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# The Geometry Awakens: on the relationship between holonomy and hyperbolic structures

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# The Geometry Awakens:
# On the relationship between holonomy and hyperbolic structures

*Gianluca Faraco*

*to anyone believe in me*

*I do not have any particular recipe. It is just like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks, and with some luck you might find a way out.*

Maryam Mirzakhani

# Abstract

This thesis examines the relationship between branched hyperbolic structures on surfaces and representations of the fundamental group into $\mathrm{PSL}_2\mathbb{R}$. A branched hyperbolic structure on a surface $S$ is a hyperbolic cone-structure such that the angle around any interior cone point is an integer multiple of $2\pi$.

Any such structure determines a holonomy representation of the fundamental group $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$. We ask, conversely, when a representation of the fundamental group arises as holonomy of a branched hyperbolic structure. We consider only $2-$dimensional hyperbolic structures.

In this work, we take into account Mathews's theorems and we improve them. Let $S$ be a closed surface of genus 2, then we show that any representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with Euler class $\mathcal{E}(\rho) = \pm 1$ is the holonomy of a branched hyperbolic structure $\sigma$ on $S$. In order to show this, we prove that any such representation sends a simple non-separating curve to an elliptic element. Also, we need to consider separately a special class of representations, namely representations with virtually abelian pairs. This class of representations turns out to problematic, and we need to deal with them in a different way. The same ideas we used in the genus 2 case can be used in all genus. Using them we may prove that any representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with Euler class $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$ which sends a non-separating simple curve to a non-hyperbolic element, is the holonomy of a branched hyperbolic structure $\sigma$ on $S$ with one cone point of angle $4\pi$.

# Abstract in Italian

Questa tesi esamina la relazione tra strutture iperboliche ramificate su superfici e rappresentazioni del gruppo fondamentale in $\mathrm{PSL}_2\mathbb{R}$. Una struttura iperbolica ramificata su una superficie $S$ è una struttura iperbolica conica tale che l'angolo attorno a qualsiasi punto conico è un multiplo intero di $2\pi$.
Qualsiasi struttura di questo tipo determina una rappresentazione del gruppo fondamentale $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ detta olonomia. Ci chiediamo, al contrario, quando una rappresentazione del gruppo fondamentale sorge come olonomia di una struttura iperbolica ramificata. In questo lavoro considereremo solo strutture iperboliche $2-$dimensionali.
Sia $S$ una superficie chiusa di genere 2, e sia $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ una rappresentazione con classe di Eulero $\mathcal{E}(\rho) = \pm 1$. Allora $\rho$ è olonomia di una struttura iperbolica ramificata $\sigma$ su $S$ con un punto conico $4\pi$. Le stesse tecniche usate nel caso del genere 2 possono essere adoperate anche in genere più alto. In questo modo possiamo provare che qualsiasi rappresentazione $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ con la classe Euler $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$ che applica una curva semplice non separante ad un elemento non iperbolico, è l'olonomia di una struttura iperbolica ramificata *sigma* su $S$ con un punto cono di angolo $4\pi$.

# Introduction

## Into the problem

**The protagonists of this work.** A hyperbolic structure on a surface $S$ is a geometric structure locally modelled on the geometry of the hyperbolic plane $\mathbb{H}^2$ with its isometry group $\mathrm{PSL}_2\mathbb{R}$ acting by Möbius transformations,

$$\mathrm{PSL}_2\mathbb{R} \times \mathbb{H}^2 \longrightarrow \mathbb{H}^2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \longmapsto \frac{az+b}{cz+d}$$

More precisely, any hyperbolic structure is the datum of a maximal atlas, whose charts take value in open set of the hyperbolic plane, and such that changes of coordinates are given by restriction on elements in $\mathrm{PSL}_2\mathbb{R}$. Equivalently a hyperbolic structure on a surface $S$ may be seen as the datum of a triangulation $\tau$ and a metric $d$ such that any simplex of the triangulation is isometric to a geodesic triangle in the hyperbolic plane.

Viewing a hyperbolic structure as a $(\mathrm{PSL}_2\mathbb{R}, \mathbb{H}^2)-$structure on a surface $S$, any such structure induces a map $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ which is well-defined up to post-composition with an element of $\mathrm{PSL}_2\mathbb{R}$. Such map is commonly known as *developing map.* For any given hyperbolic structure on $S$, the developing map develops the geometry on $S$ into $\mathbb{H}^2$ so that all geometric information can be read on the hyperbolic plane. Any developing map has an equivariance property with respect to the action of the fundamental group $\pi_1 S$ on the universal cover $\widetilde{S}$ of $S$. Indeed for any element $\gamma \in \pi_1 S$ the composition $\mathsf{dev} \circ \gamma$ is another developing map for $S$. Thus there exists an element $g \in \mathrm{PSL}_2\mathbb{R}$ such that

$$\mathsf{dev} \circ \gamma = g \cdot \mathsf{dev}$$

because any developing map is well-defined up to post-composition with an element in $\mathrm{PSL}_2\mathbb{R}$. The map $\gamma \longmapsto g$ defines a homomorphism $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ such that the following equation holds

$$\mathsf{dev} \circ \gamma = \rho(\gamma) \cdot \mathsf{dev}$$

The representation $\rho$ is known as *holonomy for the structure*. Geometric data about the structure is encapsulated in this representation; but how much information? For any complete hyperbolic structure on $S$, its holonomy representation encodes all necessary information about the structure, that is any complete hyperbolic structure is completely determined by its holonomy representation. The holonomy of a complete hyperbolic structure is called *Fuchsian representation* and it turns out to be a faithful and discrete representation.

However, the picture changes completely as soon as we consider *branched hyperbolic structures*. These are non-complete hyperbolic structures on which we allow cone points of angle $2k\pi$ for $k \in \mathbb{N}$. Around any cone point the geometry is not necessarily modeled on $\mathbb{H}^2$ by a local diffeomorphism, but possibly by a finite branched cover; in other words, a local chart can now be a map of the form $z \to z^{k+1}$.

From a given (possibly) branched hyperbolic structure, it is an easy matter to obtain the holonomy representation. However, the reverse problem to recover a hyperbolic structure from given a representation $\rho$, is longer, arduous and not always possible! The difficulties arise because a holonomy representation of a branched hyperbolic structure does not encode enough information about the structure. For the same reason, the same non-Fuchsian representation $\rho$ may arise as holonomy of non-isomorphic branched hyperbolic structure.

**Historical view.** The story starts in the first year of 80's. Goldman, in its doctoral dissertation [13], consider the representation space $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ for a closed orientable surface $S$ of genus $g \geq 2$. He shows that such space turns out to be a real analytic variety of dimension $6g-6$ with $4g-3$ connected components. Any component is parametrized by the Euler number $\mathcal{E}(\rho)$ of a representation $\rho$ inside it. The Euler number $\mathcal{E}(\rho)$ is always an integer $k$ that satisfies the so-called Milnor-Wood inequality: $|k| \leq -\chi(S)$ and it measures the obstruction to lift $\rho$ to a new representation $\widetilde{\rho}$ in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. In particular, Goldman proved that the equality characterizes those representations which are faithful and discrete; that is those representations coming from a complete hyperbolic structure on $S$. Since to any of Fuchsian representation corresponds a unique hyperbolic structure (up to isometry isotopic to the identity), the extremal components of $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ are two different copies of the Teichmüller space $\mathcal{T}(S)$ and thus they are completely understood. Nevertheless the other components are currently quite mysterious and it is natural to ask if representations that lying in the other components arise as holonomy of some geometric structure on $S$. Many developments are done so far. At the beginning of $90's$ Gallo, Goldman and Porter showed that a non-elementary representation $\rho \in \mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ with even Euler number $\mathcal{E}(\rho)$ occurs as the holonomy of $\mathbb{CP}^1-$structure[1]. A $\mathbb{CP}^1-$structure is a geometric structure locally modeled on the Riemann sphere $\mathbb{CP}^1$ with its group of holomorphic automorphism $\mathrm{PSL}_2\mathbb{C}$. At the same time, Tan showed in [27] that any representation $\rho$ with odd Euler number $\mathcal{E}(\rho)$ occurs as holonomy of a $\mathbb{CP}^1-$ structure with one conical singularity of angle $4\pi$. So far is not clear when a representation $\rho$ comes from a branched hyperbolic structure, hence we want to shed light on this problem. We know the existence of representations that cannot arise as holonomy of a branched hyperbolic structure; Tan gave in [27] an explicit example of such representation with $\mathcal{E}(\rho) = \pm 2$, see also 4.5.1.

Around $10's$ Mathews gave some improvements in [21] after many years. He showed that for surfaces of genus 2, a representation $\rho$ with $\mathcal{E}(\rho) = \pm 1$ is the holonomy of a branched hyperbolic structure with one cone point of angle $4\pi$ if it sends a simple separating curve to a non-hyperbolic element; see 4.2.5. Moreover, in the same paper Mathews used the ergodicity result to prove that almost every representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$ and such that at least one non-separating curve has elliptic image, it the holonomy of a branched hyperbolic structure with one cone point of angle $4\pi$; see 4.2.2.

## Synopsis

In this thesis, we take into account Mathews' theorems and we improve them. For genus 2 surfaces we show that *any* representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathcal{E}(\rho) = \pm 1$ arises as holonomy of a branched hyperbolic structure with one cone point of angle $4\pi$. In particular, we will prove this in two, slightly different, ways. The same ideas can be used also in higher genus case, that is for surfaces of genus $g \geq 3$; to show

[1] They proved such result around the first years of $90's$, but it remained unpublished. Actually, we may found such result as a particular case of the more general result proved by Gallo, Kapovich, and Marden in [9].

that for a closed surface any representation with Euler class $\mathcal{E}(\rho) = \pm\big(\chi(S)+1\big)$ (i.e. for almost extremal representations), which sends some non-separating simple closed curve to a non-hyperbolic element of $\mathrm{PSL}_2\mathbb{R}$, is the holonomy of a cone-manifold structure with at most one cone point with angle $4\pi$. We will conclude with a list of natural and interesting question which are strictly related to this work. In the last section, we will consider also another special type of representation, namely *purely hyperbolic representations.* We will show that under some circumstances, this type of representations arise as holonomy of a (possibly) branched hyperbolic structure. The word possibly appear because Fuchsian representations are clearly purely hyperbolic!

This thesis is organized as follows. In chapter 1 we recall well-known results about the geometry of commutator, the Lie groups $\mathrm{PSL}_2\mathbb{R}$ and $\widetilde{\mathrm{PSL}_2\mathbb{R}}$, geometric structures and deformation spaces that we will use in the sequel. We conclude the chapter with an entire section devote to examples of any type.

In chapter 2 we will branched hyperbolic structures as metric space. We will see explicitly how a branched structure on $S$ induces a distance that makes $S$ a complete geodesic space. In particular, we consider the properties of geodesics in such structures and the nature of cone singularities.

In chapter 3 we examine briefly the space of representations and the space of characters, which are objects coming from algebraic geometry; and we introduce a symplectic structure, and a measure on the character variety; however we do not say anything about how these structures arise. We introduce only the minimal notion that we need to tackle the chapter 4. After that, we introduce the Euler class, the cohomology class associated to a representation $\rho$. We give a brief description and establish a method for its calculation, via the algebra of $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. Finally, we prove the index formula. Such formula related the geometrical data of a surface with boundary endowed with a branched hyperbolic structure with, possibly non-geodesic boundary; and topological data as the relative Euler number and the Euler characteristic.

Finally in chapter 4 we improve Mathews' theorems. In the first section, we recall the main notions about the geometry and algebra of punctured torus that we will use widely in sections 4.2 and 4.3. In order to understand the *almost every* condition, an entire paragraph is devoted the strategy of Mathews' theorem. Then we show that the almost every condition may be removed. In section 4.3 we consider the same problem for surfaces of genus $g \geq 3$. Finally, in section 4.4 we consider purely hyperbolic representations.

## Main results and explanation

In this thesis, we are going to prove the following results. For the first one let $S$ be a closed surface endowed with a branched hyperbolic structure $\sigma$ with holonomy $\rho$. The following formula relates the topological data of a subsurface $C$ of $S$ with the geometric data of the branched hyperbolic structure induced by $\sigma$ on $C$. In order to do this, we must first define a new geometric invariant, namely the index of a curve (see 3.5).

**Theorem 3.5.6.** (Index Formula) *Let $\sigma$ be a branched hyperbolic structure on $S$ with holonomy $\rho$. Let $C$ be a subsurface of $S$ such that every boundary components has not elliptic holonomy and denote by $\rho_C$ the restriction of $\rho$ to $\pi_1 C$. Then the relative Euler number satisfies the following identity*

$$\mathcal{E}(\rho_C, \mathfrak{s}) = \chi(C) + k - \sum_{i=1}^{n} \mathrm{I}_{\gamma_i},$$

*where $k$ denote the total branching order on $C$, $\gamma_i$'s are the boundary components of $C$ and the indeces are computed with respect to the induced orientation by $C$ on each boundary components.*

Let now $S$ be a topological surface of genus 2, and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm 1$. We wonder under which conditions this representation arises as the holonomy of a branched hyperbolic structure.

The following result answers this question.

**Theorem 4.2.1.** *Let $S$ be a closed surface of genus two and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be any representation with $\mathcal{E}(\rho) = \pm 1$. Then $\rho$ is the holonomy for a branched hyperbolic structure with one cone point of angle $4\pi$.*

In order to prove this theorem we will make a strong use of the following new result.

**Theorem 4.2.4.** *Let $S$ be a closed surface of genus two and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be any representation with $\mathcal{E}(\rho) = \pm 1$. Then $\rho$ sends a simple non-separating curve* *to an elliptic.*

A brief explanation is in order. Let us consider the set $\mathcal{M}^{\pm 1}$ of all representations $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ satisfying the condition $\mathcal{E}(\rho) = \pm 1$, and denote by $E$ the subset of all representations that send a simple closed non-separating curve to an elliptic. In [17, Proposition 6.2], Marché, and Wolff recently proved that $E$ has full measure in $\mathcal{M}^{\pm 1}$, that is almost every representation sends a simple closed non-separating curve to an elliptic. Hence our theorem 4.2.4 is a stronger result.

Theorem 4.2.4 is interesting also for other reasons which are related to the geometrization of representations. In [3], Bowditch conjecture that every representation that sends any simple closed curve to a hyperbolic element must be Fuchsian, *i.e.* discrete and faithful. Even in [17], the authors showed that this conjecture is equivalent to the Goldman conjecture about the ergodic action of the mapping class group (see also 1), and they proved also the Bowditch conjecture for surfaces of genus 2. Our theorem 4.2.4 gives a different and independent prove of the Bowditch conjecture for representation with $\mathcal{E}(\rho) = \pm 1$.

We have tried to extend the previous results for representations $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ such that $S$ is a surface of genus at least 3 and $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$.

**Theorem 4.3.6.** *Let $S$ be a closed surface of genus $g \geq 3$ and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be any representation with $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$. Suppose $\rho$ sends a simple non-separating curve to a non-hyperbolic element. Then $\rho$ is the holonomy of a branched hyperbolic structure with one cone point of angle $4\pi$.*

If one could prove that for all representation $\rho$ with $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$, there is a simple closed non-separating curve which is not hyperbolic, then we would have a stronger result. Even better, if one could prove that such curve is elliptic we have also a proof of the Bowditch conjecture for this type of representations.

Finally, we have considered another type of representations, namely purely hyperbolic representations. For this kind of representations, we have the following characterization.

**Theorem 4.4.14.** *Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a non-Fuchsian, purely hyperbolic rep-*

*resentation and* $\rho_0 : \pi_1\Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$ *be a Fuchsian representation, where the genus of* $\Sigma$ *is strictly lower than the genus of* $S$*. Suppose there is a map* $f : S \longrightarrow \Sigma$ *such that* $\rho = \rho_0 \circ f_*$*. Then* $\rho$ *is geometrizable by a branched hyperbolic structure if and only if* $f$ *is a branched covering.*

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# CHAPTER 1

## Hyperbolic geometry

In this section we define the structures we are interested in, introducing the basic notations and terminology and proving some basic facts about them. Throughout, let $S$ be an orientable surface.

### 1.1 Hyperbolic plane

Historically, the hyperbolic plane was the result of the search for a non-euclidean plane whereon for every line $l$ and every point $p$ not in $l$, there is more than one line through $p$ do not intersect $l$. In some sense, the hyperbolic plane is the opposite of the sphere or elliptic plane.
Usually the hyperbolic plane can be described by different models, as the half-plane model $\mathcal{H}^+$, the disk model $\mathbb{D}$, or Klein model as a subset of $\mathbb{RP}^2$. Here we recall some notion about the first two models.

**Half plane model $\mathcal{H}^+$.** The half plane is defined as the locus of points of the complex plane with the positive imaginary part. $\mathcal{H}^+$ endowed with the Riemannian metric

$$g = \frac{|dz|}{\Im(z)} = \frac{\sqrt{dx^2 + dy^2}}{y}$$

becomes a simply connected Riemannian manifold of dimension two. The boundary at infinity in this model is $\mathbb{R} \cup \{\infty\}$.

For any couple of distinct points $p_0, p_1$ in $\mathcal{H}^+$, the induced distance between them is defined as

$$d_{\mathcal{H}^+}(p_0, p_1) = \inf_{\gamma} \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}dt = \inf_{\gamma} \int_0^1 \left|\frac{d\gamma}{dt}\right| \frac{dt}{y(t)}$$

where $\gamma : I \longrightarrow \mathcal{H}^+$ is any path joining $p_0$ and $p_1$.

The group of isometries of $\mathcal{H}^+$ is isomorphic to $\mathrm{PGL}_2\mathbb{R}$, and the group of orientation-preserving isometries is a subgroup of index two naturally isomorphic to $\mathrm{PSL}_2\mathbb{R}$. Indeed a straightforward computation shows that $d_{\mathcal{H}^+}(p_0, p_1) = d_{\mathcal{H}^+}(g.p_0, g.p_1)$ for every element $g$ in $\mathrm{PSL}_2\mathbb{R}$. In the next section we will give a complete characterization of orientation preserving isometries for $\mathcal{H}^+$.

For any couple distinct points there exists only one path $\gamma$ realizing the infimum; that is there exists a unique shortest path between them which is called *geodesic path.* More generally, straight lines (or geodesics) on $\mathcal{H}^+$ are given by half lines and semicircles orthogonal to the circle at infinity. In short, it is possible to see that for every couple of points $p_0, p_1$ lying on the imaginary semiaxis, the geodesic path is the segment between them in the Euclidean sense. The second step is to show that for every semicircle or half line orthogonal to the boundary at infinity there exists a transformation $g \in \mathrm{PSL}_2\mathbb{R}$ sending it to the imaginary semiaxis. Finally, use the fact that $\mathrm{PSL}_2\mathbb{R}$ preserve the hyperbolic distances to conclude.

**Disk model or Poincaré disk $\mathbb{D}$.** The second model of hyperbolic-plane we analyze is the disk model. It is defined as the subset of the complex plane whose points have magnitude less than one, *i.e.* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ endowed with the Riemannian metric $g = \frac{2|dz|}{1-|z|^2}$. Here the boundary at infinity is the unit circle $\mathbb{S}^1$. This model is clearly isometric to the half-plane model described above.

It is useful to think $\mathcal{H}^+$ and $\mathbb{D}$ as subsets of the Riemann sphere $\mathbb{CP}^1$, as shown in the picture above. With this point of view, it becomes reasonable that the isometry $j$ between these two models has to be searched in the group $\mathrm{PSU}(2) < \mathrm{PSL}_2\mathbb{C}$, which is the isometry group for $\mathbb{CP}^1$.

To map $\mathcal{H}^+$ onto $\mathbb{D}$, we invert the sphere with respect to the circle of center $-i$ and radius $\sqrt{2}$, then we reflect the sphere with respect to the real circle. The composition of these transformations provides the desired isometry, precisely

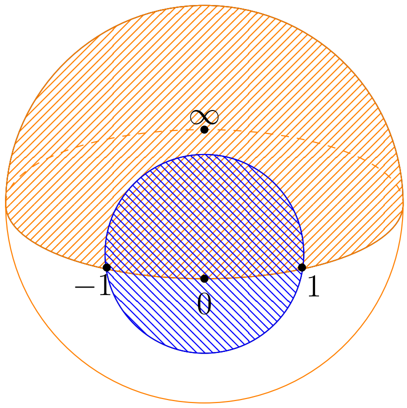


Figure 1.1: Comparison between the Half plane model $\mathcal{H}^+$ and the disk model $\mathbb{D}$ on the Riemann sphere $\mathbb{CP}^1$. The orange part represents the half-plane model, instead, the blue part represents the Poincaré disk.

$$j(z) = \frac{z-i}{1-iz}.$$

Using $j$ is possible to show that in the disc model straight lines are circular arcs orthogonal to the unit circle bounding the Poincaré disk, of course, this definition includes diameters of $\mathbb{D}$.

## 1.2 Hyperbolic isometries

We now turn on the study of hyperbolic transformation and we classify them. The group orientation-preserving isometry of the half plane model is $\mathrm{PSL}_2\mathbb{R}$ as state before. We recall that

$$\mathrm{PSL}_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R}, \text{ and } ad - bc = 1 \right\} \Big/ \{\pm I\}$$

This is the biggest subgroup of $\mathrm{PSL}_2\mathbb{C}$ which preserve the real line, the upper half plane and the lower half plane. In particular it acts on them as homeomorphism. On the other hand isometry group for the disc model is

$$\mathrm{PSU}(1,1) = \mathrm{SU}(1,1) \Big/ \{\pm I\} = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1 \right\} \Big/ \{\pm I\}$$

Clearly $\mathrm{PSL}_2\mathbb{R}$ and $\mathrm{PSU}(1,1)$ are isomorphic as groups, and the explicit isomorphism is given by the map sending every matrix $A \in \mathrm{PSL}_2\mathbb{R}$ to $JAJ^{-1}$, where $J \in \mathrm{PSL}_2\mathbb{C}$ is the matrix associate to the isometry $j$ defined above.

**Classification.** We now classify the hyperbolic transformation. In order to do this, we consider only the group $\mathrm{PSL}_2\mathbb{R}$, because the same considerations hold for $\mathrm{PSU}(1,1)$.
Let $g \in \mathrm{PSL}_2\mathbb{R}$ be any transformation different to the identity, and we wonder which are solutions of the following equation

$$g(z) = \frac{az+b}{cz+d} = z.$$

An element $g$ is said to be *elliptic* if it has one solution on the half-plane (observe that there are another solution on the lower half plane, but we do not care about it), *parabolic* if it has only one real solution (of multiplicity two), or *hyperbolic* if it has two distinct real solutions. Geometrically speaking the solution(s) of the above equation is (are) the fixed point(s) of $g$.

**Theorem 1.2.1.** *Let $g \in \mathrm{PSL}_2\mathbb{R}$ any element different to the identity. Then*

- $|\mathrm{Tr} g| < 2$ *if and only if $g$ is elliptic,*
- $|\mathrm{Tr} g| = 2$ *if and only if $g$ is parabolic,*
- $|\mathrm{Tr} g| > 2$ *if and only if $g$ is hyperbolic.*

*Proof.* Let $g$ be any element in $\mathrm{PSL}_2\mathbb{R}$ different to the idenity and consider the following equation

$$\frac{az+b}{cz+d} = z.$$

With a simple manipulation we obtain $cz^2 - (a-d)z - b = 0$ and the solutions of this equation are the fixed point of $g$. Using the classical formula to determine them we compute the discriminant $\Delta^2 = (a-d)^2 - 4bc = (a+d)^2 - 4$ (remenber that $ad - bc = 1$). Now the conclusion follows: $\Delta^2 > 0$ if and only if $(a+d)^2 = \mathrm{Tr}^2 g > 4$ hence if and only if $|\mathrm{Tr} g| > 2$. Simlarily $\Delta^2 = 0$ if and only if $|\mathrm{Tr} g| = 2$ and $\Delta^2 < 0$ if and only if $|\mathrm{Tr} g| < 2$. In the first case we have two different real solutions, in the second case only one solution of multiplicity two and in the third case two complex and conjugated solutions, in particular only one of them lies in $\mathbb{H}^2$. $\square$

Thus hyperbolic transformations are completely classified by the modulus of their trace. Even if the identity has trace equal 2, we prefer to consider it as a special element, in other words, it is the unique element which is not hyperbolic neither parabolic or elliptic. We denote by $\mathcal{E}ll$ the subset of elliptic transformations in $\mathrm{PSL}_2\mathbb{R}$ and in the same way we denote by $\mathcal{P}ar$ the subset of parabolic transformations and by $\mathcal{H}yp$ the subset of hyperbolic transformations.

**Geometry of hyperbolic transformations.** Here we recall the main properties about the geometry of hyperbolic transformations which we will use later. We begin considering the product of hyperbolic transformation because the geometry of commutators can be inferred by the geometry of the products. Indeed $[g, h] = ghg^{-1}h^{-1}$ set $\alpha = g$ and $\beta = hg^{-1}h^{-1}$ we have that $[g, h] = \alpha\beta$, so we can study the commutator in terms of $\alpha, \beta$.

**Lemma 1.2.2.** *Let $g, h$ be elliptic isometries with $g$ a rotation of $2\theta$ about $p$ and $h$ a rotation of $2\varphi$ about $q \neq p$. Supposing that they are rotation in the same sence, we have*

$$|\mathrm{Tr}\ gh| = 2(\cosh d_{\mathbb{D}}(p, q) \sin\theta \sin\varphi - \cos\theta\cos\varphi).$$

If $p = q$ then the product is still a rotation of $\theta + \varphi$ about $p$. Moreover $gh = hg$ in this case.

**Lemma 1.2.3.** *Let $g, h$ hyperbolic isometries with translation lenghts $2\lambda_g, 2\lambda_h$ respectively. Suppose further that $\mathsf{Axis}(g)$ and $\mathsf{Axis}(h)$ are disjoint. Then*

$$|\mathrm{Tr}\ gh| = 2(\cosh d_{\mathbb{D}}(\mathsf{Axis}(g), \mathsf{Axis}(h)) \sinh\lambda_g \sinh\lambda_h + \varepsilon \cosh\lambda_g \cosh\lambda_h),$$

*where $\varepsilon = 1, -1$ according to the direction of the axes. In particular, if $\varepsilon = 1$ the product $gh$ is always hyperbolic.*

**Lemma 1.2.4.** *Let $g, h$ be hyperbolic isometries and suppose that $\mathsf{Axis}(g)$ and $\mathsf{Axis}(h)$ intersect at a point $p$ in an angle $0 < \theta < \pi$, where the angle has to be consider between the half rays from $p$ to the attractive points of $g$ and $h$. Then $gh$ is hyperbolic and*

$$|\mathrm{Tr}\ gh| = 2(\cosh\lambda_g \cosh\lambda_h + \sinh\lambda_g \sinh\lambda_h \cos\theta).$$

The proofs of lemmata 1.2.2, 1.2.3, 1.2.4, may be found in [1, Chapter 7]. We turn now on commutators, beginning with the following lemma by Goldman (see [10, Lemma 3.4.5]).

**Lemma 1.2.5.** *Let $g, h$ be hyperbolic transformations. Then the following are equivalent*

- *$g, h$ are hyperbolic and their axes cross,*
- $\mathrm{Tr}[g, h] < 2$.

*Proof.* Assuming $\mathrm{Tr}[g,h] < 2$, we first show that both $g$ and $h$ must be hyperbolic. If $g$ was elliptic, up to conjugation we may assume that $g \in \mathrm{SO}_2\mathbb{R}$ and a straightforward computation show that $\mathrm{Tr}[g,h] = 2 + \sin^2\theta(a^2 + b^2 + c^2 + d^2 - 2) \geq 2$. The same holds if $g$ is parabolic, so $g \in \mathcal{H}yp$. An identical argument shows that also $h$ must be hyperbolic.
The second step is to show that $\mathrm{Tr}[g,h] < 2$ if and only if $\mathsf{Axis}(g)$ and $\mathsf{Axis}(h)$ cross. Up to conjugation we may assume that the fixed points for $g$ are $\pm 1$ and that the fixed points for $h$ are $r, \infty$. Then we write $\mathrm{Tr}[g,h]$ as function on $r$, and it easy to see that $\mathrm{Tr}[g,h] < 2$ if and only if $-1 < r < 1$. □

Before going to consider the other cases, we list some lemmata about the fixed point(s) of a commutator when $g, h$ are hyperbolic and their axes cross.

**Lemma 1.2.6.** *Suppose $g, h$ are hyperbolic and $\mathrm{tr}[g,h] < -2$, so they are hyperbolic and their axes intersect. Then $\mathsf{Axis}[g,h]$ does not intersect the axis of $g$ or $h$. Moreover the fixed points of $[g,h]$ lie on the segment of the circle at infinity between $g^+$ and $h^+$: $[g,h]^+$ is closer to $g^+$ and $[g,h]^-$ is closer to $h^+$.*

Similarly we have two similar result when $[g,h]$ is parabolic or elliptic.

**Lemma 1.2.7.** *Suppose $g, h$ are hyperbolic and $\mathrm{tr}[g,h] = -2$, so it is parabolic. Then $\mathsf{Fix}[g,h]$ lies on the segment of the circle at infinity between $g^+$ and $h^+$. The sense of rotation is clockwise if the segment from $g^+$ to $h^+$ has che clockwise orientation, otherwise the sense is counterclockwise.*

**Lemma 1.2.8.** *Suppose $g, h$ are hyperbolic and $-2 < \mathrm{tr}[g,h] < 2$, so it is elliptic. Then $\mathsf{Fix}[g,h]$ lie in the region determined by $\mathsf{Axis}(g)$, $\mathsf{Axis}(h)$ which is bounded by the arc on the circle at infinity between $g^+$ and $h^+$. The sense of rotation is clockwise if the segment from $g^+$ to $h^+$ has che clockwise orientation, otherwise the sense is counterclockwise.*

These lemmata may be proved with a direct computation. Here we offer the following proof by Matelski [18] which is more elegant and revealing. The first arguments of the proof of 1.2.6 are the same of the proofs of 1.2.7 and 1.2.8, hence we may merge the proofs of these lemmata in a unique one and then discussing case by case.

*Proofs of lemmata 1.2.6, 1.2.7 and 1.2.8.* Let $2\lambda_g, 2\lambda_h$ be the traslation distance of $g, h$ and let $p \in \mathbb{D}$ be the point of intersection of the axes of $g$ and $h$ and let $e \in \mathrm{PSU}(1,1)$ a half turn around $p$. So we have that $ege = g^{-1}$ and the same holds for $h$, further $he$ preserve $\mathsf{Axis}(h)$ but reversed its sense. Thus $he$ is an elliptic element of PSU$(1,1)$ and let $q$ be its fixed point; observe that $q \in \mathsf{Axis}(h)$ and it lies between $p$ and $h^+$ at a distance $\lambda_h$ from $p$. Now consider $ghe$, we have that $(ghe)^2 = gh(ege)(ehe) = ghg^{-1}h^{-1} = [g,h]$.So $ghe$ is a hyperbolic, parabolic or an elliptic transformation, if $[g,h]$ is hyperbolic, parabolic or elliptic respectively. Let $l_1$ be the perpendicular line from $q$ to $\mathsf{Axis}(g)$, and denote $r$ its foot. Let $l_2$ be the perpendicular line to $l_1$ passing through $q$. Let $s$ be point along $\mathsf{Axis}(g)$ between $r$ and $g^+$ at a distance $\lambda_g$ from $r$. Finally let $_3$ be the line passing through $s$ perpendicular to $\mathsf{Axis}(g)$. Denote by $R_{l_i}$ the reflection respect the line $l_i$. Then we have $he = R_{l_1}R_{l_2}$ and $g = R_{l_3}R_{l_1}$, so $ghe = R_{l_3}R_{l_2}$. Now we have the following tricotomy.

1 The axes $l_2, l_3$ do not intersect in $\mathbb{D}$ nor in the boundary at infinity. In this case $[g,h]$ is hyperbolic and $\mathsf{Axis}[g,h]$ is the common perpendicular of $l_2$ and $l_3$. In particular the fixed points are in the desired order and this conclude the proof of 1.2.6.

2 If we are not in the first case the axes cross. In particular, if $l_2, l_3$ intersect at the infinity then $[g,h]$ is parabolic and the fixed point is given by the intersection point. Moreover, the fixed point is in the desired position and this concludes 1.2.7.

3 Finally $l_2, l_3$ intersect at a point $o$ and $[g,h]$ is elliptic with wixed point $o$. By construction the fixed point lies in the desider region of $\mathbb{D}$ and this conclude 1.2.8. $\square$

We conclude with other two lemmata that summarize the remaining cases. Without loss of generality we suppose that $g$ is a parabolic element and let $p$ be the fixed point for $g$; then

**Lemma 1.2.9.** *Let $h$ any transformation, and suppose that $g, h$ have no common fixed point. Then $[g,h]$ is hyperbolic.*

Finally suppose $g$ elliptic with rotation angle $2\theta$ and let $p$ its fixed point.

