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On the space - time integrability properties of the
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Abstract

Let $u(t, x)$ be the fundamental solution to the Cauchy problem associated with the free linear inhomogeneous Schrödinger equation

$$i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad u(0, x) = 0,$$

where the forcing term F is supported on $[0, 1] \times \mathbb{R}^n$. We give counter examples that exclude the validity of the local inhomogeneous Strichartz type estimate

$$\| u \|_{L^q([2,3]; L^r(\mathbb{R}^n))} \lesssim \| F \|_{L^{\tilde{q}'}([0,1]; L^{\tilde{r}'}(\mathbb{R}^n))} \quad (1)$$

for a certain range of values of the Lebesgue exponents (q, r) and (\tilde{q}, \tilde{r}) .

In the context of seeking the optimal range of exponents values for the admissibility of the estimate (1), the new set of necessary conditions and theory of interpolation led us to consider the estimate

$$\| u \|_{L^q([2,3]; L^r(\mathbb{R}^n))} \lesssim \| F \|_{L^{\tilde{q}'}([0,1]; L^1(\mathbb{R}^n))} \quad (2)$$

with $q \leq 2n/(n-2)$ and $\tilde{q} \leq n/(n-1)$. We considered the case $q = r = n = 4$ for data of the type $F(t, x) = \delta_0(x) f(t)$ and looked at the estimate

$$\| u \|_{L^4([2,3] \times (\mathbb{R}^4))} \lesssim \| f \|_{L^4([0,1])} . \quad (3)$$

We proved a quadrilinear estimate that implies the estimate (3) with a divergence of an order less than any positive ϵ . We showed this quadrilinear estimate using multilinear interpolation tools in two different ways. In the first approach, we approximate the quadrilinear form we want to estimate via approximating the $L^{\tilde{q}'}$ data, f , by piecewise constant functions. While in the second one we approximate the quadrilinear form itself using the dominated convergence theorem.

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Chapter 1

Introduction

1.1 Main Results of the thesis

We summarise the main results that we obtained in this thesis. Consider the following Cauchy problem for the inhomogeneous free Schrödinger equation

$$i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R}^n \times \mathbb{R}, \quad u(0, x) = 0, \quad (1.1)$$

where the forcing term F is supported on $[0, 1] \times \mathbb{R}^n$.

Theorem 1.1.1. (Chapter 2) *Let $u = u(t, x)$ be the solution to the problem (1.1). If u satisfies the local inhomogeneous Strichartz type estimate*

$$\| u \|_{L^q([2,3]; L^r(\mathbb{R}^n))} \lesssim \| F \|_{L^{\tilde{q}}([0,1]; L^{\tilde{r}}(\mathbb{R}^n))},$$

then the exponents $q, r, \tilde{q}, \tilde{r}$ must satisfy the following conditions

$$\boxed{\frac{1}{q} \geq \frac{n-1}{\tilde{r}} - \frac{n}{r}, \quad \frac{1}{\tilde{q}} \geq \frac{n-1}{r} - \frac{n}{\tilde{r}}}. \quad (1.2)$$

Theorem 1.1.2. (Chapter 4) *Let the forcing term, F , in (1.1) be*

$$F(t, x) = f(t) \delta_0(x), \quad f = \sum_{k=1}^N c_k \chi_{[\frac{k-1}{N}, \frac{k}{N}]}, \quad c_k \in \mathbb{C},$$

where δ_0 is the dirac delta function at the origin and f is supported on $[0, 1]$. Then we have the following estimate

$$\boxed{\| u \|_{L^4([2,3] \times \mathbb{R}^4)} \lesssim (\log N)^{\frac{3}{4}} \| f \|_{L^4([0,1])}},$$

for large N .

Theorem 1.1.3. (Chapter 5) Let the forcing term, F , in (1.1) be

$$F(t, x) = f(t) \delta_0(x),$$

where δ_0 is the dirac delta function at the origin and f is supported on $[0, 1]$. Then we have the following estimate

$$\boxed{\| u e^{-\epsilon|x|^2} \|_{L^4([2,3] \times \mathbb{R}^4)} \lesssim |\log \epsilon|^{\frac{1}{2}} \| f \|_{L^4([0,1])}}, \quad (1.3)$$

for small ϵ .

1.2 Strichartz estimates for the Schrödinger's equation

The study of space-time integrability properties of the solutions of the Schrödinger equation has been pursued by many authors in the last thirty years. In this context, Strichartz estimates have become a fundamental and amazing tool for the study of PDEs. They have been studied in the framework of different function (distribution) spaces like Lebesgue, Sobolev, Wiener amalgam and modulation spaces and have found applications to well-posedness and scattering theory for nonlinear Schrödinger equations. See for instance [1, 3, 4, 5, 6, 8, 10, 11, 12, 13, 14, 16, 18, 21, 23, 24, 26, 27, 28].

Strichartz estimates are spacetime estimates for homogeneous and inhomogeneous linear dispersive and wave equations. They are particularly useful for solving semilinear perturbations of such equations, in which no derivatives are present in the nonlinearity. Strichartz estimates were first obtained as a consequence of Fourier restriction theorems (see for instance [4, 13, 22]) and later on they could be derived abstractly as a consequence of a dispersive inequality and an energy inequality (see [10, 16]). We give here an outline of their history.

The Fourier transform \widehat{f} of a function $f \in L^1(\mathbb{R}^n)$ can be defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx.$$

It is a well known fact (see for example [17]) that $\widehat{f}(\xi)$ is continuous and decays to 0 as $|\xi| \rightarrow \infty$. The Fourier transform is on the other hand a unitary isomorphism from L^2 onto itself. The Fourier transform of an L^2 function is no better than an L^2 function, and so

can only be defined almost everywhere and is thus completely arbitrary on sets of measure zero. If $1 < p < 2$ then it follows from the Hausdorff - Young inequality that the Fourier transform maps L^p into $L^{p'}$ where p' is the Hölder dual of p . Probably these observations are what motivated Stein to study the size of the image of an L^p function under the Fourier transform in $L^{p'}$. One way to measure such 'size' is by restricting the Fourier transform to some submanifold of \mathbb{R}^n and then studying the decay or regularity properties of the restricted functions. In the late 60's Stein question was the following:

Given a function $f \in L^p(\mathbb{R}^n)$ with $p \in [1, 2]$. For what values of $q \in [1, \infty]$ does the restriction of the Fourier transform \widehat{f} to the unit sphere (\mathbb{S}^{n-1}) satisfy that

$$\| \widehat{f}|_{\mathbb{S}^{n-1}} \|_{L^q(\mathbb{S}^{n-1})} \leq C_{n,p,q} \| f \|_{L^p(\mathbb{R}^n)}.$$

For $q = 2$, Stein [22] established the theory for $1 < p < \frac{4n}{3n+1}$. Soon after that Fefferman and Stein [9] extended this, for the dimension $n = 2$, to the range $1 < p < 6/5$. Then P. Sjolin [7] proved the theorem for $n = 3$ and $1 < p < 4/3$. All values of p and q such that the Fourier transform of an L^p function restricts to $L^q(\mathbb{S}^1)$ were determined by A. Zygmund [30]. Finally the optimal result for $q = 2$ was given by the Stein-Tomas theorem [22, 25] that reads

