrm{ISA_{oi}}$, $\mathrm{ISA_{of}}$ and $\mathrm{ISA_{if}}$ are all parallel, then all the five ISAs must be parallel.

- ISA$_{ir}$ must be parallel to ISA$_{fr}$;
- the three lines ISA$_{oi}$, ISA$_{or}$ and ISA$_{ir}$ must share a common normal and can be either all parallel or all not parallel since they are associated to the relative motion among the triplet of links {*i, o, r*} [29].

Eq. (19a) holds for all the possible ISAs' dispositions, but the one where ISA$_{oi}$, ISA$_{or}$ and ISA$_{ir}$ share a common intersection and, at the same time, $p_{or} = p_{oi}$, since coefficient b of Eq. (19a) becomes indeterminate ($\equiv$0/0) when this condition occurs. As a consequence, Eq. (19b), which has the same coefficient a as Eq. (19a), must be analyzed only with reference to this special case and uniquely to determine further singularities related to coefficient b. The analysis of Eq. (19a) reveals that a serial singularity occurs (i.e., coefficient a is equal to zero or coefficient b goes to infinity) either if $\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \| = 0$ (i.e., at mechanism configurations where $\mathbf{u}_{of} = \pm\boldsymbol{\tau}_{if}$) or if $y_{ir,oi} = 0$ (i.e., at mechanism configurations where ISA$_{ir}$ and ISA$_{oi}$ intersect one another); whereas, the same analysis brings one to the conclusion that a parallel singularity occurs (i.e., coefficient a goes to infinity or coefficient b is equal to zero) either if $y_{of,oi} = 0$ (i.e., at mechanism configurations where ISA$_{oi}$ coincides with ISA$_{of}$) or if

$$\frac{(\mathbf{u}_{or}\cdot\mathbf{u}_{oi})}{(\mathbf{u}_{ir}\cdot\mathbf{u}_{oi})}y_{ir,oi} - y_{or,oi}(\mathbf{u}_{ir}\cdot\mathbf{u}_{or}) = (p_{oi}-p_{or})\,\|\,\mathbf{u}_{ir}\times\mathbf{u}_{or}\,\| \quad\Rightarrow\quad \frac{\cos\alpha_{oi,or}}{\cos\alpha_{oi,ir}}y_{ir,oi} - y_{or,oi}\cos\alpha_{ir,or} = (p_{oi}-p_{or})\sin\alpha_{ir,or} \tag{35}$$

It is worth stressing that, if the five finite ISAs are all parallel[6] (e.g., as it happens in planar single-DOF mechanisms), Eq. (19a) becomes

$$-v_{if}\frac{\|\,\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\,\|}{y_{of,oi}} = \omega_{or}\frac{y_{ir,oi}-y_{or,oi}}{y_{ir,oi}} \equiv \omega_{or}\frac{y_{ir,or}}{y_{ir,oi}} \quad\Rightarrow\quad -v_{if}\,\|\,\mathbf{u}_{of}\times\boldsymbol{\tau}_{if}\,\|\,y_{ir,oi} = \omega_{or}y_{of,oi}y_{ir,or} \tag{36}$$

which generalizes the IOR of single-DOF planar mechanisms (see Eqs. (16') and (29) of Ref. [27]) where any translation direction (i.e., any $\boldsymbol{\tau}_{if}$) lies on the motion plane and the ISAs are all perpendicular to the same plane, that is, $\| \mathbf{u}_{of} \times \boldsymbol{\tau}_{if} \|$ is always equal to 1.

Eventually, if ISA$_{oi}$, ISA$_{or}$ and ISA$_{ir}$ share a common intersection and, at the same time, $p_{or} = p_{oi}$, the analysis of coefficient b of Eq. (19b) reveals that the conditions $\mathbf{u}_{og} \neq \pm\mathbf{u}_{ig}$ and $\mathbf{u}_{ir}\cdot(\mathbf{u}_{og}\times\mathbf{u}_{ig}) \neq 0$ can be always satisfied by suitably choosing link g. Therefore, coefficient b of Eq. (19b) never goes to infinity and is equal to zero (i.e., a further parallel singularity occurs) if ISA$_{oi}$, ISA$_{or}$ and ISA$_{ir}$ are all parallel, which, in this case, implies that they lie on the same finite line.