**Lemma 1.2.10.** *Let $h$ any transformation not fixing $p$. Then $[g,h]$ is hyperbolic and*

$$\sinh\left(\frac{\lambda}{4}\right) = \sinh\left(\frac{1}{2}d_{\mathbb{D}}(p,h(p))\right)\sin\theta,$$

*where $\lambda$ is the translation length for $[g,h]$.*

We do not report here the proofs of lemmata 1.2.2, 1.2.3, 1.2.4, 1.2.9, 1.2.10 that can be found in [1, Chapter 7].

## 1.3 About $\mathrm{PSL}_2\mathbb{R}$ and $\widetilde{\mathrm{PSL}_2\mathbb{R}}$

Coming back on $\mathrm{PSL}_2\mathbb{R}$ we study it from the topological point of view. Geometrically $\mathrm{PSL}_2\mathbb{R}$ is an open solid torus homeomorphic to $\mathbb{H}^2 \times \mathbb{S}^1$ and its universal cover is clearly homeomorphic to $\mathbb{R}^3$. Indeed we may identify $\mathrm{PSL}_2\mathbb{R}$ with the unit tangent bundle $\mathrm{UT}\mathbb{H}^2$, which is homeomorphic to $\mathbb{H}^2 \times \mathbb{S}^1$. However, this identification depends on a preliminary choice of a base point $(p_0, u_0) \in \mathbb{H}^2 \times \mathbb{S}^1$, hence is not canonical in general. More precisely we may associate to any element $g \in \mathrm{PSL}_2\mathbb{R}$ the point $\big(g(p_0), g'(u_0)\big)$ and simple arguments show that this correspondence is well defined and bijective. In order to see this let $\gamma$ be unique geodesic on $\mathbb{H}^2$ passing through $p_0$ and tangent to $u_0$. Denote by $a, b$ its points at infinity. Let $\delta$ be the normal geodesic ray coming out of $p_0$ such that its point at infinity $c$ lies in positive cyclic order with respect to $a, b$. Let now $(q, v) \in \mathbb{H}^2 \times \mathbb{S}^1$ be another point and repeat the same construction above for it to get another triple $(a', b', c')$, hence there exists a unique $g \in \mathrm{PSL}_2\mathbb{R}$ such that

$$g(a) = a' \quad g(b) = b' \quad g(c) = c'$$

because any hyperbolic transformation is uniquely defined by the image of three points on the boundary at infinity.

Of course, we have $\pi_1(\mathrm{PSL}_2\mathbb{R}) \cong \mathbb{Z}$ and the universal cover $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ is naturally identified with $\mathbb{H}^2 \times \mathbb{R}$. By classical covering theory, $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ may be seen also as the set of paths $\big\{c : [0,1] \longrightarrow \mathrm{PSL}_2\mathbb{R}\big\} = \big\{c : [0,1] \longrightarrow \mathrm{UT}\mathbb{H}^2\big\}$ up to homotopy starting from the base point. Roughly speaking any element of the universal cover may be seen as a path with a unit tangent vector attached to any point that changes continuously, regardless of where it started because the base point is arbitrary. By construction the projection of $c \in \widetilde{\mathrm{PSL}_2\mathbb{R}}$ to $\mathrm{PSL}_2\mathbb{R}$ is the unique isometry sending the unit tangent vector at $c(0)$ to the unit tangent vector at $c(1)$.

Any element $c \in \widetilde{\mathrm{PSL}_2\mathbb{R}}$ is elliptic, parabolic or hyperbolic accordingly as is its projection. The identity element lifts to an infinite cyclic subgroup generated by $\mathbf{z}$, namely the center of $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ which is isomorphic to $\mathbb{Z}$. In particular, these lifts correspond to those paths starting and ending at basepoint $(p_0, u_0)$ of the following form

$$c(t) = (p_0, e^{2nt\pi i})$$

for some $n \in \mathbb{Z}$. With this notation $\mathbf{z} = (p_0, e^{2t\pi i})$.

**Preferred lift.** Any element $g \in \mathrm{PSL}_2\mathbb{R}$ has infinetely many lifts that differ by a power of $\mathbf{z}$. However we may wonder if there is a nicest lift of $g$ in some sense and the answer turns out to be positive if $g$ is hyperbolic or parabolic.
Suppose $g$ is hyperbolic, hence it is a translation along its axis $\mathsf{Axis}(g)$ by some distance $d$. Then there exists a unique one parameter subgroup $c : \mathbb{R} \longrightarrow \mathrm{PSL}_2\mathbb{R}$ (with a little abuse of notation) such that $c(t)$ is a hyperbolic translation along $\mathsf{Axis}(g)$ of distance $|t|d$. In particular $c(0) = \mathrm{id}$ and $c(1) = g$. Its restriction to $[0,1]$ gives a unique path in $\mathrm{PSL}_2\mathbb{R}$ which we define as the *preferred or simplest* lift of $g$. Choose the basepoint such that $p_0 \in \mathsf{Axis}(g)$ and $u_0$ pointing out along the axis. Then the preferred lift may be thought as the path $\big(c(t)(p_0), c'(t)(u_0)\big)$ where $c(t)(p_0)$ lies in $\mathsf{Axis}(g)$ and the unit tangent vector $c'(t)(u_0)$ pointing out along $\mathsf{Axis}(g)$ for every time $t \in [0,1]$.
A similar argument works also for parabolic isometries. Indeed if $g$ is parabolic then it translates along a horocircle $h$ by some distance $d$ with respect to the Euclidean metric induced by the hyperbolic one on $h$. As above there exists a unique one parameter subgroup $c : \mathbb{R} \longrightarrow \mathrm{PSL}_2\mathbb{R}$ such that $c(t)$ is a parabolic translation along $h$ of distance $|t|d$ and $c(0) = \mathrm{id}$ and $c(1) = g$. Again its restriction to $[0,1]$ gives a unique path in $\mathrm{PSL}_2\mathbb{R}$ which we consider as preferred lift and also in this case the preferred lift may be thought as the path $\big(c(t)(p_0), c'(t)(u_0)\big)$ where $c(t)(p_0)$ lies in $h$ and the unit tangent vector $c'(t)(u_0)$ pointing out along $h$ for every time $t \in [0,1]$ in the direction of translation.
On the other hand the situation changes drammatically when we consider elliptic elements. If $g$ is an elliptic isometry then there are infinitely many one parameter subgroups $c : \mathbb{R} \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $c(1) = g$, and this is reflected by the fact that anyone of them contains the cyclic subgroup generated by $\mathbf{z}$, *i.e.* the center of $\widetilde{\mathrm{PSL}_2}\mathbb{R}$. More precisely suppose $g$ is a rotation of angle $\theta \in ]0, 2\pi[$ measured counterclockwise about a point $o \in \mathbb{H}^2$ and consider the following family of one parameter subgroups

$$\begin{aligned} c_n(t) : \mathbb{R} &\longrightarrow \mathrm{PSL}_2\mathbb{R} \\ t &\longmapsto \begin{pmatrix} \cos(\theta + 2\pi n - 2\pi)t & -\sin(\theta + 2\pi n - 2\pi)t \\ \sin(\theta + 2\pi n - 2\pi)t & \cos(\theta + 2\pi n - 2\pi)t \end{pmatrix} \end{aligned}$$

when $n > 0$ and

$$\begin{aligned} c_n(t) : \mathbb{R} &\longrightarrow \mathrm{PSL}_2\mathbb{R} \\ t &\longmapsto \begin{pmatrix} \cos(\theta + 2\pi n)t & -\sin(\theta + 2\pi n)t \\ \sin(\theta + 2\pi n)t & \cos(\theta + 2\pi n)t \end{pmatrix} \end{aligned}$$

when $n < 0$. In this case for any $c_n(1) = g$ for any $n \in \mathbb{N} \setminus \{0\}$ because everyone of them projects to the isometry fixing $o$ and sending the unit tangent vector $c_n'(0)$ to $c_n'(1)$. Note that $c_n(t) = \mathbf{z}^m$ for some values of $t$ and some $m \in \mathbb{Z}$.
The restrictions of such homomorphisms define different paths for each $n \neq 0$ in $\mathrm{UT}\mathbb{H}^2$. Indeed for any fixed $n$ the angle between $c_n'(0)$ and $c_n'(1)$ is always $\theta$ but the unit vectors spin exactly $n-1$ times (and not $n$) as the path is traversed. Hence this paths are all different and are pairwise not homotopic relative to endpoints. Thus a simplest lift does not exists in this sense but we have two simplest lifts in this case which are respectively the simplest counterclockwise lift $c_1$ and the simplest clockwise lift $c_{-1}$.
We denote the set of simplest lift of hyperbolic and parabolic elements by $\mathrm{Hyp}_0$ and $\mathrm{Par}_0$. For any hyperbolic element $c \in \widetilde{\mathrm{PSL}_2\mathbb{R}}$ there exists a unique $m \in \mathbb{Z}$ such that $\mathbf{z}^{-m}c \in \mathrm{Hyp}_0$, thus we define $\mathrm{Hyp}_m = \mathbf{z}^m\mathrm{Hyp}_0$. In the same way $\mathrm{Par}_m = \mathbf{z}^m\mathrm{Par}_0$.

*Remark* 1.3.1. It is classical to see the existence of two different classes of parabolic isometries in $\mathrm{PSL}_2\mathbb{R}$ because rotations about points at infinity may be clockwise and counterclockwise. Precisely every parabolic isometry is conjugated with only one of the following transformations

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

however these latter are not conjugated between them in $\mathrm{PSL}_2\mathbb{R}$ even if they are in $\mathrm{PSL}_2\mathbb{C}$.

This remark allows to a further distinction of $\mathrm{Par}_m$ in $\mathrm{Par}_m^-$ and $\mathrm{Par}_m^+$ of parabolic elements which are clockwise and countrclockwise rotations about a point at infinity respectively.
Finally we define $\mathrm{Ell}_1$ the set of simplest counterclockwise lifts of elliptic elements in $\mathrm{PSL}_2\mathbb{R}$. Similarly $\mathrm{Ell}_{-1}$ is the set of simplest clockwise lifts of elliptic elements in $\mathrm{PSL}_2\mathbb{R}$. For any $m > 0$ we define as in the other case $\mathrm{Ell}_m$ as $\mathbf{z}^m\mathrm{Ell}_1$ and $\mathrm{Ell}_{-m}$ as $\mathbf{z}^{-m+1}\mathrm{Ell}_{-1}$.

*Remark* 1.3.2. Since the set $\mathrm{Ell}_0$ is not define we have $\mathrm{Ell}_1 = \mathbf{z}\mathrm{Ell}_{-1}$.

**Topology of $\mathrm{PSL}_2\mathbb{R}$ and $\widetilde{\mathrm{PSL}_2\mathbb{R}}$.** We already know that $\mathrm{PSL}_2\mathbb{R}$ is a solid torus, but we do not know how yet the topology of the subsets $\mathcal{H}\mathrm{yp}$, $\mathcal{P}\mathrm{ar}$ and $\mathcal{E}\mathrm{ll}$.
Following Goldman in [12] the set of geodesic axes in $\mathbb{H}^2$ is a Möbius band $M$ and the function $\mathcal{H}\mathrm{yp} \longrightarrow M \times \mathbb{R}^+$ that associates to any hyperbolic element its axis and the translation distance $d$ is a double covering (remember that for any axis

there two hyperbolic functions translating along it with distance $d$ that are opposite to each other). Thus the sets $\mathcal{H}\mathrm{yp}$ in $\mathrm{PSL}_2\mathbb{R}$ and $\mathrm{Hyp}_n$ in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ (for any $n$) are homeomorphic to an open solid tube. Similarly, we may define a function $\mathcal{E}\mathrm{ll} \longrightarrow \mathbb{H}^2\times ]0,2\pi[$ that associates to any elliptic isometry its fixed point and it's rotation angle which turns out to be a homeomorphism for every $n \neq 0$. The sets $\mathcal{H}\mathrm{yp}$ and $\mathcal{E}\mathrm{ll}$ form two disjoint $3-$dimensional open subset in $\mathrm{PSL}_2\mathbb{R}$ and their common boundary is a $2-$dimensional space that represents the set of parabolic isometries.

**Relationship between lifts and traces.** We have seen in 1.2.1 that any hyperbolic isometry is characterized by its trace. Here we see that a similar characterization holds also for elements in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. Since $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ is the universal cover of $\mathrm{PSL}_2\mathbb{R}$ it covers also $\mathrm{SL}_2\mathbb{R}$, hence the notion of *trace* is well-defined in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. We may see the trace Tr as a continuous $\mathbb{R}-$valued function defined $\mathrm{SL}_2\mathbb{R}$ and taking in mind the previous paragraph we claim the following.

**Lemma 1.3.3.** *Let* $\widetilde{\mathrm{Tr}}$ *be composition of the covering projection* $\widetilde{\mathrm{PSL}_2\mathbb{R}} \longrightarrow \mathrm{SL}_2\mathbb{R}$ *with the trace function* Tr. *Then it is continuous and*

*1* $\widetilde{\mathrm{Tr}}(\mathbf{z}^n) = 2(-1)^n$

*2* $\widetilde{\mathrm{Tr}}(\mathrm{Par}_n) = 2(-1)^n$

*3* $\widetilde{\mathrm{Tr}}(\mathrm{Hyp}_n)$ *is the open interval* $]2,\infty[$ *is* $n$ *is even or the open interval* $]-\infty,-2[$ *if* $n$ *is odd.*

*Proof.* The function $\widetilde{\mathrm{Tr}}$ is continuous because composition of continuous function. Thus we only need to show that it has the claimed behaviour. Let $\{\pm\mathrm{id}\}$ be the center of $\mathrm{SL}_2\mathbb{R}$ that turns out to be a subgroup of order two. By the covering projection $\widetilde{\mathrm{PSL}_2\mathbb{R}} \longrightarrow \mathrm{SL}_2\mathbb{R}$ it lifts to the center of $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ which is isomorphic to $\mathbb{Z}$. In particular the identity $\mathrm{id} \in \mathrm{SL}_2\mathbb{R}$ lifts to a subgroup of $\mathbb{Z}$ of index two, that is the subgroup generated by $\mathbf{z}^2$. Thus the first item follows. Of course the trace of a parabolic element in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ is $\pm 2$. Since the set $\mathrm{Par}_n \cup \{\mathbf{z}^n\}$ is connected for any $n$ the trace on it is uniquely defined by its value on $\mathbf{z}^n$, hence $\widetilde{\mathrm{Tr}}(\mathrm{Par}_n) = 2(-1)^n$. Also the set $\mathrm{Hyp}_n$ is connected for any $n$ thus the trace function takes values in $]-\infty,-2[$ or $]2,\infty[$. Since it is bounded by $\mathrm{Par}_n \cup \{\mathbf{z}^n\}$ on which the trace is $2(-1)^n$ the third claim follows. $\square$

*Remark* 1.3.4. By continuity $\widetilde{\mathrm{Tr}}(\mathrm{Ell}_n) \in ]-2,2[$. Indeed for every $n \neq 0$ the set $\mathrm{Ell}_n$ is connected and its boundary is given by $\mathrm{Par}^+_{n-1} \cup \mathrm{Par}^-_n$ if $n > 0$ and by $\mathrm{Par}^-_n \cup \mathrm{Par}^+_{n+1}$ if $n < 0$.

**Lifts of commutators.** In this last paragraph we consider the commutators of elements in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ and we explain briefly the relation with their trace. Following [20, 22] we give the following lemma.

**Lemma 1.3.5.** *Let $g, h \in \mathrm{PSL}_2\mathbb{R}$. Then $[g, h]$ has a well-defined lift to $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. That is, any couple of lifts $\widetilde{g}_1$, $\widetilde{h}_1$ and $\widetilde{g}_2$, $\widetilde{h}_2$ satisfy $\left[\widetilde{g}_1, \widetilde{h}_1\right] = \left[\widetilde{g}_2, \widetilde{h}_2\right]$.*

*Proof.* Let $\widetilde{g}_2 = \mathbf{z}^n\widetilde{g}_1$ and $\widetilde{h}_2 = \mathbf{z}^m\widetilde{h}_1$. Since $\mathbf{z}$ commutes with every element of $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ we notice that

$$\left[\widetilde{g}_1, \widetilde{h}_1\right] = \left[\widetilde{g}_2, \widetilde{h}_2\right]$$

as desired. □

Even if the lift of a commutator $[g, h]$ of two elements $g, h \in \mathrm{PSL}_2\mathbb{R}$ is well-defined, it may differ from the simplest lift. More precisely its simplest lift belongs to $\mathrm{Hyp}_0$, however for any couples of lifts $\widetilde{g}$, $\widetilde{h}$ there exists an integer $n$ such that

$$\left[\widetilde{g}, \widetilde{h}\right] = \mathbf{z}^n\widetilde{[g, h]}$$

The previous lemma says that this integer does not dipend on the choice of the lifts and the next proposition tell us all possible values of $n$. We state it without proof that can be found in [22].

**Proposition 1.3.6.** *Let $g, h \in \mathrm{PSL}_2\mathbb{R}$, then $[g, h]$ is well-defined and belongs*

$$\left[\widetilde{g}, \widetilde{h}\right] \in \{1\} \cup \left(\bigcup_{n=-1}^{1} \mathrm{Hyp}_n \cup \mathrm{Ell}_n\right) \cup \mathrm{Par}_0 \cup \mathrm{Par}_{-1}^{+} \cup \mathrm{Par}_{1}^{-}$$

*where $\mathrm{Ell}_0$ is the empty set for convenience.*

Combining 1.3.3 with 1.3.6 we get the following corollary.

**Corollary 1.3.7.** *Let $g, h \in \mathrm{PSL}_2\mathbb{R}$ then*

*1* $\mathrm{Tr}[g, h] > 2 \Longrightarrow [g, h] \in \mathrm{Hyp}_0$,

*2* $\mathrm{Tr}[g, h] = 2 \Longrightarrow [g, h] \in \mathrm{Par}_0$,

*3* $\mathrm{Tr}[g, h] \in ]-2, 2[ \Longrightarrow [g, h] \in \mathrm{Ell}_{-1} \cup \mathrm{Ell}_1$,

*4* $\mathrm{Tr}[g, h] = -2 \Longrightarrow [g, h] \in \mathrm{Par}_{-1}^{+} \cup \mathrm{Par}_{1}^{-}$,

*5* $\mathrm{Tr}[g, h] < -2 \Longrightarrow [g, h] \in \mathrm{Hyp}_{-1} \cup \mathrm{Hyp}_1$.

## 1.4 Geometric structures on manifolds

Let $X$ be a model space, that is a homogeneous space and let $G$ be a Lie group acting transitively on $X$.
Let $U$ be an open set in $S$, a *local chart* is a map $\varphi : U \longrightarrow X$ which is a homeomorphism onto its image. If we have two local charts $\varphi_i$ and $\varphi_j$ defined on overlapping open sets $U_i$ and $U_j$, these are said to be *compatible* if there is an element $g_{ij}$ such that $\varphi_i \circ g_{ij} = \varphi_j$. A set of compatible charts whose domains cover $S$ is called *atlas.* Let $\mathcal{A}$ be an atlas and let $\psi : V \longrightarrow X$ be a local chart not in $\mathcal{A}$. We say that $\psi$ is compatible with $\mathcal{A}$ if it is compatible with every local chart in $\mathcal{A}$. Adjoin $\psi$ to $\mathcal{A}$ we get a new larger atlas and by a recursive procedure, we may adjoin every compatible chart to produce a *maximal atlas.* A $(G, X)-$structure on $S$ is the datum of a maximal atlas $\mathcal{A}$.

**Lemma 1.4.1.** *Any atlas $\mathcal{A}$ is contained in a unique maximal atlas.*

*Proof.* Adjoin to $\mathcal{A}$ all chart $\psi$ that are compatible with $\mathcal{A}$ to produce a new atlas $\mathcal{M}$. This is maximal because any local chart compatible with the new atlas must be also compatible with $\mathcal{A}$ so by construction it belongs to $\mathcal{M}$. Thus the new atlas $\mathcal{M}$ is maximal. It remains to show that this is unique. Suppose $\mathcal{M}'$ is another maximal atlas, thus any chart in it must be compatible with $\mathcal{A}$ hence they belong to $\mathcal{M}$, that is $\mathcal{M}' \subset \mathcal{M}$. A similar argument shows $\mathcal{M} \subset \mathcal{M}'$, thus $\mathcal{M}' = \mathcal{M}$. □

Suppose $X$ is a Riemannian homogeneous space, upon which a Lie group $G$ acts transitively by isometries. Under this condition, any $(G, X)-$structure on $S$ may be seen as a metric such that every point of $S$ has a neighborhood isometric to a standard ball in our model geometry $X$. Indeed every $(G, X)-$manifold $S$ inherits a Riemannian metric $g_S$ locally isometric to the Riemannian metric $g_X$ on $X$. Any local chart $\varphi : U \longrightarrow X$ determines a Riemannian metric $\varphi^*(g_X)$ on the open set $U$; since $G$ acts isometrically on $X$ with respect to $g_X$, any couple of compatible charts agree on the intersection of overlapping sets. Hence there exists a metric $g_S$ on $S$ whose restriction to any open set $U$ equals with the pullback metric $\varphi^*(g_X)$.

In particular, since the kind of surfaces we are interested in are orientable, any $(G, X)-$ structure on a surface $S$ can be reduced to a $(G^+, X)-$structure, where $G^+$ denotes the group of orientation-preserving isometries. Instead of continuing with general arguments; all definitions and basic facts are given for the structure we are interested in, though they have obvious generalization to other geometries.

### 1.4.1 Hyperbolic structures

We begin with the classical notion of hyperbolic structure. Thus let $\mathbb{H}^2$ be our model space with its group of orientation-preserving isometry $\mathrm{PSL}_2\mathbb{R}$. Following the previous discussion, we give the following definition.

**Definition 1.4.2.** A *hyperbolic structure* $\sigma$ on a surface $S$ is the datum of a maximal atlas whose charts are orientation preserving homeomorphisms onto an open subset of $\mathbb{H}^2$ and such that transition functions are restrictions of elements in $\mathrm{PSL}_2\mathbb{R}$.

This kind of structures was studied in the past and actually are completely known. In the case of the structure is complete as metric space ( for instance in the closed case ), by the Hopf-Rinow theorem they arise as the quotient of the action of a Fuchsian group $\Gamma$ on the hyperbolic plane. We recall that a subgroup of $\mathrm{PSL}_2\mathbb{R}$ is called *Fuchsian* if it is a discrete torsion-free subgroup with respect to the induce topology by $\mathrm{PSL}_2\mathbb{R}$ as Lie group, that acts discontinuously on $\mathbb{H}^2$.

**Branched hyperbolic structures.** We are now going to define the main structure we are interested in, that is *branched hyperbolic structures.* They arise as a natural generalization of the hyperbolic structure defined above, where conical singularities are allowed. In order to introduce them, we need to define local charts in full generality.

**Definition 1.4.3.** Let $p \in S$ be an inner point, that is $p \notin \partial S$. A local chart in $p$ on $S$ is a finite degree orientation preserving branched covering map $\varphi : U \longrightarrow \mathbb{H}^2$ onto an open subset of $\mathbb{H}^2$, which we consider as a local coordinate on $U$.

In particular, any local chart of degree $k$ is an isometry between the domain $U$ and the hyperbolic wedge (with sides glued) of angle $2k\pi$, that is a hyperbolic cone. The same idea can be used to define local charts around boundary point.

**Definition 1.4.4.** Let $p \in S$ be a boundary point. A local chart in $p$ on $S$ is an isometry between an open neighborhood $U$ of $p$ and a hyperbolic wedge of angle $\theta$ (without sides glued), where $\theta$ is a positive real number.

Notice that local charts of degree one are homeomorphisms. Suppose $\varphi_1 : U_1 \longrightarrow \mathbb{H}^2$ and $\varphi_2 : U_2 \longrightarrow \mathbb{H}^2$ are two different charts with non-empty intersection, that is $U_1 \cap U_2 \neq \phi$. We would like to point out that the writing

$$g_{12} = \varphi_2 \circ \varphi_1^{-1}, \quad \text{where } g_{12} \in \mathrm{PSL}_2\mathbb{R},$$

does not make sense in general, because the function $\varphi_1^{-1}$ may fail to be well-defined (for instance if it is a branched chart of degree at least 2). We bypass this problem saying that two charts are *compatible* if there exists $g_{21}$ such that $\varphi_2 = g_{21} \circ \varphi_1$ on the intersection, and everything works.

**Definition 1.4.5.** A *branched hyperbolic structure* $\sigma$ on a surface $S$ is the datum of a maximal atlas whose charts are defined as in 1.4.3 and 1.4.4 around inner points and boundary points respectively, and such that transition functions are restrictions of elements in $\mathrm{PSL}_2\mathbb{R}$.

In some sense the definition 1.4.5 contains 1.4.2. Indeed if all charts are local homeomorphisms then $\sigma$ is a hyperbolic structure like in the previous definition 1.4.2 where all points are regular. Otherwise, some charts are local isometries onto some hyperbolic wedge (or hyperbolic cone) and they define special points on $S$ which are called *cone points* or *corner points* depending on whether they lie inside $S$ or on the boundary. The next definition explains this more precisely.

**Definition 1.4.6.** An inner point $p$ is called *cone point of order $k$* if any open neighborhood around $p$ is isometric to a hyperbolic wedge of angle $2k\pi$. Similarly a boundary point $p$ is called *corner point of order $s$* if any open neighborhood $U$ is isometric to a hyperbolic wedge of angle $(2s+1)\pi$.

Let $ds^2_P$ be the Poincaré metric on $\mathbb{H}^2$, then for any local chart $\varphi : U \longrightarrow \mathbb{H}^2$, the pullback metric $\varphi^*(ds^2_P)$ defines a Riemannian metric on $U$, which is singular if $\varphi$ is a local branched chart. Combine all of them we may define a singular Riemannian metric and the induced distance makes $S$ endowed with $\sigma$ a metric space (see 2.2.1). Thus the notion of angle and distance on the surface are well-defined, in particular if a point $p$ is regular then the angle around it is exactly $2\pi$ but if it is a cone (or corner) point then the angle around $p$ is $2(k+1)\pi$ (or $(2s+1)\pi$), where $k$ (or $s$) is $\mathsf{ord}(p)$.

**Definition 1.4.7.** A cone point $p$ is said to be *simple* if its order is exactly one. If there are more than one points of orders $k_1, \ldots, k_n$, the sum of all orders is called *total branch order* for the structure.

*Remark* 1.4.8. If $S$ is closed that is compact without boundary the number of cone points if finite. Otherwise, we found a convergent sequence of cone points that accumulate to a limit point $q$ because $S$ is a compact metric space. However, every chart centered in $q$ contains infinitely many cone points and then it fails to be a finite degree branched covering over an open subset of $\mathbb{H}^2$.

### 1.4.2 Developing-holonomy pairs

A branched hyperbolic structure $\sigma$ on $S$ lifts to a branched hyperbolic structure $\widetilde{\sigma}$ on the universal cover $\widetilde{S}$.

**Definition 1.4.9.** A *developing map* $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ for $\sigma$ is a smooth orientation-preserving map with isolated critical points and such that its restriction to any sufficiently small open set is a (possibly branched) chart for $\widetilde{\sigma}$.

A developing map always exists and it may be constructed performing analytic continuation of local charts of $\sigma$ along all paths in $S$. Commencing with $\varphi_1 : U_1 \longrightarrow \mathbb{H}^2$, let $\varphi_2 : U_2 \longrightarrow \mathbb{H}^2$ another chart such that $U_1 \cap U_2 \neq \phi$. Replacing $\varphi_2$ with $\varphi_2' = g_{12} \circ \varphi_2$ we get two charts which agree on the intersection $U_1 \cap U_2$ and a well defined map in their union. Continuing this process along any path in $S$ we obtain a developing map $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$.

**Lemma 1.4.10.** *Let $\sigma$ be a branched hyperbolic structure on $S$ with maximal atlas $\mathcal{M}$. A developing map $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ for $\sigma$ is well-defined up to post composition with a Möbius transformation.*

*Proof.* Let $\varphi : U \longrightarrow \mathbb{H}^2$ be any chart for $\sigma$ and let $g \in \mathrm{PSL}_2\mathbb{R}$ be any isometry. The post-composition map $g \circ \varphi$ is another chart for $\sigma$ and it belongs to $\mathcal{M}$ because the atlas is maximal. Let $\psi : V \longrightarrow \mathbb{H}^2$ be any chart compatible with $\varphi : U \longrightarrow \mathbb{H}^2$ and let $h$ be transition function. Then $g \circ \varphi$ is compatible with $\psi$ and the transition function is given by $gh$. Keeping this in mind and commencing with $g \circ \varphi$ the analytic continuation along paths in $S$ we obtain $g \circ \mathsf{dev}$. $\square$

Hence the developing map gives a way to read the geometry of $\sigma$ on the hyperbolic plane; since $\mathrm{PSL}_2\mathbb{R}$ is the group of orientation-preserving isometries for $\mathbb{H}^2$, any element acts on the hyperbolic plane without change the information encoded on the developed image.

The developing map $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ of branched hyperbolic structure $\sigma$ satisfies also an equivariance property with respect to a representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$. Walking around a closed loop $\gamma$ in $S$ we passing through several local neighborhoods $U_1, \dots, U_m = U_1$ and we are subjected to several changes of coordinates, namely $g_{12}, \dots, g_{m1}$ which are isometries for $\mathbb{H}^2$. Let $p$ any point in $\gamma \cap U_1$, after one turn in $S$ we are again in $p$, but image of $\gamma$ under the developing map ends at $g(p) = g_{12} \cdots g_{m1}(p)$. This isometry depends only on the homotopy class of the loop $\gamma$. The map $\gamma \longmapsto g$ defines a homomorphism $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ which is called *holonomy* of the developing map.

**Lemma 1.4.11.** *Let $\mathsf{dev}_1$ and $\mathsf{dev}_2$ be two developing maps for $\sigma$ equivariant with respect to $\rho_1$ and $\rho_2$ respectively. Then $\mathsf{dev}_2 = g \circ \mathsf{dev}_1$ for some $g \in \mathrm{PSL}_2\mathbb{R}$ and $\rho_2 = g\rho_1 g^{-1}$.*

*Proof.* By lemma 1.4.10 the developing map for a structure $\sigma$ is unique up to post composition with a Möbius transformation. If $\mathsf{dev}_1$ and $\mathsf{dev}_2$ are two developing maps for $\sigma$ there exists $g$ such that $\mathsf{dev}_2 = g \circ \mathsf{dev}_1$. For the second part let $\gamma \in \pi_1 S$ and $\widetilde{p}$ be any point in $\widetilde{S}$. Then $\mathsf{dev}_2(\gamma.\widetilde{p}) = \rho_2(\gamma)\mathsf{dev}_2(\widetilde{p}) = \rho_2(\gamma)g\mathsf{dev}_1(\widetilde{p})$. On the other hand $\mathsf{dev}_2(\gamma.\widetilde{p}) = g\mathsf{dev}_1(\gamma.\widetilde{p}) = g\rho_1(\gamma)\mathsf{dev}_1(\widetilde{p})$. Thus $\rho_2 = g\rho_1 g^{-1}$. □

Thus it makes sense to consider for any structure the conjugacy class of $\rho$ which is called *holonomy for the structure.*

*Remark* 1.4.12. Note that if the holonomy $\rho$ has non-trivial centralizer, then we can find different developing maps which are equivariant with respect to the same representation. However, in the sequel, we never consider this kind of representations because they cannot arise as the holonomy of any hyperbolic geometry (branched or not) on a surface.

Thus every branched hyperbolic structure can be seen as an equivalent class of couples $(\mathsf{dev}, \rho)$ called *developing-holonomy pair* that encode all necessary information about the structure. Conversely for any map $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ equivariant with respect to a representation $\rho$ defines a possibly branched hyperbolic structure. Indeed an atlas is obtained by pre-composition with the local inverse of the covering projection. That is we have the following proposition the summarize this paragraph.

**Proposition 1.4.13.** *Any branched hyperbolic structure $\sigma$ on $S$ defines a smooth, orientation-preserving, map $\mathsf{dev} : \widetilde{S} \to \mathbb{H}^2$ with isolated critical points, equivariant with respect to a representation $\rho$ of the fundamental group. Conversely: any map developing map $\mathsf{dev}$ equivariant with respect to a representation $\rho$ defines a maximal atlas, whose charts take values on $\mathbb{H}^2$ and transition functions are restrictions of global isometries of the hyperbolic plane, that is a branched hyperbolic structure $\sigma$.*

**About the holonomy.** Not all representations can be the holonomy for a branched hyperbolic structure, for instance, Tan gave in [27] an explicit example of representation which cannot arise as holonomy of a branched hyperbolic structure. The general problem of understanding which representations are holonomies of branched structure is essentially open and partial develops are done by Mathews in [20, 21]. This force to introduce the following definition.