Theorem 1.2.1. *If $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \frac{2n+2}{n+3}$ then*

$$\| \widehat{f}|_{\mathbb{S}^{n-1}} \|_{L^2(\mathbb{S}^{n-1})} \leq C_{n,p} \| f \|_{L^p(\mathbb{R}^n)}.$$

In 1977, inspired by this work of E. Stein, P. Tomas and I.E. Segal [20], Robert Strichartz, in his pioneering paper [21], posed and addressed the following question:

Given a subset S of \mathbb{R}^n , a positive measure $d\mu$ supported on S and with temperate growth at infinity and a function $f \in L^2(\mathbb{R}^n, \mu)$, determine the values of q for which the tempered distribution $f d\mu$ has a Fourier transform that satisfies

$$\| \widehat{f d\mu} \|_{L^q(\mathbb{R}^n)} \leq C_q \| f \|_{L^2(\mathbb{R}^n)}. \tag{1.4}$$

Strichartz [21] provided the complete solution of this problem when S is a quadratic surface in \mathbb{R}^n . As an important application to this, and by choosing the correct quadratic surface, he obtained spacetime estimates for the solutions of certain linear dispersive equations. In particular, if we consider the Cauchy problem for the inhomogeneous free Schrödinger equation in n -space dimensions

$$\begin{aligned} i\partial_t u(t, x) + \Delta_x u(t, x) &= F(t, x), & (t, x) \in \mathbb{R}^n \times \mathbb{R}, \\ u(0, x) &= f(x) \end{aligned} \quad (1.5)$$

we get the following estimate

$$\| u \|_{L^{\frac{2(n+2)}{n}}(\mathbb{R}^{n+1})} \lesssim \| f \|_{L^2(\mathbb{R}^n)} + \| F \|_{L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^{n+1})}. \quad (1.6)$$

To obtain this estimate, Strichartz essentially noticed that the solution

$$e^{it\Delta} f(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi - it|\xi|^2} \widehat{f}(\xi) d\xi$$

of the homogeneous problem, that is the problem 1.5 with $F = 0$, can equivalently be represented as $(\widehat{f}\mu)^\vee$ where μ is the measure supported on the parabola $\{\tau = -|\xi|^2\} \subset \mathbb{R}^{n+1}$ and is defined by

$$\int_{\mathbb{R}^{n+1}} g(\tau, \xi) d\mu(\tau, \xi) = \int_{\mathbb{R}^n} g(-|\xi|^2, \xi) d\xi$$

for all continuous functions g on \mathbb{R}^{n+1} . The parabola is a smooth hypersurface and has non vanishing Gaussian curvature but it is not compact. If we take a function $\phi \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\phi = 1$ on $|\tau| + |\xi| \leq 1$, and consider the measure $\phi\mu$ now supported on a compact subset of the parabola then for any Schwartz function f on whose Fourier transform is supported on the unit ball in \mathbb{R}^n we have that $\widehat{f}\phi = \widehat{f}$. Applying the "dual" version of Stein-Tomas theorem 1.2.1 in the dimension $n + 1$ with $q = \frac{2n+4}{n}$ to f and using (1.4) yield that

$$\| e^{it\Delta} f(x) \|_{L^q(\mathbb{R}^{n+1})} = \| (\widehat{f}\phi\mu)^\vee \|_{L^q(\mathbb{R}^{n+1})} = \| (\widehat{f}\mu)^\vee \|_{L^q(\mathbb{R}^{n+1})} \lesssim \| \widehat{f} \|_{L^2(\mathbb{R}^n)} = \| f \|_{L^2(\mathbb{R}^n)}. \quad (1.7)$$

The solution of the inhomogeneous problem 1.5 with initial data $f = 0$ is

$$v(t, x) = -i \int_0^t e^{i(t-s)\Delta} F(s, x) ds.$$

Strichartz obtained the spacetime estimate

$$\| v \|_{L^q(\mathbb{R}; L^q(\mathbb{R}^n))} \lesssim \| F \|_{L^{q'}(\mathbb{R}; L^{q'}(\mathbb{R}^n))}$$

first by observing that since

$$e^{it\Delta} h(x) = ct^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} h(y) dy$$

then by interpolating between the energy estimate

$$\| e^{it\Delta} h \|_{L^2(\mathbb{R}^n)} = \| h \|_{L^2(\mathbb{R}^n)}$$

that we get using Plancherel's theorem and the dispersive estimate

$$\| e^{it\Delta} h \|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-\frac{n}{2}} \| h \|_{L^1(\mathbb{R}^n)}$$

that we get from the explicit integral representation of the Schrödinger operator $e^{it\Delta}$, we obtain

$$\| e^{it\Delta} h \|_{L^q(\mathbb{R}^n)} \lesssim t^{-n(\frac{1}{2} - \frac{1}{q})} \| h \|_{L^{q'}(\mathbb{R}^n)}, \quad 2 \leq q \leq \infty.$$

Thus obtaining that

$$\| v(t, \cdot) \|_{L^q(\mathbb{R}^n)} \lesssim \int_0^t (t-s)^{-n(\frac{1}{2} - \frac{1}{q})} \| F(\cdot, s) \|_{L^{q'}(\mathbb{R}^n)} ds$$

to which the fractional integration theorem may be applied because

$$1 - n\left(\frac{1}{2} - \frac{1}{q}\right) + \frac{1}{q} = \frac{1}{q'}, \quad \text{that is, } q = \frac{2(n+2)}{n}$$

to eventually get the estimate

$$\| v \|_{L^q(\mathbb{R} \times \mathbb{R}^n)} \lesssim \| F \|_{L^{q'}(\mathbb{R} \times \mathbb{R}^n)}. \tag{1.8}$$

From the estimates (1.7) and (1.8) follows the Strichartz estimate (1.6).

Spacetime estimates of this type that followed after were all named in honor of Robert Strichartz. More examples of such estimates and their relation to the restriction theorem can be found in [22]. If we consider the general spacetime estimate

$$\| v \|_{L^q(\mathbb{R}; L^r(\mathbb{R}^n))} \lesssim \| F \|_{L^{q'}(\mathbb{R}; L^{r'}(\mathbb{R}^n))} \quad (1.9)$$

in mixed Lebesgue norms, we find that the very first result by R. Strichartz corresponds to the case

$$r = \tilde{r} = q = \tilde{q} = \frac{2(n+2)}{n}.$$

Right after that, J. Ginibre and G. Velo in 1985 (see [13]), K. Yajima in 1987 (see [29]), and T. Cazenave and F. B. Weissler in 1988 (see [3]) proved the estimates when (q, r) and (\tilde{q}, \tilde{r}) are admissible pairs with $q \neq 2$ and $\tilde{q} \neq 2$. M. Keel and T. Tao (see [16]) obtained the result for admissible pairs with $q = 2$ and $\tilde{q} = 2$. In 1998, the complete solution of the problem for the homogeneous part was achieved by Keel and Tao [16]. They proved a Strichartz estimate in an abstract setting and gave as an application the concrete examples of the Strichartz estimates for the wave and Schrödinger equations. Their approach was to consider a measure space (X, dx) and a Hilbert space H and assume that at each time $t \in \mathbb{R}$ we have an operator $U(t) : H \rightarrow L^2(X)$ which obeys the energy estimate

$$\| U(t)f \|_{L^2(X)} \lesssim \| f \|_H \quad (1.10)$$

for all t and all $f \in H$ and satisfies that for all $g \in L^1(X)$,

$$\| U(s)(U(t))^* \|_\infty \lesssim |t-s|^{-\sigma} \| g \|_1 \quad (\text{untruncated decay}) \quad (1.11)$$

whenever $s \neq t$ and

$$\| U(s)(U(t))^* \|_\infty \lesssim (1+|t-s|)^{-\sigma} \| g \|_1 \quad (\text{untruncated decay}) \quad (1.12)$$

whenever $s = t$. They went further introducing the following concept of σ -admissibility.