2.1.4. Tra-Tra case

In this case, if link r coincides with link f, then $\dot{q}$ and $\dot{s}$ are v_{if} and v_{of}, respectively, and Eq. (5a) must be considered together with the ISAs' disposition (see Table 1 and Fig. 2d); otherwise (i.e., if $r \neq f$), $\dot{q}$ and $\dot{s}$ are v_{if} and v_{or}, respectively, and Eq. (25a) must be considered together with the ISAs' disposition (see Table 2 and Fig. 6).

If link r coincides with link f, the analysis of Eq. (5a), where link j is any fourth link chosen so that it performs a helical motion with respect to link f with an ISA$_{jf}$ that is neither parallel to $\boldsymbol{\tau}_{if}$ nor parallel to $\boldsymbol{\tau}_{of}$, reveals that coefficient a is equal to zero (i.e., a serial singularity occurs) only when $y_{jf,jo} = 0$, that is, at mechanism configurations where ISA$_{jf}$ coincides with ISA$_{jo}$; whereas, coefficient b of Eq. (5a) is equal to zero (i.e., a parallel singularity occurs) only when $y_{jf,ji} = 0$, that is, at mechanism configurations where ISA$_{jf}$ coincides with ISA$_{ji}$.

If link r is different from link f and cannot be freely chosen, the analysis of Eq. (25a) reveals that coefficient a is equal to zero (i.e., a serial singularity occurs) when either $y_{rf,of} = 0$ (i.e., at mechanism configurations where ISA$_{rf}$ coincides with ISA$_{of}$) or $\| \mathbf{u}_{oi} \times \boldsymbol{\tau}_{if} \| = 0$ (i.e., at mechanism configurations where $\mathbf{u}_{oi} = \pm\boldsymbol{\tau}_{if}$); whereas, coefficient b of Eq. (25a) is equal to zero (i.e., a parallel singularity occurs) when either $y_{rf,ri} = 0$ (i.e., at mechanism configurations where ISA$_{rf}$ coincides with ISA$_{ri}$) or $\| \mathbf{u}_{oi} \times \boldsymbol{\tau}_{or} \| = 0$ (i.e., at mechanism configurations where $\mathbf{u}_{oi} = \pm\boldsymbol{\tau}_{or}$).

2.3. Procedure for determining the singularities

The serial-/parallel-singularity conditions identified in Section 2.2 are all the possible ones that can occur in a single-DOF spatial mechanism. They include also all the special cases in which the mechanism is of a particular type (e.g., spherical or planar). Of course, a mechanism configuration that simultaneously satisfies a parallel-singularity condition and a serial-singularity condition is a type-III singularity [2]. Such conditions are listed in Table 3 for the case $r = f$ and in Table 4 for the case $r \neq f$.

These conditions are geometric and analytic at the same time, since they are written by using geometric parameters of ISAs' locations that can be implicitly or explicitly expressed as functions of the generalized coordinate, q, of the single-DOF mechanism. Accordingly, they can be used by implementing either geometrically or analytically the following singularity analysis procedure:

i) according to the IOR type (i.e., Hel-Hel, Hel-Tra, Tra-Hel, Tra-Tra) that must be analyzed, the specific singularity conditions are selected among the ones reported in Tables 3 and 4 and the ISAs that appear in them are identified;
ii) the geometric parameters of the ISAs identified at step (i) are geometrically (e.g., through a 3D drawing of a CAD system) or analytically (e.g., by solving the closure equations of the mechanism) related to the mechanism configuration;

[6] It is worth stressing that, since ISA$_{ir}$ is parallel to ISA$_{rf}$ and ISA$_{oi}$ is parallel ISA$_{of}$, if ISA$_{oi}$, ISA$_{or}$ and ISA$_{ir}$ are all parallel, then all the five ISAs must be parallel.

Table 3
Enumeration of the singularity conditions for the case in which link r coincides with link f (i.e., $r = f$).