**Definition 1.4.14.** A representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is said to be *geometrizable by a (branched) hyperbolic structures* if it is the holonomy of a (branched) hyperbolic structure $\sigma$ on $S$. Equivalently a representation is geometrizable if there exists a possibly branched developing map $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ which is $\rho$-equivariant.

Of course, Fuchsian representations are geometrizable by a unique genuine hyperbolic structure, whereas elementary representations are never geometrizable by a (possibly) branched hyperbolic structure (see 3.2.6).

*Remark* 1.4.15. As pointed out above if $\rho$ is a geometrizable representation then every conjugated representation is geometrizable by the same structure on $S$ and then it makes sense to consider its conjugacy class $[\rho]$. However, in the sequel, we will work with a fixed representant instead of the entire conjugacy class.

So far a complete characterization is known only for the punctured torus. Precisely in [20] Mathews proved the following theorem.

**Theorem 1.4.16** (Mathews)**.** *Let $S$ be the punctured torus and $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ a homomorphism. The following are equivalent:*

1 *$\rho$ is geometrizable by a branched hyperbolic structure on $S$ with geodesic boundary, except for at most one corner point, and no interior cone points;*

2 *$\rho$ is not virtually abelian, that is its image does not contain an abelian subgroup of finite index.*

Further develops was done for closed surfaces in [21], but we will come back on this later.
We will adopt from now on the term *geometrizable* only for those representations coming from branched hyperbolic structures, unless otherwise specified.

### 1.4.3 Deformation space - part I

Here we study the moduli space of branched hyperbolic structures on a given surface. Throughout this section $S$ is a closed connected surface of genus at least 2.

**Definition 1.4.17.** Two different structures $\sigma_1$ and $\sigma_2$ on $S$ are *isomorphic* if there exists an orientation preserving diffeomorphism $f : \sigma_1 \longrightarrow \sigma_2$ that pulls back charts for $\sigma_2$ to charts for $\sigma_1$.

In order to define the moduli space of branched hyperbolic structures we in particular interested on *marked isomorphism.* We give the following definition.

**Definition 1.4.18.** A *marked branched hyperbolic structure* on $S$ is a couple $(\sigma, f)$ where $\sigma$ is a branchd hyperbolic structure on $S$ and $f : S \longrightarrow \sigma$ is an orientation preserving diffeomorphism called *marking.* Two marked structures $(\sigma_1, f)$ and $(\sigma_2, g)$ are said to be *equivalent* if there exists an isomorphism $h : \sigma_1 \longrightarrow \sigma_2$ such that $g^{-1}hf : S \longrightarrow S$ is homotopic to the identity. We denote by $\mathcal{H}(S)$ the moduli space of isomorphism classes of branched hyperbolic structures on $S$.

We may consider $\mathcal{H}(S)$ also from a different point of view. We have shown before that every branched hyperbolic structure may be seen as an equivalence class of couples $[\mathsf{dev}, \rho]_{\mathrm{PSL}_2\mathbb{R}}$. However, the set of such classes is not yet the moduli space $\mathcal{H}(S)$. To obtain it from this description we need to consider also the quotient by the action of the group $\mathsf{Diff}_0(S)$ of diffeomorphism of $S$ isotopic to the identity. Any element of this group acts on the set of such equivalence classes by composition with its inverse on the first factor and trivially on the second one. More precisely two couples $[\mathsf{dev}_1, \rho_1]$ and $[\mathsf{dev}_2, \rho_2]$ are related by $f \in \mathsf{Diff}_0(S)$ if and only if there exists $g \in \mathrm{PSL}_2\mathbb{R}$ such that $\mathsf{dev}_1 \circ \widetilde{f}^{-1} = g \circ \mathsf{dev}_2$ and $\rho_1 = g\rho_2 g^{-1}$. The following lemma explains the reason why we need to consider also the action of $\mathsf{Diff}_0(S)$ on such couples.

**Lemma 1.4.19.** *Let $\sigma$ be a branched hyperbolic structure on $S$ with developing map* $\mathsf{dev}$ *and holonomy $\rho$. Let $f \in \mathsf{Diff}_0(S)$, then $f^*\sigma$ is another branched hyperbolic structure on $S$ isometric to $\sigma$ with holonomy $\rho$.*

*Proof.* By definition $f^*\sigma$ is the unique structure that makes $f$ an isometry and its developing map is $\mathsf{dev} \circ \widetilde{f}^{-1}$ for some $\widetilde{f}$. Different lifts of $f$ produce different developing map that differ by some deck transformation, hence $\mathsf{dev} \circ \widetilde{f}^{-1}$ is well-defined up to post-composition with some Möbius transformation. We claim that $\mathsf{dev} \circ \widetilde{f}^{-1}$ is equivariant with respect to $\rho \circ f_*^{-1}$. Indeed

$$\mathsf{dev} \circ \widetilde{f}^{-1}(\gamma p) = \mathsf{dev}(f_*^{-1}(\gamma)\widetilde{f}^{-1}(p)) = \rho(f_*^{-1}(\gamma))\mathsf{dev}(\widetilde{f}^{-1}(p)) = \rho \circ f_*^{-1}(\gamma)\mathsf{dev} \circ \widetilde{f}^{-1}(p)$$

From this and the fact that $f_* = \mathrm{id}_{\pi_1 S}$ we deduce that the structure $f^*\sigma$ has holonomy $\rho$. $\square$

Hence $f \in \mathsf{Diff}_0(S)$ realized a marked equivalence between $\sigma$ and $f^*\sigma$, thus they must considered to be the same structure. The quotient set is $\mathcal{H}(S)$.
This description gives a way to put a topology on $\mathcal{H}(S)$, namely the compact-open topology on the set of couples $(\mathsf{dev}, \rho)$ and then consider the quotient topology wit respect to the action of $\mathrm{PSL}_2\mathbb{R}$ and $\mathsf{Diff}_0(S)$.

With this new perspective we may define a projection map from such moduli space of branched hyperbolic structure on $S$ to the $\mathrm{PSL}_2\mathbb{R}-$character variety of $\pi_1 S$

$$\mathfrak{hol} : \mathcal{H}(S) \longrightarrow \mathcal{X}(S)$$
$$\sigma = [\mathsf{dev}, \rho] \longmapsto [\rho]$$

which associates any branched hyperbolic structure its holonomy representation. In this work we are interested in studying the image and the fibers of the holonomy map, then we will come back to them very soon. Actually we know that the holonomy map $\mathfrak{hol}$ is not surjective for any surface $S$ and its image is essentially unknown.

### 1.4.4 Deformation space - part II

**Movement of cone points.** Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a geometrizable representation, that is $\rho$ is the holonomy for some branched hyperbolic structure $\sigma$ on $S$. It is natural to ask if $\rho$ can also be the holonomy for exotic branched hyperbolic structures on $S$. In other words we are interested to study the fibres of the holonomy map $\mathfrak{hol} : \mathcal{H}(S) \longrightarrow \mathcal{X}(S)$. For this reason we are now going to define a powerful tool to produce new branched hyperbolic structures from a given one without change the holonomy.
Let $\sigma$ be a branched hyperbolic structure on $S$ with holonomy $\rho$ and developing map $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$. We begin with the following definition:

**Definition 1.4.20.** Let $p$ be a cone point for $\sigma$ on $S$ and let $\tau_1, \tau_2 : [0,1] \longrightarrow S$ be two distinct embedded geodesic path such that: $\tau_1(0) = \tau_2(0) = p$ and they do not share any other points. Then they are called *embedded twin paths* if they are injectively developed and overlap once developed, that is $\mathsf{dev}(\tau_1(t)) = \mathsf{dev}(\tau_2(t)) = \widehat{\tau}(t)$ for every $t \in [0,1]$.

The movement of cone points is a surgery on $S$ that changes the structure only locally around a cone point $p$ and maintain the structure unchanged outside an appropriate neighborhood of $p$. This operation can be done using two embedded twin paths.

Geometrically speaking, we are cutting $S$ along $\tau_1 \cup \tau_2$ producing a surface $S'$ with a boundary $b$. Observe that $b$ is not smooth and it is formed by four segments given by two copies $\tau_1^1$ and $\tau_1^2$ of $\tau_1$ and two copies $\tau_2^1$ and $\tau_2^2$ of $\tau_2$ (see 1.2).

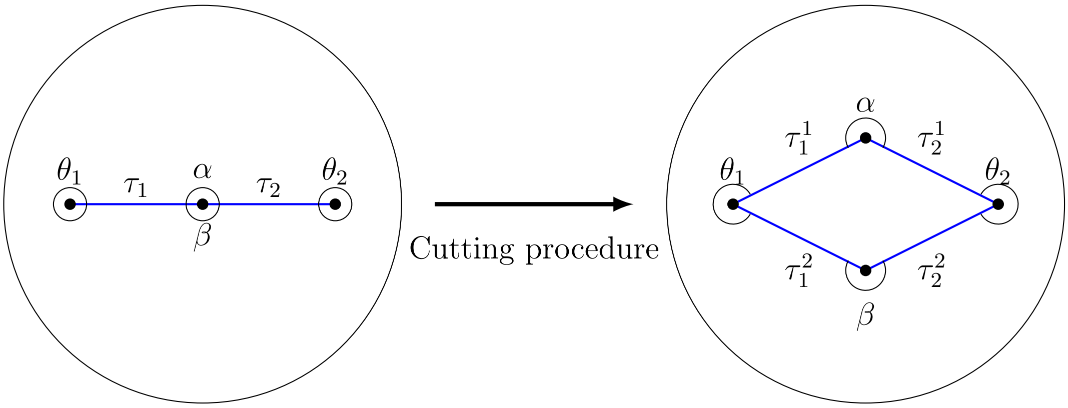


Figure 1.2: Cutting $S$ along $\tau_1 \cup \tau_2$ we get a surface with a piecewise geodesic boundary and four corner angles of magnitude $\alpha, \beta, \theta_1, \theta_2$.

Since the embedded twin paths have the same developed image, all segments of $b$ have the same developed image. Hence we can glue back them in a different way: precisely we glue $\tau_1^1$ with $\tau_2^1$ and $\tau_1^2$ with $\tau_2^2$ (see 1.3).

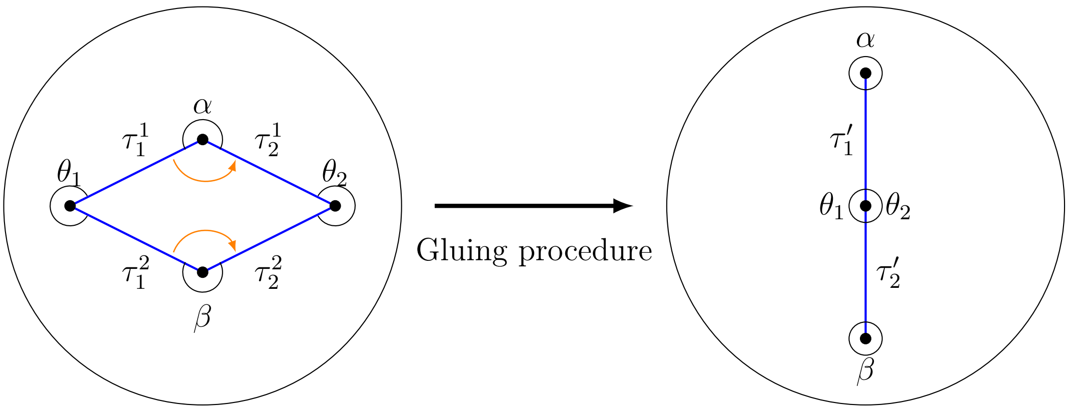


Figure 1.3: The gluing process produces a new branched hyperbolic structure. The angles $\alpha$ and $\beta$ are both multiples of $2\pi$. In particular if $\tau_1(0) = \tau_2(0)$ was a simple cone point, then $\alpha = \beta = 2\pi$. Similarly the angles $\theta_1$ and $\theta_2$ may be different from $2\pi$, but they are also integer multiples of $2\pi$. Hence, the extremal points $\tau_1(1)$ and $\tau_2(1)$ collapse to a cone point of angle $\theta_1 + \theta_2$.

The result of this operation is a closed surface of the same type of $S$, but with a new branched hyperbolic structure, namely $\sigma'$. We will say that $\sigma'$ is a *local deformation* of $\sigma$.

**Definition 1.4.21.** Let $\sigma$ be a branched hyperbolic structure on $S$ and $\sigma'$ be a local deformation of $\sigma$. We will say that $\sigma$ and $\sigma'$ are *connected by a movement of cone points.*

Since this surgery takes place in a small simply connected neighborhood of the cone point the holonomy of the structure is preserved. The new structure $\sigma'$ is globally different from $\sigma$ because the developing map is change, and *one* developed image of the cone point $p'$ for the structure $\sigma'$ is $\widehat{\tau}(1)$. All other developed images of $p'$ are given by the images of $\widehat{\tau}(1)$ via the holonomy of the structure.

**The holonomy fibre.** We have seen in the previous paragraph that the map $\mathfrak{hol}$ is non-injective since every local deformation produces a new branched hyperbolic structure with the same holonomy. Therefore the fibre over a geometrizable representation $\rho$ is a moduli space defined as follow.

$$\mathcal{H}_{k,\rho}(S) = \left\{ \begin{array}{c} \textsf{hyperbolic structures with total} \\ \textsf{branching order } k \textsf{ and holonomy } \rho \end{array} \right\} \Big/ \sim_{\textsf{iso}}.$$

Let $\sigma$ be a branched hyperbolic structure on $S$, and let $\textsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ its developing map. Post-composing the developing map $\textsf{dev}$ with natural inclusion $i : \mathbb{H}^2 \hookrightarrow \mathbb{CP}^1$ and post-composing the holonomy $\rho$ with the inclusion $\mathrm{PSL}_2\mathbb{R} \hookrightarrow \mathrm{PSL}_2\mathbb{C}$; a branched hyperbolic structure may be seen also as *branched projective structure*; that is a structure locally modelled on the Riemann sphere and such that transition functions are restriction of elements in $\mathrm{PSL}_2\mathbb{C}$. Denote with $\mathcal{M}_{k,\rho}$ the moduli space of branched projective structures on $S$; which may be defined in the same way of $\mathcal{H}_{k,\rho}$. As soon as the representations $\rho$ was not elementary, we have the following result established in [5].

**Theorem 1.4.22.** *$\mathcal{M}_{k,\rho}(S)$ is a complex manifold of dimension $k$.*

However there are branched projective structures with holonomy$\rho$, which are not branched hyperbolic structure (see for instance [5]); hence $\mathcal{H}_{k,\rho}$ is just a subset of $\mathcal{M}_{k,\rho}$.

**Lemma 1.4.23.** *The moduli space $\mathcal{H}_{k,\rho}$ is an open subset of $\mathcal{M}_{k,\rho}$. In particular $\mathcal{H}_{k,\rho}(S)$ turns out to be a complex manifold of the same dimension $k$.*

*Proof.* Let $\sigma$ be a branched hyperbolic structure on $S$ with holonomy $\rho$. Moving any cone points of $\sigma$ in any possible direction we produce a neighborhood of $\sigma$ in $\mathcal{M}_{k,\rho}$. Since the developing map for branched hyperbolic structures is stable under movements of cone points, *i.e* the developed image does not change for local deformations; the conclusion follows. □

Actually we do not know if $\mathcal{H}_{k,\rho}(S)$ is the union of some connected components of $\mathcal{M}_{k,\rho}(S)$. What we know so far is that $\mathcal{H}_{k,\rho}(S)$ is locally path-connected, indeed when we say that two structures $\sigma_1$ and $\sigma_2$ are connected by a movement of cone points, the word "connected" is not casual. Using the same notation above, $\widehat{\tau} : [0,1] \longrightarrow \mathbb{H}^2$ is a parametrization of the developed image of the geodesics twin paths $\tau_1, \tau_2$. For any $t \in [0,1]$ the lifts of the segment parametrized by the interval $[0,t]$ are again a couple of geodesics twin paths, because they lie in $\tau_1, \tau_2$. Cut and paste along these path we get a new structure $\sigma_t$. Following $t \in [0,1]$ we obtain a path of branched hyperbolic structures in $\mathcal{H}_{k,\rho}(S)$ joining the structures $\sigma_0$ and $\sigma$.

*Remark* 1.4.24. Even if any cone point $p$ can be moved locally around a small neighborhood of $p$; a priori there is no guarantee that one can move a cone point along any given path. The reason is that for any embedded geodesic segment $\tau$ starting from $p$ its twin could be a self-intersect geodesic or something worse, or could be embedded but may intersect $\tau$ at some regular point $q$. Hence finding a couple of geodesic twin paths is very hard.
Even worse there are movements that are not allowed even when a couple of embedded twin paths can be found. This is due to the presence of elliptic elements in the image of the representation $\rho$. Indeed if we move a cone point $p$ to another point $q$ such that one of its developed image (and then all) is the fixed point for an elliptic in $\Gamma = \rho(\pi_1 S)$ the structure $\sigma$ degenerates to a nodal curve. The same problem occurs if, moving $p$ to another point $r$, the developed image of the geodesic twins contain a fixed point for an elliptic. This problem can be easily solved moving $p$ to $r$ avoiding the fixed points, indeed under some circumstances, a couple of embedded twin paths may be found.

We conclude with a concrete example of the second pathology describe in the previous remark. We construct explicitly a branched hyperbolic structure on a surface of genus 2 with a closed separating curve with elliptic holonomy. After that, we show the existence of movements that produce a nodal curve.

**Example 1.4.25.** Let $T$ be the one-holed torus, $\langle \alpha, \beta \rangle$ be a basis for the fundamental group $\pi_1 T$ and let $\rho : \pi_1 T \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation such that $tr([g,h]) = 0$, where $g = \rho(\alpha), h = \rho(\beta)$. Then $\rho$ is the holonomy of a hyperbolic cone-manifold structure on $T$ with geodesic boundary $\gamma$, no cone points inside $T$ and exactly one cone point of angle $2\pi$ on the boundary [20, Theorem 1.1, Proposition 5.3]. In short to see this, let $r = \mathsf{Fix}[g,h]$ and let $p$ be a point sufficiency near to $r$. The segments

$$p \to g^{-1}h^{-1}ghp \to hp \to ghp \to h^{-1}ghp \to p$$

gives a non-degenerate pentagon, which is the fundamental domain for the action of $\rho$. *Different choice of p produce different non-isometric pentagons.* Let $T_1$ and $T_2$ be

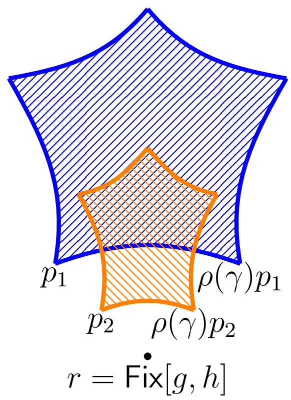


Figure 1.4: When the point $p$ approches to the fix point of $\rho(\gamma)$ the blue pentagon degenerates to the orange quadrilateral.

two copies of the same structure defined above with and glue them along the geodesic boundary identifying the cone-points. We get a branched hyperbolic structure $\sigma$ on a closed surface $S$ of genus 2, with one cone point of angle $4\pi$. Different choice of $p$ get different branched hyperbolic structures on $S$. When $p$ approches to $r$ the structure $\sigma$ degenerates to a nodal curve because the length of $\gamma$ becomes shorter and shorter (see the picture to understand what happen to the fundamental domains for $T_1$ and $T_2$). In particular, when $p = r$ the segment $p \to g^{-1}h^{-1}ghp$ is just a point and the pentagon is infact a quadrilateral and $S$ is obatined by gluing two tori with the same orbifold structure at their orbifold point.

## 1.5 Examples

We conclude this chapter with an entire section devotes to examples of this kind of structures. In the first and the second example, we show how to obtain a branched hyperbolic structure on a closed surface $S$ by gluing the sides of a regular polygon.

**Example 1.5.1.** Let $S$ be the surface of genus $g$ by gluing the sides of a $4g$-gon with the usual labelling $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$. In the hyperbolic plane, there are infinitely many regular $4g-$gons, the angles on each vertex have the same value strictly between 0 and the Euclidean one $\frac{2g\pi}{2g+1}$. Therefore we obtain hyperbolic cone-structures with one cone point of any angle strictly between 0 and $(4g-2)\pi$; in particular we obtain branched hyperbolic structures with one cone point of angle $2k\pi$ for any integer $1 \le k \le 2g-2$, i.e. branched hyperbolic structures with one cone point of order $k$ for any integer $0 \le k \le 2g-3$. For instance in the case of $g=2$ we have the unbranched hyperbolic structure coming from the regular octagon of angles $\frac{\pi}{4}$, and a branched hyperbolic structure with a cone point $4\pi$ (i.e. a cone point of order $k=1$) coming from the right-angled octagon.

**Example 1.5.2.** Here we do a little variation of the previous example. We obtain $S$ by gluing opposite sides of a $(4g+2)$-gon. The vertices are grouped into two sets, hence the quotient has two special points. Precisely we have a $(4g+2)$-gon where any vertex has the same angle $0<\theta<\frac{2\pi g}{2g+1}$ and we get a cone hyperbolic structures with a pair of cones each having any angle $0<\alpha<2g\pi$. In particular, we obtain cone hyperbolic structures in genus $g$ with two cone points of angle $2k\pi$ for any integer $1\le k\le g-1$, i.e. branched hyperbolic structures with two cone points of order $k$ for any integer $0\le k\le g-2$. In this example, $g$ must be at least 3. Indeed if $g=2$ then $0<\alpha<4\pi$, hence we get a hyperbolic structure if and only if $\alpha=2\pi$, that is an unbranched structure. In the sequel, we justify this in a different way.

**Example 1.5.3.** Let $S$ be any surface of genus at least 2. Then any hyperbolic structure $\sigma$ on $S$ is a very particular example of a branched hyperbolic structure where every local chart is a branched cover of degree one, that is a homeomorphism.

**Example 1.5.4.** Let $S$ be a surface with a genuine hyperbolic structure $h$ and let $T$ be a topological surface of genus $g$. Let $f: T \longrightarrow S$ be a branched covering, then pulling back the structure $h$ we get a branched hyperbolic structure $\sigma$ on $T$. This geometric construction of $\sigma$ provides also an example of discrete but not faithful representation $\rho$ that come from a branched hyperbolic structure.

There are many other structures that are not described in the example 1.5.4, for instance, we can never define a branched hyperbolic structure on a surface of genus three using a branched cover on a surface of genus 2 because they do not exist. However, branched structures on surfaces of genus 3 exist as shown in the following example.

**Example 1.5.5.** Let $\Sigma_1$ and $\Sigma_2$ be two topological surfaces having a hyperbolic metric $\sigma_1$ and $\sigma_2$ each one with a cone-singularity $p_1, p_2$ of angles $\theta_1$, $\theta_2$ respectively and such that $\theta_1 + \theta_2 = 2\pi$. Let now $\gamma_1$, $\gamma_2$ be two geodesics path of the same length starting from $p_1, p_2$ respectively. Cutting each surface along its slit we produce two surfaces with one boundary component $b_1$ and $b_2$ and a corner point of angle $\theta_1$ and $\theta_2$ respectively. We cut the surfaces along these path producing two slits and then glue them isometrically identifying the cone-points.

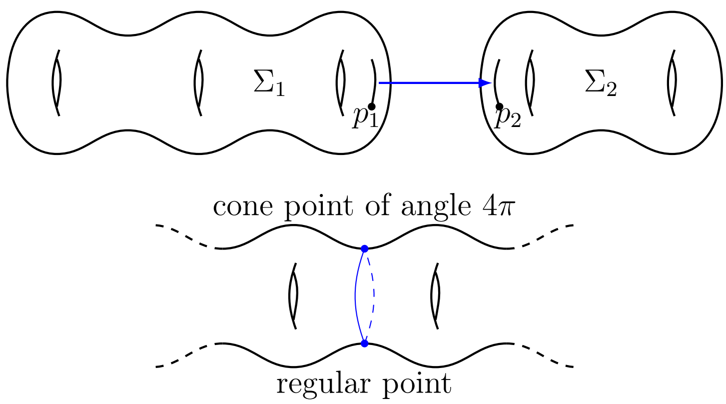


Figure 1.5: We cut the surfaces along their slits and then weglue them isometrically identifying the cone-points.

Gluing these isometrically along their boundaries identifying the corner points we get a new surface with exactly one cone point of angle $4\pi$ and the holonomy of the blue curve in the picture above is an elliptic transformation in $\mathrm{PSL}_2\mathbb{R}$, precisely it is a hyperbolic rotation of angle $\theta_1$. In particular the image is not discrete if $\theta_1$, and then $\theta_2$, are irrationals modulo $2\pi$.

The next examples are less geometrical. In the following one, we produce a large family of non-isomorphic branched hyperbolic structures having the same holonomy.

**Example 1.5.6.** Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a holonomy representation. Let $\mathcal{F}_\rho$ be the $\mathbb{H}^2$-bundle over $S$ with holonomy $\rho$, that is the quotient of $\widetilde{S} \times \mathbb{H}^2$ by the action of the fundamental group, where $\pi_1 S$ acts on $\widetilde{S}$ by deck transformations, and on $\mathbb{H}^2$ by isometries via the representation $\rho$. This product is a foliation whose leaves are $\widetilde{S} \times \{z\}$ for every $z \in \mathbb{H}^2$, and this foliation descend to a foliation on $\mathcal{F}_\rho$. A branched hyperbolic structure $\sigma$ on $S$ is defined by a transverse section $s$ to this foliation, infact its lift defines a $\pi_1 S$-equivariant section $\widetilde{s} : \widetilde{S} \longrightarrow \widetilde{S} \times \mathbb{H}^2$ and the projection to the second factor gives us a map $\widetilde{S} \to \mathbb{H}^2$ equivariant with respect to $\rho$; thus a hyperbolic structure (possibily branched) on $S$. If the section $s$ is transverse to any leaf $\widetilde{S} \times \{z\}$ then the structure is hyperbolic without cone points. Otherwise if $s$ tangent to some leaf, the tangency point projects to cone point on $S$, and its order is just the order of tangency.

**Example 1.5.7.** In [28, Theorem A], the author gives a sufficient condition for the existence of a branched hyperbolic structure. Precisely let $S$ be a closed surface. Let $p_1, \ldots, p_n$ be points of $S$ and $\theta_1, \ldots, \theta_n \in 2\pi\mathbb{Z}^+$ be positive real numbers. Assume that

$$2\pi\chi(S) + \sum_{i=1}^{n} (\theta_i - 2\pi) < 0,$$

and $\theta_i \neq 2\pi$ for some $i$. Then there exists a unique branched hyperbolic structures on $S$ having at $p_i$ a conical singularity of angle $\theta_i$. In the next chapter we will see that this condition is also necessary.

# CHAPTER 2

## Geometry of branched hyperbolic structures

In this chapter, we are interested in studying the branched hyperbolic structure from the topological and metrical point of view. As above $S$ is any orientable surface possibly with boundary and punctures. However from now on punctures correspond to cusps or funnels, that is the holonomy around any puncture is neither elliptic or trivial.

### 2.1 Topological constraints

Let $S$ be a compact, connected and oriented surface. By a classical theorem of geometry and topology, an oriented surface $S$ admits a hyperbolic structure in the sense of definition 1.4.2 if and only if its Euler characteristic $\chi(S)$ is negative (see for instance [2, Theorem B.3.5],[26]). We recall that for any compact surface $S$, the Euler characteristic may be computed by the well-known formula $\chi(S) = 2-2g-b$, where $g$ denotes the genus of $S$ and $b$ the number of boundary components of $S$. In this work we are mainly interested in branched hyperbolic structures, that is hyperbolic structures where cone points are allowed. Cone points can be considered as points on which the curvature is concentrated. As we will see in 2.1.4 the topology of the surface imposes limits on the curvature concentrated on them.

**Definition 2.1.1.** Let $S$ be a compact, connected and oriented surface. A *hyperbolic triangulation* $\tau$ on $S$ is the datum of a triangulation of $S$ and a metric, such that

1 the link of each simplex is piecewise linear homeomorphic to a circle, and

2 the restriction of the metric to each simplex is isometric to a geodesic simplex in hyperbolic space.

We have the following result.

**Lemma 2.1.2.** *Let $S$ be a triangulated surface endowed with a (possibly branched) hyperbolic structure $\sigma$. Then $\sigma$ induces a hyperbolic triangulation $\tau$ on $S$. In particular if $\sigma$ is branched, we may suppose that all cone points are vertices of $\tau$.*

Conversely, a hyperbolic triangulation $\tau$ on $S$ does not define a branched hyperbolic structure in general. However, if the magnitude of any cone point is $2k\pi$, any hyperbolic triangulation defines a maximal atlas whose charts (possibly branched) take values on $\mathbb{H}^2$, and transition functions are restrictions of global isometries of the hyperbolic plane.

*Remark* 2.1.3. The trianguation $\tau$ makes $S$ a simplical complex of dimension two.

Let $\sigma$ be a branched hyperbolic structure on $S$. By 2.1.2; the structure $\sigma$ induces a hyperbolic triangulation $\tau$ on $S$. As we have shown before any cone point $p$ for $\sigma$ may be considered as a vertex of $\tau$, and around it, there are finitely many triangles because any local charts containing $p$ has a finite degree. Note that the sum of angles insisting on $p$ is exactly $2k\pi$ for some positive integer $k$.

**Proposition 2.1.4** (Gauß-Bonnet condition)**.** *Let $S$ be a compact orientable surface (possibly with geodesic boundary). Any branched hyperbolic structure $\sigma$ on $S$ with cone points of order $k_p$ satisfies the following relation*

$$\chi(S) + \sum_{p\in S} k_p < 0, \quad \textit{where } k_p = \mathsf{ord}(p).$$

*Indeed the left hand side is $2\pi$ times the opposite of the hyperbolic area of $S$.*

Notice that the sum above is well-defined because there are finitely many cone points on $S$. Moreover, the relation gives a bound on the orders of the cone points; roughly speaking we can not have too much curvature concentrated on a single point and if we want more than once, then we need to share it on each one.

*Proof.* Suppose first $S$ is closed. We write the left hand side of the relation above in a different way so that it becomes

$$2\pi\chi(S) - \sum_{q\in S} 2\pi - \theta_q < 0.$$

Let $\tau$ be a triangulation of $S$ induced by the lemma 2.1.2, such that any cone point $p_i$ for $\sigma$ is also vertex for $\tau$. For every vertex $q$ the angle around it could be:

$$\theta_q = \begin{cases} 2\pi & \text{if } q \text{ is regular,} \\ \theta_i & \text{if } q = p_i \text{ for some } i. \end{cases}$$

The Euler characteristic of $S$ can be computed by the well-known formula $\chi(S) = V - E + F$, where $V, E, F$ are the numbers of vertices, edges and faces respectively. Since $\tau$ is a triangulation $2E = 3F$, thus the formula becomes $2\chi(S) = 2V - F$. From the fact that every triangle is hyperbolic we deduce that $\pi F > \sum \theta_q$, because the area of a hyperbolic triangle with angles $\alpha, \beta, \gamma$ is $\pi - \alpha - \beta - \gamma$. This imply that

$$2\pi\chi(S) = 2\pi V - \pi F < 2\pi V - \sum_{q \in S} \theta_q = \sum_{q \in S} 2\pi - \theta_q.$$

If $S$ has geodesic boundary we doubling $S$ (where corner points are identified) to get a closed surface $S'$. Notice that the previous argument applies word-by-word to $S'$ even if some point is neither regular or cone point of angle $2k\pi$ for some $k$. Hence

$$2\pi\chi(S') - \sum_{q \in S'} 2\pi - \theta_q < 0.$$

By symmetry we get the desired result. $\square$

Let $S$ be any compact surface. By the Gauß-Bonnet condition, the number of cone points is bounded from above. A straightforward computation shows that the maximal number of simple cone points is $2g - 3$ where $g$ is the genus of $S$. We will say that a branched hyperbolic structure $\sigma$ has *maximal branching order* if it attains this upper bound, *i.e.* $2g - 3 = -\chi(S) - 1$.

## 2.2 Metric properties

This section is devoted to study the branched hyperbolic structures on surfaces as metric space. Even if around any sufficiently small open neighborhood of a regular point they are similar to a genuine hyperbolic structure, cone points make them quite different in general. The main reference for this section is [4].

For the sake of reading we recall the basic notions we use in this section about metric spaces. Let $(X, d)$ be a metric space, the length of a curve $\gamma : [0, 1] \longrightarrow X$ is given by

$$l(\gamma) = \sup_{0 = t_0 \le \cdots \le t_k = 1} \sum_{i=0}^{k-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big)$$

where the supremum is taken amoung all possible partitions of $[0, 1]$. Curves with finite length are called *rectifiable*. A curve $\gamma$ is called *local geodesic* if $d\big(\gamma(t), \gamma(s)\big) = |t - s|$ for any $t, s \in [0, 1]$ sufficiently closed. In particular it is called *geodesic* is such condition holds for any couples of times $t, s$. A metric space $(X, d)$ in which every couple of points are joined by a geodesic is called *geodesic space*. Finally a metric space $(X, d)$ is called *length space* if the distance between any couple of points is equal to the infimum of length of rectifiable curves joining them.