Definition 1.2.1. The exponent pair (q, r) is said to be σ -admissible if $q \geq 2$, $r \geq 2$, $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}. \quad (1.13)$$

If equality holds in (1.13) then (p, q) is said to be sharp σ -admissible otherwise it is nonsharp σ -admissible.

Setting this abstract framework they proved the following theorem

Theorem 1.2.2. *If $U(t)$ obeys (1.10) and (1.11) then the estimates*

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_H, \quad (1.14)$$

$$\left\| \int (U(s))^* F(s) ds \right\|_H \lesssim \|F\|_{L_t^{q'} L_x^{r'}}, \quad (1.15)$$

$$\left\| \int_{s<t} U(s)(U(s))^* F(s) ds \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \quad (1.16)$$

hold for all sharp σ -admissible exponent pairs (q, r) and (\tilde{q}, \tilde{r}) . Furthermore, if the decay hypothesis is strengthened to (1.12), then (1.14), (1.15) and (1.16) hold for all σ -admissible exponents (q, r) , (\tilde{q}, \tilde{r}) .

As a consequence of Theorem 1.2.2, Keel and Tao were able to prove the endpoint Strichartz estimates for the wave and Schrödinger equation in higher dimensions thus solving the problem of determining the possible homogeneous Strichartz estimates for the wave and Schrödinger equations in higher dimensions completely. For the Schrödinger equation specifically, they provided the following result

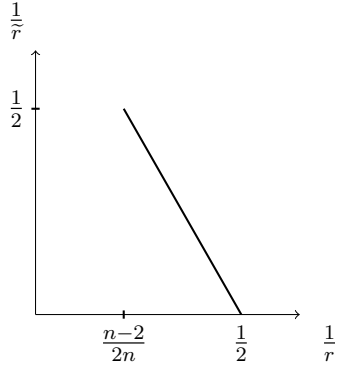
Corollary 1.2.3. *Suppose that $n \geq 1$ and (q, r) and (\tilde{q}, \tilde{r}) are Schrödinger admissible pairs ((q, r) is sharp $\frac{n}{2}$ -admissible). If u is a (weak) solution to the problem*

$$\begin{aligned} i\partial_t u(t, x) + \Delta_x u(t, x) &= F(t, x), & (t, x) \in \mathbb{R}^n \times \mathbb{R}, \\ u(0, \cdot) &= f \end{aligned}$$

for some data f , F and time $0 < T < 1$, then

$$\|u\|_{L_t^q([0, T]; L_x^r)} + \|u\|_{C([0, T]; L^2)} \lesssim \|f\|_{L^2} + \|F\|_{L_t^{\tilde{q}'}([0, T]; L_x^{\tilde{r}'})}.$$

Conversely, if the above estimate holds for all f , F and T , then (q, r) and (\tilde{q}, \tilde{r}) must be Schrödinger-admissible.



For $n \geq 3$, the closed line segment is Schrödinger-admissible.

After M. Keel and T. Tao [16] obtained the result for admissible pairs with $q = 2$ or $\tilde{q} = 2$, T. Cazenave and F. Weissler [4] following the ideas of R. Strichartz gave the first result for exponents pairs that are not Schrödinger-admissible. It was in 1992 when they proved that (see Figure (CW) below) if

$$r = \tilde{r}, \quad \frac{n}{r} + \frac{1}{q} < \frac{n}{2}$$

and the scaling condition

$$\frac{n}{r} \left(1 - \frac{1}{r} + \frac{1}{\tilde{r}} \right) = \frac{1}{q} + \frac{1}{\tilde{q}} \tag{1.17}$$

is satisfied, then the inhomogeneous Strichartz estimate (1.9) for the Schrödinger equation holds when

$$\begin{aligned} 2 < r \leq \infty, & \quad n = 1, \\ 2 < r < \infty, & \quad n = 2, \\ 2 < r < \frac{2n}{n-2}, & \quad n \geq 3. \end{aligned}$$

This was the first result for pairs different from the admissible pairs.

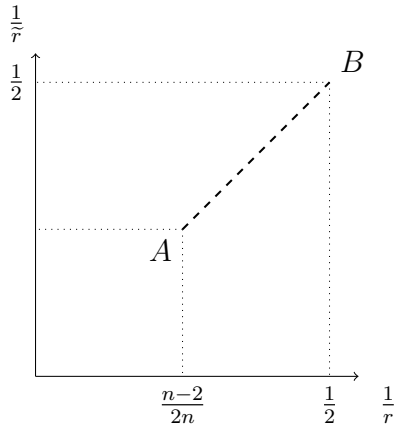


Figure (CW): For $n \geq 3$, Cazenave and Weissler proved the estimate for the open line segment \overline{AB} .

Later, in 1994, T. Kato [14] proved the estimates inside the square ABCD in Figure (K) below whenever

$$\frac{n}{r} + \frac{1}{q} < \frac{n}{2} \quad \text{and} \quad \frac{n}{\tilde{r}} + \frac{1}{\tilde{q}} < \frac{n}{2}$$

and (1.17) are satisfied.

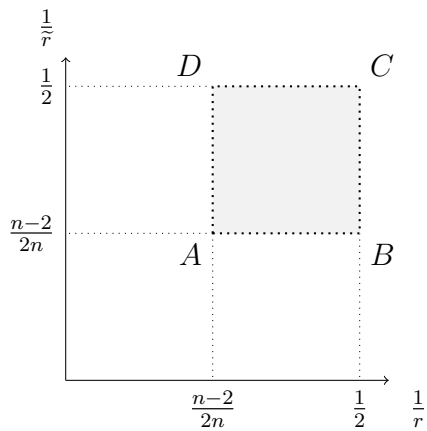


Figure (K): For $n \geq 3$, Kato proved the estimate for the open square ABCD.

The proofs of J. Ginibre and G. Velo, and T. Cazenave and F. Weissler were based on the work of R. Strichartz.

The problem of finding the optimal range of Lebesgue exponents (q, r) and (\tilde{q}, \tilde{r}) for the

Strichartz estimates for the solution of the Cauchy problem associated with the inhomogeneous Schrödinger equation with initial data identically zero (the problem (1.5) with $f = 0$) is still open for dimensions $n \geq 3$, see for example [10, 26, 28] and the references therein.

Adopting an abstract setting and interpolation techniques, Foschi [10] and Vilela [26] independently obtained almost equivalent results for the solutions of inhomogeneous Schrödinger equations. In particular, Foschi [10], as an application to a general local inhomogeneous Strichartz estimates result, obtained the up to the moment most general result for this problem. These results coincide with the results with Vilela [26] that focused on the concrete example of the inhomogeneous Schrödinger equation.