Case	Serial-Singularity Conditions ($r = f$)	Parallel-Singularity Conditions ($r = f$)
Hel-Hel	s3.1)$(\mathbf{u}_{of} \neq \pm\mathbf{u}_{fi})\&\left(\frac{\|\mathbf{u}_{of} \times \mathbf{u}_{fi}\|}{\mathbf{u}_{of} \cdot \mathbf{u}_{fi}} = \frac{y_{if,oi}}{p_{if} - p_{oi}}\right)$ s3.2)$(\mathbf{u}_{of} = \pm\mathbf{u}_{fi})\&(y_{if,oi} = 0)$ s3.3)$(\mathbf{u}_{og} \neq \pm\mathbf{u}_{ig})\&(\mathbf{u}_{if} \cdot (\mathbf{u}_{og} \times \mathbf{u}_{ig}) = 0)$	p3.1)$y_{of,oi} = 0$ p3.2)$(\mathbf{u}_{og} \neq \pm\mathbf{u}_{ig})\&(\mathbf{u}_{of} \cdot (\mathbf{u}_{og} \times \mathbf{u}_{ig}) = 0)$
Hel-Tra	s3.4)$y_{if,oi} = 0$ (i.e., $\mathrm{ISA_{if}} \equiv \mathrm{ISA_{oi}}$)	p3.3)$\mathbf{u}_{if} = \pm\boldsymbol{\tau}_{of}$
Tra-Hel	s3.5)$\mathbf{u}_{of} = \pm\boldsymbol{\tau}_{if}$	p3.4)$y_{of,oi} = 0$ (i.e., $\mathrm{ISA_{of}} \equiv \mathrm{ISA_{oi}}$)
Tra-Tra	s3.6)$y_{jf,jo} = 0$ (i.e., $\mathrm{ISA_{jf}} \equiv \mathrm{ISA_{jo}}$)	p3.5)$y_{jf,ji} = 0$ (i.e., $\mathrm{ISA_{jf}} \equiv \mathrm{ISA_{ji}}$)

iii) the singular configurations of the studied mechanism are identified by imposing the singularity conditions selected at step (i) either geometrically (e.g., by moving the 3D model of the mechanism in the CAD system) or analytically (i.e., by using the singularity conditions to write equations with the generalized coordinate, q, as unknown and, then, solving such equations).

3. Results

The proposed technique for identifying singular configurations of single-DOF spatial mechanisms is applied to an RCCC mechanism (Fig. 7). An RCCC mechanism features four binary links sequentially connected through one revolute (R) pair and three cylindrical (C) pairs to form a single loop. In the literature [29,36], RCCC mechanisms are somehow a benchmark for testing ISA-based algorithms since they have been extensively studied (see, for instance, Refs. [38–41]).

With reference to Fig. 7, links 2 and 4 are chosen as input and output links, respectively; link 1 is chosen as reference link both for the input and the output links, and link 3, which imposes $\mathbf{u}_{43} \neq \pm\mathbf{u}_{32}$ by construction (see Fig. 7), plays the role of the possible fourth link to use when applying the above-deduced formulas. In short, the above-reported formulas can be particularized to the studied case with the following positions: i=2, o=4, f=r=1, and g=3; also, ω_{21} and ω_{41} are the input and output rates, respectively. The step-by-step implementation of the procedure presented in Section 2.3 is reported below.

3.1. Step (i)

Since f=r=1, and since the involved kinematic pairs impose that the input motion (i.e., motion 21) and the output motion (i.e., motion 41) are both helical motions, the singularity conditions to apply must be sought in the row of the Hel-Hel case of Table 3. Moreover, the availability of link 3 with $\mathbf{u}_{43} \neq \pm\mathbf{u}_{32}$ brings one to select, in Table 3, condition (s3.3), as serial-singularity condition, which, in the studied case, becomes

$$\mathbf{u}_{21} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{32}) = 0, \tag{37}$$

and condition (p3.2), as parallel-singularity condition, which, in the case under study, becomes

$$\mathbf{u}_{41} \cdot (\mathbf{u}_{43} \times \mathbf{u}_{32}) = 0 \tag{38}$$

The analysis of Eqs. (37) and (38) reveals that they involve the following ISAs: ISA_{21}, ISA_{41}, ISA_{32}, and ISA_{43}.