### 2.2.1 Completeness

Any geometric structure $\sigma$ on $S$ may be complete in two different ways. Any structure $\sigma$ induces a distance $d_\sigma$ that makes $(S, d_\sigma)$ a metric space. It is natural to ask if such metric space is complete or not. On the other hand, a geometric structure may be also complete in Thurston sense, and this happens when the developing map is a covering map.

*Remark* 2.2.1. In this thesis $\sigma$ is always a (possibly branched) hyperbolic structure, however, the previous discussion holds for other types of geometry.

**Completeness as metric space.** Let $\sigma$ be any branched hyperbolic structure on $S$ and $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ its developing map. Let $ds^2_P$ be the Poincaré metric on the model space $\mathbb{H}^2$. The pullback metric $ds^2_\sigma$ via $\mathsf{dev}$ is a Riemannian metric which is singular around cone points on $S$. Thus we may define a metric $d_\sigma$ by letting $d_\sigma(x, y)$ be the infimum of Riemannian length of all piecewise differentiable path between $x$ and $y$.
Here we use an alternative approach suggested by Mathews in [19]. By 2.1.2 $S$ can be triangulated by geodesic $2-$simplexes. Each simplex inherits a metric from the hyperbolic space and the idea is to combine all of these simplexes to define a pseudometric $d_\sigma$ that turns out to be a global distance on $S$. In order to do this, we introduce the *m-strings*.

**Definition 2.2.2.** Let $x, y$ be two points of $S$. A $m-$string in $S$ from $x$ to $y$ is a sequence $\zeta = (x_0, ..., x_m)$ of points of $S$ such that $x_0 = x$ and $x_m = y$ and for each $i = 0, ..., m-1$ there exists a simplex $s_i$ containing $x_i$ and $x_{i+1}$. $m$ is defined as *size* of the sequence, and define the length of $\zeta$ to be

$$l(\zeta) = \sum_{i=0}^{m-1} d_{s_i}\big(x_i, x_{i+1}\big)$$

where $d_{s_i}$ is the hyperbolic distance on $s_i$ (note that $s_i$ may be a face of a $2-$simplex of the triangulation $\tau$).

Every $m-$strings determines a path in $S$, given by concatenation of segments. Every segment lies in a particular simplex $s_i$ and it is geodesic with respect to the induced distance on $s_i$ (remember that each $2-$simplex is isometric to a geodesic triangle in $\mathbb{H}^2$).
The *intrinsic pseudometric* $d_\sigma$ is defined as

$$d_\sigma(x, y) = \inf\{l(\zeta) : \zeta \text{ is a string from } x \text{ to } y\}$$

and if there is no string from $x$ to $y$, then $d_\sigma(x, y) = \infty$.

**Lemma 2.2.3.** *Every pair of points $x, y \in S$ can be joined by a $m-$string for some $m$.*

*Proof.* Let $\gamma$ be any path between $x$ and $y$ since $S$ is connected it is also path connected hence such path exists. Now take a judicious partition of $\gamma$ such that any pair of nodes lies in the same simplex $s$. □

The intrinsic pseudometric will not be a metric in general (see for instance [4, Chapter I.7, example 7.7]) and we would need a further condition to ensure that $d_\sigma$ is a metric. However under our conditions $d_\sigma$ is a metric and its restriction to any simplex $s$ coincides with $d_s$.

**Lemma 2.2.4.** *The pseudometric $d_\sigma$ is a distance and it coincide with $d_s$ for any simplex $s$. In particular $(S, d_\sigma)$ is a length space.*

*Proof.* Let $x, y$ be any pair of points in $S$ and let $\zeta = (x, ..., y)$ be any string between them. If $x \in s_i$ and $y \in s_j$ for some $i, j$ such that $i \neq j$, then $d_\sigma(x, y)$ is strictly positive because sum of strictly positive real numbers. On the other hand if $x, y$ belong to the same simplex $s$, then $d_\sigma(x, y) = d_s(x, y) > 0$, becasue $d_s$ on the simplex $s$ is induced by the hyperbolic distance. Hence $d_\sigma(x, y) > 0$ for any pair of distinct points $x, y$. For the second part notice that any string is a rectifiable curve. Since the distance between any pair of points is defined as the infimum of rectifiable curves joining them, $(S, d_\sigma)$ is also a length space. □

**Lemma 2.2.5.** *The length space $(S, d_\sigma)$ is complete.*

In the compact case the result follows immediately because any compact metric space is totally bounded and complete. In particular, any Cauchy sequence converges in $S$. Even if the result is false for non-compact metric spaces (see [4] for counterexamples), in our case it holds for any surface $S$.

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be any Cauchy sequence in $S$. By definition for any $\varepsilon > 0$ there exists a $n_\varepsilon \in \mathbb{N}$ such that $d_\sigma(x_n, x_m) < \varepsilon$ for any $n, m > n_\varepsilon$, and we can choose it such that the open neighbourhood $B_{2\varepsilon}(x_{n_\varepsilon+1})$ is a simply connected open ball containing all terms of the sequence $(x_{n_\varepsilon+m})_{m\in\mathbb{N}}$. Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers converging to zero, then the open balls $B_{2\varepsilon_k}(x_{n_{\varepsilon_k}+1})$ form a nested sequence of non-empty sets in $S$. We claim that the following holds: there is a point $p$ such that

$$p \in \bigcap_{k\in\mathbb{N}} \overline{B_{2\varepsilon_k}(x_{n_{\varepsilon_k}+1})}$$

In this case the sequence $x_n$ converges to $p$ (in particular the intersection is exactly $p$). Such intersection can not be empty, otherwise there exists $k \in \mathbb{N}$ such that every open ball $B_{2\varepsilon_k}(x_{n_{\varepsilon_k}+1})$ contains a puncture. On the other hand any puncture corresponds to cusp or funnel whose ideal boundary (possibly a point) lies at infinite distance from any point of $S$. Hence there is a point $p$ in the intersection; and since $d_\sigma(x_{n_{\varepsilon_k}}, p) < \varepsilon_k$, we see that $x_n \to p$. □

*Remark* 2.2.6. The initial enforcement of the holonomy around punctures in $S$ ensure that any branched hyperbolic structure is complete. Indeed without such assumption is possible to create non-complete structures. For instance, let $\sigma$ be any branched hyperbolic structure on a closed surface of genus at least 2. Let $\in S$ be any point and consider $S \setminus \{p\}$ with the structure induced by $\sigma$. The new structure is still branched hyperbolic but now is not complete.

Finally, a branched hyperbolic structure is a geodesic space by the Hopf-Rinow theorem. Precisely we have the following theorem.

**Theorem 2.2.7** (Hopf-Rinow). *Let $(X, d)$ be a complete, locally compact length space. Then every bounded subset of $X$ is compact. Moreover, for any pair of points in $X$, there exists a shortest geodesic between them, that is $X$ is a geodesic space.*

We refer to [4] for the proof. This theorem can be applied to our structures by previous results. Thus any branched hyperbolic structure $\sigma$ induces a distance $d_\sigma$ that makes $(S, d_\sigma)$ a complete geodesic space.

**The induced distance and geodesic behaviour.** The lack of previous description is that the local behaviour of geodesics when they pass through a cone point is not clear. In order to clarify this, following [4] and [19], we give an explicit description of an open neighborhood of a cone point. Let $\sigma$ be a branched hyperbolic structure on $S$, again by lemma 2.1.2, $\sigma$ induces a hyperbolic triangulation $\tau$ on $S$

on which any cone point is also a vertex of the triangulation. Let $p$ be any point of $S$ and $B$ =Lk$(p, S)$ the link of $p$. The hyperbolic cone of radious $r$ is defined as the quotient space: $\mathcal{C}(B, r) = B \times [0, r[\Big/ B \times \{0\}$ and the equivalence class $B \times \{0\}$ is called *vertex of the cone.* Such cone $\mathcal{C}(B, r)$ may be consider as an open neighborhood of $p$ where the vertex is identified with $p$. We write any element of the cone as $bt$ for simplicity, where $b \in B$ and $t \in [0, r[$.

Let $d_B$ be the induced distance on $B$ by $d_\sigma$, and let

$$d_B^\pi(x, y) = \min\{\pi, d_B(x, y)\}$$

be another distance defined by truncation. For any $q_1 = b_1t_1, q_2 = b_2t_2 \in \mathcal{C}(B, r)$ we define $d(q_1, q_2)$ to be the non-negative number such that

$$\cosh\big(d(q_1, q_2)\big) = \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos\big(d_B^\pi(b_1, b_2)\big).$$

As espected the distance $d$ is equivalent to $d_\sigma$ on the open cone.

**Lemma 2.2.8.** *On the open cone $\mathcal{C}(B, r)$ the distance $d$ coincide with $d_\sigma$, that is $d = d_\sigma|_{\mathcal{C}(B,r)}$ on $\mathcal{C}(B, r)$.*

*Proof.* For points $q_1, q_2$ separated by sufficiently small angles, the metric is just that of hyperbolic geometry, given by the hyperbolic cosine rule. If they belong to the same simplex $s$ the distances between these point coincide, because $d_\sigma$ coincide with the hyperbolic metric on any simplex. In particular the distances $d$ and $d_s$ coincide on any simplex $s$, thus $d = d_\sigma|_{\mathcal{C}(B,r)}$. □

*Remark* 2.2.9. If $p$ is a regular point in the interior of $S$, the hyperbolic cone $\mathcal{C}(B, r)$ is isometric to an open ball in $\mathbb{H}^2$.

Let $\gamma$ be any geodesic passing through the vertex $p$ and consider the following reparametrization

$$\gamma : ] - r, r[ \longrightarrow \mathcal{C}(B, r)$$

such that $\gamma(t) = b_1t$ if $t < 0$ and $\gamma(t) = b_2t$ if $t > 0$. Let $\varepsilon$ be a positive real number sufficiently small, then we have

$$\cosh d(\gamma(-\varepsilon), \gamma(\varepsilon)) = \cosh^2 \varepsilon - \sinh^2 \varepsilon \cos\big(d_B^\pi(b_1, b_2)\big)$$

On the other hand if $\gamma$ is a geodesic, then the following holds $\cosh d(\gamma(-\varepsilon), \gamma(\varepsilon)) = 2\varepsilon$. Hence $d_B^\pi(b_1, b_2) = \pi$ necessarily, that is $d_B(b_1, b_2) \geq \pi$. Thus any geodesic makes an angle of at least $\pi$ at $p$. This means that if $p$ is a regular point such angle is exactly $\pi$ and there is only one direction on which $\gamma$ can continue. When $p$ is a cone point the situation change completely because any geodesic passing through $p$ can continue in infinitely many directions.

*Remark* 2.2.10. The same consideration holds also for a hyperbolic structure with cone points of angle less than $2\pi$. In those cases, a geodesic path avoids the cone point.

**Completeness in Thurston sense.** Even if branched structures are metrically complete, they are not in Thurston sense. Indeed the developing map of a branched hyperbolic structure fails to be a global diffeomorphism because it has some critical points corresponding to cone points for the $\widetilde{\sigma}$ on $\widetilde{S}$. Let $\sigma$ be any structure on $S$ with developing map $\mathsf{dev}$ and holonomy $\rho$. The following characterization is well-known in literature.

$$\text{completeness} \iff \mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2 \text{ is diffeomorphism}$$

$$\iff \rho(\pi_1 S) \circlearrowleft \mathbb{H}^2 \text{ is free and properly discontinuous}$$

$$\iff \mathsf{dev} \text{ induces } S \cong \widetilde{S}\big/\pi_1 S \xrightarrow{\sim} \rho(\pi_1 S)\big\backslash \mathbb{H}^2.$$

As a consequence the action of $\pi_1 S$ is not properly discontinuous in general, because the holonomy representation may send closed loops (simple or not) to elliptic elements. Even worse a *fundamental domain* for the action of the fundamental group does not exist and then the developing map is an overlapping tessellation of $\mathbb{H}^2$. In particular these structures are never uniformizable in the sense that there not exists an open domain $\Omega \subseteq \mathbb{H}^2$ such that $\sigma \cong \rho(\pi_1 S)\big\backslash \Omega$.

## 2.2.2 Geometric features

In this paragraph, we introduce many geometric features about branched hyperbolic structures. This section does not to be intended original in the following sense. Results without proof are well-known facts in literature and a good reference for them is [4]. Regarding the other results reported, I believe that they are already known, even if I did not find a reference for them in literature. Hence for the sake of completeness for each one of them I give a personal proof.

Let now $\pi : \widetilde{S} \longrightarrow S$ be the covering map. The singular metric $ds^2_\sigma$ on $S$ can be lifted to a singular metric on $\widetilde{S}$ that makes the universal cover a metric space as well. Using the same argument of [5, Lemma 3.6] we are able to show that the universal cover $\widetilde{S}$ is a CAT$(-1)$-space. It follows immediately that $S$ is a locally CAT$(-1)$-space. Precisely we have the following.

**Proposition 2.2.11.** *Let $\sigma$ be a branched hyperbolic structure on $S$. Then $\widetilde{S}$ is a* CAT$(-1)$ *spaces. In particular its geometric boundary of $\widetilde{S}$ is an oriented circle.*

*Proof.* Let $ds^2_{\widetilde{\sigma}}$ on $\widetilde{S}$ be the lift of the singular metric $ds^2_\sigma$ on $S$. Equivalently this metric may be defined pulling back the Poincaré metric $ds^2_P$ via the developing map. This metric is smooth with curvature $-1$ away from the cone points. Since all singularities have angles greater than $2\pi$, the metric is hyperbolic everywhere and it can be approximated by smooth metrics of curvature less than $-1$, thus $\widetilde{S}$ is a CAT$(-1)$-space. Since $ds^2_\sigma$ is smooth outside a sufficiently small closed neighborhood of cone points, there exists a smooth metric $ds^2$ with curvature less than $-1$ on $S$ that agrees with $ds^2_\sigma$ outside such neighborhood and let $d$ be the distance induced by $ds^2$. Then the identity map between $(\widetilde{S}, d_{\widetilde{\sigma}})$ and $(\widetilde{S}, d)$ is a quasi-isometry and hence they have the same boundary, that is a circle. $\square$

From the previous proposition, we deduce some interesting consequences. The first one regards the developing map.

**Proposition 2.2.12.** *Let $S$ be a surface endowed with a branched hyperbolic structure $\sigma$. Then the developing map* $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ *is uniformly open, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that $B_\delta(\mathsf{dev}(\widetilde{q})) \subseteq \mathsf{dev}(B_\varepsilon(\widetilde{q}))$ for any $q \in S$. Moreover the developing map* $\mathsf{dev}$ *is surjective.*

In order to prove such proposition, we may suppose that all cone points are simple by the following lemma.

**Lemma 2.2.13** (Splitting lemma)**.** *Let $S$ be a closed surface endowed with a branched hyperbolic structure $\sigma$. Then there exists a local deformation $\sigma'$ of $\sigma$ such that all cone points are simple.*

*Proof of Lemma 2.2.13.* If all cone points of $\sigma$ are simple there is nothing to prove and $\sigma = \sigma'$. Otherwise let $p$ a cone point for $\sigma$ of order $k \geq 2$. Let $\tau_1$, $\tau_2$ be a couple of embedded twin paths defined on $[0, 1]$ (we remember that a couple of embedded twin path always exists see 1.4.24). Suppose $\tau_i(p)$ is a regular point for $i = 1, 2$.

We denote by $\theta_1$ and $\theta_2$ the angles between them at $p$. Of course both $\theta_1, \theta_2$ are multiples of $2\pi$. Performing the cut and paste procedure the point $p$ splits in two cone points whose angles around them are exactly $\theta_1$ and $\theta_2$ and the points $\tau_1(1)$ and $\tau_2(1)$ collapse to a simple cone point. A recursive argument produce a branched hyperbolic structure whose cone points are all simple. $\square$

Hence we can reduce the proof of proposition 2.2.12 to only those structure with only simple cone points. Indeed by the splitting lemma 2.2.13, any structure $\sigma$ may be deformed to a branched hyperbolic structure $\sigma'$ whose cone points are simple. Since the developed image is invariant for local deformations, then $\mathsf{dev}_\sigma$ is surjective if and only if $\mathsf{dev}_{\sigma'}$ is.

*Proof of Proposition 2.2.12.* Let $p_1, \dots, p_n$ be the cone points for $\sigma$ and define $2\varepsilon = \min\{d_\sigma(p_i, p_j)\}$. If $\widetilde{q}$ is a regular point and $B_\varepsilon(\widetilde{q})$ does not contain cone points, the claim follows because the developing map restricted to it is just an isometry. Hence suppose that $B_\varepsilon(\widetilde{q})$ contains a cone point $\widetilde{p}$. By assumption $\widetilde{p}$ is simple and $B_\varepsilon(\widetilde{q})$ does not contain any cone points other than $\widetilde{p}$. Let be $r$ the geodesic ray between $\widetilde{p}$ and $\widetilde{q}$ and let $\tau_1$ and $\tau_2$ be geodesic twin rays from $\widetilde{p}$ until the boundary of $B_\varepsilon(\widetilde{q})$ such that both of them form a straight angle with $r$. Note that $\tau_1 \cap \tau_2 = \{\widetilde{p}\}$ because $B_\varepsilon(\widetilde{q})$ is simply connected and then they should bound a disk in the hyperbolic plane. Hence $\tau_1 \cup \tau_2$ divide the open ball in two open sets and the developing map restricted to the one containing $r$ is a local isometry and its image contains $B_\varepsilon(\mathsf{dev}(\widetilde{q}))$. Then $B_\varepsilon(\mathsf{dev}(\widetilde{q})) \subseteq \mathsf{dev}(B_\varepsilon(\widetilde{q}))$ for any $\widetilde{q} \in \widetilde{S}$.
Suppose now that the developing map is not surjective and let $\delta > 0$ be a fixed real number such that $0 < \delta < \varepsilon/2$. Let $\widetilde{q}$ be any point at distance $\delta$ from the boundary $\partial\mathsf{dev}[\widetilde{S}]$. Then $\mathsf{dev}(B_\varepsilon(\widetilde{q})) \subset B_\varepsilon(\mathsf{dev}(\widetilde{q}))$ hence a contradiction. $\square$

Other nice geometric features are listed in the following lemmata.

**Lemma 2.2.14.** *For any path in $S$, there exists a unique geodesic in the homotopy class with fixed end-points. Moreover, for any closed path which is not null-homotopic, there exists a unique geodesic closed path in the free homotopy class.*

**Corollary 2.2.15.** *Let $S$ be a closed surface endowed with a branched hyperbolic structure $\sigma$. Then $(S, d_\sigma)$ is a locally uniquely geodesic space; i.e. for any point $p$ there exists a positive real number $r$ such that any $q \in B_r(p)$, can be joined to $p$ with a unique geodesic. In particular the injectivity radious* $\mathsf{injrad}(\sigma)$ *of $S$ is strictly positive.*

*Proof.* Let $p$ any point in $S$ and let $(U, \varphi)$ any chart centered at $p$. Up to shrink $U$ enough we can suppose it is simply connected and let $r$ be a positive real number such that $B_r(p)$ is contained in $U$. Let $q$ be any other point inside the ball $B_r(p)$, then any couple of paths between $p$ and $q$ are homotopic because $B_r(p)$ is simply connected, hence there exists a unique geodesic segment between them by the previuous lemma. For the second part we define the following function: $\mathsf{rad} : S \longrightarrow \mathbb{R}$ that associates to any point $p$ the real number $\sup\{r \in \mathbb{R} : B_r(p) \text{ is uniquely geodesic}\}$. This function is continuous and since $S$ is compact it has minimum. This is by definition the injectivity radious of $(S, d_\sigma)$, and from the first part this is not zero.

□

**Lemma 2.2.16.** *Geodesic in $\widetilde{S}$ are simple path. In particular, between any two points, there is a unique geodesic. Further, any couple of non-disjoint geodesics intersect each other either transversally, or in a geodesic segment having two cone point as end-points.*

Let $\gamma$ be a simple separating curve in $S$. It is better to point out that its geodesic representative in its free homotopy class may be not smooth in general. The following lemmata give a complete overview of the situation.

**Lemma 2.2.17.** *Let $S$ be a closed surface endowed with branched hyperbolic structure and let $\gamma$ be a simple separating curve on $S$ with non-trivial holonomy and let $\gamma_{\mathsf{geod}}$ be the geodesic representative in its free homotopy class. Suppose the total branching order is maximal, that is $2g-3$, then $\gamma_{\mathsf{geod}}$ pass through at least one cone point.*

*Proof.* Suppose $\gamma_{\mathsf{geod}}$ does not contain any cone point, then it is smooth and divide $S$ in two incompressible subsurfaces $C_1$ and $C_2$ of genus $h_1, h_2$ respectively and with one boundary. Of course $h_1 + h_2 = g$. The easier case is when there is only one cone point of order $2g-3$ (that is of maximal order), then a direct computation show that

$$\chi(C_1) + 2g - 3 = 2 - 2h_1 - 1 + 2g - 3 = 2(g - h_1 - 1) \geq 0$$

contradicting the Gauß-Bonnet condition. More generally if we have finitely many points, some of them belong to $C_1$ and the others (if any) belong to $C_2$. Denote with $k_1, k_2$ the total branching order in $C_1$, $C_2$ respectively. Now it easy to see that $\chi(C_1) + k_1 < 0$ if and only if $\chi(C_2) + k_2 > 0$. □

*Remark* 2.2.18. When the branching order is not maximal, the picture is quite different. It is possible to construct branched hyperbolic structures with simple separating curve whose geodesic representative is smooth, as shown in the example below. However, any curves whose geodesic representative is smooth (in any) have hyperbolic holonomy.

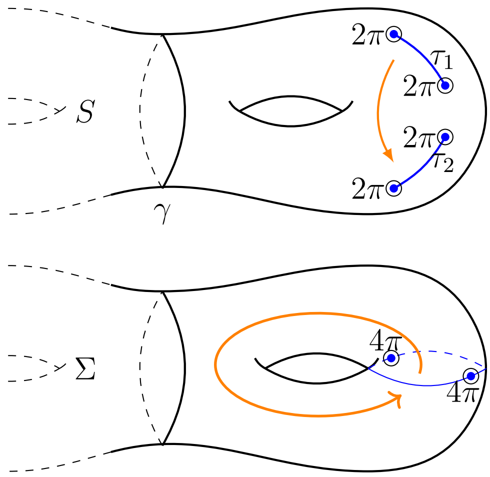


Figure 2.1: $S$ has genus at least 2 hence the total branch order on $\Sigma$ is not maximal. This can be check with a straighforward computation using the Gauß-Bonnet condition. Note that this example is very similar to 1.5.5.

**Example 2.2.19.** Let $S$ be a surface of genus $g$ with a complete hyperbolic structure and let $\gamma$ be a separating curve on $S$. Let $\tau_1, \tau_2$ be a geodesic segment in the same subsurface defined by $\gamma$ such that they have the same length and do not intersect $\gamma$. Cut them to obtain a new surface with the same genus and two boundaries, namely $\tau_1^1 \cup \tau_1^2$ and $\tau_2^1 \cup \tau_2^2$. Now we glue $\tau_1^i$ with $\tau_2^i$ for $i = 1, 2$ obtaining a new surface $\Sigma$ of genus $g+1$ and with two cone points of angle $4\pi$ (see picture 2.1). The branched structure on $\Sigma$ is the same of $S$ outside an open neighborhood of the new handle, in particular, $\gamma$ is still a simple separating geodesic smooth curve for the new structure.

**Lemma 2.2.20.** *Let $S$ be closed surface endowed with a branched hyperbolic structure and let $\gamma$ be a simple curve on $S$ with elliptic or parabolic holonomy and let $\gamma_{\mathsf{geod}}$ be the geodesic representative in its free homotopy class. Then $\gamma_{\mathsf{geod}}$ pass through at least one cone point.*

*Proof.* Suppose the converse, then $\gamma_{\mathsf{geod}}$ is a smooth geodesic on $S$, and let $\widetilde{\gamma_{\mathsf{geod}}}$ be a lift of $\gamma_{\mathsf{geod}}$ to $\widetilde{S}$. Then the developed image $\widehat{\gamma_{\mathsf{geod}}} = \mathsf{dev}(\widetilde{\gamma_{\mathsf{geod}}})$ is a smooth geodesic on $\mathbb{H}^2$ invariant for the action of $\rho(\gamma)$. Indeed the developed image is a priori a piecewise geodesic on the model space $\mathbb{H}^2$ because the developing map is local isometry outside the branch locus on $\widetilde{S}$. Hence, for every point $\widetilde{q} \in \widetilde{\gamma_{\mathsf{geod}}}$ the angle around it is always $2\pi$, because $\gamma_{\mathsf{geod}}$ never meet cone points, then the angle around any point $\mathsf{dev}(\widetilde{q})$ is always $2\pi$, that is $\widehat{\gamma_{\mathsf{geod}}}$ is a smooth geodesic on $\mathbb{H}^2$. Since the only transformation with an invariant geodesic are hyperbolic we get a contradiction. $\square$

# CHAPTER 3

## Euler class of representations

We are going to introduce the Euler class of a representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$, that is a cohomology class associated to any representation. In a previous subsection 1.4.2 we showed that any (possibly branched) hyperbolic structure defines a representation $\rho$ encoding the geometric data on $S$. Often the representation itself is not sufficient to recover the geometry on $S$, however, from the Euler class, we may extract some information. It will be interesting to see how the Euler class imposes conditions on the possible geometries on $S$, in particular, we will establish well-known results that will be used in the next chapters 4.

### 3.1 The character variety

We start with some general facts about the character variety. Let $S$ be any surface, the representation variety $\mathsf{Hom}(\pi_1 S, \mathrm{SL}_2\mathbb{R})$ is defined as the set of all homomorphisms $\rho : \pi_1 S \longrightarrow \mathrm{SL}_2\mathbb{R}$. Fix a presentation of $\pi_1 S$, then we may associate to any generator a matrix in $\mathrm{SL}_2\mathbb{R}$ such that the matrices satisfy the condition of any relators. Considering the entries of matrices as coordinates variables, the set $\mathsf{Hom}(\pi_1 S, \mathrm{SL}_2\mathbb{R})$ may be seen as the solution set of some polynomial equations, then it is a closed algebraic variety. In general, this variety has singularities. For a closed surface of genus $g \geq 2$ we have the following theorem.

**Theorem 3.1.1** (Goldman [12])**.** *Let $S$ be a closed surface of genus $g$ at least 2. Then the space $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ has $4g - 3$ connected components.*

To have another perspective of this space we may consider the representation set $\mathsf{Hom}(\pi_1 S, \mathrm{SL}_2\mathbb{R})$ of all homomorphisms $\rho : \pi_1 S \longrightarrow \mathrm{SL}_2\mathbb{R}$. As in the case of $\mathrm{PSL}_2\mathbb{R}$

it is a closed algebraic variety. In such space we can define the following equivalence relation

$$\rho_1 \sim \rho_2 \text{ if and only if } \pi \circ \rho_1 = \pi \circ \rho_2$$

where $\pi : \mathrm{SL}_2\mathbb{R} \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is the canonical projection. The quotient space is still the representation set $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$. The $\mathrm{PSL}_2\mathbb{R}-$*character variety* $\mathcal{X}(S)$ is defined as the GIT quotient $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})//\mathrm{PSL}_2\mathbb{R}$ by conjugation.
The quotient space has still $4g-3$ connected components, where two of them are diffeomorphic copies of the Teichmüller space of $S$. We call such components *extremal components* of $\mathcal{X}(S)$. The reason of such name will be explained later in 3.2, and we will refer to the other components as *non-extremal components*.

**Algebraic description of the character variety.** For a general surface (possibily with boundary), the *character* $\chi_\rho$ of a representation $\rho$ is the function defined as follows: $\mathrm{Tr}\circ\rho : \pi_1 S \longrightarrow \mathbb{R}$ given by $\mathrm{Tr}\circ\rho(\alpha) = \mathrm{Tr}\big(\rho(\alpha)\big)$. By using well-known trace relations, is possible to see that the function $\chi_\rho$ is determined by its values at only finitely many elements $\alpha_1, \dots, \alpha_n$ (see for instance [6]).
Then we may define a function $T$ :$\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R}) \longrightarrow \mathbb{R}^n$ that sends any representation $\rho \longmapsto \Big(\mathrm{Tr}\big(\rho(\alpha_1)\big), \dots, \mathrm{Tr}\big(\rho(\alpha_n)\big)\Big)$ and we may define the *character variety* to be the image of such function, that is $\mathcal{X}(S) = T\Big(\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})\Big)$. Away from the singularities the image of $T$ and the GIT quotient described above can be identified.

**Actions on the representation variety.** There is an action of $\mathrm{PSL}_2\mathbb{R}$ on the set of homomorphisms $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ by conjugation. The quotient space may be identified with the moduli space of isomorphism classes of flat principal $\mathrm{PSL}_2\mathbb{R}-$bundles over $S$. Similarly, $\mathrm{SL}_2\mathbb{R}$ acts on the set $\mathsf{Hom}(\pi_1 S, \mathrm{SL}_2\mathbb{R})$ by conjugation and the quotient set may be identified with the moduli space of isomorphism classes of flat principal $\mathrm{SL}_2\mathbb{R}-$bundles over $S$. Away from the singularities, this quotient space can be identified with the $\mathrm{SL}_2\mathbb{R}-$*character variety.*
There is also an action of $\pi_1 S$ on the representation and character variety from the right given by pre-composition, that is every element $\phi \in \mathrm{Aut}\ \pi_1 S$ acts on a representation $\rho \in \mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ to give $\rho \circ \phi$. This action clearly descends to the character variety. Here the action of the inner automorphism $\mathrm{Inn}\ \pi_1 S$ is trivial because traces are invariant under conjugation. Thus it makes sense to consider the action of the quotient group $\mathrm{Out}\ \pi_1 S = \dfrac{\mathrm{Aut}\ \pi_1 S}{\mathrm{Inn}\ \pi_1 S}$.
We need to consider these actions for the following reason. Fix a presentation of

$\pi_1 S$ and let $\rho \in \mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ be a geometrizable representation, that is $\rho$ is the holonomy of a possibly branched hyperbolic structure with developing map $\mathsf{dev}$. Take any element $g \in \mathrm{PSL}_2\mathbb{R}$, then $g \circ \mathsf{dev}$ is the developing map for another structure isomorphic to the first one and it is equivariant with respect to $g\rho g^{-1}$. If we want to consider isomorphic structures as the same structure on $S$, then we have to work with the conjugacy of $\rho$ which is an element of $\mathcal{X}(S)$.
On the other hand, the action of any element $\phi$ of $\mathsf{Aut}\ \pi_1 S$ have not any effect on the geometry on $S$, but it acts as change basis automorphism. The representation changes, but the geometry is still the same. In other words, there exists an orientation preserving isometry whose induced group isomorphism is $\phi$. Hence points in $\mathcal{X}(S)$ which are related under the action of Out $\pi_1 S$ have to consider the same in terms of underlying geometry.
When $S$ is a closed surface or the punctured torus the quotient group Out $\pi_1 S$ has a geometric interpretation summarize by the following theorem.

**Theorem 3.1.2.** *Let $S$ be a closed surface or the punctured torus there is an isomorphism*

$$\mathrm{Out}\ \pi_1 S \cong \frac{\mathrm{Aut}\ \pi_1 S}{\mathrm{Inn}\ \pi_1 S} \cong \frac{\mathrm{Homeo}(S)}{\mathrm{Homeo}_0(S)} = \mathrm{MCG}(S)$$

Here $\mathrm{MCG}(S)$ denotes the mapping class group of $S$. Indeed any homeomorphism of $S$ determines an automorphism of the fundamental group which is unique up to conjugation, that is an outer automorphism. On the other hand, by Dehn-Nielsen theorem, every automorphism of $\pi_1 S$ is induced by a homeomorphism, in particular, if two automorphisms are conjugated they determine isotopic homeomorphisms.

**Symplectic form and invariant measure.** There is a smooth symplectic structure on the quotient space $\mathcal{X}(S)$, that is a $2-$form $\omega$ which is closed and non-degenerate, outside the singular locus. We do not explain here how this structure arises, hence for further details, we refer the reader to Goldman [11].

**Proposition 3.1.3.** *The $2-$form $\omega$ is invariant under the action of* $\mathrm{MCG}(S)$.

By taking an appropriate power of $\omega$, we obtain an area form on $\mathcal{X}(S)$ hence a measure $\mu_S$. Viewing $\mathcal{X}(S)$ as a subset of some $\mathbb{R}^{2n}$, away from the singularities $\omega^n$ is some multiple of the standard Euclidean area form, in particular, $\mu_S$ is absolutely continuous with respect to the Lebesgue measure.