We give an outline of the results in [10] and their proof that follows and extends the ideas of M. Keel and T. Tao [16] and generalizes the results therein. The argument in [10] goes like this:

Let $(X, d\mu)$ be a measure space, H a Hilbert space and $\sigma > 0$. Consider a family of linear operators $U(t) : H \rightarrow L_X^2$ defined for each $t \in \mathbb{R}$. Let $U^*(t) : L_X^2 \rightarrow H$ be the adjoint of $U(t)$. Assume in addition that the family $U(t)$ satisfies the energy estimate

$$\| U(t)h \|_{L_X^2} \lesssim \| h \|_H, \quad \forall t \in \mathbb{R}, h \in H. \quad (1.18)$$

and the dispersive inequality

$$\| U(t)U^*(s)f \|_{L_X^\infty} \lesssim |t-s|^{-\sigma} \| f \|_{L_X^1}, \quad \forall s \neq t, f \in L_X^1 \cap L_X^2. \quad (1.19)$$

The energy estimate allows us to consider the operator $T : H \rightarrow L_t^\infty(\mathbb{R}; L_X^2)$ defined as $Th(t) = U(t)h$, for $t \in \mathbb{R}$ and $h \in H$. Its formal adjoint is the operator $T^* : L_t^1(\mathbb{R}; L_X^2) \rightarrow H$ is given by the H -valued integral

$$T^*F = \int U^*(s)F(s)ds.$$

The composition TT^* is the operator

$$TT^*F(t) = \int U(t)U^*(s)F(s)ds,$$

which can be decomposed as the sum of its retarded and advanced parts,

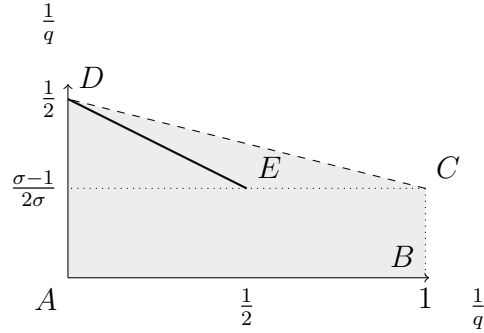
$$(TT^*)_R F(t) = \int_{s < t} U(t)U^*(s)F(s)ds,$$

$$(TT^*)_A F(t) = \int_{s > t} U(t)U^*(s)F(s)ds.$$

The operator T usually solves the initial value problem for a linear homogeneous differential equation, while the retarded operator $(TT^*)_R$ solves the corresponding inhomogeneous problem with zero initial conditions (Duhamel's principle). Then besides the definition of σ -admissible exponents (q, r) given in [16], the following conception of σ -acceptable exponents was introduced

Definition 1.2.2. The exponents pair (q, r) are said to be σ -acceptable if

$$1 \leq q, r \leq \infty, \quad \frac{1}{q} < 2\sigma \left(\frac{1}{2} - \frac{1}{r} \right), \quad \text{or} \quad (q, r) = (\infty, 2).$$



The shaded trapezoid ABCD represents the range of the σ -acceptable exponents

The line segment \overline{DE} represents of the σ -admissible exponents

As it was already remarked in [16], we expect the inhomogeneous estimate

$$\| (TT^*)_R F \|_{L_t^q(\mathbb{R}; L_X^r)} \lesssim \| F \|_{L_t^{\tilde{q}}(\mathbb{R}; L_X^{\tilde{r}})} \quad (1.20)$$

to have a wider range of admissibility than the one given by sharp σ -admissible pairs. For example, in the context of the inhomogeneous Schrödinger equation, the results obtained by Kato [14] correspond to the case when the pairs (q, r) and (\tilde{q}, \tilde{r}) are σ -acceptable and satisfy the conditions

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \sigma \left(1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right), \quad \frac{1}{r}, \frac{1}{\tilde{r}} > \frac{\sigma - 1}{2\sigma}.$$

The goal is to find the largest range for the pairs (q, r) and (\tilde{q}, \tilde{r}) which guarantees the validity of the inhomogeneous estimates (1.20), and which can be deduced by assuming only the energy estimate (1.18) and dispersive property (1.19). The main result obtained in [10] is summarized by the following theorem.

Theorem 1.2.4. (*Global inhomogeneous estimates*). *Let $1 \leq q, \tilde{q}, r, \tilde{r} \leq \infty$. If $U(t)$ obeys (1.18) and (1.19), then the estimate (1.20) holds when the exponent pairs (q, r) and (\tilde{q}, \tilde{r}) are σ -acceptable, verify the scaling condition*

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \sigma \left(1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right) \quad (1.21)$$

and satisfy one of the following sets of conditions:

- if $\sigma < 1$, there are no further conditions;
- if $\sigma = 1$, we also require that $r, \tilde{r} < \infty$;
- if $\sigma > 1$, we distinguish two cases,

– the non-sharp case

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad (1.22)$$

$$\frac{\sigma - 1}{r} \leq \frac{\sigma}{\tilde{r}}, \quad \frac{\sigma - 1}{\tilde{r}} \leq \frac{\sigma}{r}, \quad (1.23)$$

– the sharp case

$$\frac{1}{q} + \frac{1}{\tilde{q}} = 1, \quad (1.24)$$

$$\frac{\sigma - 1}{r} < \frac{\sigma}{\tilde{r}}, \quad \frac{\sigma - 1}{\tilde{r}} < \frac{\sigma}{r}, \quad (1.25)$$

$$\frac{1}{r} \leq \frac{1}{q}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{q}}. \quad (1.26)$$

Theorem 1.2.4 comes with the remarks that

- Conditions (1.22) and (1.23) which appear in the non sharp case for $\sigma > 1$ are always trivially satisfied if $\sigma < 1$ or if $\sigma = 1$ and $r, \tilde{r} < \infty$.
- Condition (1.21) together with $\frac{1}{q} + \frac{1}{\tilde{q}} \leq 1$ have the following interpretation: if $(\frac{1}{Q}, \frac{1}{R})$ is the midpoint between the points $(\frac{1}{q}, \frac{1}{r})$ and $(\frac{1}{\tilde{q}}, \frac{1}{\tilde{r}})$, then (Q, R) is a sharp σ -admissible pair.

- Formally TT^* coincides with its dual $(TT^*)^*$ while $(TT^*)_R^* = (TT^*)_A$. Moreover, $(TT^*)_A$ becomes $(TT^*)_R$ if we invert the direction of time. These duality relations explain why all conditions must be invariant under the symmetry $(q, r) \leftrightarrow (\tilde{q}, \tilde{r})$.
- When $\sigma > 1$, Theorem 1.2.4 improves on Kato's result [14]. Kato's theorem required r and \tilde{r} to be less than $\frac{2\sigma}{\sigma-1}$. This restriction is replaced with a condition which can be read as

$$\frac{\sigma - 1}{\sigma} \leq \frac{r}{\tilde{r}} \leq \frac{\sigma}{\sigma - 1}.$$

Foschi's proof of Theorem 1.2.4 for the global inhomogeneous Strichartz estimates makes use of the techniques of Keel and Tao [16] and is based on the localized version given in Theorem 1.2.5 below of the inhomogeneous estimates.