3.2. Step (ii)

Since the four ISAs (i.e., ISA_{21}, ISA_{41}, ISA_{32}, and ISA_{43}) identified in step (i) are all joint axes, their poses are immediately found either geometrically through a simple inspection of the mechanism drawing (Fig. 7) or analytically after having solved the system of constraint equations of the mechanism.

From a geometric point of view, Fig. 7 implements this step since it shows both the mechanism configuration and how these four ISAs are related to the mechanism configuration. From an analytic point of view, the constraint-equation system of an RCCC mechanism can be solved in closed form as reported, for instance, in [29,42] and provides at most two solutions; the solution of the constraint-equation system immediately gives the locations of the sought-after ISAs.

3.3. Step (iii)

From a geometric point of view, Eq. (37) is satisfied (i.e., a serial-singularity occurs), if $\mathbf{u}_{21}$ (i.e., ISA_{21}), $\mathbf{u}_{32}$ (i.e., ISA_{32}), and $\mathbf{u}_{43}$ (i.e., ISA_{43}) are all parallel to a unique plane. With reference to Fig. 7, such a condition can occur, if and only if $\mathbf{n}_{123} = \pm\mathbf{n}_{432}$. Fig. 8 shows an RCCC configuration where this condition is satisfied.

Differently, Eq. (38) is satisfied (i.e., a parallel-singularity occurs), if $\mathbf{u}_{41}$ (i.e., ISA_{41}), $\mathbf{u}_{32}$ (i.e., ISA_{32}), and $\mathbf{u}_{43}$ (i.e., ISA_{43}) are all

Table 4
Enumeration of the singularity conditions for the case in which link r is different from link f (i.e., $r \neq f$).