**Conjecture 1** (Goldman)**.** *Let $S$ be a closed surface. Then* $\mathrm{MCG}(S)$ *acts ergodically on the non extremal components of $\mathcal{X}(S)$.*

By recent works of Marché and Wolff [17] and [16], we know that the conjecture holds for closed surfaces of genus 2, whereas it is still open for higher genus surfaces.

## 3.2 The Euler class

In this section $S$ always be a closed surface of genus at least 2. For every representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ we may define a $\mathbb{RP}^1-$bundle over $S$ as the associate fibre bundle to the principal $\mathrm{PSL}_2\mathbb{R}-$bundle over $S$ equipped with its flat connection. The Euler class $e(\rho)$ of $\rho$ arises naturally as an obstruction to finding global sections of this bundle.

Let $\tau$ be a topological triangulation, then a section $s_0$ can be easily found on the $0-$skeleton choosing an element of $\mathbb{RP}^1$ above every vertex. This section can be extended to a section $s_1$ over the $1-$skeleton joining the $0-$sections by paths of $\mathbb{RP}^1-$elements. Since $\pi_1(\mathbb{RP}^1) = \mathbb{Z}$ there infinitely many extensions of $s_0$ up to homotopy. Over any $2-$cell $T$, the section over $1-$skeleton defines a $\mathbb{RP}^1-$vector field along $\partial T$, hence a map $\mathfrak{s}_T : \partial T \longrightarrow \mathbb{RP}^1$ of degree $d_T$ that corresponds to the number of times the vector field spins along $\partial T$. We may assign to every $2-$cell the integer $d_T$ giving a $2-$cochain $e(\rho) \in H^2(S, \mathbb{Z})$. In determining $e(\rho)$ we made different choices as the triangulation $\tau$ and the $1-$section over the $1-$skeleton. Adjustment by a $2-$coboundary corresponds to altering the amount of spin chosen along each particular edge. Hence the cohomology class of this $2-$cochain does not depend on the choice of $1-$section. Moreover, it can be seen that this cohomology class does not depend on the cellular decomposition of our surface $S$ chosen. Thus $e(\rho)$ is well-defined $2-$cochain called *Euler class of* $\rho$ of $\mathcal{F}_\rho$.
Since $H^2(S, \mathbb{Z}) \cong \mathbb{Z}$ we can associate to $e(\rho)$ the integer $\mathcal{E}(\rho)$ using the Kronecker pairing. We define $\mathcal{E}(\rho)$ as the *Euler number* associate to $\rho$.

**Lemma 3.2.1.** *The Euler number satisfies the following equality*

$$\mathcal{E}(\rho) = \sum_{T \in \tau} d_T.$$

*Proof.* Let $[S]$ be the fundamental class of $S$, that is a generator of $H_2(S, \mathbb{Z})$. Now $[S] = [T_1] + \cdots + [T_n]$, then

$$\mathcal{E}(\rho) = e(\rho)[S] = \sum_{T \in \tau} e(\rho)[T] = \sum_{T \in \tau} d_T$$

where the last equality holds by definition of $e(\rho)$. □

The Euler number may be seen as a function $\mathcal{E} : \mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R}) \longrightarrow \mathbb{Z}$ that associates to any representation $\rho$ its Euler number $\mathcal{E}(\rho)$. It is natural to ask which is the image of such function. In [29] Wood, based on earlier work by Milnor [23], showed that the Euler number satisfies the following inequality (which is actually known as Milnor-Wood inequality)

$$|\mathcal{E}(\rho)| \leq -\chi(S).$$

We have the following theorem by Goldman.

**Theorem 3.2.2.** *The connected component of* $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ *are the preimages of* $\mathcal{E}\big(k\big)$*, where* $k$ *is an integer satisfying* $|k| \leq -\chi(S)$*.*

The equality holds as soon as the representation is Fuchsian, that is faithful and discrete, and they always arise as the holonomy of a unique and complete hyperbolic structure.

**Corollary 3.2.3** (Goldman [12])**.** *Let* $S$ *be a closed orientable surface with* $\chi(S) < 0$*, and let* $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$*. Then* $\rho$ *is the holonomy of a complete hyperbolic structure on* $S$ *if and only if* $\mathcal{E}(\rho) = \pm\chi(S)$*.*

Now suppose $\rho$ is a geometrizable representation, that is $\rho$ is the holonomy of a possibly branched hyperbolic structure on $S$. Let $p_1, \dots, p_n$ be the cone points of orders $k_1, \dots, k_n$, respectively.
From a more general point of view, the Euler number of the holonomy of a (possibly branched) hyperbolic structure can be computed in terms of the orders of the cone points.

**Proposition 3.2.4.** *Let* $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ *be a representation which is the holonomy of a (possibly branched) hyperbolic structure on a closed surface* $S$*. Then Euler number satisfies the identity*

$$\mathcal{E}(\rho) = \pm\bigg(\chi(S) + \sum_{i=1}^{n} k_i\bigg)$$

*where the sign depends on the orientation of* $S$*.*

*Proof.* Let $\tau$ be a hyperbolic triangulation, such that every cone point is a vertex of the triangulation, so we have a simplicial decomposition of $S$ with hyperbolic triangles. There is a $\mathbb{RP}^1-$vector field $V$ on $S$ with one singularity for every vertex, edge and face of $S$. The orders of the singularities are $1 + k_i$ at any vertex (remember

that for regular points $k = 0$), $-1$ on every edge, and 1 on every face. By the Hopf-Poincaré theorem the sum of the indices of the singularities equals the sum of the indices of the singularities, then $\chi(S) + \sum k_i$.
Now perturb the vector field so that the singularities lie off the $1-$skeleton. Then the spin of $V$ around a triangle $T \in \tau$ is equal to the sum of the indices of singular points of $V$ inside $T$, or its negative, depending on whether the orientation induced by $\mathsf{dev}$ is the same as the orientation induced by the fundamental class $[S]$. For now, assume these orientations agree; otherwise, all the cohomology classes must be multiplied by $-1$. Hence the spin of $V$ around any triangle $T \in \tau$ is equal to the sum of indices of the singular points of $V$ inside $T$ which is in turn equal to the degree of the map $\mathfrak{s}_T : \partial T \longrightarrow \mathbb{RP}^1$ defined above. By 3.2.1 the sum of all indices of singular points is equal to $\mathcal{E}(\rho)$, hence

$$\mathcal{E}(\rho) = \chi(S) + \sum_{i=1}^{n} k_i. \qquad \square$$

*Remark* 3.2.5. As expected if $\rho$ is the holonomy of hyperbolic structure on $S$ without cone points, then every point in $S$ is regular and we found again the above equality $\mathcal{E}(\rho) = \pm\chi(S)$.

*Remark* 3.2.6. If $\sigma$ is a branched hyperbolic structure on $S$, the Gauß-Bonnet condition implies that Euler number is never zero. In particular, the Euler number of elementary representations is always zero (see [17]). It follows that they never arise as the holonomy of a branched hyperbolic structure on $S$.

*Remark* 3.2.7. The Euler number of a representation that is the holonomy of a branched hyperbolic structure is negative because the developing map of a branched hyperbolic structure is assumed to be orientation-preserving.

## 3.3 The relative case

Let now $S$ be a surface with boundary. We may define the *relative Euler class* in the same way described above, but we first need to define a trivialization over the boundary. In the case of surfaces without boundary, it does not matter how we extend the $0-$section along the $1-$skeleton since each edge belongs to two faces, different choices cancel each other out. Here $S$ has boundary, and again it does not matter how we extend the $0-$section over edges lying in the interior of our surface. However each boundary edge belongs to only one face, then here it does matter. Hence the right thing to do is to define a trivialization along the boundary, that is a $1-$section, and extend such section to a $1-$section over the $1-$skeleton.

Let $\gamma \subset \partial S$ be a boundary component and suppose that $\rho(\gamma)$ has not elliptic holonomy. A special trivialization along $\gamma$ is the datum of a section $\mathfrak{s} : \gamma \longrightarrow \mathcal{F}_\rho|_\gamma$ defined by following a fixed point of $\rho(\gamma) \in \mathbb{RP}^1$ along $\gamma$ using the flat connection associate to $\mathbb{RP}^1-$bundle. Note that a special trivialization exists whenever $\rho(\gamma)$ has non-elliptic holonomy and it does not depend on the choice of the fixed point.

*Remark* 3.3.1. Since $\rho(\gamma)$ is not elliptic there is a preferred lift $\widetilde{\rho(\gamma)}$ in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. We may see such lift as a path in $\mathrm{UT}\mathbb{H}^2 \cong \mathbb{H}^2 \times \mathbb{RP}^1$ and the projection of such path to the second factor gives a section along $\gamma$ which is homotopic (relative to basepoint) to the special trivialization $\mathfrak{s}$.

Thus the relative Euler class is a $2-$cochain $e(\rho, \mathfrak{s}) \in H^2(S, \partial S, \mathbb{Z})$, and it measures the obstruction to extend the special trivialization along the boundary over $S$. In the same way, the *relative Euler number* is an integer $\mathcal{E}(\rho, \mathfrak{s})$ defined using the Kronecker pairing, and the Milnor-Wood inequality is satisfied as well (for further details see [12]).

**Definition 3.3.2.** Let $S$ be a compact connected orientable surface with boundary. We define *Fuchsian* those representations $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ such that $|\mathcal{E}(\rho, \mathfrak{s})| = -\chi(S)$ with respect to the special trivialization $\mathfrak{s}$.

As in the closed case Fuchsian representations arise as holonomy of a complete hyperbolic structure on $S$, precisely we have the following result which was proved by Goldman in [12] when $S$ has the boundary with hyperbolic holonomy and more generally in the non-compact case by Mathews in [21].

**Theorem 3.3.3.** *Let $S$ be a compact connected orientable surface with $\chi(S) < 0$, and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$. If $S$ has boundary, assume $\rho$ takes each boundary curve to a non-elliptic element, so the relative Euler class $\mathcal{E}(\rho, \mathfrak{s})$ is well-defined. Then $\rho$ is the holonomy of a complete hyperbolic structure on $S$ with totally geodesic or cusped boundary components (respectively as each boundary curve is taken by $\rho$ to a hyperbolic or parabolic) if and only if $\mathcal{E}(\rho, \mathfrak{s}) = -\chi(S)$.*

Let $S$ be a surface (possibly with boundary), $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation and $\gamma$ be a separating simple closed loop with non-elliptic holonomy. Decomposing $S$ in two pieces the Euler number of $\rho$ can be computed in terms of the relative Euler numbers of each piece with respect to the special trivialization. More generally we have the following lemma whose proof is immediate.

**Lemma 3.3.4.** *Let $\mathcal{F}_\rho$ be a $\mathbb{RP}^1-$bundle over $S$ with holonomy $\rho$, and let $\{l_k\}$ be a finite family of disjoint simple closed curves in $S$ containing also the boundary curves of $S$. Let $\bar{\mathfrak{s}}$ be a section of $\mathcal{F}$ defined on $\{l_k\}$. Denote by $\{C_j\}$ the family of the closure of the connected components of $S \setminus \{l_k\}$, then*

$$\mathcal{E}(\rho, \bar{\mathfrak{s}}_{|\partial C}) = \sum_j \mathcal{E}(\rho_{C_j}, \bar{\mathfrak{s}}_{|\partial C_j})$$

*Proof.* It is sufficient to observe that if two subsurfaces of $S$ are joined along a common boundary then the spins along the common boundary cancel out so that the relative Euler class is additive. □

Finally suppose $\rho$ is the holonomy of a possibly branched hyperbolic structure on $S$ without corner points and with (possibly) cone points lying in the interior of $S$. Then we may apply the same arguments of the proposition 3.2.4 to get a relative version of the same result. That is we have the following proposition.

**Proposition 3.3.5.** *Let $S$ a surface with boundary. Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation such that takes every boundary curve to a non-elliptic element. Suppose $\rho$ is the holonomy of a (possibly branched) hyperbolic structure on a closed surface $S$ with no corner points. Then relative Euler number satisfies the identity*

$$\mathcal{E}(\rho) = \pm\left(\chi(S) + \sum_{i=1}^{n} k_i\right)$$

*where the sign depends on the orientation of $S$.*

## 3.4 An algebraic point of view

There is also an algebraic interpretation of the (possibily relative) Euler class. Let $S$ be a surface with genus $k$ and with $n$ boundary components ($n$ could be eventually zero) and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation such that $\rho(b_i)$ is not elliptic for every $i \in \underline{n}$ (if any).

Let $p$ be a base point in $S$, and let $\widetilde{p}$ be a lift of $p$ in its universal cover, then the fundamental group $\pi_1(S, p)$ has the following presentation

$$\big\langle a_1, b_1, \ldots, a_k, b_k, c_1, \ldots, c_n | [a_1, b_1] \ldots [a_k, b_k] b_1 \ldots b_n = 1 \big\rangle.$$

and defines a fundamental $(4k + n)-$gon in $S$ which is simply connected based at $p$. Set $g_i = \rho(a_i)$, $h_i = \rho(b_i)$ and $c_i = \rho(b_i)$.

Let $(p_0, u_0)$ be a basepoint in $\mathrm{UT}\mathbb{H}^2 \cong \mathbb{H}^2 \times \mathbb{RP}^1$ and draw geodesic between points which are joined by edge in $S$ starting from $p_0$. This gives a $(4k+n)-$polygon in $\mathbb{H}^2$ that may be concave, have self-intersection, or even worse it may be degenerate. We may think this point $(p_0, u_0)$ as a $0-$section over $p$, indeed the projection to the second factor gives an element of $\mathbb{RP}^1$ that we take as $0-$section over $p$. We now extend it to a $1-$section in $S$ in the following way. First notice that there is a bijective corrispondence between edges of the fundamental $(4k+n)$-gon in $S$ and edges of the respective polygon in $\mathbb{H}^2$ defined as above. We begin extending the $0-$section to $1-$section along $a_1$, the respective edge in $\mathbb{H}^2$ is the geodesic segment between $p_0$ and $\rho(a_1)(p_0)$. Consider the points $(p_0, u_0)$ and $(g_1(p_0), g_1'(u_0))$, where $g_1 = \rho(a_1)$, then any lift $\widetilde{g_1}$ of $g_1$ gives a unique path in $\mathrm{UT}\mathbb{H}^2$ (up to homotopy relative to endpoints) of tangent vectors between these endpoints. We take as $1-$section along $a_1$ the projecton to the second factor of such path in $\mathrm{UT}\mathbb{H}^2$. We can play the same game for the other edges to define a $1-$section over $1-$skeleton (where along any boundary edge $c_i$ we consider the section given by the special lift of $\rho(c_i)$). Moving anticlockwise around the polygon in $S$, we now obtain a loop in $\mathrm{UT}\mathbb{H}^2$ which is represented by

$$[\widetilde{g_1}, \widetilde{h_1}] \dots [\widetilde{g_k}, \widetilde{h_k}]\widetilde{c}_1 \dots \widetilde{c}_n$$

where $\widetilde{g_i} = \widetilde{\rho}(a_i)$ and $\widetilde{h_i} = \widetilde{\rho}(b_i)$ are arbitrarily lifts of $g_i, h_i$ and $\widetilde{c_i} = \widetilde{\rho}(c_i)$ are the simplest lifts in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$.

Since $[a_1, b_1] \dots [a_k, b_k]c_1 \dots c_n = 1$ that product is equal to $\mathbf{z}^m$ for some $m \in \mathbb{Z}$. Geometrically $m$ is the number of spin of tangent vectors around the fundamental $(4k+n)-$gon in $S$. We have the following result.

**Theorem 3.4.1** (Goldman [12])**.** *Let $S$ be an orientable surface with $\chi(S) < 0$. Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation, and let $\pi_1(S)$ have the presentation given above, where no $c_i$ is elliptic. The (possibly relative) Euler class $e(\rho)$ takes the fundamental class $[S]$ to $m \in \mathbb{Z}$ where the unique lift of the relator*

$$[\widetilde{g_1}, \widetilde{h_1}] \dots [\widetilde{g_k}, \widetilde{h_k}]\widetilde{c}_1 \dots \widetilde{c}_n \in \widetilde{\mathrm{PSL}_2\mathbb{R}}$$

*is equal to $\mathbf{z}^m$.*

**Corollary 3.4.2.** *Under the same assumptions, the integer $m$ satisfies the following inequality*

$$|m| < -\chi(S).$$

Then the Euler number of a representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ measure also the obstruction to lift it to a representation in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$. In particular, a representation $\rho$ lifts to a representation in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ if and only if there exists a nowhere zero section of the associated $\mathbb{RP}^1-$ bundle with holonomy $\rho$.

*Remark* 3.4.3. If $S$ has boundaries a representation $\rho$ lifts to a representation in $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ if and only if there exists a nowhere zero section of the associated $\mathbb{RP}^1-$ bundle with holonomy $\rho$ with respect to the special trivialization $\mathfrak{s}$ along the boundaries.

In the sequel, we will work with puncture torus, and we make a strong use of the following result.

**Proposition 3.4.4.** *Let $S$ be a punctured torus and $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation such that the relative Euler class is well-defined. Then*

*1* $\mathrm{Tr}[g,h] \leq -2$ *if and only if* $\mathcal{E}(\rho,\mathfrak{s}) = -1$,

*2* $\mathrm{Tr}[g,h] \geq 2$ *if and only if* $\mathcal{E}(\rho,\mathfrak{s}) = 0$,

*where $\mathfrak{s}$ is the special trivalization along the boundary.*

*Proof.* Suppose first $\mathrm{Tr}[g,h] \leq -2$, then $[g,h] \in \mathrm{Hyp}_{\pm 1} \cup \mathrm{Par}_1^- \cup \mathrm{Par}_{-1}^+$ by 1.3.7. We may suppose without loss of generality that $[g,h] \in \mathrm{Hyp}_{-1} \cup \mathrm{Par}_1^+$, the other case occur reverting the orientation of $S$. Since $[g,h]c = 1$ in $\mathrm{PSL}_2\mathbb{R}$, then $c^{-1} = [g,h]$ and its simple lift $\widetilde{c}^{-1} \in \mathrm{Hyp}_0 \cup \mathrm{Par}_0^+$ and the following holds $\widetilde{c}^{-1} = \mathbf{z}[g,h]$, thus $[g,h]\widetilde{c} = \mathbf{z}^{-1}$ and $\mathcal{E}(\rho,\mathfrak{s}) = -1$.
Now suppose $\mathrm{Tr}[g,h] \geq 2$, then $[g,h] \in \mathrm{Hyp}_0 \cup \mathrm{Par}_0 \cup \{1\}$ by 1.3.7. Since $[g,h]c = 1$ in $\mathrm{PSL}_2\mathbb{R}$, then $c^{-1} = [g,h]$ then $[g,h]\widetilde{c} = 1$. Thus $\mathcal{E}(\rho,\mathfrak{s}) = 0$. □

## 3.5 Index formula

Let $S$ be a closed surface endowed with a branched hyperbolic structure $\sigma$ with holonomy $\rho$. Let $C$ be a subsurface of $S$ with boundary, then $\rho$ induces a representation $\rho_C : \pi_1 C \longrightarrow \mathrm{PSL}_2\mathbb{R}$ in a natural way. Precisely we must choose a base point $p$ on $C$, connected to a base point $q$ on the overall surface $S$ by a particular path, then we may define a representation $\rho_C : \pi_1(C,p) \longrightarrow \mathrm{PSL}_2\mathbb{R}$.

In this section we introduce a new index formula for branched hyperbolic structures that relates the relative Euler number of $\rho_C$ with the Euler characteristic of $C$ and the total branch order $k_C$ on $C$. However the relative Euler class and the Euler characteristic are topological invariants whereas the branching order depends on the geometry on $S$ thus, in order to relate them, we first need to introduce another invariant that takes into account geometry and topology at the same time. This invariant is defined in the following paragraph.

**Index of a curve.** Let $\sigma$ be a branched hyperbolic structure on $S$, with holonomy $\rho$ and let $\mathsf{dev} : \widetilde{S} \longrightarrow \mathbb{H}^2$ be the developing map associate to $\sigma$. Let $C$ be a subsurface of $S$ with boundary $\gamma$ and suppose $\rho(\gamma)$ is non-elliptic. Let $\rho_C$ be the induced representation on $\pi_1 C$ and let $\mathcal{F}_{\rho_C}$ be the flat $\mathbb{RP}^1-$bundle associated to $\rho_C$.
Work with $\mathcal{F}_{\rho_{|C}}$ is quite hard, thus we begin defining a bundle isomorphism between it and the unit tangent bundle $E = \mathrm{UT}C$ outside the cone points using the developing map. In order to do this, and let $p_1, \dots, p_k$ be the cone points on $C$. Let $C' = C \setminus \{p_1, ..., p_k\}$ and let $\widetilde{C'}$ be a lift to $\widetilde{S}$, then the restriction of the developing map $\mathsf{dev}$ to $\widetilde{C'}$ defines an orientation preserving local diffeomorphism $\mathsf{dev}_{|\widetilde{C'}} : \widetilde{C'} \longrightarrow \mathbb{H}^2$. This map induces in natural way a map between the unit tangent bundle over $\widetilde{C'}$ and the unit tangent bundle over $\mathbb{H}^2$

$$\begin{aligned} \mathrm{UT}\widetilde{C'} &\longrightarrow \quad \widetilde{C'} \times \mathrm{UT}\mathbb{H}^2 \\ (\widetilde{p}, v_p) &\longmapsto \quad (\widetilde{p}, d\mathsf{dev}_p(v_p)) \end{aligned}$$

Let $\alpha$ be the unique geodesic passing through $\mathsf{dev}(p)$ and tangent to $d\mathsf{dev}_p(v_p)$. We define *attractive point of* $v_p$ to be the point at infinity $a_\alpha$ obtained by following $\alpha$ in the direction $d\mathsf{dev}_p(v_p)$. Similarly we define *repulsive point of* $v_p$ to be the point at infinity $r_\alpha$ obtained following $\alpha$ in the direction $-d\mathsf{dev}_p(v_p)$.
Consider now the map $\mathrm{UT}\mathbb{H}^2 \longrightarrow \mathbb{RP}^1$ that associates to any unit vector $d\mathsf{dev}_p(v_p)$ the point at infinity $a_\alpha$. It is well-defined because Poincaré metric on $\mathbb{H}^2$ is complete and equivariant with respect to the action of $\mathrm{PSL}_2\mathbb{R}$. Post composing with the map defined above we get a map

$$\widetilde{\mathsf{D}} : \mathrm{UT}\widetilde{C'} \longrightarrow \widetilde{C'} \times \mathbb{RP}^1$$

which is $(\pi_1 C, \rho \times \mathsf{id})-$equivariant. Hence the quotient by the action defines a bundle isomorphism $\mathsf{D} : E_{|C'} \longrightarrow \mathcal{F}_{\rho_{|C}}$.

Let $\dot{\gamma}$ be the unit vector field along $\gamma$ and define $\mathfrak{u} = \mathsf{D}(\dot{\gamma})$. Of course, it is a section of $\mathcal{F}_{\rho_{|C}}$ along $\gamma$ and then it makes sense to compare it with the special trivialization $\mathfrak{s}$. This discussion can be easily extended to subsurface with more than one boundary components.

*Remark* 3.5.1. Let $j$ be the isometry between the upper half plane $\mathbb{H}^2$ and the disk model $\mathbb{D}^2$. Thus $j$ can be extended to an identification between the boundary at infinity $\mathbb{RP}^1$ and $\mathbb{S}^1$ of the two models respectively. Using this identification the $\mathbb{RP}^1-$bundle $\mathcal{F}_\rho$ can be identified with the $\mathbb{S}^1-$bundle associate to $\rho$ over $C$. In this way $\mathcal{F}_{\rho_{|\gamma}} \simeq \gamma \times \mathbb{S}^1$ and the section $\mathfrak{s}$ becomes $\gamma \times \{p\}$ where $p \in \mathbb{S}^1$. Similarly the section $\mathfrak{u}$ is identified with a section of a $\mathbb{S}^1-$bundle over $C$ which we still denote by $\mathfrak{u}$.

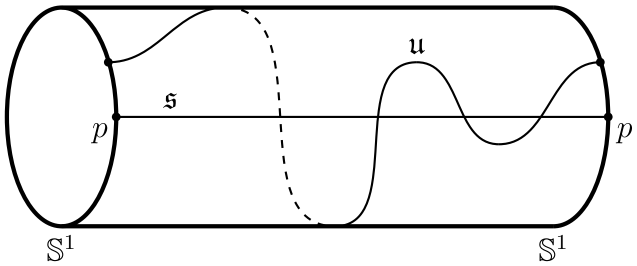


Figure 3.1: The $\mathbb{S}^1-$ bundle over $\gamma$ is a torus, but here we draw a cylinder for simplicity, where the extremal circles are identified via the holonomy $\rho$. The picture show the comparison between the sections $\mathfrak{s}$ and $\mathfrak{u}$. The same picture also shows the comparison between them on the $\mathbb{RP}^1-$bundle.

**Definition 3.5.2.** Let $\mathfrak{s}$ be the special trivialization along $\gamma$ and $\mathfrak{u}$ be the section defined above. Let $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the map such that $\mathfrak{s} = f \cdot \mathfrak{u}$. We define *index* $\mathrm{I}_\gamma$ of $\gamma$ to be the degree of the map $f$ with respect to the orientation of $\gamma$ induced by $C$.

Roughly speaking, the index of a curve $\gamma$ measure how many times the unit vector field along $\gamma$ spin with respect to the special trivialization. The sign of the index depends on the orientation of $\gamma$ because the section $\mathfrak{u}$ does.

**Lemma 3.5.3.** *The sign of the index* $\mathrm{I}_\gamma$ *depends on the orientation of* $\gamma$. *Precisely* $\mathrm{I}_{-\gamma} = -\mathrm{I}_\gamma$, *where* $-\gamma$ *means the curve* $\gamma$ *with the opposite orientation induced by* $C$.

*Proof.* Let $\mathfrak{u}^+$ be a parametrization of the $\mathbb{S}^1-$vector field $\mathfrak{u}$ along $\gamma$ with respect to the orientation on $\gamma$ induced by $C$

$$\begin{aligned} \mathfrak{u}^+ : [0,1] &\longrightarrow \quad \gamma \times \mathbb{S}^1 \\ t &\longmapsto \quad \big(\gamma(t), a_{\alpha(t)}\big) \end{aligned}$$

where $a_{\alpha(t)}$ is the attractive point of $\dot\gamma(t)$. Let $\mathfrak{u}^-$ be a parametrization of the $\mathbb{S}^1$−vector field $\mathsf{D}(-\dot\gamma)$ along $-\gamma$. Then

$$\begin{aligned} \mathfrak{u}^- : [0,1] &\longrightarrow \quad \gamma\times\mathbb{S}^1 \\ t &\longmapsto \big(\gamma(1-t), r_{\alpha(t)}\big) \end{aligned}$$

because the attractive point of $-\dot\gamma(1-t)$ is the repulsive point of $\dot\gamma(t)$. Let $\psi$ be the map such that

$$\big(\gamma(t), a_{\alpha(t)}\big) \longmapsto \big(\gamma(1-t), r_{\alpha(t)}\big)$$

Of course it is an involution. Observe that $\mathfrak{u}^- = \psi\circ\mathfrak{u}^+$, and the deg$\psi$ is $-1$ because it reverses the orientation. Finally let $\mathfrak{s}$ be the special trivialization along $\gamma$, and $f$ be the map such that $\mathfrak{s} = f\cdot\mathfrak{u}^+$. Writing $\mathfrak{s} = f\cdot\psi^2\circ\mathfrak{u}^+$ we obatin $\mathfrak{s} = (f\cdot\psi)\circ\mathfrak{u}^-$, and deg$f\cdot\psi$ is $-$deg$f$, that is $-\mathrm{I}_\gamma$ $=\mathrm{I}_{-\gamma}$ by definition. □

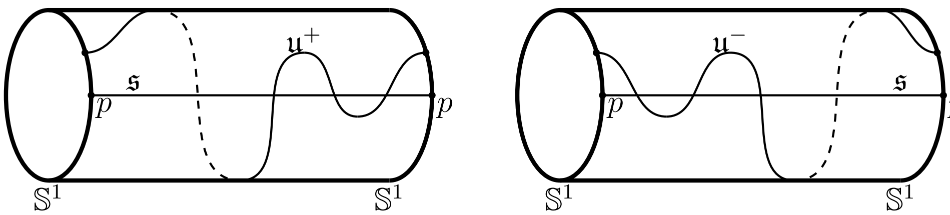


Figure 3.2: On the left we see the comparison between the sections $\mathfrak{s}$ and $\mathfrak{u}^+$, and on the right between the sections $\mathfrak{s}$ and $\mathfrak{u}^-$. In particular the section $\mathfrak{u}^+$ turns clockwise, whereas $\mathfrak{u}^-$ turns counterclockwise. It follows that the degrees are opposite.

**Definition 3.5.4.** Let $A$ be an annulus and $\rho_A : \pi_1 A \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation. We will say $A$ *has hyperbolic holonomy* if there exists a hyperbolic element $g\in\mathrm{PSL}_2\mathbb{R}$ such that $\rho_A\big(\pi_1 A\big) = \langle g\rangle$.

**Lemma 3.5.5.** *Let $A = \mathbb{S}^1\times[0,1]$ be an annulus with hyperbolic holonomy and let $s = \{\mathfrak{s},\mathfrak{u}\}$ be the section of the associate $\mathbb{RP}^1$−bundle $\mathcal{F}_{\rho_A}$ over $\mathbb{S}^1\times\{i\}$ for $i = 0,1$. Then $\mathcal{E}(\rho_A, s) = -degf$, where $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is such that $\mathfrak{s} = f\cdot\mathfrak{u}$ and the degree of $f$ is computed with respect to the orientation induced by $A$ on the component where $\mathfrak{s}$ is defined.*

*Proof.* Let $\gamma_i = \mathbb{S}^1\times\{i\}$ and suppose that $\mathfrak{s}$ is defined over $\gamma_0$ and $\mathfrak{u}$ over $\gamma_1$. Let $\varphi : \gamma_0 \longrightarrow \gamma_1$ be the homeomorphism induced by the isotopy joining $\gamma_0$ with $\gamma_1$. Then $\mathfrak{u}\circ\varphi$ is another section defined on $\gamma_0$ with the opposite orientation. Let

$f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the function such that $\mathfrak{s} = f \cdot (\mathfrak{u} \circ \varphi)$, then by definition of index we get the conclusion $\mathcal{E}(\rho_A, s) = -\text{deg} f$ (the sign minus appear because $\mathfrak{s}$ and $\mathfrak{u}$ are defined on $\gamma_0$ with respect to opposite orientations). □

Now we can finally state the main result of this section.

**Index Formula.** Let $\sigma$ be a branched hyperbolic structure on $S$, and let $C \subset S$ be a subsurface. Then the holonomy representation $\rho$ for $\sigma$ induces a representation $\rho_C : \pi_1 C \longrightarrow \text{PSL}_2\mathbb{R}$ in the natural way. As in the closed case, starting from a surface with boundary $C$ and a representation $\rho_C$ we can associate a flat $\mathbb{RP}^1-$bundle $E_{\rho_C}$ over $C$. We assume that each boundary components $\gamma \subset C$ has not elliptic holonomy. In this case for each of them we can define a section $s : \gamma \longrightarrow E_{\rho_C}|_\gamma$ by following a fixed point of $\rho(\gamma) \in \mathbb{RP}^1$ along $\gamma$ using the flat connection associate to $\mathbb{RP}^1-$bundle.

**Theorem 3.5.6.** *Let $\sigma$ be a branched hyperbolic structure on $S$ with holonomy $\rho$. Let $C$ be a subsurface of $S$ such that every boundary components has not elliptic holonomy and denote by $\rho_C$ the restriction of $\rho$ to $\pi_1 C$. Then the relative Euler number satisfies the following identity*

$$\mathcal{E}(\rho_C, \mathfrak{s}) = \chi(C) + k - \sum_{i=1}^{n} \mathrm{I}_{\gamma_i},$$

*where $k$ denote the total branching order on $C$, $\gamma_i$'s are the boundary components of $C$ and the indeces are computed with respect to the induced orientation by $C$ on each boundary components.*

*Remark* 3.5.7. Observe that we can always assume that no cone point belongs to $\partial C$. In fact, we do not require that a boundary component $\gamma$ is the geodesic representative in its homotopy class.

*Remark* 3.5.8. We prove the theorem in the case of all cone points are simple. This is not a strong assumption because every cone point of order greater than one can be split into two or more simple cone points.