Theorem 1.2.5. (*Local inhomogeneous estimates*). *Assume $U(t)$ obeys (1.18) and (1.19), and let I and J be two time intervals of unit length $|I| = |J| = 1$ separated by a distance of scale 1, $\text{dist}(I, J) \approx 1$. Then, the estimate*

$$\| TT^* F \|_{L_t^q(J; L_X^r)} \lesssim \| F \|_{L_t^{\tilde{q}}(I; L_X^{\tilde{r}})}, \quad F \in L_t^{\tilde{q}}(I; L_X^{\tilde{r}}), \quad (1.27)$$

holds for all pairs (q, r) and (\tilde{q}, \tilde{r}) which verify the following conditions:

$$1 \leq q, \tilde{q} \leq \infty, \quad 2 \leq r, \tilde{r} \leq \infty, \quad (1.28)$$

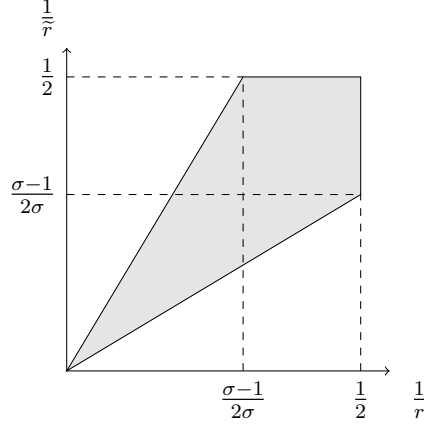
$$\frac{\sigma - 1}{r} \leq \frac{\sigma}{\tilde{r}}, \quad \frac{\sigma - 1}{\tilde{r}} \leq \frac{\sigma}{r}, \quad (1.29)$$

$$\frac{1}{q} \geq \sigma \left(\frac{1}{\tilde{r}} - \frac{1}{r} \right), \quad \frac{1}{\tilde{q}} \geq \sigma \left(\frac{1}{r} - \frac{1}{\tilde{r}} \right), \quad (1.30)$$

and if $\sigma = 1$, we must also require $r, \tilde{r} < \infty$.

For the local estimates of Theorem 1.2.5, the exponent pairs (q, r) and (\tilde{q}, \tilde{r}) are not required to be σ -acceptable. Now we sketch the proof given in [10] of the local inhomogeneous

estimates.



Admissible range for the exponents r and \tilde{r} for the local Strichartz estimates.

Proof of Theorem 1.2.5.

Let ε_{local} be the set of points $(\frac{1}{q}, \frac{1}{r}; \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}})$ in $[0, 1]^4$ corresponding to the pairs (q, r) , (\tilde{q}, \tilde{r}) for which the estimate (1.27) is valid. Start by observing that the dispersive estimate (1.19) immediately yields the case $q = r = \tilde{q} = \tilde{r} = \infty$,

$$\begin{aligned} \| TT^*F \|_{L_t^\infty(J; L_X^\infty)} &\lesssim \int_I \| U(t)U^*(s)F(s) \|_{L_t^\infty(J; L_X^\infty)} ds \\ &\lesssim \int_I \| F(s) \|_{L_X^1} ds \\ &= \| F \|_{L_t^1(I; L_X^1)}. \end{aligned} \tag{1.31}$$

Hence, $(0, 0; 0, 0) \in \varepsilon_{local}$. On the other hand, exploiting the factorization TT^* , we can apply the homogeneous Strichartz estimates

$$\| Th \|_{L_t^q(\mathbb{R}; L_X^r)} \lesssim \| h \|_H, \quad \| T^*F \|_H \lesssim \| F \|_{L_t^{\tilde{q}'}(\mathbb{R}; L_X^{\tilde{r}'})},$$

previously obtained in [16] and get the estimate

$$\| TT^*F \|_{L_t^q(J; L_X^r)} \lesssim \| T^*F \|_H \lesssim \| F \|_{L_t^{\tilde{q}'}(I; L_X^{\tilde{r}'})}. \tag{1.32}$$

Hence $(\frac{1}{q}, \frac{1}{r}; \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}}) \in \varepsilon_{local}$ whenever (q, r) and (\tilde{q}, \tilde{r}) are sharp σ -admissible pairs.

By standard L^p interpolation [2] between (1.31) and (1.32), we obtain that ε_{local} contains

the convex hull of the set

$$\{(0, 0; 0, 0)\} \cup \left\{ \left(\frac{1}{q}, \frac{1}{r}; \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}} \right) : (q, r) \text{ and } (\tilde{q}, \tilde{r}) \text{ are sharp } \sigma - \text{admissible pairs} \right\}.$$

Since we have restricted F and TT^*F to unit time intervals, it follows from Hölder's inequality that when $q \geq Q$, $\tilde{q} \geq \tilde{Q}$, and $(\frac{1}{q}, \frac{1}{r}; \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}}) \in \varepsilon_{local}$ then $(\frac{1}{Q}, \frac{1}{r}; \frac{1}{\tilde{Q}}, \frac{1}{\tilde{r}}) \in \varepsilon_{local}$. If we apply this property to the points of the above convex hull we obtain that ε_{local} contains a set ε_* exactly described by the conditions appearing in Theorem 1.2.5. More details of this computation and the proof of the global inhomogeneous estimates are given in [10].

Our main interest in this thesis is the local inhomogeneous estimates for the Shrödinger's equation (1.5) with initial data f identically zero. As an application of Theorem 1.2.5, the following theorems for the Shrödinger's equation were proved in [10].

Theorem 1.2.6. *[Sufficient conditions] Consider the following Cauchy problem for the inhomogeneous free Schrödinger equation in n -space dimensions*

$$i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad u(0, x) = 0. \quad (1.33)$$

Let the forcing term F in (1.33) be supported on $[0, 1] \times \mathbb{R}^n$. Then the corresponding solution u satisfies the estimate

$$\| u \|_{L^q([2,3]; L^r(\mathbb{R}^n))} \lesssim \| F \|_{L^{\tilde{q}'}([1,2]; L^{\tilde{r}'}(\mathbb{R}^n))} \quad (1.34)$$

whenever the exponents $q, r, \tilde{q}, \tilde{r}$ satisfy the following conditions

$$\begin{aligned} 1 \leq q, \tilde{q} \leq \infty, \quad 2 \leq r, \tilde{r} \leq \infty, \\ \frac{n-2}{r} \leq \frac{n}{\tilde{r}}, \quad \frac{n-2}{\tilde{r}} \leq \frac{n}{r}, \\ \frac{n}{2} \left(\frac{1}{\tilde{r}} - \frac{1}{r} \right) \leq \frac{1}{q}, \quad \frac{n}{2} \left(\frac{1}{r} - \frac{1}{\tilde{r}} \right) \leq \frac{1}{\tilde{q}}, \end{aligned}$$

and if $n > 2$, we must also require that $r, \tilde{r} < \infty$.