	$r \neq f$	
Case	**Serial-Singularity Conditions**	**Parallel-Singularity Conditions**
Hel-Hel	s4.1) $(\mathbf{u}_{of} \neq \pm\mathbf{u}_{ir})\&(\mathbf{u}_{of} \cdot (\mathbf{u}_{if} \times \mathbf{u}_{ir}) = 0)$ s4.2) $\left\{\begin{array}{c} \neg((p_{oi} = p_{ri} = p_{fi})\&(Q'_{oi} \equiv Q'_{if} \equiv Q_{of})\&(Q_{oi} \equiv Q'_{ri} \equiv Q_{or})) \\ \& \\ (\mathbf{u}_{of} \neq \pm\mathbf{u}_{ir}) \\ \& \\ \left(\begin{array}{c} (p_{oi} - p_{fi}) \parallel \mathbf{u}_{of} \times \mathbf{u}_{fi} \parallel + y_{if,oi}(\mathbf{u}_{of} \cdot \mathbf{u}_{fi}) = y_{of,oi}\dfrac{\mathbf{u}_{if} \cdot \mathbf{u}_{oi}}{\mathbf{u}_{of} \cdot \mathbf{u}_{oi}} \\ \vee \\ \dfrac{\parallel \mathbf{u}_{or} \times \mathbf{u}_{ir} \parallel}{\mathbf{u}_{or} \cdot \mathbf{u}_{ir}} = \dfrac{y_{ir,oi}}{p_{ri} - p_{oi}} \end{array}\right) \end{array}\right.$ s4.3) $\left\{\begin{array}{c} \neg((p_{oi} = p_{ri} = p_{fi})\&(Q'_{oi} \equiv Q'_{if} \equiv Q_{of})\&(Q_{oi} \equiv Q'_{ri} \equiv Q_{or})) \\ \& \\ (\mathbf{u}_{if} \parallel \mathbf{u}_{of} \parallel \mathbf{u}_{oi} \parallel \mathbf{u}_{ir} \parallel \mathbf{u}_{or} \parallel) \\ \& \\ ((y_{ir,oi} = 0) \vee (y_{if,of} = 0)) \end{array}\right.$	p4.1) $(\mathbf{u}_{of} \neq \pm\mathbf{u}_{ir})\&(\mathbf{u}_{or} \cdot (\mathbf{u}_{of} \times \mathbf{u}_{ir}) = 0)$ p4.2) $\left\{\begin{array}{c} \neg((p_{oi} = p_{ri} = p_{fi})\&(Q'_{oi} \equiv Q'_{if} \equiv Q_{of})\&(Q_{oi} \equiv Q'_{ri} \equiv Q_{or})) \\ \& \\ (\mathbf{u}_{of} \neq \pm\mathbf{u}_{ir}) \\ \& \\ \left((y_{of,oi} = 0) \vee \left((p_{oi} - p_{ri}) \parallel \mathbf{u}_{or} \times \mathbf{u}_{ir} \parallel + y_{ir,oi}(\mathbf{u}_{or} \cdot \mathbf{u}_{ir}) = y_{or,oi}\dfrac{\mathbf{u}_{ir} \cdot \mathbf{u}_{oi}}{\mathbf{u}_{or} \cdot \mathbf{u}_{oi}}\right)\right) \end{array}\right.$ p4.3) $\left\{\begin{array}{c} \neg((p_{oi} = p_{ri} = p_{fi})\&(Q'_{oi} \equiv Q'_{if} \equiv Q_{of})\&(Q_{oi} \equiv Q'_{ri} \equiv Q_{or})) \\ \& \\ (\mathbf{u}_{if} \parallel \mathbf{u}_{of} \parallel \mathbf{u}_{oi} \parallel \mathbf{u}_{ir} \parallel \mathbf{u}_{or} \parallel) \\ \& \\ ((y_{of,oi} = 0) \vee (y_{ir,or} = 0)) \end{array}\right.$
Hel-Tra	s4.4) $(y_{ir,oi} = 0) \vee \left(\dfrac{(\mathbf{u}_{if} \cdot \mathbf{u}_{oi})\, y_{of,oi}}{(\mathbf{u}_{of} \cdot \mathbf{u}_{oi})(\mathbf{u}_{of} \cdot \mathbf{u}_{if})} - y_{if,oi} = (p_{oi} - p_{if})\dfrac{\parallel \mathbf{u}_{of} \times \mathbf{u}_{if} \parallel}{(\mathbf{u}_{of} \cdot \mathbf{u}_{if})}\right)$ s4.5) $\mathrm{ISA_{if}} \equiv \mathrm{ISA_{oi}} \equiv \mathrm{ISA_{of}}$	p4.4) $(\mathbf{u}_{ir} = \pm\boldsymbol{\tau}_{or}) \vee (y_{of,oi} = 0)$
Tra-Hel	s4.6) $(\mathbf{u}_{of} = \pm\boldsymbol{\tau}_{if}) \vee (y_{ir,oi} = 0)$	p4.5) $(y_{of,oi} = 0) \vee \left(\dfrac{(\mathbf{u}_{or} \cdot \mathbf{u}_{oi})\, y_{ir,oi}}{(\mathbf{u}_{ir} \cdot \mathbf{u}_{oi})(\mathbf{u}_{ir} \cdot \mathbf{u}_{or})} - y_{or,oi} = (p_{oi} - p_{or})\dfrac{\parallel \mathbf{u}_{ir} \times \mathbf{u}_{or} \parallel}{(\mathbf{u}_{ir} \cdot \mathbf{u}_{or})}\right)$ p4.6) $\mathrm{ISA_{ir}} \equiv \mathrm{ISA_{oi}} \equiv \mathrm{ISA_{or}}$
Tra-Tra	s4.7) $(y_{rf,of} = 0) \vee (\mathbf{u}_{oi} = \pm\boldsymbol{\tau}_{if})$	p4.7) $(y_{rf,ri} = 0) \vee (\mathbf{u}_{oi} = \pm\boldsymbol{\tau}_{or})$