*Proof of Theorem 3.5.6.* Let $\mathcal{F}_\rho$ be the $\mathbb{RP}^1-$bundle over $S$ with holonomy $\rho$ and we denote by $\mathcal{F}_{\rho|_C}$ its restriction over $C$. We will get the result by a brute force computation of the relative Euler class with respect to the special trivialization $\mathfrak{s}$ along the boundary of $C$. Let $\gamma_1, \ldots, \gamma_n$ the boundary of $C$, and let $p_1, \ldots, p_k$ be the cone points on $C$. For any $j \in \{1, ..., k\}$ we consider a small open disc $D_j$ centered at $p_j$ and we call $b_j$ its boundary curve. Finally let $\gamma_i^*$ be a curve isotopically equivalent to $\gamma_i$ in $C \setminus \bigcup_{i\in\underline{k}} D_j$. Let $C' = C \setminus \{p_1, ..., p_k\}$ as above. We define a section $\bar{\mathfrak{s}}$ over the family of curves $\{\gamma_i, \gamma_i^*, b_j\}_{i\in\underline{n}, j\in\underline{k}}$ in the following way. Over each boundary component $\gamma_i$, the section $\bar{\mathfrak{s}}$ agree with the special trivialization defined along $\gamma_i$ following the fixed point $\rho_C(\gamma_i)$ with flat connection associate to the $\mathbb{RP}^1-$bundle. Over any other curve of the family the section $\bar{\mathfrak{s}}$ is the $\mathbb{RP}^1-$vector field $\mathfrak{u}_\mathfrak{i}$ along $\gamma_i^*$ and $\mathfrak{u}_\mathfrak{j}$ along $b_j$ defined as above. The complement of the family $\{\gamma_i, \gamma_i^*, b_j\}_{i\in\underline{n}, j\in\underline{k}}$ is the disjoint union of $n$ annuli $A_i$, $k$ discs $D_j$ and a component $C''$. By lemma 3.3.4

$$\mathcal{E}(\rho_C, \mathfrak{s}) = \mathcal{E}(\rho_{C''}, \bar{\mathfrak{s}}_{|\partial C''}) + \sum_{j=1}^{k} \mathcal{E}(\rho_{D_j}, \bar{\mathfrak{s}}_{|\partial D_j}) + \sum_{i=1}^{n} \mathcal{E}(\rho_{A_i}, \bar{\mathfrak{s}}_{|\partial A_i})$$

Since $\mathsf{dev}$ is a local diffeomorphism when restricted to $C''$, the map $\psi_{|C''}$ is still a bundle isomorphism, then $\mathcal{E}(\rho_{C''}, \bar{\mathfrak{s}}_{|\partial C''}) = \chi(C'')$ by a classical result, and $\mathcal{E}(\rho_{D_j}, \bar{\mathfrak{s}}_{|\partial D_j}) = \deg\bar{\mathfrak{s}}_{|\partial D_j} = 2$ since we are supposing that all cone points are simple.
Finally, for any annulus $A_i$, let $g_i : \gamma_i \longrightarrow \gamma_i^*$ be the homeomorphism induced by the isotopy joining $\gamma_i$ with $\gamma_i^*$, and let $f_i$ the map such that $\mathfrak{s} = f_i \cdot (\bar{\mathfrak{s}}_{|\gamma_i^*} \circ g_i)$, then by 3.5.5

$$\mathcal{E}(\rho_{A_i}, \bar{\mathfrak{s}}_{|\partial A_i}) = -\deg f_i = -\mathrm{I}_{\gamma_i}$$

Summing up we get the desire result

$$\mathcal{E}(\rho_C, \mathfrak{s}) = \chi(C) + k - \sum_{i=1}^{n} \mathrm{I}_{\gamma_i}.$$ □

*Remark* 3.5.9. The relative Euler number is negative as well as the Euler number of $\rho$. Changing the orientation of $S$ the same formula holds with opposite sign. In fact, if we change the orientation of $S$ the branched structure $\sigma$ is locally modeled on the lower half plane. Hence the developing map $\mathsf{dev}$ becomes an orientation reversing local diffeomorphism, and then all addendum has opposite sign.

We are now focusing our attention on the following corollaries.

**Corollary 3.5.10.** *Let $\sigma$ be a branched hyperbolic structure on $S$ with holonomy $\rho$. Let $C_1, C_2$ two subsurfaces with boundary such that $S = C_1 \cup C_2$ and no boundary components has elliptic holonomy. Then*

$$\mathcal{E}(\rho) = \mathcal{E}(\rho_{C_1}, \mathfrak{s}) + \mathcal{E}(\rho_{C_2}, \mathfrak{s}) = \chi(S) + \sum_{i=1}^{n} k_i.$$

*where $\mathfrak{s}$ is the special trivialization defined along each boundary component.*

*Proof.* It is sufficient to observe that two adjacent components induce opposite orientations on their common boundary components and then we can suppose without loss of generality that:

$$\mathcal{E}(\rho_{C_1},\mathfrak{s}) = \chi(C_1) + k_1 - \sum_{i=1}^{n} \mathrm{I}_{\gamma_i}, \quad \mathcal{E}(\rho_{C_2},\mathfrak{s}) = \chi(C_2) + k_2 - \sum_{i=1}^{n} \mathrm{I}_{-\gamma_i}.$$

Summing up and using 3.5.3 we get the initial formula defined in 3.2.4, that is:

$$\mathcal{E}(\rho) = \chi(C_1) + \chi(C_2) + k_1 + k_2 = \chi(S) + \sum_{i=1}^{n} k_i. \qquad \square$$

**Corollary 3.5.11.** *Let $A$ be an annulus with hyperbolic holonomy $\rho_A$, then $\mathcal{E}(\rho_A,\mathfrak{s})$ is zero with respect to the special trivialization along the boundary components. Moreover if $\gamma_1,\gamma_2$ are the boundary components of $A$, then $|\mathrm{I}_{\gamma_1} - \mathrm{I}_{\gamma_2}|$ coincide with the number of cone points inside $A$.*

*Proof.* The first statement follows by the definition of Euler class and applying the Milnor-Wood inequality. For the second one it is sufficient to observe that $A$ induces opposite orientations to its boundary components, then apply theorem 3.5.6 to $A$. $\square$

*Remark* 3.5.12. The index formula gives also a bound on the possible values of the index $\mathrm{I}_\gamma$ for a curve $\gamma$. Infact with respect to the special trivialization of $\gamma$, the relative Euler number $\mathcal{E}(\rho_C,\mathfrak{s})$ satisfies the Milnor-Wood inequality that is $|\mathcal{E}(\rho_C,\mathfrak{s})| \leq -\chi(C)$. This implies immediately the following chain inequalites

$$\chi(C) + k \leq \mathrm{I}_\gamma \leq k.$$

# CHAPTER 4

## Geometrization of Representantions

In this chapter, we continue our investigation about the relationship between hyperbolic structures and their holonomy representation. The main question here is the following: let $S$ be a closed surface of genus at least 2

*under which condition a representation $\rho$ is the holonomy of a possibly branched hyperbolic structure?*

## 4.1 Representations of the punctured torus

Throughout this section, let $H$ be a punctured torus. We prefer to use the letter $H$ here, instead of $S$ of $T$, because in the sequel will be useful to think punctured torus as a handle attached to a surface of lower genus than the original surface $S$.

Let $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation. We already know from 1.4.16 that $\rho$ is the holonomy of a branched hyperbolic structure with no cone point and at most one corner point on the boundary of $H$ if and only if it is not virtually abelian. The reason for this section is to give some background materials that we will use widely in other sections.

### 4.1.1 Character variety of punctured torus

We start analyzing the character variety of representations $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ without consider geometric structures. Let $p \in H$ be a basepoint for the fundamental group and let $(\alpha, \beta)$ be a basis. Any representation $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is uniquely determined by the images $\rho(\alpha)$ and $\rho(\beta)$. A representation into $\mathrm{PSL}_2\mathbb{R}$ obviously lifts to $\mathrm{SL}_2\mathbb{R}$, and we have two choices, each for the lifts of $\rho(\alpha)$ and $\rho(\beta)$. For now consider $\rho$ as a representation into $\mathrm{SL}_2\mathbb{R}$ and denote $\rho(\alpha) = g$ and $\rho(\beta) = h$. The character of $\rho$ is determined by the value of $\mathrm{Tr}\circ\rho$ at finitely many elements of $\pi_1 H$. For the punctured torus with $\pi_1 H = \langle \alpha, \beta \rangle$, it is sufficient to consider only the three elements $\alpha, \beta, \alpha\beta$. Any word $w$ of $\pi_1 H$ may be written in terms of $\alpha, \beta$ and their inverses, and the trace of $\rho(w)$ can be expressed as a polynomial in $(x, y, z) = (\mathrm{Tr}g, \mathrm{Tr}h, \mathrm{Tr}gh)$. In our case we have the important relation

$$\mathrm{Tr}\ [g,h] = \ \mathrm{Tr}^2 g + \ \mathrm{Tr}^2 h + \ \mathrm{Tr}^2 gh - \ \mathrm{Tr}g\ \mathrm{Tr}h\ \mathrm{Tr}gh - 2$$

and hence we define the polynomial

$$k(x, y, z) = x^2 + y^2 + z^2 - xyz - 2.$$

*Remark* 4.1.1. It is a classical result that if $\rho_1$ is irreducible and defines the same triple $(x, y, z)$ as another representation $\rho_2$, then $\rho_1$ and $\rho_2$ are conjugate; so that the triple $(\mathrm{Tr}g, \mathrm{Tr}h, \mathrm{Tr}gh)$ defines the pair $g, h \in \mathrm{SL}_2\mathbb{R}$ uniquely up to conjgacy.

The set of all $(x, y, z) = (\mathrm{Tr}g, \mathrm{Tr}h, \mathrm{Tr}gh)$ is the character variety $\mathcal{X}(H)$ of the punctured torus $H$.

**Theorem 4.1.2** (Goldman [10])**.** *Given $(x, y, z) \in \mathbb{R}^3$, there exist $g, h \in \mathrm{SL}_2\mathbb{R}$ such that $(x, y, z) = (\mathrm{Tr}g, \mathrm{Tr}h, \mathrm{Tr}gh)$ if and only if*

$$\mathrm{Tr}\ [g,h] = \ \mathrm{Tr}^2 g + \ \mathrm{Tr}^2 h + \ \mathrm{Tr}^2 gh - \mathrm{Tr}g\ \mathrm{Tr}h\ \mathrm{Tr}gh - 2 \geq 2$$

*or at least one of $|x|, |y|, |z|$ is greater than* 2.

For representations $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$, the character variety may be described starting from the character variety of representations into $\mathrm{SL}_2\mathbb{R}$. There are four different ways to lift the couple $\rho(\alpha), \rho(\beta)$ into $\mathrm{SL}_2\mathbb{R}$, which are related by sign changes. Thus we simply take the character variety $\mathcal{X}(H)$ of representations into $\mathrm{SL}_2\mathbb{R}$ modulo the equivalence relation

$$(x, y, z) \sim (-x, -y, z) \sim (-x, y, -z) \sim (x, -y, -z)$$

induced by these four possible lifts. We have now described the character variety $\mathcal{X}(H) \subset \mathbb{R}^3$. Points with $k(x, y, z) = 2$ describe reducible representations, which include also abelian representations, as shown by the following lemma. We recall that a representation $\rho$ is said to be reducible if its image is a set of matrices such that, acting as linear transformations on $\mathbb{C}^2$, leaves invariant a line in $\mathbb{C}^2$.

**Lemma 4.1.3.** *A representation* $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ *(or* $\mathrm{SL}_2\mathbb{R}$*) is reducible if and only if the character of* $\rho$ *is such that* $k(x, y, z) = 2$*.*

*Proof.* Following Goldman [10], this can be checked by a direct computation. Let $g, h \in \mathrm{SL}_2\mathbb{R}$. We may write

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $ad - bc = 1$. By applying an inner automorphism, we may assume that $g$ is the Jordan canonical form. Suppose first $g$ is diagonal, *i.e*

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

then $\mathrm{Tr}[g, h] = 2 + bc(\lambda - \lambda^{-1})^2$. Hence either $g = \pm\mathrm{id}$, or $bc = 0$ *i.e* $h$ is upper-triangular or lower-triangular. If $g$ is not diagonal, then

$$g = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

with $s \neq 0$. In such case $\mathrm{Tr}[g, h] = 2 + s^2c^2 \geq 2$, so the condition $\mathrm{Tr}[g, h] = 2$ implies that $c = 0$ and $h$ is upper-triangular. $\square$

Points with $k(x, y, z) \neq 2$ describe irreducible representations, hence describe a conjugacy class of representations precisely. For any $t \neq 2$, we define the *relative character variety* as the space of all representations (up to conjugacy) with $\mathrm{Tr}[g, h] = t$ by $\mathcal{X}_t(H) = k^{-1}(t) \cap \mathcal{X}(H)$.

**Virtually abelian representations.** We finally consider a special type of representation, namely *virtually abelian representations*. We dedicate an entire paragraph to them because we will need to consider them in the next section.

**Definition 4.1.4.** A representation $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is said to be virtually abelian it its image contains an abelian subgroup of finite index.

Consider the following set of $\mathbb{R}^3$:

$$V = \{0 \times 0 \times \mathbb{R} \setminus [-2, 2]\} \cup \{0 \times \mathbb{R} \setminus [-2, 2] \times 0\} \cup \{\mathbb{R} \setminus [-2, 2] \times 0 \times 0\}.$$

Of course $V \subset \mathcal{X}(H)$. The following result gives a complete characterization of this type of representations (see [20] for the proof).

**Lemma 4.1.5.** *Let $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation and let $(\alpha, \beta)$ be any basis of $\pi_1 H$. Then $\rho$ is virtually abelian (but not abelian) if and only if $(\mathrm{Tr}g, \mathrm{Tr}h, \mathrm{Tr}gh) \in V$, where $g = \rho(\alpha)$ and $h = \rho(\beta)$.*

Moreover this type of representations has a nice geometric description given by the following lemma.

**Lemma 4.1.6.** *With the same notation of the previous lemma; a representation $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is virtually abelian representation if and only if two of $\{g, h, gh\}$ are half-turns about distinct points $q_1, q_2$ and the third is a non-trivial translation along the unique axis passing through $q_1$ and $q_2$.*

*Proof.* The sufficient condition follows immediately. Indeed half-turns have trace 0 and a non-trivial translation has trace greater than 2 in magnitude, hence if $\{g, h, gh\}$ are isometries of the required type, the triple $(\mathrm{Tr}g, \mathrm{Tr}h, \mathrm{Tr}gh) \in V$.
We need to show the necessary condition, and we may suppose that $\mathrm{Tr}g = 0$, $\mathrm{Tr}h = 0$ and $|\mathrm{Tr}gh| > 2$ since the other cases are similar. Hence $g$ and $h$ are half-turns about two points $q_1, q_2$. Suppose $q_1 = q_2$ then $gh = \mathrm{id}$, that is $\mathrm{Tr}gh = \pm 2$ hence a contradiction. Since the points $q_1, q_2$ are distinct there is a unique geodesic line $q_1 q_2$ passing through them. Both $g$ and $h$ preserve such line reversing its orientation. Of course also the composition $gh$ preserve the line $q_1 q_2$ maintaining the orientation (because the orientation is reversed two times). Since $gh \neq$id, we can conclude that it is a non-trivial translation along $q_1 q_2$. □

### 4.1.2 The action of MCG$(H)$

We are going to consider the effect of changing basis $(\alpha_1, \beta_1) \longrightarrow (\alpha_2, \beta_2)$ on a representation $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$ by pre-composition, as discussed in section 3.1. As we said before, the underlying geometry does not change but the character does. Since trace is invariant under conjugation, this action descends to an action of Out $\pi_1 H \cong \mathrm{MCG}(H)$. We have the following theorem by Nielsen, see [24] and [15].

**Theorem 4.1.7** (Nielsen)**.** *An automorphism $\psi$ of $\pi_1 H = \langle \alpha, \beta \rangle$ takes $[\alpha, \beta]$ to a conjugate of itself or its inverse.*

By the previous theorem $[\alpha_1, \beta_1]$ is conjugate to $[\alpha_2, \beta_2]^{\pm 1}$, so $\mathrm{Tr}[g_1, h_1] = \mathrm{Tr}[g_2, h_2]$ and $k(x_1, y_1, z_1) = k(x_2, y_2, z_2)$. That is, the triples $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ lie on the same level set of the polynomial $k$. Hence the action of the mapping class group of the punctured torus $\mathrm{MCG}(H)$ preserves the level set $k(x,y,z) = t$, that is preserves the relative character variety $\mathcal{X}_t(H)$.

**Ergodicity.** The action of $\mathrm{MCG}(H)$ becomes interesting on the level sets $\mathcal{X}_t(H)$ for $t > 2$. Since $\mathcal{X}_t(H)$ is a $2-$dimensional closed subspace of $\mathcal{X}(H)$, the symplectic $2-$form $\omega$ induces a symplectic structure $\omega_t$ on each level, which is also an area form. The action of $\mathrm{MCG}(H)$ is somewhat of wild on each level for $t > 2$. If $2 < t \leq 18$ the action is known to be ergodic, see [10]. On the other hand, when $t > 18$, wandering domains appear in $\mathcal{X}_t(H)$, arising from the intersection of $\mathcal{X}_t(H)$ with the Fricke space $\mathcal{F}(P)$ of a pair of pants $P$.
Inside wandering domains the action of $\mathrm{MCG}(H)$ is known to properly and freely, whereas outside the action is known to be ergodic. In [10] is shown that, for any $t > 2$, we may found in $\mathcal{X}_t(H)$ characters of two type of representations, namely:

1. **Pants representation**: that is $(x,y,z)$ is the character of discrete representation $\rho : \pi_1 H \longrightarrow \mathrm{PSL}_2\mathbb{R}$, which it may be considered the holonomy of complete hyperbolic structure on a pair of pants. In particular there are no elliptics in the image of $\rho$. In this case: let $\overline{\rho} : \pi_1 H \longrightarrow \mathrm{SL}_2\mathbb{R}$ be any lift of $\rho$, then up to change the basis of $\pi_1 H$ we may suppose that the character $\big(\mathrm{Tr}\overline{\rho}(\alpha), \mathrm{Tr}\overline{\rho}(\beta), \mathrm{Tr}\overline{\rho}(\alpha\beta)\big)$ lies in the octant $]-\infty, -2]^3$.
2. **Representation with elliptics**: that is $(x,y,z)$ is equivalent to another character with some coordinate in the interval $]-2,2[$. In this case $(x,y,z)$ is the character of a representation $\rho$ which sends a simple closed curve to an elliptic transformation. We denote this subset with $\Omega_t$.

**Theorem 4.1.8.** *For any $t > 2$, the action of* $\mathrm{MCG}(H)$ *on $\Omega_t$ is ergodic and, properly and freely on $\mathcal{X}_t(H) \setminus \Omega_t$.*

With this theorem we conclude this section because we have all necessary background material to tackle the next.

## 4.2 Geometrizable representation

In this section, we will enter into the heart of this work. Throughout next subsections, $S$ always be a closed surface of genus 2 and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be any representation.

### 4.2.1 State of art, discussion, and results

The main goal of this section is to prove the following result.

**Theorem 4.2.1.** *Let $S$ be a closed surface of genus two. Then any representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathcal{E}(\rho) = \pm 1$ is geometrizable by a branched hyperbolic structure with one cone point of angle $4\pi$.*

In [21], Mathews proved the following theorem, on which we rely the strategy of the proof of our main theorem 4.2.1.

**Theorem 4.2.2** (Mathews)**.** *Let $S$ be a closed surface of genus two. Then almost every representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathcal{E}(\rho) = \pm 1$, which sends a non-separating curve $\gamma$ on $S$ to an elliptic is the holonomy of a branched hyperbolic structure on $S$ with one cone point of angle $4\pi$.*

*Remark* 4.2.3. As we see below this theorem holds in any genus. However here we prefer to emphasize it in the case of surfaces with genus 2.

We now explain briefly the strategy of our proof: precisely we will show that the assumptions of 4.2.2 are not really necessary. In order to understand why the assumptions of 4.2.2 can be removed, we need to analyze them first. By Goldman's theorem 3.1.1 and 3.2.2, the set of representations with $\mathcal{E}(\rho) = \pm 1$ is formed by two connected components of the representation space, hence characters of these representations form two connected components of character variety $\mathcal{X}(S)$. Since the singular locus in $\mathcal{X}(S)$ is given by only elementary representations and since they have Euler class zero, it follows that these components are smooth and the non-degenerate $2-$form $\omega_S$ is well-defined everywhere and defines a measure $\mu_S$.
Let us denote by $E$ the subset of all representations with $\mathcal{E}(\rho) = \pm 1$, that send a simple non-separating curve to an elliptic. Then the previous claim of 4.2.2 may be restated in the following way: *almost every representation in $E$ is the holonomy of a branched hyperbolic structure with a single cone point of angle $4\pi$.* Now, two simple questions may naturally arise:

1. how big is $E$? That is, what is the measure of the set $E$?
2. where does the *almost every* condition come from?

Actually $E$ is a subset of full measure by a recent work of Marché and Wolff [17, Proposition 6.2 $\big($case of $(g, k) = (2, 1)\big)$] . However, this is not enough in the sense that there might be a subset of measure zero of representations with $\mathcal{E}(\rho) = \pm 1$ that does not send a simple non-separating curve to an elliptic element. In such case, the condition that $\rho$ sends a simple non-separating curve to an elliptic is necessary and can not be removed in any way. Here we show the following result, which is stronger with respect to [17, Proposition 6.2 $\big($case of $(g,k)=(2,1)\big)$].

**Theorem 4.2.4.** *Let $S$ be a surface of genus 2 and $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ any representation such that $\mathcal{E}(\rho) = \pm 1$. Then $\rho$ sends a simple non-separating curve to an elliptic.*

It remains to understand where the *almost every* condition comes from. As we see very soon, there are some representations that contain virtually abelian (but not abelian) subgroups which are free and generated by two elements. We will say that such representations contain a virtually pair (see 4.2.8 below for a precise definition). This kind of representations is problematic in the sense that we will explain in the next section and, as we see, Mathews' proof does not apply to them. However, we show that they arise as holonomy of a branched hyperbolic structure on $S$ in a different way; that is we prove the following.

**Proposition 4.2.17.** *Let $S$ be a surface of genus 2 and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm 1$ whose image contains a virtually abelian pair. Then $\rho$ is geometrizable by a branched hyperbolic structure with a single cone point of angle $4\pi$.*

Hence, by this proposition, the almost every condition can be removed as well. We will see also that such representations have a nice geometrical description.

The conclusion of the main theorem 4.2.1 may be reached with a second, slightly different, approach. In [21], Mathews was able to show also the following result.

**Theorem 4.2.5** (Mathews)**.** *Let $S$ be a closed surface of genus two and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm 1$. Suppose that there is a separating curve $\gamma$ on $S$ such that $\rho(\gamma)$ is not hyperbolic, then $\rho$ is the holonomy of a branched hyperbolic structure on $S$ with one cone point of angle $4\pi$.*

One might prove that any representation with $\mathcal{E}(\rho) = \pm 1$ sends a simple close separating curve to a non-hyperbolic element, and it would be enough to reach the conclusion of 4.2.1. On the other hand, putting theorems 4.2.2 and 4.2.5 together, a representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathcal{E}(\rho) = \pm 1$ is geometrizable by a branched hyperbolic structure with one cone point of angle $4\pi$ if

1. $\rho$ send a separating simple closed loop $\gamma$ to an elliptic element by 4.2.5; or
2. $\rho$ send a separating simple closed loop $\gamma$ to a parabolic element by 4.2.5; or
3. $\rho$ send a non-separating simple closed loop $\gamma$ to an elliptic element by 4.2.2 and 4.2.17.

Recently Marché and Wolff proved in [16] and [17] the Bowditch conjecture for representations of genus two surfaces; more precisely they were able to show the following result.

**Theorem 4.2.6** (Marché-Wolff, 2015). *Every non-Fuchsian representation sends some simple closed curve $\gamma$ to a non-hyperbolic element of* $\mathrm{PSL}_2\mathbb{R}$*.*

Hence we may reach the conclusion of the main theorem 4.2.1 by showing the following propositions.

1. There are no representations $\rho$ with $\mathcal{E}(\rho) = \pm 1$ and such that it sends a simple closed curve to the identity (proposition 4.2.26), and
2. suppose $\rho$ sends a non-separating closed curve to a parabolic and suppose there exists a simple closed non-separating curve $\beta$ such that $i(\alpha, \beta) = 1$ and $\mathsf{Fix}(\alpha) \cap \mathsf{Fix}(\beta) = \phi$. Then $\rho$ sends a non-separating curve to an elliptic element (proposition 4.2.28).

Indeed such propositions, together with 4.2.2, 4.2.5 and 4.2.17, imply the following claim:

> Let $S$ be a closed surface of genus two and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm 1$. Suppose that there is a simple curve $\gamma$ on $S$ such that $\rho(\gamma)$ is not hyperbolic, then $\rho$ is the holonomy of a branched hyperbolic structure on $S$ with one cone point of angle $4\pi$.

By Marché-Wolff's theorem, our main theorem 4.2.1 follows.

*Remark* 4.2.7. Observe that the case with $\mathcal{E}(\rho) = 0$ is ruled out from the discussion because there are not representations coming from a branched hyperbolic structure since they contradict the Gauß-Bonnet condition. Compare with 3.2.6.

### 4.2.2 Virtually abelian representations are problematic

We would like to distinguish a special class of representations, namely representations containing a *virtually abelian subgroup*. We recall that a group $G$ is said to be *virtually abelian* if it contains an abelian subgroup of finite index and similarly a representation $\rho$ in $\mathrm{PSL}_2\mathbb{R}$ is virtually abelian if its image is virtually abelian.

**Definition 4.2.8.** We will say that a representation contains a *virtually abelian pair* if there are two simple non-separating closed loops with intersection number one and their images via $\rho$ is given by two elliptic elements of order 2 with different fixed points. In this case, their commutator is a hyperbolic transformation along the axis passing through their fixed points.

In order to understand the issues of this kind of representations, we need to explain the proof of 4.2.2. Let $q$ be any point on $S$ and let $\rho : \pi_1(S, q) \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be any representations with $\mathcal{E}(\rho) = -1$ and suppose it sends a non-separating curve $\alpha$ to an elliptic. Starting from $\alpha$ we may found a separating curve $\gamma$ that split $S$ into two pieces, namely we may cut off the handle $H$ containing $\alpha$ from $S$.
More precisely let $\beta$ be a closed non-separating curve such that $i(\alpha, \beta) = 1$ and denote by $\gamma$ their commutator. Since $\alpha$ is elliptic, then by lemma 1.2.10 the commutator $\gamma$ has hyperbolic holonomy, in particular $\mathrm{Tr}\rho(\gamma) > 2$ by 1.2.5. Let $\rho_H$ be the representation of $\pi_1(H, q)$ induced by $\rho$ from the following inclusion $\pi_1(H, q) \hookrightarrow \pi_1(S, q)$, then $\mathcal{E}(\rho_H, \mathfrak{s}) = 0$ with respect to the special trivialization along the boundary $\gamma$ by the proposition 3.4.4. Hence we have localized the deficiency of the Euler class of $\rho$, moreover, by additivity, the relative Euler class of the representation induced on the other piece is extremal.
Let us denote by $\Sigma$ the closure of the subsurface $S \setminus H$ in $S$ and by $\rho_\Sigma$ the induced representation by $\rho$ from the inclusion $\pi_1(\Sigma, q) \hookrightarrow \pi_1(S, q)$. Hence $\rho_\Sigma$ is the holonomy of a complete hyperbolic structure with totally geodesic boundary by the theorem 3.3.3.
Suppose $\rho$ does not contain virtually abelian pairs, hence the representation $\rho_H$ is not virtually abelian and, by Theorem 1.4.16, it is the holonomy of a branched hyperbolic structure $\sigma_H$ on $H$ with geodesic boundary, except for at most one corner point of angle $\theta$ and no cone points inside $H$. Since $\mathrm{Tr}\rho(\gamma) > 2$ the magnitude of angle of the corner point is greater than $2\pi$ and does not exceed $3\pi$ (see [20, Proposition 5.8]).

Now the idea is to find a branched hyperbolic structure on $\Sigma$ with geodesic boundary except for at most corner point of angle $\theta_1 \in ]\pi, 2\pi[$, without cone points inside $\Sigma$ and holonomy $\rho_\Sigma$; that fits together with a branched hyperbolic structure on $H$ with one corner point of angle $\theta_2 = 4\pi - \theta_1 \in ]2\pi, 3\pi[$. Precisely we may identify the corner points and then glue these structures along their boundary. Topologically the resulting surface turns out to be the original surface $S$; geometrically we get $S$ endowed with a branched hyperbolic structure with one cone point of angle $4\pi$ and holonomy $\rho$.

In order to do this, we wish to truncate flares inside the convex core of $\Sigma$, to find a branched hyperbolic structure $\sigma_\Sigma$ on $\Sigma$ with a corner point of angle $\theta_1$. However this truncation cannot be done too far inside the convex core, but we may cut inside the collar of the geodesic boundary. We recall that the collar width $w(t)$ depends only on the trace $t$ of $\rho(\gamma)$ and it may be computed by the following formula

$$\sinh w(t) = \frac{1}{\sinh\left(\frac{d(t)}{2}\right)} \qquad \text{where } d(t) = 2\cosh^{-1}\left(\frac{t}{2}\right) \text{ is the translation distance}$$

Let $p$ be a point inside the collar, consider the geodesic representative of $\gamma$ based at $p$ and cut along it to obtain a branched structure on $\Sigma$ with one corner point of angle $\theta_2$. The developed image $\widehat{p}$ of $p$ lies inside the $w(t)-$neighbourhood of the axis of $\rho(\gamma)$. Using classical notion of hyperbolic geometry we may see that the magnitude of $\theta_2$ depends only on the distance of the point $\widehat{p}$ from the axis of $\rho(\gamma)$; that is the distance of $p$ from the geodesic boundary of the (unique) complete hyperbolic structure on $\Sigma$ with holonomy $\rho_\Sigma$ Hence the possible values of $\theta_1$ are bounded from above by the value $\theta_{\max}$ which depends on the width of the collar.
What remains to do is to find a branched hyperbolic structure on $H$ with one corner point of angle $4\pi - \theta_1$ that fits together with $\Sigma$ endowed with $\sigma_\Sigma$. Despite theorem 1.4.16 ensures the existence of branched hyperbolic structure on $H$ with holonomy $\rho_H$, we do not know a priori if $\rho_H$ may be the holonomy of a branched hyperbolic structure with one corner point of angle lying in the range $]\theta_{\max}, 3\pi[$.

*Remark* 4.2.9. The condition that $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ does not contain virtually abelian pair is necessary. Indeed when $\rho$ contains a virtually abelian pair the representation $\rho_H$ turns out to be virtually abelian, and such representation does not arise as holonomy of a branched hyperbolic structure on $H$.

**Good representations.** Before continuing the discussion we need some preliminaries. Following Mathews, we give the following definition.

**Definition 4.2.10.** Let $H$ be a punctured torus, with a basis $\alpha, \beta$ for $\pi_1(H, q)$, where $q$ is a point on the boundary of $H$. Let $\rho : \pi_1(H, q) \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathrm{Tr}[g, h] > 2$, where $g = \rho(\alpha)$ and $h = \rho(\beta)$. For any point $p$ in $\mathbb{H}^2$, we define $\mathcal{P}(g, h; p)$ to be the (possibily degenerate) hyperbolic pentagon given by the following polygonal

$$p \to [g^{-1}, h^{-1}](p) \to h(p) \to gh(p) \to h^{-1}gh(p) \to p$$

With the same notation of the previous definition we state the following lemma (the proof may be found in [20, Lemma 3.4]).

**Lemma 4.2.11.** *Let $\rho_H : \pi_1(H, q) \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation. The representation $\rho_H$ is the holonomy of a branched hyperbolic structure on $H$ with no interior cone points and at most one corner point if and only if there exist a free basis $(\alpha, \beta)$ of $\pi_1(H, q)$ and a point $p \in \mathbb{H}^2$ such that $\mathcal{P}(g, h; p)$ is a non-degenerate pentagon bounding an immersed open disc in $\mathbb{H}^2$.*

We then come back to previous discussion. The representation $\rho_H$ and a basis $(\alpha, \beta)$ of $\pi_1(H, q)$ are given by the construction above. Hence to find a branched hyperbolic structure on $H$, with holonomy $\rho_H$ and such that fits with the branched hyperbolic structure on $\Sigma$, means to find a basis $(\alpha', \beta')$ (possibly different to the given one!) and point $p$ inside the $w(t)-$neighbourhood of the axis $\rho_H\big([\alpha', \beta']\big)$ such that the pentagon $\mathcal{P}(g', h'; p)$ is a non-degenerate pentagon bounding an immersed open disc in $\mathbb{H}^2$; where $g' = \rho_H(\alpha')$ and $h' = \rho_H(\beta')$.

*Remark* 4.2.12. We remember that the change of basis of $\pi_1(H, q)$ changes the character of $\rho_H$, but it has no effect on the geometry on $H$. This reasoning may be extended to the entire surface $S$. Changing the base of the handle $H$ we change the presentation of the fundamental group $\pi_1(S, q)$, but again the change of basis does not effect on the geometry on $S$. Hence we do not care if we need to change the basis of $\pi_1(H, q)$ in order to find a *good* branched hyperbolic structure with holonomy $\rho_H$.