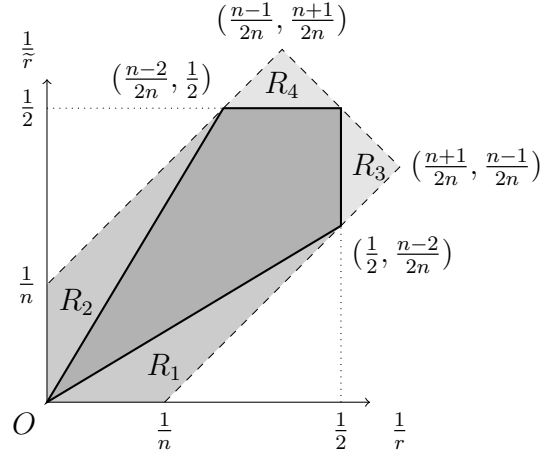
Theorem 1.2.7. *[Necessary conditions] Let the solution u and the forcing term F be as in Theorem 1.2.6. If u satisfies the estimate (1.34) then the exponents $q, r, \tilde{q}, \tilde{r}$ must satisfy the following conditions*

$$\begin{aligned} \frac{1}{r} + \frac{1}{\tilde{r}} \leq 1, \quad \left| \frac{1}{r} - \frac{1}{\tilde{r}} \right| \leq \frac{1}{n}, \\ \frac{2}{q} \geq \frac{n-2}{r} - \frac{n}{\tilde{r}}, \quad \frac{2}{\tilde{q}} \geq \frac{n-2}{\tilde{r}} - \frac{n}{r}, \end{aligned} \quad (1.35)$$

$$\frac{1}{q} \geq \frac{n}{2} \left(\frac{1}{\tilde{r}} - \frac{1}{r} \right), \quad \frac{1}{\tilde{q}} \geq \frac{n}{2} \left(\frac{1}{r} - \frac{1}{\tilde{r}} \right). \quad (1.36)$$

The range of values for the exponents $(q, r, \tilde{q}, \tilde{r})$ which is described by the necessary conditions described in Theorem 1.2.7 is larger than the corresponding one given by the sufficient conditions of Theorem 1.2.6.

The regions R_1, R_2, R_3 and R_4 represent the differences between the necessary and sufficient conditions for the exponents r and \tilde{r} .



1.3 Scope of the thesis

In this section, we introduce the research problem we are going to study throughout this work.

As we have seen in Section 1.2, the largest range of admissibility for the Lebesgue exponents known for the local inhomogeneous Strichartz type estimate (1.34) associated with the problem (1.33) is the one given in both [10] and [26] independently. Nevertheless, the optimal range of the exponents (q, r) and (\tilde{q}, \tilde{r}) for this estimate is still unknown for the higher dimensions $n \geq 3$. This is because the counter examples in [10] and [26] don't exclude the possibility of the validity of the estimate (1.34) outside the admissibility range obtained there. In other words, the range of values for the exponents $(q, r, \tilde{q}, \tilde{r})$ described by the necessary conditions given in Theorem 1.2.7 is larger than the corresponding one given by the sufficient conditions of Theorem 1.2.6. In particular, the already known counter examples do not deny the possibility that local estimates could be satisfied for some (q, \tilde{q}) when (r, \tilde{r})

are in one of the ranges R_j , $j = 1, 2, 3, 4$, described below.

$$\begin{aligned}
 R_1 & : \frac{1}{r} > \frac{1}{2}, & \frac{1}{r} + \frac{1}{\tilde{r}} &\leq 1, & \frac{1}{r} - \frac{1}{\tilde{r}} &\leq \frac{1}{n}, \\
 R_2 & : \frac{1}{\tilde{r}} > \frac{1}{2}, & \frac{1}{r} + \frac{1}{\tilde{r}} &\leq 1, & \frac{1}{\tilde{r}} - \frac{1}{r} &\leq \frac{1}{n}, \\
 R_3 & : \frac{n-2}{r} > \frac{n}{\tilde{r}}, & \frac{1}{r} - \frac{1}{\tilde{r}} &\leq \frac{1}{n}, \\
 R_4 & : \frac{n-2}{\tilde{r}} > \frac{n}{r}, & \frac{1}{\tilde{r}} - \frac{1}{r} &\leq \frac{1}{n}.
 \end{aligned}$$

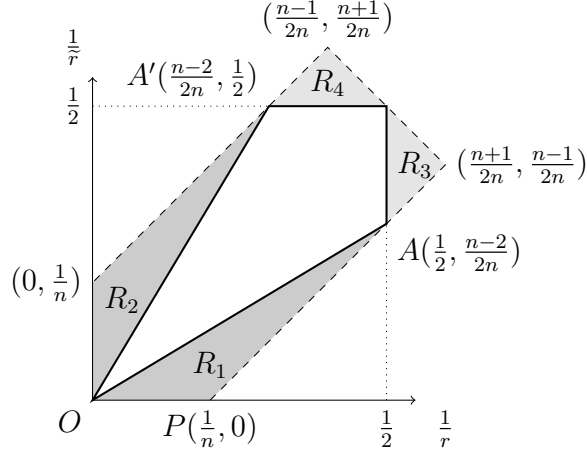


Figure (R): The shadowed regions represent the gap between the necessary and sufficient conditions

In Chapter 2

We do give new counter examples that exclude a certain range of values for the exponents (q, \tilde{q}) in the regions R_j . The counter examples are based on highly oscillatory functions in the time variable multiplied by certain concentration functions in the spatial variable. The region that optimizes the mixed Lebesgue norm of the solution and hence the Strichartz estimate depends on the frequency of the oscillations.

In chapters 3-5

We consider the inhomogeneous Strichartz type estimate (1.34) for the the necessary values of the exponents q and \tilde{q} taking into account the new restrictions obtained in Chapter 2. We actually look at the estimate

$$\| u \|_{L^q([2,3], L^n(\mathbb{R}^n))} \lesssim \| F \|_{L^{\tilde{q}}([0,1], L^1(\mathbb{R}^n))}. \quad (1.37)$$

The reason we are interested in the estimate (1.37) lies in the fact that when $r = n$ and $\tilde{r}' = \infty$ we are standing at the point $P(\frac{1}{n}, 0)$ in the $\frac{1}{r} - \frac{1}{\tilde{r}}$ plane shown in the figure (R) above. If the estimate (1.37) was proven to be satisfied for some values of q and \tilde{q} , then standard L^p interpolation techniques [2] help us to recover the triangle AOP . Not only would follow the estimate (1.34) on the triangle AOP , but it also would be valid throughout the triangle $A'OP'$ by the "duality" properties of the operator that gives the solution. These properties make all the conditions on the exponents values invariant under the symmetry $(q, r) \leftrightarrow (\tilde{q}, \tilde{r})$, (see Chapter 1 section 1.1 or [10] for more details). A special case of the estimate (1.37) is the following estimate

$$\boxed{\| u \|_{L^q([2,3], L^n(\mathbb{R}^n))} \lesssim \| f \|_{L^{\tilde{q}'([0,1])}} \| g \|_{L^1(\mathbb{R}^n)}}. \quad (1.38)$$

Indeed, the estimate (1.37) reduces to the estimate (1.38) when the inhomogeneity F is of the form $F(t, x) = f(t)g(x)$ where $f \in L^{\tilde{q}'([0,1])}$ and $g \in L^1(\mathbb{R}^n)$. We investigate the forcing term $F(t, x) = f(t)\delta_0(x)$ where δ_0 is the dirac delta function at the origin. The delta function δ_0 represents a full concentration of the mass at the origin. Although δ_0 is not a function on \mathbb{R}^n , and thus does not belong to $L^p(\mathbb{R}^n)$ for any p , we will sacrifice some rigor. The main reason behind this is that the delta function is the limit in the sense of distributions of a sequence of functions $f_\epsilon \in L^1(\mathbb{R}^n)$ with $\|f_\epsilon\|_{L^1(\mathbb{R}^n)} = 1$. Therefore, if the estimate (1.38) held true then so should the estimate

$$\boxed{\| u \|_{L^q([2,3], L^n(\mathbb{R}^n))} \lesssim \| f \|_{L^{\tilde{q}'([0,1])}}}. \quad (1.39)$$