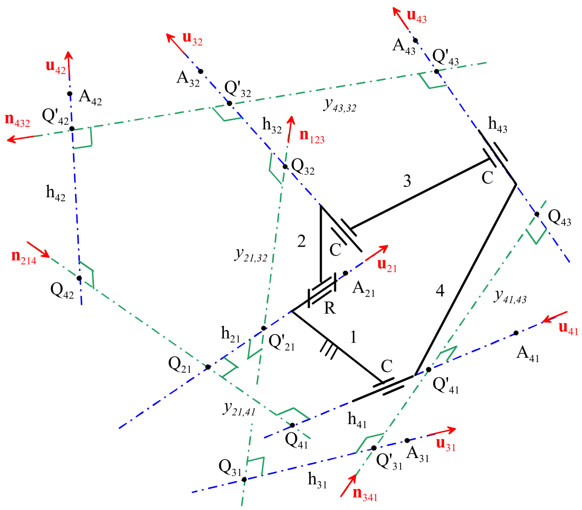

Fig. 7. An RCCC mechanism at a non-singular configuration (the six blue lines are the six ISAs associated to the six possible relative motions among the four links of the mechanism).

parallel to a unique plane. With reference to Fig. 7, such a condition can occur, if and only if $\mathbf{n}_{341} = \pm\mathbf{n}_{432}$. Fig. 9 shows an RCCC configuration where this condition is satisfied.

From an analytic point of view, $\mathbf{u}_{21}$ and $\mathbf{u}_{41}$ are known constant unit vectors, since ISA$_{21}$ and ISA$_{41}$ are lines fixed to link 1 (see Fig. 7). The solution of the constraint-equation system (i.e., step (ii)) has provided the explicit expressions of $\mathbf{u}_{32}$ and $\mathbf{u}_{43}$ as functions of the joint variable of the R-pair, say θ_{21}, which, here, plays the role of input variable. Such explicit expressions, which are reported in [29], when introduced into Eq. (37) (Eq. (38)) together with $\mathbf{u}_{21}$ ($\mathbf{u}_{41}$), transform Eq. (37) (Eq. (38)) into one scalar equation in one unknown (i.e., θ_{21}) whose solution provides the values of θ_{21} corresponding to the serial (parallel) singularities.

4. Discussion

In the above-presented case study, at a serial singularity, since a=0, ω_{41} (i.e., the output rate) is equal to zero whatever be the value of ω_{21} (i.e., the input rate), which brings the conclusion that, the C-pair connecting links 4 and 1 either makes link 4 translate with

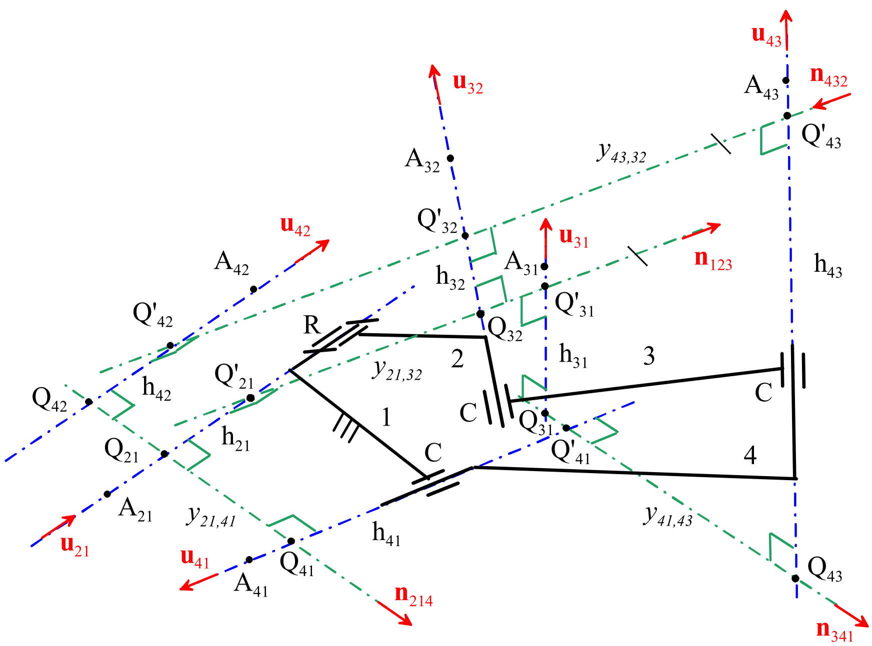

Fig. 8. An RCCC mechanism at a configuration that satisfies Eq. (37) (i.e., at a serial singularity).