With this spirit, we give the following definition.

**Definition 4.2.13.** Consider a punctured torus $H$, with a basis $(\alpha, \beta)$ for $\pi_1(H, q)$, where $q$ is a point on the boundary of $H$; and a representation $\rho : \pi_1(H, q) \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathrm{Tr}[g, h] > 2$, where $g = \rho(\alpha)$ and $h = \rho(\beta)$. Define $\rho$ to be $\varepsilon-$good for a specified orientation of $H$, if there exists a basis $\alpha', \beta'$, of the same orientation as $\alpha, \beta$, and a point $p$ at distance less than $\varepsilon$ from $\mathsf{Axis}\rho([\alpha', \beta'])$, such that the pentagon $\mathcal{P}(g', h'; p)$ is non-degenerate, bounds an embedded disc, and is of the specified orientation.

We will say that a character is $\varepsilon-$good for a specified orientation of $H$ if it is the character of a $\varepsilon-$good representation. Note that since $\mathrm{Tr}[g,h] > 2$ the representation $\rho$ is irreducible, then any character correspond to a unique conjugacy class of representations. Thus

$$\text{a character is } \varepsilon - \text{good} \iff \text{all corresponding representations are } \varepsilon - \text{good}.$$

We define $\varepsilon-$*bad* representations (characters) those representations (characters) which are not $\varepsilon-$good. We will define *bad*-representations, those representations which $\varepsilon-$bad for any $\varepsilon$.

By the theorem 1.4.16, any non-virtually abelian representation is a $\varepsilon-$good representation for some $\varepsilon$, whereas virtually abelian representation are $\varepsilon-$bad for any $\varepsilon$ as we see below 4.2.14. In particular they are $w(t)-$bad for any $t > 2$. On the other hand a non-virtually abelian representation may be $w(t)-$bad even if it is good for other values of $\varepsilon$. Indeed the "goodness" is a weaker condition of $w(t)-$goodness because we want the point $p$ at a certain distance from $\mathsf{Axis}\rho([\alpha', \beta'])$. The main aim of this section is to show that for any $t > 2$ the *only* $w(t)-$bad representation are virtually abelian.

**Lemma 4.2.14.** *Let $\rho : \pi_1(H, q) \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a virtually abelian representation. Then $\rho$ is $\varepsilon-$bad for any $\varepsilon$.*

We recall for convenience the following characterization of virtually abelian representations. Let $G \subset \mathrm{PSL}_2\mathbb{R}$ be a subgroup generated by two elements $g, h$, then $G$ is virtually abelian (but not abelian) if and only if two of $\{g, h, gh\}$ are half-turns about points $q_1 \neq q_2 \in \mathbb{H}^2$ and the third is a non trivial translation along the axis $q_1q_2$. In particular, their commutator is also a translation along the same axis of $gh$.

*Proof.* We may suppose without loss of generality that $g, h$ are elliptics of order two and $gh$ is hyperbolic. If we take $p$ in $\mathsf{Axis}\ gh$, then all vertices of $\mathcal{P}(g, h; p)$ lies on the axis and the pentagon does not bound a disc. Thus we may suppose $p$ lies outside the axis, in particular: the points $p$, $gh(p)$ and $[g, h](p)$ lie in the same side and at the same distance from the axis, whereas the points $hgh^{-1}(p)$ and $h(p)$ lie on the other side. Since the segment $p \to [g, h](p)$ lies between $\mathsf{Axis}\ gh$ and the point $gh(p)$ the pentagon $\mathcal{P}(g, h; p)$ does not bound a disc. □

**Bad-representations are virtually abelian.** It is natural to ask if there are $w(t)-$bad representations which are not virtually abelian. Let $t > 2$ be a fixed real number. We consider the following subset of the relative character variety $\mathcal{X}_t(H)$:

- $\Omega_t$ the subset of characters of representations taking some simple closed curve to an elliptic;
- $B_t$ the set of $w(t)-$bad characters in $\Omega_t$ (where $w(t)$ is the quantity defined above). It turns out to be closed and nowhere dense in $\Omega_t$; and
- $V_t$ the closed subset of characters of virtually abelian representations in $\Omega_t$.

By Lemma 4.2.14 the following inclusion holds $V_t \subset B_t$. We recall for convenience that the symplectic $2-$form $\omega$ on $\mathcal{X}(H)$ induces on a symplectic form $\omega_t$ on any level set $\mathcal{X}_t(H)$. In particular, since $\mathcal{X}_t(H)$ is $2-$dimensional, $\omega_t$ turns out to be an area form $\mu_t$ which is invariant with respect to the action of the mapping class group $\text{MCG}(H)$. Finally, since we are considering representations $\rho$ that sends a simple non-separating curve to an elliptic element; any representation $\rho_H$ defined as above is a representation with elliptics; hence it belongs to open set $\Omega_t$.

**Proposition 4.2.15.** [21, Proposition 6.2] *For all $t > 2$, $\mu_t(B_t) = 0$ where $\mu_t$ is the measure induced by $\mu_H$ on the level set $\mathcal{X}_t(H)$. That is: $\mu_t-$almost every character in $\Omega_t$ is $w(t)-$good.*

The proof of such proposition relies on the following idea. It is always possible to construct by hand a representation $\rho^\star : \pi_1 H \longrightarrow \text{PSL}_2\mathbb{R}$ which is $w(t)/2-$good for any $t > 2$ and we consider its character which is of the form $(x^\star, y^\star, z^\star)$. Little perturbation of such character in $\mathcal{X}_t(H)$ is still the character of a $w(t)/2-$representation. The set $V$ of perturbations turns out to be an open subset of $\mathcal{X}_t(H)$ of positive measure and since the action of $\Gamma = \text{MCG}(H)$, is ergodic on $\Omega_t$ the claim follows because invariant sets have null, or conull, measure. For further details, we invite the reader to read [19, 21]. We are going to show the following result.

**Proposition 4.2.16.** *For all $t > 2$, $B_t = V_t$. Equivalently: if $\rho$ is a $w(t)-$bad representation, then it is virtually abelian.*

**Notation:** for the sake of readability and simplicity we will make a little abuse of notation confusing a character $(x, y, z)$ with the conjugacy class of representations having it as character. Indeed, under assumption, any representation we are considering is irreducible, hence any character corresponds to a unique conjugacy class of representation.

*Proof of proposition 4.2.16.* The strategy of the proof is the following. Let $\rho_0 \in \mathcal{X}_t(H)$ be a non-virtually abelian representation, hence by Theorem 1.4.16 it is the holonomy of branched hyperbolic structure on $H$ with geodesic boundary, except for at most one corner point, and no interior cone points. Hence there is a basis $(\alpha, \beta)$ for $\pi_1(H, q)$ and a point $p \in \mathbb{H}^2$ such that the pentagon $\mathcal{P}(g, h; p)$ bounds a disc in $\mathbb{H}^2$, where $g = \rho_0(\alpha)$ and $h = \rho_0(\beta)$. Clearly $\rho_0$ is a $d-$good representation where $d$ is the distance of $p$ from the axis of $\rho_0([\alpha, \beta])$. If $d < w(t)$ there is nothing to prove, otherwise we show that there is a particular basis and a particular point such that $d < w(t)$.

Little perturbations of $\rho_0$ (that is of its character) form an open set $B_\varepsilon(\rho_0)$ of $d-$good characters in $\mathcal{X}_t(H)$, thus the set of $d-$good characters has full measure in $\mathcal{X}_t(H)$ because the mapping class group $\mathrm{MCG}(H)$ acts ergodically on each level of the character variety of $H$. Let $\rho_1 \in B_\varepsilon(\rho_0)$ be a $w(t)/2-$good representation and consider a path $\alpha : [0, 1] \longrightarrow \mathcal{X}_t(H)$ such that

1 $\alpha(0) = \rho_0$;

2 $\alpha(1) = \rho_1$;

3 $\alpha(s)$ is $w(t)/2-$good for every $s \in ]0, 1]$.

We may observe that such path exists because $w(t)/2-$good representations form an open set of full measure. The representation $\alpha(1) = \rho_1$ is a $w(t)/2-$good, hence there is a basis $(\alpha_1, \beta_1)$ and a point $p_1$ within the axis of $\rho_1([\alpha_1, \beta_1])$ such that the pentagon $\mathcal{P}(g_1, h_1; p_1)$ bounds a disc. Let $\delta_1$ be the distance between the axis of $\rho_1([\alpha_1, \beta_1])$ and the point $p_1$. We now perturb $\rho_1$ along $\alpha$, and for any $s$ sufficiently near to 1 the character $(x_s, y_s, z_s)$ is the character of a $w(t)/2-$good representation, in particular we may choose $p_s$ near to $p_1$ so that $\mathcal{P}(g_s, h_s; p_s)$ still bound a disc. Moving $s$ to 0 the pentagon $\mathcal{P}(g_s, h_s; p_s)$ changes at any time. If for any time $s$ the pentagon $\mathcal{P}(g_s, h_s; p_s)$ bounds a disc, then we may choose $p_0$ near to $p_1$ and such that $\mathcal{P}(g_0, h_0; p_0)$ bounds a pentagon, hence $\rho_0$ is $w(t)/2-$good as desidered. Suppose there exists $s_1$ such that $\mathcal{P}(g_s, h_s; p_s)$ does not bound a disc for any $s \in [0, s_1]$. The representation $\rho_{s_1}$ is still $w(t)/2-$good, hence we may found another basis $(\alpha_{s_1}, \beta_{s_1})$ and another point $p_{s_1}$ at distance less than $w(t)/2$ from the axis of $\rho_1([\alpha_{s_1}, \beta_{s_1}])$ such that $\mathcal{P}(g_{s_1}, h_{s_1}; p_{s_1})$ bounds a disc. Set $\delta_2$ to be maximum between the distance $d_{\mathbb{H}^2}\big(p_{s_1}, \mathsf{Axis}(\rho_{s_1}([\alpha_{s_1}, \beta_{s_1}])\big)$ and $\delta_1$. We may iterate this procedure. If it is not end after a finite number of steps, we find an increasing sequence $\delta_k \in [0, w(t)/2]$, which has limit $\delta \leq w(t)/2$. We may note that all representation along the path $\alpha$ are in particular $\delta-$good (and $\delta \leq w(t)/2$).

Suppose now that $\rho_0$ is not $\delta-$good, that is $\delta-$bad. Hence for any basis $(\alpha, \beta)$ and any point $p$ at distance less than $\delta$ from the axis $\rho_0([\alpha, \beta])$, the pentagon $\mathcal{P}(g, h; p)$

does not bounds a disc in $\mathbb{H}^2$, where $g = \rho_0(\alpha)$ and $h = \rho_0(\beta)$. Fix a particular basis $(\alpha', \beta')$ and a particular point $p'$ at distance less than $\delta$ from the axis $\rho_0([\alpha', \beta'])$ and we define the quantity $\xi(\alpha', \beta'; p')$ as the maximal radious for a Euclidean ball centered in $\rho_0$ such that any representation $\rho$ inside such ball satisfy the following condition

> $\mathcal{P}(\rho(\alpha), \rho(\beta); p')$ bounds a disc if and only if the pentagon $\mathcal{P}(g', h'; p')$ does

The quantity $\xi(\alpha', \beta', p')$ depends on the choices of the basis and the point; and we define the following

$$\xi = \inf_{\substack{(\alpha,\beta) \text{ basis of } \pi_1(H,q); \\ p \in \delta-\text{neighbourhood of axis } \rho([\alpha,\beta])}} \xi(\alpha', \beta', p')$$

We may easily see that $\xi = 0$ if and only if $\rho_0$ is virtually abelian. Since $\rho_0$ is not virtually abelian by assumption, $\xi > 0$, hence any representation inside the Euclidean ball $B_\xi(\rho_0)$ is $\delta-$bad, in particular any representation in $\alpha \cap B_\xi(\rho_0)$ is $\delta-$bad. Hence a contradiction. Thus $\rho$ is $w(t)-$good and $B_t = V_t$ and this conclude the proof of 4.2.16. $\square$

### 4.2.3 The final solution - first approch

We are going to prove the main theorem 4.2.1. We begin removing the 'almost every' condition from the theorem 4.2.2, that is the existence of a non-separating curve with elliptic holonomy becomes a sufficient condition for a representation to be the holonomy of a branched hyperbolic structure.

**Representations with virtually abelian pairs.** In this paragraph $\rho$ always be a representation containing a virtually abelian pair (see 4.2.8 for the definition of such representations). We prove the following result.

**Proposition 4.2.17.** *Let $S$ be a surface of genus 2 and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm 1$ whose image contains a virtually abelian pair. Then $\rho$ is geometrizable.*

We remember that although $g$ and $h$ are defined up to a sign in $\mathrm{SL}_2\mathbb{R}$ the commutator $[g, h]$ is a well-defined element in $\mathrm{SL}_2\mathbb{R}$. In particular, lifting $g$ and $h$ in $\mathrm{SL}_2\mathbb{R}$ with either sign does not change the trace of the commutator, thus it makes sense to consider the trace of $[g, h] \in \mathrm{PSL}_2\mathbb{R}$.

*Proof of proposition 4.2.17.* Up to change the orientation of $S$, we may suppose that $\mathcal{E}(\rho) = -1$. We divide the proof into three lemmata. In the first one, we show the existence of a simple closed separating curve $\gamma$ dividing $S$ in two punctured torus $H$ and $\Sigma$, and such that one of the induced representations is virtually abelian. In order to do this, we fix a base point $q \in S$ for the fundamental group $\pi_1(S, q)$.

**Lemma 4.2.18.** *There exists a simple separating curve with hyperbolic holonomy such that the induced representation on $H$ is virtually abelian.*

*Proof.* Since $\rho$ has virtually abelian pairs there are two simple non-separating curves $\alpha_1$ and $\beta_1$ based at $q$ such that $g_1 = \rho(\alpha_1)$ and $h_1 = \rho(\beta_1)$ are elliptics of order 2 and their intersection number is one. Their commutator $\gamma$ is a separating simple curve with hyperbolic holonomy and divides $S$ in two punctured torus $H$ and $\Sigma$, each of them with a marked point $q_1$ and $q_2$ respectively on the boundary (note that they coincide with $q$ on the overall surface $S$). We define $H$ to be the once containing $\alpha_1$ and $\beta_1$, and $\rho_1$ to be the induced representation of $\pi_1(H, q_1)$ via $\rho$. By construction $\rho_1$ is virtually abelian. $\square$

Let $\Sigma$ be the second piece and define $\rho_2$ to be the induced representation of $\pi_1(\Sigma, q_2)$. Let $\alpha_2, \beta_2$ be a basis for $\pi_1(\Sigma, q_2)$ so that $[\alpha_2, \beta_2]$ is homotopic to $\gamma$ but traversed in opposite direction with respect to $[\alpha_1, \beta_1]$. Set $g_2 = \rho_2(\alpha_2)$ and $h_2 = \rho_2(\beta_2)$.

**Lemma 4.2.19.** *The representation $\rho_2 : \pi_1(\Sigma, q_2) \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is Fuchsian.*

*Proof.* Consider the basis for $\pi_1(H, q_1)$ and $\pi_1(\Sigma, q_2)$ defined above. Since $[\alpha_1, \beta_1] = \gamma$ and $[\alpha_2, \beta_2] = \gamma^{-1}$, the fundamental group of $S$ has the following standard presentation

$$\pi_1 S = \big\langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] = \mathrm{id} \big\rangle$$

The relation between the generators may be read in terms of hyperbolic transformations, namely $[g_1, h_1][g_2, h_2] = id$. It lifts to the relation $[g_1, h_1][g_2, h_2] = -id$ in $\mathrm{SL}_2\mathbb{R}$ because $\mathcal{E}(\rho) = -1$, and then $\mathrm{Tr}[g_1, h_1] = -\mathrm{Tr}[g_2, h_2]$. Since $g_1$ is elliptic, the trace of $[g_1, h_1]$ is greater than 2 by 1.2.5 and 1.2.10, it follows that $\mathrm{Tr}[g_2, h_2] < -2$; moreover $g_2$ and $h_2$ are both hyperbolic by 1.2.5 and their axis cross. The relative Euler class $\mathcal{E}(\rho_2, \mathfrak{s}) = -1$ by 3.4.4 then $\rho_2$ is Fuchsian and it is the holonomy of a complete hyperbolic structure with geodesic boundary. $\square$

**Lemma 4.2.20.** *There exists a fundamental domain for $\rho$. It turns out to be a pentagon, namely a degenerate octagon with four sides aligned.*

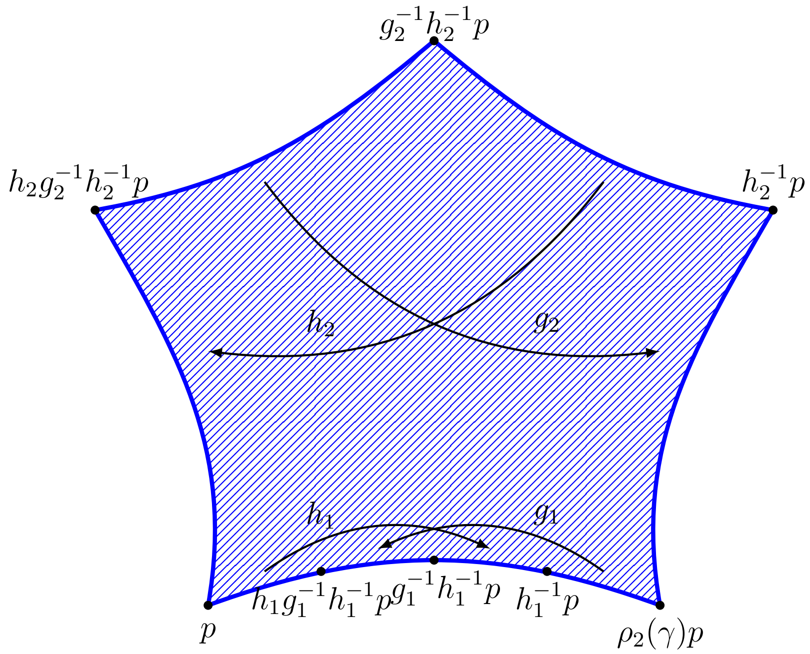


Figure 4.1: The fundamental domain for $\rho_2$ in blue.

*Proof.* We are now going to construct a fundamental domain for $\rho$. So let $p$ be a point on Axis $\rho_2(\gamma)$. Since $\rho_2$ is Fuchsian we may start from $p$ to define a pentagonal fundamental domain for $\rho_2$ such that the sum of all inner angles is exactly $\pi$. Observe that $\rho_2(\gamma)p \in$Axis $\rho_2(\gamma)$ so the entire segment joining them lies on the axis of $\rho_2(\gamma)$. Now we use the representation $\rho_1$ to divide such segment in four small pieces, like in the picture 4.1 to obtain a degenerate octagon with three straight angles, so that the sum of all interior angles is exactly $4\pi$. □

We finally glue the correspondent sides using $\rho$ to obtain a closed surface of genus 2 endowed with a branched hyperbolic structure with exactly one cone point of angle $4\pi$, and this concludes the proof of 4.2.17 □

*Remark* 4.2.21. A more geometrical interpretation of the proof above is the following. Let $S$ be a puncture torus endowed with a complete hyperbolic structure with totatlly geodesic boundary $\gamma$ and holonomy $\rho_2$. Divide $\gamma$ in four segments $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ arranged in cyclic order with lenght $l_1, l_2, l_1, l_2$ respectively and then glue $\gamma_1$ with $\gamma_3$ and $\gamma_2$ with $\gamma_4$ reverting the orientation. The result of this operation is a closed surface of genus 2 with a branched hyperbolic structure with one cone point of angle $4\pi$ and holonomy $\rho$.

**Finding curve with elliptic holonomy.** In order to conclude the proof of the main theorem 4.2.1 we are going to show 4.2.4, that is: *any representation with* $\mathcal{E}(\rho) = \pm 1$ *sends a simple non-separating curve to an elliptic.*

In [9]: Gallo-Kapovich-Marden show that for any non-elementary representation $\pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$, where $S$ is a closed surface with $\chi(S) < 0$, there is a system of simple closed curves decomposing $S$ in pants and such that every curve has hyperbolic holonomy.

**Theorem 4.2.22** (Gallo-Kapovich-Marden [9])**.** *Let $S$ be an oriented surface with $\chi(S) < 0$ and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a non-elementary representation. Then there exists a system of disjoint curves $\alpha_i$ decomposing $S$ into pants and punctured tori, such that each $\rho(\alpha_i)$ is hyperbolic.*

Explicitly, they cut $g-1$ handles, one at a time, so that the genus decreases by 1 at each step; and from the remaining piece (of genus 1), cut off pants until they get a punctured torus, then this too is cut into pants. This clearly implies the existence of a separating closed curve with hyperbolic holonomy.
Applying they result in our case we may deduce that any representation $\rho$ with $\mathcal{E}(\rho) = \pm 1$ sends a simple separating curve to a hyperbolic element. Let $\gamma$ be such curve. Then $\gamma$ splits $S$ in two punctured torus $\Sigma_1$ and $\Sigma_2$ with representations $\rho_1$ and $\rho_2$ defined in the usual way. The relative Euler numbers $\mathcal{E}(\rho_1, \mathfrak{s})$ and $\mathcal{E}(\rho_2, \mathfrak{s})$ are well-defined with respect to the special trivialization defined along $\gamma$ because it has hyperbolic holonomy. We may suppose without loss of generality that $\mathcal{E}(\rho) = -1$ (because the other case occur reversing the orientation), in this case only one of the following holds

**1** $\mathcal{E}(\rho_1, \mathfrak{s}) = -1$ and $\mathcal{E}(\rho_2, \mathfrak{s}) = 0$, or

**2** $\mathcal{E}(\rho_1, \mathfrak{s}) = 0$ and $\mathcal{E}(\rho_2, \mathfrak{s}) = -1$.

Up to rename components and representations we may suppose to be in the first case. Let $(\alpha, \beta)$ be a basis for $\pi_1(\Sigma_2, q_2)$, where $q_2$ is a point on the boundary of $\Sigma_2$ and $[\alpha, \beta] = \gamma$. Note that $\mathrm{Tr}\ \rho_2([\alpha, \beta]) > 2$ by 3.4.4. Since $\mathrm{Tr}\ \rho_2(\gamma) > 2$, Goldman showed in [10] that only one of the following possibilities occur.

1. $\rho_2$ is a pants representation,
2. $\rho_2$ is a representation with elliptics.

We have the following result.

**Lemma 4.2.23.** *Let $\psi$ be any pants representation. Then it has non-trivial relative Euler class; equivalently the relative Euler number is not zero.*

*Proof.* We argue by contradiction. Let $P$ be a pair of pants and

$$\pi_1 P = \langle b_1, b_2, b_3 | b_1 b_2 b_3 = 1 \rangle$$

be a presentation of the fundamental group. Let $\psi : \pi_1 P \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a pants representation, up to change the basis of $\pi_1 P$ we may suppose that $\rho(b_i) = g_i$ is parabolic or hyperbolic for any $i \in \{1, 2, 3\}$. Let $\widetilde{c_i}$ be the simplest lift of $g_i$ and assume that $\widetilde{c_1}\widetilde{c_2}\widetilde{c_3} = 1$. As $\widetilde{c_3}^{-1} = \widetilde{c_1}\widetilde{c_2} \in \mathrm{Hyp}_0$, then $\mathrm{Tr}\widetilde{c_1} \geq 2$, $\mathrm{Tr}\widetilde{c_2} \geq 2$ and $\mathrm{Tr}\widetilde{c_1}\widetilde{c_2} \geq 2$, hence $\mathrm{Tr}g_1\mathrm{Tr}g_2\mathrm{Tr}g_1g_2 \geq 8$. Note that this make sense because lifting $g_i$ to $\mathrm{SL}_2\mathbb{R}$ with either sign does not change the product. On the other hand $\psi$ is a pants representation, thus $\mathrm{Tr}g_1\mathrm{Tr}g_2\mathrm{Tr}g_1g_2 \leq -8$: a contradiction. ☐

We finally show that the following.

**Lemma 4.2.24.** *$\rho_2$ is a representation with elliptics.*

*Proof.* By lemma 4.2.23, if $\rho_2$ were a pant representation then

$$\widetilde{\rho_2(\alpha)}\widetilde{\rho_2(\beta)}\widetilde{\rho_2(\alpha\beta)}^{-1} = \mathbf{z}$$

On the other hand we have the following chain of equalities, because $\rho_2$ lifts to a representation in $\widetilde{\mathrm{PSL}_2}\mathbb{R}$

$$\widetilde{\rho_2(\alpha)}\widetilde{\rho_2(\beta)}\widetilde{\rho_2(\alpha\beta)}^{-1} = \widetilde{\rho_2(\alpha)}\widetilde{\rho_2(\beta)}\widetilde{\rho_2(\alpha)}^{-1}\widetilde{\rho_2(\beta)}^{-1}\widetilde{\rho_2(\gamma)}^{-1} = [\widetilde{\rho_2(\alpha)}, \widetilde{\rho_2(\beta)}]\widetilde{\rho_2(\gamma)}^{-1} = 1$$

Hence $\rho_2$ is a representation with elliptics, then it send a simple closed loop $\delta$ to an elliptic element in $\mathrm{PSL}_2\mathbb{R}$. ☐

This conclude the proof of 4.2.4, indeed the proof of theorem 4.2.1.

### 4.2.4 The final solution - second approch

In order to prove the theorem 4.2.1 a different approch is possible. Recently it was proved the Bowditch conjecture for surfaces of genus 2; that is:

**Theorem 4.2.25** (Marché-Wolff 2015 [16])**.** *Every non-Fuchsian representation sends some simple closed loop $\gamma$ to a non-hyperbolic element of* $\mathrm{PSL}_2\mathbb{R}$.

Representations $\rho$ with Euler number $\mathcal{E}(\rho) = \pm 1$ are never Fuchsian representation, hence there are five possibilities:

1. $\rho$ send a simple closed loop to the identity;
2. $\rho$ send a separating simple closed loop $\gamma$ to an elliptic element;
3. $\rho$ send a separating simple closed loop $\gamma$ to a parabolic element;
4. $\rho$ send a non-separating simple closed loop $\gamma$ to an elliptic element;
5. $\rho$ send a non-separating simple closed loop $\gamma$ to a parabolic element.

In fact the case (1) never occurs as shown by the following lemma.

**Proposition 4.2.26.** *Let $S$ be a surface of genus 2 and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation such that $\mathcal{E}(\rho) = \pm 1$. Then no simple closed loop is sent to the identity.*

*Proof.* Up to change the orientation of $S$, we suppose that $\mathcal{E}(\rho) = -1$. Suppose that $\alpha$ is a simple curve such that $\rho(\alpha) = id$. If $\alpha$ is a non-separating curve, let $\beta$ be any non-separating simple curve such that $i(\alpha, \beta) = 1$, and denote by $\gamma$ their commutator. Of course $\rho(\gamma) = id$, thus we can suppose that $\rho$ send a separating simple curve $\gamma$ to the identity. Since $\gamma$ is separating we may cut $S$ in two punctured torus $\Sigma_1$ and $\Sigma_2$, and we obtain two representations $\rho_1$ and $\rho_2$. The relative Euler numbers are well-defined because $\rho(\gamma) = id$, and by additivity $\mathcal{E}(\rho_1, \mathfrak{s}) + \mathcal{E}(\rho_2, \mathfrak{s}) = \mathcal{E}(\rho) = -1$. We suppose without loss of generality that $\mathcal{E}(\rho_1, \mathfrak{s}) = -1$ and $\mathcal{E}(\rho_2, \mathfrak{s}) = 0$. Then $\rho_1$ is the holonomy of a complete hyperbolic structure on $\Sigma_1$ with totally geodesic or cusped boundary (see 3.3.3). In particular the holonomy of the boundary $\gamma$ must be hyperbolic or parabolic, then a contradiction. □

The cases (2) and (3) are completely covered by the theorem 4.2.5. Hence we may assume $\rho$ sends a non-separating simple curve $\alpha$ to a non-hyperbolic element. Suppose first $\rho(\alpha)$ is an elliptic element. Let $\beta$ be another simple non-separating closed curve such that $i(\alpha, \beta) = 1$, and consider their commutator $\gamma = [\alpha, \beta]$. By the classical theory of geometry of hyperbolic transformations the commutator has hyperbolic or trivial holonomy (see 1.2.10). In particular, $\rho(\gamma)$ cannot be trivial by the previous lemma 4.2.26. Hence $\rho(\gamma)$ is hyperbolic. If $\rho$ does not contain virtually abelian pairs, then $\rho$ is geometrizable by the theorem 4.2.5. On the other hand, if $\rho$ contains virtually abelian pairs, then it is geometrizable by proposition 4.2.17. Hence also the case (4) is completely covered. In order to finish the proof of 4.2.1 we need to show the following result.

**Proposition 4.2.27.** *Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm 1$. Suppose it sends a non-separating simple curve to a parabolic element. Then $\rho$ is geometrizable by a branched hyperbolic structure with one cone point of angle $4\pi$.*

Let $\alpha$ be a non-separating simple closed curve with parabolic image and let $\beta$ be a simple closed loop such that $i(\alpha, \beta) = 1$. Denote by $\gamma$ their commutator. If $h = \rho(\beta)$ is elliptic we have done by 4.2.5, then we may assume it is a parabolic or hyperbolic transformation. Since $g = \rho(\alpha)$ is a parabolic transformation, it may share a fixed point with $h$ and in such case $\rho(\gamma)$ is a parabolic transformation by 1.2.9. Since $\gamma$ is a separating curve we have done again by 4.2.2, thus we may assume that $g$ and $h$ have not a common fixed point and their commutator is hyperbolic. Hence proposition 4.2.27 may be reduced to the following one.

**Proposition 4.2.28.** *Let $S$ be a surface of genus $2$ and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm 1$. Suppose $\rho$ sends a non-separating simple closed curve $\alpha$ to a parabolic element and there exists a simple closed curve $\beta$ such that $i(\alpha, \beta) = 1$ and $\mathsf{Fix}(\rho(\alpha)) \cap \mathsf{Fix}(\rho(\beta)) = \phi$. Then $\rho$ sends a non-separating closed loop $\delta$ to an elliptic element.*

*Proof of proposition 4.2.28.* Let $q$ be a point on $S$ and consider $\pi_1(S, q)$, that is we may consider that all curves are based at $q$. Let $\alpha$ be a non-separating curve with parabolic image and $\beta$ a simple non-separating curve such that $i(\alpha, \beta) = 1$ and $\mathsf{Fix}(\rho(\alpha)) \cap \mathsf{Fix}(\rho(\beta)) = \phi$. Define $\gamma$ their commutator. Since $\gamma$ is a simple closed separating curve, it splits $S$ in two pieces and let $\Sigma$ be the one containing $\alpha$. Of course $\Sigma$ also contains the curves $\beta$, and $\gamma$ as its boundary component. Let $\psi : \pi_1 \Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$ the induced representation on $\pi_1(\Sigma, q_1)$ by $\rho$, where the point $q_1$ coincide with $q$ on the overall surface. The trace of $\psi(\gamma)$ is greater than 2 by 1.2.5, hence the relative Euler class $\mathcal{E}(\psi, \mathfrak{s}) = 0$ by 3.4.4. Then $\psi$ is a representation with elliptics by 4.2.23 and 4.2.24 and we are done. □

Thus any representation sending a non-separating curve to a parabolic element sends also a non-separating closed curve to an elliptic element in $\mathrm{PSL}_2\mathbb{R}$ and this concludes the proof of 4.2.1.

## 4.3 Higher genus cases

A natural question now may be: *what about surfaces of genus at least 3?* In the previous section we stated the Mathews' theorem 4.2.2 for surfaces of genus 2 but it holds for any genus, indeed the strategy of the proof does not care about the genus of $S$. Thus we report here the original version of such theorem.

**Theorem 4.3.1** (Mathews)**.** *Let $S$ be a closed surface of genus $g \geq 2$. Then almost every representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$, which sends a non-separating curve $\gamma$ on $S$ to an elliptic is the holonomy of a branched hyperbolic structure on $S$ with one cone point of angle $4\pi$.*

Actually we are not able to generalize it as in the case of genus 2 surfaces, but little improvements are possible. More precisely the 'almost every' condition may be removed as in the previous case. However, we are not able to show that *any* representation sends a simple non-separating curve to an elliptic. Also, the proposition 4.2.28 may be reformulated for all closed surfaces because there is nothing special about the genus 2 in the proof.

**The first improvement of 4.3.1.** Like in the genus 2 case the 'almost every' condition arises because there are representations containing virtually abelian pairs. Those representations are bad representation in the sense of the definition 4.2.13, as we see in the previous section they are not the holonomy of a branched hyperbolic structure obtained by gluing two surfaces with boundary, both endowed with a hyperbolic structure with a corner point on the boundary. However, they are the holonomy of a branched hyperbolic structure obtained from a complete hyperbolic structure on a surface with totally geodesic boundary, where the boundary is glued with itself like in the remark 4.2.21.