Another reason for choosing the delta function comes from the fact that many counter examples for the estimate (1.34) involve concentration in balls or spherical shells centered at the origin. This means that concentration at the origin should be the most difficult case to deal with among data of this type. Treating this "worst" case with success would be encouraging to try to prove the more general estimate (1.38). We choose the extreme necessary values for the exponents q and \tilde{q} when $(r, \tilde{r}) = (n, \infty)$. These would be $q = \frac{2n}{n-2}$

and $\tilde{q}' = n$. That is we study the estimate

$$\boxed{\| u \|_{L^{\frac{2n}{n-2}}([2,3], L^n(\mathbb{R}^n))} \lesssim \| f \|_{L^n([0,1])}}. \quad (1.40)$$

At this point, we choose a certain dimension n so that $\frac{2n}{n-2} = n =$ an even integer. The dimension $n = 4$ fulfills this criterion. This particular choice has the technical advantage of enabling us to replace the estimate of the norm $\| \cdot \|_{L_t^4 L_x^4}$ with the estimate of a multilinear form. Indeed, using the explicit formula

$$u(t, x) = \frac{1}{(4\pi)^2} \int_0^1 \frac{f(s)}{(t-s)^2} e^{i\frac{|x|^2}{4(t-s)}} ds \quad (1.41)$$

for the fundamental solution u of (1.33) that corresponds to the inhomogeneity

$F(t, x) = f(t) \delta_0(x)$ we get that

$$\| u \|_{L^4([2,3]; L^4(\mathbb{R}^4))}^4 = \frac{1}{2\pi^2} \int_2^3 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{iP(t, s_1, s_2, s_3, s_4)|x|^2}}{\prod_{l=1}^4 (t-s_l)^2} f(s_1) f(s_2) f(s_3) f(s_4) ds_1 ds_2 ds_3 ds_4 dx dt \quad (1.42)$$

with

$$P(t, s_1, s_2, s_3, s_4) = \frac{1}{4} \sum_{l=1}^4 \frac{(-1)^l}{t-s_l}.$$

Then using the change of variables

$$t-2 \longrightarrow t, \quad 1-s \longrightarrow s$$

in (3.13) we immediately see that

$$\| u(t+2, x) \|_{L^4([0,1]; L^4(\mathbb{R}^4))}^4 = \| u(t, x) \|_{L^4([2,3]; L^4(\mathbb{R}^4))}^4 = T(f, f, f, f) \quad (1.43)$$

where $T : L^4([0,1]) \times L^4([0,1]) \times L^4([0,1]) \times L^4([0,1]) \rightarrow \mathbb{C}$ is the quadrilinear form given by

$$T(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{iA(t, s_1, s_2, s_3, s_4)|x|^2}}{\prod_{l=1}^4 (1+t+s_l)^2} f_1(1-s_1) f_2(1-s_2) f_3(1-s_3) f_4(1-s_4) ds_1 ds_2 ds_3 ds_4 dx dt.$$

with

$$A(\tau, \sigma_1, \sigma_2, \sigma_3, \sigma_4) = P(2 + \tau, 1 - \sigma_1, 1 - \sigma_2, 1 - \sigma_3, 1 - \sigma_4) = \sum_{l=1}^4 \frac{(-1)^l}{1 + \tau + \sigma_l}.$$

Moreover, and because $\|f(1 - s)\|_{L^4([0,1])} = \|f(s)\|_{L^4([0,1])}$, the estimate (1.40) becomes

$$\boxed{\|u(t + 2, x)\|_{L^4([0,1] \times \mathbb{R}^4)} \lesssim \|f\|_{L^4([0,1])}} \quad (1.44)$$

We will see in Chapter 3 Section 5 that it is enough to prove the estimate (1.44) for functions f that are realvalued.

The basic idea here is that the estimate (1.44) is a consequence of the quadrilinear estimate

$$|T(f_1, f_2, f_3, f_4)| \lesssim \prod_{j=1}^4 \|f_j\|_{L^4([0,1])}. \quad (1.45)$$

Therefore, to prove the special estimate (1.44), we try to prove the estimate (1.45) using the multilinear interpolation method. This is what we do following two different approaches that we give in detail in chapters 4 and 5. We get to prove the estimate (1.45) and hence the estimate (1.44) with a divergence of an order less than any positive ϵ . In Chapter 3 Section 5, we summarize these approaches and briefly discuss the ideas of the proofs given in the next two chapters.

Chapter 2

New necessary conditions

Consider the following Cauchy problem for the inhomogeneous free Schrödinger equation in n -space dimensions

$$i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad u(0, x) = 0. \quad (2.1)$$

Using the Fourier transform, we get the following explicit formula for the solution

$$u(t, x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \int_{\mathbb{R}^n} \frac{e^{i\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{\frac{n}{2}}} F(s, y) dy ds. \quad (2.2)$$

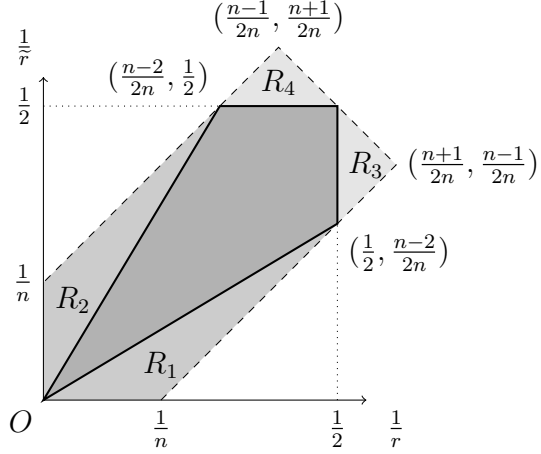
We are interested in the local inhomogeneous Strichartz-type estimate (2.3) below.

$$\| u \|_{L^q([2,3]; L^r(\mathbb{R}^n))} \lesssim \| F \|_{L^{\tilde{q}}([0,1]; L^{\tilde{r}}(\mathbb{R}^n))}, \quad (2.3)$$

Recall from chapter 1 section 1.3 that the counter examples found by Foschi [10] gave the following necessary conditions:

$$\begin{aligned} \frac{1}{r} + \frac{1}{\tilde{r}} &\leq 1, & \left| \frac{1}{r} - \frac{1}{\tilde{r}} \right| &\leq \frac{1}{n}, \\ \frac{2}{q} &\geq \frac{n-2}{r} - \frac{n}{\tilde{r}}, & \frac{2}{\tilde{q}} &\geq \frac{n-2}{\tilde{r}} - \frac{n}{r}, \end{aligned} \quad (2.4)$$

$$\frac{2}{q} \geq \frac{n}{\tilde{r}} - \frac{n}{r}, \quad \frac{2}{\tilde{q}} \geq \frac{n}{r} - \frac{n}{\tilde{r}}. \quad (2.5)$$



The estimate (2.1) is known to hold in the dark shaded kite-like region for some values of q and \tilde{q} . The regions R_1 , R_2 , R_3 and R_4 represent the difference between the necessary and sufficient conditions for the exponents r and \tilde{r} .