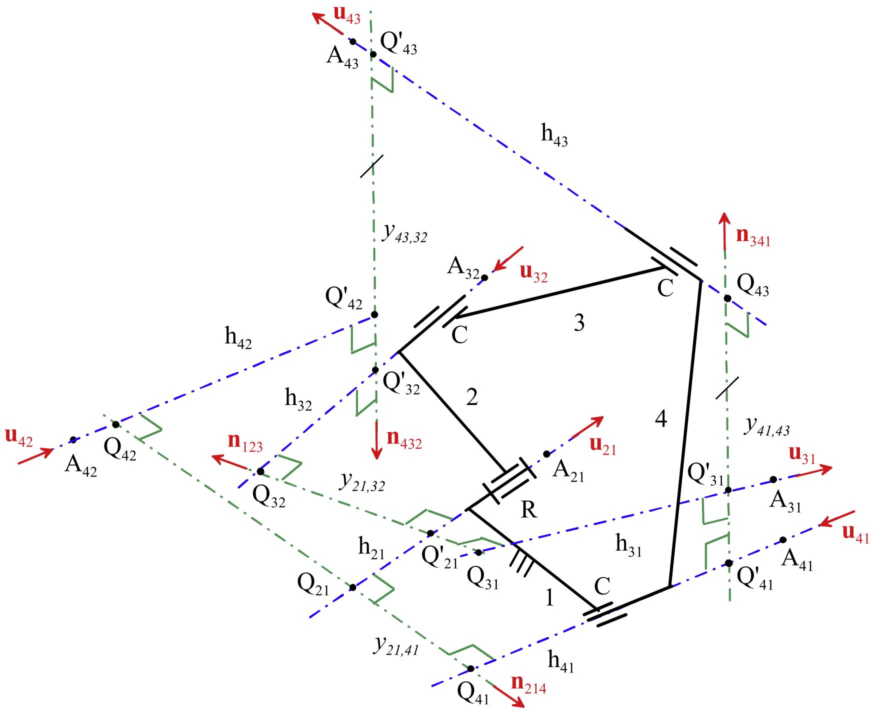

Fig. 9. An RCCC mechanism at a configuration that satisfies Eq. (38) (i.e., at a parallel singularity).

respect link 1 with $\boldsymbol{\tau}_{41}\equiv\mathbf{u}_{41}$ and $\nu_{41}\neq0$, or keeps link 4 at rest with respect to link 1. Consequently (see Figs. 2b and 8), at a serial singularity, ISA$_{42}$ and ISA$_{21}$ must be parallel with $\omega_{42}=-\omega_{21}$ for the relative motion theorems [30]. Differently, at a parallel singularity, since b=0, ω_{21} (i.e., the input rate) is equal to zero whatever be the value of ω_{41} (i.e., the output rate), which brings the conclusion that the R-pair connecting link 2 and 1 is idle. Consequently (see Figs. 2c and 9), at a parallel singularity, ISA$_{42}$ and ISA$_{41}$ must be parallel with $\omega_{42}=\omega_{41}$ for the relative motion theorems [30].

This result can be generalized to all the Hel-Hel cases with f=r as follows: by including condition ν_{of}=0 (ν_{if}=0) among the possible translations, Fig. 2b, that is, ISA$_{oi}$ || ISA$_{if}$ and $\omega_{oi}=-\omega_{if}$, (Fig. 2c, that is, ISA$_{oi}$ || ISA$_{of}$ and $\omega_{oi}=\omega_{of}$,) geometrically identifies a serial (parallel) singularity that satisfies condition (s3.3) (condition (p3.2)) of Table 3.

In addition, the analysis of the singularity conditions reported in Tables 3 and 4 reveals that most of them correspond to possible intersections or parallelisms or even coincidences of two or more ISAs. This observation is coherent and extends to spatial single-DOF mechanisms what is reported in the literature for planar and spherical single-DOF mechanisms [8,9,25–27,37].