**Proposition 4.3.2.** *Let $S$ be a surface of genus $g \geq 2$ and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$ whose image contains a virtually abelian pair. Then $\rho$ is geometrizable by a branched hyperbolic structure with a cone point of angle $4\pi$.*

*Proof.* This proof is a summary of the arguments of the proof of 4.2.17. As before, up to change the orientation of $S$ we may suppose $\mathcal{E}(\rho) = \chi(S) + 1$. Let $q$ be any point on $S$ and consider $\pi_1(S, q)$. Since $\rho$ contains virtually abelian pairs there are two simple non-separating curves $\alpha_1$ and $\beta_1$ based at $q$ such that $g_1 = \rho(\alpha_1)$ and $h_1 = \rho(\beta_1)$ are elliptics of order 2 and their intersection number is one. Their commutator $\gamma$ is a separating simple curve with hyperbolic holonomy and divides $S$ into two pieces and one of them is a punctured torus. We define $H$ the punctured torus containing $\alpha_1$ and $\beta_1$, and $\rho_1$ to be the induced representation of $\pi_1(H, q_1)$ via

$\rho$, where $q_1$ is the point on the boundary of $H$ that coincide with $q$ on the overall surface. By construction $\rho_1$ is virtually abelian.
Let $\Sigma$ be the second piece and define $\rho_2$ to be the induced representation of $\pi_1(\Sigma, q_2)$, where, as above, where $q_2$ is the point on the boundary of $\Sigma$ that coincide with $q$ on the overall surface. We claim, like in the previous case, that the representation $\rho_2 : \pi_1(\Sigma, q_2) \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is Fuchsian. Indeed, since $g_1$ is elliptic, the trace of $[g_1, h_1]$ is greater than 2 by 1.2.5 and 1.2.10, thus the relative Euler class $\mathcal{E}(\rho_1, \mathfrak{s}) = 0$ by 3.4.4 then $\mathcal{E}(\rho_2, \mathfrak{s}) = \chi(\Sigma)$, that is $\rho_2$ is a Fuchsian representation by 3.3.3 and it is the holonomy of a complete hyperbolic structure with geodesic boundary on $\Sigma$.
Finally, we are going to construct a fundamental domain for $\rho$. So let $p$ be a point on $\mathsf{Axis}\ \rho_2(\gamma)$. Since $\rho_2$ is Fuchsian we may start from $p$ to define a fundamental domain for $\rho_2$ such that the sum of all inner angles is exactly $\pi$ that turns out to be a $4g-3$-gon in $\mathbb{H}^2$. Observe that $\rho_2(\gamma)p \in \mathsf{Axis}\ \rho_2(\gamma)$ so the entire segment joining them lies on the axis of $\rho_2(\gamma)$. Now we use the representation $\rho_1$ to divide such segment into four smaller pieces so that the sum of all interior angles is exactly $4\pi$. Gluing the correspondent sides using $\rho$ we get a closed surface of genus $g$ endowed with a branched hyperbolic structure with exactly one cone point of angle $4\pi$, and this conclude the proof of 4.3.2 □

**A second improvement.** In this paragraph, we will relax furtherly the hypothesis of 4.3.1. As before $S$ is a surface of genus greater than two and $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is a representation such that $\mathcal{E}(\rho) = \chi(S) + 1$.
Even if we are not able to show that any representation sends a simple non-separating curve to an elliptic, we are able to show that any representation $\rho$ that sends a non-separating curve to a non-hyperbolic element is the holonomy of a branched hyperbolic structure on $S$. First of all, we notice that the following fact still holds in the higher genus case, indeed using the same arguments we may show the following lemma.

**Lemma 4.3.3.** *Let $S$ be a surface of genus $g \geq 2$ and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation such that $\mathcal{E}(\rho) = \pm\big(\chi(S) + 1\big)$. Then no simple closed loop is sent to the identity.*

Suppose $\rho$ sends a non-separating simple curve $\alpha$ to a parabolic element, let $\beta$ be a simple curve such that $i(\alpha, \beta) = 1$ and denote by $\gamma$ their commutator, by the previous lemma $\beta$ and $\gamma$ have not trivial holonomy. If $h = \rho(\beta)$ is elliptic we have done by 4.3.1, then we may assume it is a parabolic or hyperbolic transformation.

Since $g(\alpha)$ is a parabolic transformation, it might share a fixed point with $h$. In this case the commutator $\rho(\gamma)$ is a parabolic transformation by lemma 1.2.9. The following lemma shows that we can always find a non-separating curve $\beta$, such that $i(\alpha,\beta)=1$ and $h=\rho(\beta)$ does not share any fixed point with $\rho(\alpha)$.

**Lemma 4.3.4.** *Let $\rho:\pi_1 S\longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho)=\pm\big(\chi(S)+1\big)$. Suppose $\rho$ sends a non-separating curve $\alpha$ to a parabolic element. Then there exists a simple non separating curve $\beta$ such that $i(\alpha,\beta)=1$ and $\mathsf{Fix}(\rho(\alpha))\cap\mathsf{Fix}(\rho(\beta))=\phi$. In particular $\rho\big([\alpha,\beta]\big)$ is hyperbolic.*

*Proof.* Let $\alpha$ and $\beta'$ as in the discussion above and suppose they share a fixed point $q$. Since $\rho$ is non-elementary, there is a simple curve $\xi$ such that does not meet $\alpha$ and $\beta'$ and it does not fix $q$. Take $\xi$ with the orientation so that $\beta=\beta'\xi$ is homotopic to a simple curve, then it not fix $q$ because $\xi$ does not and $i(\alpha,\beta)=1$ by construction. □

Thus we may assume that $g=\rho(\alpha)$ and $h=\rho(\beta)$ have not a common fixed point and their commutator is hyperbolic by 1.2.9. Now the proposition 4.2.28 may be restate for any genus, precisely:

**Proposition 4.3.5.** *Let $S$ be a surface of genus $g\geq 2$ and let $\rho:\pi_1 S\longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a representation with $\mathcal{E}(\rho)=\pm\big(\chi(S)+1\big)$. Suppose $\rho$ sends a non-separating simple closed loop $\alpha$ to a parabolic element and there exists a simple closed curve $\beta$ such that $i(\alpha,\beta)=1$ and $\mathsf{Fix}(\rho(\alpha))\cap\mathsf{Fix}(\rho(\beta))=\phi$. Then $\rho$ sends a non-separating closed loop $\delta$ to an elliptic element.*

*Proof of proposition 4.3.5.* Let $q$ be a point on $S$ and consider $\pi_1(S,q)$, that is we may consider that all curves are based at $q$. Let $\alpha$ be a non-separating curve with parabolic image and $\beta$ a simple non-separating curve such that $i(\alpha,\beta)=1$ and $\mathsf{Fix}(\rho(\alpha))\cap\mathsf{Fix}(\rho(\beta))=\phi$. Define $\gamma$ their commutator. Since $\gamma$ is a simple closed separating curve, it splits $S$ in two pieces and let $\Sigma$ be the one containing $\alpha$. Of course $\Sigma$ also contains the curves $\beta$, and $\gamma$ as its boundary component. Let $\psi:\pi_1(\Sigma,q_1)\longrightarrow \mathrm{PSL}_2\mathbb{R}$ the induced representation of $\pi_1(\Sigma,q_1)$ by $\rho$; where $q_1$ is a point on the boundary that coincide with $q$ on the overall surface. The trace of $\psi(\gamma)$ is greater than 2 by 1.2.5, hence the relative Euler class $\mathcal{E}(\psi,\mathfrak{s})=0$ by 3.4.4. Then $\psi$ is a representation with elliptics by 4.2.23 and 4.2.24 and we are done. □

Thus Mathews' theorem 4.3.1 may be improve in the following way:

**Theorem 4.3.6.** *Let $S$ be a closed surface of genus $g\geq 3$. Then every representation $\rho:\pi_1 S\longrightarrow \mathrm{PSL}_2\mathbb{R}$ with $\mathcal{E}(\rho)=\pm\big(\chi(S)+1\big)$, which sends a non-separating simple* *curve $\gamma$ on $S$ to a non-hyperbolic element is the holonomy of a branched hyperbolic structure on $S$ with one cone point of angle $4\pi$.*

## 4.4 Purely hyperbolic representations

We are going to introduce a particular type of representations, namely *purely hyperbolic representations*. Of course, Fuchsian representations are purely hyperbolic, but also some non-Fuchsian representations are; in the previous example 1.5.4 we may found examples of such representations. From now on we will deal with surfaces of genus $g \geq 2$. Motivated by the example 1.5.4 we give the following definition.

**Definition 4.4.1.** We will say that a non-elementary representations $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is *purely hyperbolic* if its image consist only of hyperbolic elements other than the identity.

We may wonder if purely hyperbolic representations arise as holonomy of a hyperbolic cone-structure. The Fuchsian case is well-known in literature, indeed Goldman's theorem [12, Corollary D] characterize them completely. On the other hand in the following paragraph, we will give examples of purely hyperbolic representations which never arise as the holonomy of a hyperbolic cone-structure.

We recall, for the reader convenience, that a representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is said to be *discrete* if its image is a discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$ (with respect to the induced topology of the Lie group structure). A generic non-elementary, discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$ contains hyperbolic elements, but it might contain also parabolic elements or elliptic elements of finite order (see [1, Theorem 8.4.1]). More precisely there is the following characterization.

**Proposition 4.4.2.** *A subgroup $\Gamma$ of $\mathrm{PSL}_2\mathbb{R}$ is discrete if and only if each elliptic element (if any) has finite order.*

By the previous proposition, we may note that any purely hyperbolic representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ is discrete, *i.e.* the image of $\rho$ is a discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$. In [12], Goldman shows that faithful and discrete representations are Fuchsian. Hence non-Fuchsian, purely hyperbolic representations are discrete and not faithful representations.

There are examples of purely hyperbolic representations which do not arise as the holonomy of a branched hyperbolic structure (see 4.5.1 in the sequel). On the other hand, there are purely hyperbolic representations which are the holonomy of a branched hyperbolic structure as shown in the following example.

**Example 4.4.3.** Let $\Sigma$ be a surface of genus 2 with a complete hyperbolic structure $\sigma_0$ with Fuchsian holonomy $\rho_0$. Consider a topological surface $S$ of genus $g \geq 4$ and a branched covering $f : S \longrightarrow \Sigma$; hence the structure $\sigma_0$ can be pulled back to a hyperbolic cone-structure $\sigma$ on $S$. The holonomy $\rho$ of $\sigma$ turns out to be a discrete, non-faithful representation of $\pi_1 S$. In particular the image of $\rho$ consists only of hyperbolic transformations other than the identity because $\rho = \rho_0 \circ f_*$.

Hence the following question naturally arises.

**Question 1.** Let $\rho$ be a non-Fuchsian, purely hyperbolic representation. Under which condition $\rho$ arise as holonomy of a hyperbolic cone-structure?

### 4.4.1 A necessary condition

In order to give an answer to the question 1; a necessary condition for a purely hyperbolic representation $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ to be the holonomy of a hyperbolic cone-structure is the following: the quotient space $\mathbb{H}^2/\rho(\pi_1 S) = \Sigma$ must be closed (hence compact without boundary); *i.e.* the group $\rho\big(\pi_1 S\big)$ is a cocompact subgroup of $\mathrm{PSL}_2\mathbb{R}$. More precisely we have the following lemma.

**Lemma 4.4.4.** *Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a purely hyperbolic representation. Suppose $\rho$ arises as holonomy of a hyperbolic cone-structure $\sigma$ on $S$; then $\rho\big(\pi_1 S\big)$ is a cocompact subgroup of* $\mathrm{PSL}_2\mathbb{R}$.

*Proof.* By assumption there exists a hyperbolic cone-structure $\sigma$ with holonomy $\rho$. Since $\rho$ is purely hyperbolic, the image $\rho(\pi_1 S)$ is a discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$ by 4.4.2 and it acts freely and properly discontinuous on the hyperbolic plane. Hence the quotient space $\mathbb{H}^2/\rho(\pi_1 S) = \Sigma$ is a complete hyperbolic surface (in particular connected). It remains to show that $\Sigma$ is compact. Let $\mathsf{dev}_\sigma : \widetilde{S} \longrightarrow \mathbb{H}^2$ be the developing map for $\sigma$; since it is $\big(\pi_1 S, \rho(\pi_1 S)\big)-$equivariant, it descends to a branched map $f : S \longrightarrow \mathbb{H}^2/\rho(\pi_1 S) = \Sigma$. We may note that $f$ turns out to be a proper orientation preserving map between surfaces. In particular it is a local isometry outside the branch points. According to [25, Exercise 8.21] we claim that any proper orientation preserving map $f$ between $2-$surfaces, with at least one regular point, is surjective. Indeed this is a mapping degree matter. We may pick any regular value $q \in \Sigma$ and look at the sum

$$\deg(f) = \sum_{f(p)=q} \text{sign}, d_p f,$$

where the sign is $+1$ if $d_p f$ preserves orientation, $-1$ otherwise. Any $q \notin \mathsf{Im}(f)$ is trivially a regular value and the sum is of course null. Since there is some regular value $q \in \mathsf{Im}(f)$ with $\text{sign}\, d_p f = +1$ for all $f(p) = q$ then the sum cannot be zero. Hence the conclusion; *i.e.* $\Sigma$ is compact. □

Since $\rho$ is non-elementary, then by [1, Theorem 5.2.1] the image $\rho\big(\pi_1 S\big)$ of $\rho$ is a Fuchsian group and the invariant set for the action of such group is the entire hyperbolic plane. It follows from [14, Corollary 4.2.7] that, under our condition, $\rho\big(\pi_1 S\big)$ is a cocompact subgroup if and only if it a Fuchsian group of the first kind, *i.e.* the limit set is the entire circle at infinity.

From now on we will deal only with non-elementary purely hyperbolic representations $\rho$ such that $\rho\big(\pi_1 S\big)$ is a cocompact subgroup of $\mathrm{PSL}_2\mathbb{R}$.

### 4.4.2 Main results

In this paragraph we give a complete characterization of those purely hyperbolic representations that arise as holonomy of a hyperbolic cone-structure. Some preliminaries are in order.

Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a purely hyperbolic representation; by definition, its image contains only hyperbolic elements other than the identity. By the discussion of the previous section 4.4.1, we may assume that $\rho(\pi_1 S)$ is a purely hyperbolic cocompact subgroup of $\mathrm{PSL}_2\mathbb{R}$, *i.e.* a Fuchsian subgroup of the first kind. In particular, it is discrete by proposition 4.4.2 and acts freely and properly discontinuous on the hyperbolic plane.

**Lemma 4.4.5.** *The quotient space $\mathbb{H}^2/\rho(\pi_1 S) = \Sigma$ is a complete hyperbolic closed surface with holonomy representation $\rho_0 : \pi_1\Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$.*

*Proof.* The group $\pi_1 S$ is finitely generated, hence also its image $\rho\big(\pi_1 S\big)$ is. By [14, Theorem 4.6.1] the group $\rho\big(\pi_1 S\big)$ is geometrically finite; *i.e.* there exists a convex fundamental region for $\rho\big(\pi_1 S\big)$ with finitely many sided. By [14, Theorem 4.5.1], the fundamental region has finite hyperbolic area, and since $\rho(\pi_1 S)$ has no parabolics, such region is also compact. Then by [14, Corollary 4.2.3] the quotient space $\Sigma$ is a compact surface endowed with a complete hyperbolic structure of finite volume. □

We may note that $\rho$ and $\rho_0$ have the same image, hence there exists a map $f_* : \pi_1 S \longrightarrow \pi_1 \Sigma$ such that $\rho = \rho_0 \circ f_*$. Now surfaces are $K(\pi, 1)$-spaces, thus any map between them is uniquely determined up to homotopy by the induced map between the fundamental groups. Thus there exists a map $f : S \longrightarrow \Sigma$. That is we have shown the following proposition.

**Proposition 4.4.6.** *Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a non-Fuchsian purely hyperbolic representation. Then there exists a closed surface $\Sigma$ of genus lower than $S$, a Fuchsian representation $\rho_0 : \pi_1 \Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$ and a map $f : S \longrightarrow \Sigma$ such that $\rho = \rho_0 \circ f_*$.*

Suppose from now on that $\Sigma$ is closed, *i.e.* it has finite volume. We can now state the following lemma.

**Lemma 4.4.7.** *Let $f : S \longrightarrow \Sigma$ be a branched covering between surfaces. Let $\sigma_0$ be a complete hyperbolic structure on $\Sigma$ with Fuchsian holonomy $\rho_0$. Then the pull-back structure $\sigma = f^*\sigma_0$ is hyperbolic cone-structure on $S$ with purely hyperbolic holonomy $\rho$.*

*Proof.* The hyperbolic structure $\sigma_0$ may be pulled-back to hyperbolic cone-structure $\sigma$ by standard arguments and cone points correspond to branch points of $f$; that is points where $f$ fails to be a local homeomorphism. The map $f$ induces a homomorphism $f_* : \pi_1 S \longrightarrow \pi_1 \Sigma$; and the holonomy $\rho$ for $\sigma$ is given by the composition map $\rho_0 \circ f_* : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$. Hence the image of $\rho$ is contained in the Fuchsian group $\rho_0(\pi_1 S)$ which is purely hyperbolic. In particular, if $\deg f \geq 2$, then $\rho$ is a discrete, non-faithful representation, that is not Fuchsian. □

*Remark* 4.4.8. We may note that if $f$ were a covering map in the usual sense, the same arguments show that $\rho$ is Fuchsian. Indeed in such case, by classical covering theory, the homomorphism $f_*$ turns out to be a monomorphism.

This lemma provides a sufficient condition for a purely hyperbolic representation to be holonomy of hyperbolic cone-structure. Is it also necessary? We introduce the following definition.

**Definition 4.4.9.** Let $f : S \longrightarrow \Sigma$ be a map between surfaces. We will say that $f$ is a pinch map if there are two simple closed, non-contractible, curves $\alpha$ and $\beta$ meeting trasversally on a single point such that $f(\alpha)$ and $f(\beta)$ are contractible in $\Sigma$.

We now state the following result.

**Lemma 4.4.10.** *Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a non-Fuchsian and purely hyperbolic representation. Let $\rho_0 : \pi_1\Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a Fuchsian representation, where the genus of $\Sigma$ is strictly lower than the genus of $S$. Suppose there is a map $f : S \longrightarrow \Sigma$ such that*

*1 $f$ is a pinch map,*

*2 $f_* : \pi_1 S \longrightarrow \pi_1\Sigma$ is such that $\rho = \rho_0 \circ f_*$.*

*Then $\rho$ does not aries as holonomy of a hyperbolic cone-structure.*

*Proof.* First of all we may notice that $f_* : \pi_1 S \longrightarrow \pi_1\Sigma$ is surjective because $f$ is a pinch map; thus $\rho\big(\pi_1 S\big) = \rho_0\big(\pi_1\Sigma\big)$. Suppose there exists a hyperbolic cone-structure $\sigma$ with holonomy $\rho$ and consider its developing map $\mathsf{dev}_\sigma : \widetilde{S} \longrightarrow \mathbb{H}^2$. Since it is $\big(\pi_1 S, \rho(\pi_1 S)\big)-$equivariant, it descends to a branched map $b : S \longrightarrow \mathbb{H}^2/\rho(\pi_1 S) = \Sigma$. The induced map $b_*$ on the fundamental groups is such that $\rho = \rho_0 \circ b_*$, thus it is just that of the pinching map because coincides with $f_*$. Hence $\deg\big(S \longrightarrow \mathbb{H}^2/\rho(\pi_1 S)\big) = 1$, implying that such map is a branched map of degree one, that is a homeomorphism, a contradiction. $\square$

The following corollary is immediate.

**Corollary 4.4.11.** *Let $f : S \longrightarrow \Sigma$ be a pinch map, and $\sigma_0$ be a complete hyperbolic structure on $\Sigma$. Then $\sigma_0$ can not be pulled-back to a hyperbolic cone-structure on $S$.*

**Corollary 4.4.12.** *Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a purely hyperbolic representation. Suppose there is a pinch map $f : S \longrightarrow \Sigma$ and a purely hyperbolic holonomy $\rho_0 : \pi_1\Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$ such that $\rho = \rho_0 \circ f$. Then $\rho$ does not aries as holonomy of a hyperbolic cone-structure on $S$.*

*Proof.* First we may notice that since $\rho_0$ is purely hyperbolic, the subgroup $\rho_0\big(\pi_1\Sigma\big)$ of $\mathrm{PSL}_2\mathbb{R}$ is discrete; hence it acts freely and properly discontinuous on the hyperbolic plane, in particular the quotient space $\mathbb{H}^2/\rho_0\big(\pi_1\Sigma\big)$ is a surface endowed with a complete hyperbolic structure. As before the map $f_* : \pi_1 S \longrightarrow \pi_1\Sigma$ is surjective because $f$ is a pinch map; thus $\rho\big(\pi_1 S\big) = \rho_0\big(\pi_1\Sigma\big)$. Suppose that $\sigma$ is a hyperbolic cone-structure on $S$ with holonomy $\rho = \rho_0 \circ f_*$, then we may consider its developing map $\mathsf{dev}_\sigma : \widetilde{S} \longrightarrow \mathbb{H}^2$. From the relation $\rho\big(\pi_1 S\big) = \rho_0\big(\pi_1\Sigma\big)$, we may deduce that $\mathsf{dev}_\sigma$ is $\big(\pi_1 S, \rho_0(\pi_1\Sigma)\big)-$equivariant, and it descends to a branched map $S \longrightarrow \mathbb{H}^2/\rho_0(\pi_1\Sigma) = \Sigma$. The induced map on the fundamental group is just that of the pinching map where some handles are pinched to a point, hence the above map is homotopic to a pinch map of degree 1, implying that $S \longrightarrow \mathbb{H}^2/\rho(\pi_1 S)$ is a homeomorphism, hence a contradiction. $\square$

In order to prove the main theorem we will use the following result.

**Theorem 4.4.13** (Edmonds, [7]). *If $f : S \longrightarrow \Sigma$ is a map of nonzero degree between closed orientable surfaces, then there is a pinch map $\pi : S \longrightarrow T$ and there is a branched covering $b : T \longrightarrow \Sigma$ sucht hat the composition $b \circ \pi$ is homotopic to $f$.*

Using Edmonds' theorem togheter with the lemmata 4.4.7 and 4.4.10, we are able to prove our main theorem.

**Theorem 4.4.14.** *Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a non-Fuchsian, purely hyperbolic representation and $\rho_0 : \pi_1\Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a Fuchsian representation, where the genus of $\Sigma$ is strictly lower than the genus of $S$. Suppose there is a map $f : S \longrightarrow \Sigma$ such that $\rho = \rho_0 \circ f_*$. Then $\rho$ is geometrizable by a hyperbolic cone-structure if and only if $f$ is a branched covering.*

*Proof.* By Edmonds' theorem 4.4.13; there exists an intermediate surface $T$, a pinch map $\pi : S \longrightarrow T$ and a branched covering $b : T \longrightarrow \Sigma$ such that the composition $b \circ \pi$ is homotopic to $f$. Now the sufficient condition comes from 4.4.7, whereas the necessary condition follows from 4.4.10 and its corollaries. $\square$

*Remark* 4.4.15. By a recent result of Marché-Wolff in [17], for closed surfaces of genus 2 there are no purely hyperbolic representations which are not Fuchsian.

The following result immediately follows.

**Corollary 4.4.16.** *Let $\sigma$ be a hyperbolic cone-structure on a closed surface $S$ and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ its holonomy representation. Then $\rho$ is purely hyperbolic if and only if $\sigma$ is the lift of a complete hyperbolic structure by a branched covering.*

*Remark* 4.4.17. Let $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ be the representation variety of all representation $\pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$. This space turns out to be a disjoint union of $4g-3$ connected components; which are parametrized by the Euler number (see [12]). In [8, Proposition 1.2], the authors show that the set of discrete and non-faithful representations form a nowhere dense closed subset in each component of the representation variety. Hence purely hyperbolic representations are essentially rare. In the following paragraph, we show that they do not appear in each component of the representation variety.

### 4.4.3 Euler number of purely hyperbolic representations

Let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be a purely hyperbolic representation. It is natural to ask which are the possible values of the Euler number $\mathcal{E}(\rho)$. The following result follows from a straightforward computation.

**Lemma 4.4.18.** *Let $f : S \longrightarrow \Sigma$ be a branched covering map, and $\rho_0$ be a Fuchsian representation. Consider the representation $\rho = \rho_0 \circ f_* : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$; then*

$$\mathcal{E}(\rho) = d \cdot \mathcal{E}(\rho_0)$$

*where $d$ is the degree of $f$.*

Hence the following lemma follows immediately from the previous one.

**Lemma 4.4.19.** *Let $\rho$ be a non-Fuchsian, purely hyperbolic representation, then $\mathcal{E}(\rho)$ is even.*

*Proof.* By lemma 4.4.6 there exists a closed surface $\Sigma$ of genus lower than $S$, a Fuchsian representation $\rho_0 : \pi_1\Sigma \longrightarrow \mathrm{PSL}_2\mathbb{R}$ and a map $f : S \longrightarrow \Sigma$ such that $\rho = \rho_0 \circ f_*$. By lemma 4.4.18 we have that $\mathcal{E}(\rho) = d \cdot \mathcal{E}(\rho_0)$; where $d$ is the degree of $f$. Since $\rho_0$ is Fuchsian, then $\mathcal{E}(\rho_0) = \pm\chi(\Sigma) = \pm\big(2 - 2g_\Sigma\big)$. Hence $\mathcal{E}(\rho)$ is always even. □

Following remark 4.4.17 we give an example of non-purely hyperbolic representation with even Euler number.

**Example 4.4.20.** Let $S$ be a closed surface of genus 2 with a hyperbolic cone-structure $\sigma_0$ with a single cone point of angle $4\pi$ and let $\rho_0$ be its holonomy representation. Consider a geodesic segment of length $l$ on $S$ and cut along it to get a new surface homotopically equivalent to $S$ with an open disc removed. Geometrically the new surface inherits the hyperbolic cone-structure coming from $S$ and has a piecewise geodesic boundary $\gamma$ with two corner points of angle $2\pi$. As in 1.5.5, take two copies $S_1$ and $S_2$ of the new surface and glue the resulting surfaces as in picture 1.5 to get a closed surface of genus 4 endowed with a hyperbolic cone-structure $\sigma$ with four cone point of angle $4\pi$. The holonomy $\rho$ of $\sigma$ is given by $\rho : \pi_1 S *_{\langle\gamma\rangle} \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$. We have shown before that for surfaces of genus two any representation with $\mathcal{E}(\rho) = \pm 1$ sends a simple non-separating curve to an elliptic element. The image of $\rho$ clearly coincide with the image of $\rho_0$, hence we may conclude that $\rho$ is not purely hyperbolic.

## 4.5 Questions and open problems

In this last section, we collect some natural and interesting questions. The first one regards the type of representations we have considered so far.

### 4.5.1 About representations with $\mathcal{E}(\rho) = \pm\big(\chi(S)+1\big)$

With the new formulation 4.3.6 of theorem 4.3.1, the existence of a non-separating curve with non-hyperbolic image is a sufficient condition for a representation $\rho$, with $\mathcal{E}(\rho) = \pm\big(\chi(S)+1\big)$, to be the holonomy of a branched hyperbolic structure with one cone point of angle $4\pi$ on $S$. Hence it is natural to ask the following:

**Question 2.** Let $\rho$ be any representation with $\mathcal{E}(\rho) = \pm\big(\chi(S)+1\big)$, does it send a simple non-separating curve to a non-hyperbolic element?

As far as it is true for representations of surfaces of genus 2 by our theorem 4.2.4, so that 4.3.6 becomes 4.2.1 in such case.
Despite a comment by Tan in [27], actually we do not if any representation may be the holonomy of a branched hyperbolic structure with a single cone point of angle $4\pi$. On the other hand, there are no counter-examples known in the literature, then the following conjecture seems reasonable:

**Conjecture 2.** *Let $S$ be a closed surface of genus $g \geq 2$. Then any representation with Euler class $\mathcal{E}(\rho) = \pm\big(\chi(S)+1\big)$ is geometrizable by a branched hyperbolic structure with a single cone point of angle $4\pi$.*

### 4.5.2 Other types of representations

So far we took into account only representations with Euler number close to the extremal value $\pm\chi(S)$. For other values of $\mathcal{E}(\rho)$ there is no theorem like 4.2.1 and 4.3.6, but the theory in these case is not completely mysterious.

**Are all representations geometrizable?** Clearly, none representation with $\mathcal{E}(\rho) = 0$ arises as holonomy of a branched hyperbolic structure because it contradicts the Gauß-Bonnet condition. Moreover Tan gave in [27] an explicit example of representation which is not the holonomy of any branched hyperbolic structure. Even if his example was given for a representation of a genus 3 closed surface with $\mathcal{E}(\rho) = -2$, it can be generalized in any genus greater than 3, thus the analogous of conjecture 2 might be false for different values of $\mathcal{E}(\rho)$.

**Example 4.5.1** (Tan's counter-example)**.** Let $S$ be a genus $g$ surface, obtained by attaching $h$ handles to a surface of genus $g-h$, where $g-h \geq 2$. We define a representation $\rho$ in the following way: $\rho$ is discrete and faithful on the original surface, and trivial on each handle we have attached. In this way $\rho(\pi_1 S)$ is a discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$ and the quotient $\mathbb{H}^2/\rho(\pi_1 S)$ is a genus $g-h$ surface. Then $\rho$ cannot be the holonomy of a branched hyperbolic structure on $S$.
First of all, we may notice that $\mathcal{E}(\rho) = 2+2h-2g$. Suppose now that $S$ admits a hyperbolic structure with holonomy $\rho$, then its developing map $\mathsf{dev}: \widetilde{S} \longrightarrow \mathbb{H}^2$ is a $\big(\pi_1 S, \rho(\pi_1 S)\big)-$equivariant map that passes down to branch map

$$f : S \longrightarrow {}_{\rho(\pi_1 S)}\Big\backslash^{\mathbb{H}^2}$$

Consider the induced map of fundamental groups. This is the same map induced by the map that pinches to a point each handle we have attached before, hence the map $f$ is homotopic to a pinching map of degree one. Since any branch cover of degree, one is just a homeomorphism we found a contradiction, that is $\rho$ cannot be the holonomy of a branched hyperbolic structure.

From this previous example we may deduce that for each even $k$ in $2-2g \leq k \leq -1$ the connected component $\mathcal{E}^{-1}(k)$ of $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$ contains a representation which does not arise as the holonomy of a branched hyperbolic structure. This leads to the following question.

**Question 3.** For each odd values of $k$ in $2-2g \leq k \leq -1$ consider the connected component $\mathcal{E}^{-1}(k)$ of $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$. Is any representation in $\mathcal{E}^{-1}(k)$ geometrizable by branched hyperbolic structure on $S$ with $|\chi(S)-k|$ simple cone point?

We may note that a positive answer to conjecture 2 gives also a positive answer to this question in the case of $k = 3-2g$. It is easy to see that little perturbations of the representation $\rho$ of 4.5.1 produce new representations which are holonomy of a branched hyperbolic structure on $S$; hence the following question arises immediately.

**Question 4.** For each even values of $k$ in $2-2g \leq k \leq -1$ consider the connected component $\mathcal{E}^{-1}(k)$ of $\mathsf{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})$. Does geometrizable representations constitute a dense, or conull, subset of $\mathcal{E}^{-1}(k)$?

**Relationship with the Bowditch's question.** In this last paragraph, we want to point out how the previous questions are related with the Bowditch's question, that we report here for the reader convenience, which is known to be true for the genus 2, but still open for all other cases.

**Bowditch's question.** Let $S$ be a closed surface and let $\rho : \pi_1 S \longrightarrow \mathrm{PSL}_2\mathbb{R}$ be any representation. Suppose $\rho$ sends any simple closed curve to a hyperbolic element, then is it Fuchsian?

In order to understand which non-Fuchsian representations arise as holonomy of a branched hyperbolic structure a positive answer to the Bowditch's question is certainly a big advantage. The existence of a non-separating simple curve with non-hyperbolic holonomy permits us to localize the deficiency of the Euler number and then cut off handles from the original surface. If we are able to localize all deficiencies of $\mathcal{E}(\rho)$, we cut them off, and the rest of the representation has extremal Euler class.

If the representation sends a simple close separating curve to a non-hyperbolic element, to localize the deficiency of the Euler class becomes much arduous, but still possible in some cases. After a positive answer to the Bowditch's question, the question 2 becomes equivalent to the following one.

**Question 5.** Let $\rho$ be a non-Fuchsian and suppose it sends a simple separating curve to a non-hyperbolic element. Does it send a simple non-separating curve to a non-hyperbolic element?

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