Regarding the implementation of step (ii), sometimes, among the ISAs that are necessary to identify the singularity conditions, some ISAs, named "secondary" ISAs [29], that are not immediately localizable through a simple inspection of the mechanism configuration (e.g., ISA$_{42}$ and ISA$_{31}$ in Fig. 7) might be included. In these cases, the geometric or analytic relationships that relate them to the mechanism configuration and how they move during the mechanism motion are determinable directly through the technique illustrated in [29] or, indirectly, through the other ISA-determination techniques presented in the literature [35,36,43,44]

Moreover, since, in a single-DOF mechanism with holonomic and scleronomic constraints, any secondary variable, s, can be implicitly or explicitly expressed as a function, $s(q)$, of the unique generalized coordinate q, the following relationship holds (see Eq. (6))

$$\dot{s}=\frac{ds}{dq}\dot{q} \quad \Rightarrow \quad \frac{ds}{dq}=\frac{\dot{s}}{\dot{q}}=\frac{a(q)}{b(q)} \tag{39}$$

which leads to the conclusion that serial (parallel) singularities correspond to the extreme values of the output (input) variable[7]. As a consequence, the proposed methodology can be used to identify the extreme poses of the output (input) link during design of spatial single-DOF mechanisms and extends those techniques already used for planar single-DOF mechanisms (see, for instance, Refs. [25, 26]). The determination of links' or joints' extreme positions is a relevant problem in mechanism synthesis and, for single-DOF mechanisms, it could be considered the main practical application of their singularity analysis. For the determination of such extrema in single-DOF spatial mechanisms, the literature provides a great number of ad-hoc analyses devoted to particular mechanism

[7] The fact that single-DOF mechanisms' singularities correspond to extreme positions of links or joints was first stressed by Hunt's works (see [8] for details and Refs.).

types (see, for instance, Refs. [45–54]) and a few of general techniques [55–60]. Most of the proposed techniques algebraically manipulates the mechanism closure equations to obtain a unique polynomial equation that contains only two motion variables, say q (input variable) and s (output variable); then, they determine the conditions that make two or more solutions of this equation, $f(s,q)=0$, coincide (e.g., by equating to zero the discriminant of a quadratic equation). Such a procedure is cumbersome to implement even for a specific mechanism type and cannot be generalized to all the mechanisms. Differently, by exploiting the fact that, at an extreme position, the output link stops (i.e., $\dot{s} = 0$) and inverts its motion, Baker et al. [55–57] formulated general techniques that use velocity-loop equations and screw systems to write them for any mechanisms. Since velocity-loop equations are always linear and homogeneous in the motion-variable rates, they are easy to manipulate for eliminating the rates of the motion variables that are neither input nor output variables, thus finding the analytic condition that identifies the extreme. With respect to these approaches, the one proposed in this paper further facilitates the determination of extrema by providing the exhaustive enumeration of the geometric conditions ISAs have to satisfy at extreme positions, which allows purely geometric analyses based on 3D CAD systems.

Eventually, it is worth stressing that the case study clearly shows that the exhaustive enumeration of singularity conditions reported in Tables 3 and 4 makes the proposed technique really fast to implement by using 3D drawings generated through a CAD system.

5. Conclusions

The fact that the poses of instantaneous screw axes (ISAs) depend only on the mechanism configuration in single-DOF mechanisms has been exploited to devise a geometric and analytic technique for identifying the singular configurations of these mechanisms.

The proposed technique extends to spatial single-DOF mechanisms those techniques, proposed for single-DOF planar mechanisms that are based on instant centers' locations. As far as this author is aware, this extension is novel.

In particular, the exhaustive enumeration of all the singularity conditions of these mechanisms has been deduced and used to identifies singularities either through 3D drawings or through analytic computations. To better illustrate the proposed technique, it has been applied to identify the singularities of an RCCC mechanism.

The proposed methodology facilitates the determination of the extreme poses of links in spatial single-DOF mechanisms, which makes it of particular interest in mechanism synthesis.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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