Recall also that these conditions do not deny the possibility that the estimate (2.3) holds in the regions R_j below for some certain values of the exponents q and \tilde{q} .

$$\begin{aligned}
R_1 & : \quad \frac{n-2}{r} > \frac{n}{\tilde{r}}, & \frac{1}{r} - \frac{1}{\tilde{r}} & \leq \frac{1}{n}, \\
R_2 & : \quad \frac{n-2}{\tilde{r}} > \frac{n}{r}, & \frac{1}{\tilde{r}} - \frac{1}{r} & \leq \frac{1}{n}, \\
R_3 & : \quad \frac{1}{r} > \frac{1}{2}, & \frac{1}{r} + \frac{1}{\tilde{r}} & \leq 1, & \frac{1}{r} - \frac{1}{\tilde{r}} & \leq \frac{1}{n}, \\
R_4 & : \quad \frac{1}{\tilde{r}} > \frac{1}{2}, & \frac{1}{r} + \frac{1}{\tilde{r}} & \leq 1, & \frac{1}{\tilde{r}} - \frac{1}{r} & \leq \frac{1}{n}.
\end{aligned}$$

In this chapter we give new necessary conditions that restrict the already-known possible range for the values of the exponents q and \tilde{q} to a smaller one inside the regions R_j . Precisely, our new necessary conditions are stronger than the necessary conditions (2.5) in the regions R_1 and R_2 . We shall prove the following theorem

Theorem 2.0.1. (*The new necessary conditions*) *Let $u = u(t, x)$ be the solution to the problem (2.1) where the inhomogeneous term F is supported on $[0, 1] \times \mathbb{R}^n$. If u satisfies the estimate*

$$\|u\|_{L^q([2,3];L^r(\mathbb{R}^n))} \lesssim \|F\|_{L^{\tilde{q}'}([0,1];L^{\tilde{r}'}(\mathbb{R}^n))},$$

then the exponents $q, r, \tilde{q}, \tilde{r}$ must satisfy the following conditions

$$\boxed{\frac{1}{q} \geq \frac{n-1}{\tilde{r}} - \frac{n}{r}, \quad \frac{1}{\tilde{q}} \geq \frac{n-1}{r} - \frac{n}{\tilde{r}}} \quad (2.6)$$

First we obtain a weaker version of the new necessary condition (2.6), namely

$$\boxed{\frac{1}{q} \geq \frac{n-2}{\tilde{r}} - \frac{n}{r}, \quad \frac{1}{\tilde{q}} \geq \frac{n-2}{r} - \frac{n}{\tilde{r}}} \quad (2.7)$$

This is still stronger than the known conditions (2.4) and (2.5) in the same regions described before. Then we show how to improve the counter example and modify its defining parameters so that we get the stronger new condition (2.6).

To prove the new necessary conditions (2.7) and the improved ones (2.6), we use two different sets of counter examples in two different methods. In the first approach, we basically use the method of stationary phase described in Lemma 2.1.1 below, (see [22], Chapter VIII) and apply it to the first set of counter examples. In the second one we simplify the proof and give the same results using merely the idea of the Knapp's counter example for the second set of counter examples. However, the inhomogeneity used in all the counter examples consists of a concentration in both the spatial and temporal variables multiplied by an oscillation of certain high frequency.

2.1 Counter Examples

We apply the stationary phase method to the solutions corresponding to forcing terms of the form $F(s, y) = e^{iNs} \chi_{[0, \epsilon]}(s) \chi_{\Omega_{\epsilon, N}}(y)$, where ϵ , N and $\Omega_{\epsilon, N}$ are chosen in such a way to satisfy the assumptions required to apply Lemma 2.1.1. In the first counter example, we show in a rather detailed way how we made the choice of the parameters in the inhomogeneity term used to get the best we can from applying this method.

Lemma 2.1.1. *Consider the oscillatory integral $I(N) = \int_a^b e^{iN\phi(s)} \chi(s) ds$. If the phase $\phi \in C^5([a, b])$ such that $\phi'(s) = 0$ for a point $s_* \in [a + \delta, b - \delta]$ with $\delta > 0$ and $\phi''(s) \geq 1$ and the amplitude $\chi \in C^3([a, b])$ then*

$$I(N) = \frac{e^{\frac{\pi}{4}i} \sqrt{\frac{2\pi}{\phi''(s_*)}} \chi(s_*) e^{iN\phi(s_*)}}{\sqrt{N}} + O\left(\frac{1}{N}\right),$$

where the implicit constant in the O -symbol depend on $b - a$, δ , $\chi(a)$, $\chi(b)$ and bounds for $|\chi^{(j)}|$, $j = 0, 1, 2$ and $|\phi^{(k)}|$, $k = 2, 3, 4, 5$.

For the sake of completeness, and since the phase and amplitude and their derivatives in the counter examples involve many parameters, we give a detailed proof of Lemma 2.1.1 in the appendix (section 2.3). This is useful in controlling the contribution from these parameters to the error term in particular.

Now, let $J(t, x, y)$ be the oscillatory integral defined by

$$J(t, x, y) = \int_0^1 e^{i\lambda\xi(s;t,x,y)} \zeta(s; t, x, y) ds, \quad (t, x, y) \in [2, 3] \times \mathbb{R}^n \times \mathbb{R}^n.$$

We define the region $L_J \subset [2, 3] \times \mathbb{R}^n \times \mathbb{R}^n$ to be the region where Lemma 2.1.1 can be applied to $J(t, x, y)$ to give an estimate of it when its phase $\xi(s; t, x, y)$ has a nondegenerate stationary point. A point (t, x, y) belongs to L_J if and only if

- (i) The phase $\xi(s; t, x, y)$ is stationary so that $\partial_s \xi(s; t, x, y) = 0$ for a point $s = s_* \in [\delta, 1 - \delta]$, with $\delta > 0$ a small (fixed) positive number,
- (ii) The second derivative of the phase satisfies $|\partial_s^2 \xi(s; t, x, y)| \gtrsim 1$,
- (iii) The derivatives $\partial_s^{(j+3)} \xi(s; t, x, y)$ and $\partial_s^{(j)} \zeta(s; t, x, y)$, $j = 0, 1, 2$ have uniform upper bounds.

Now, consider the explicit formula (2.2) for the solution $u(t, x)$ and take the forcing term (the nonhomogeneity) given by

$$F(s, y) = e^{iNs} \chi_{[0, \epsilon]}(s) \chi_{\Omega_{\eta, N}}(y),$$

where $N > 1$, $0 < \epsilon < 1$ and η is a small but fixed positive number. We later will define the region $\Omega_{\eta, N}$ and give additional conditions on N and ϵ as well in order to optimize the ratio $\|u\|_{L_t^q L_x^r} / \|F\|_{L_s^{\tilde{q}'} L_y^{\tilde{r}'}}$. Applying Fubini's theorem and then the rescaling $\frac{s}{\epsilon} \rightarrow s$, (2.2), can be

written as follows

$$\begin{aligned}
u(t, x) &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\epsilon \int_{\Omega_{\eta, N}} \frac{e^{i\left[Ns - \frac{|x-y|^2}{4(t-s)}\right]}}{(t-s)^{\frac{n}{2}}} dy ds \\
&= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{\Omega_{\eta, N}} \int_0^\epsilon \frac{e^{i\left[Ns - \frac{|x-y|^2}{4(t-s)}\right]}}{(t-s)^{\frac{n}{2}}} ds dy \\
&= \frac{\epsilon}{(4\pi)^{\frac{n}{2}}} \int_{\Omega_{\eta, N}} \int_0^1 \frac{e^{i\left[N\epsilon s - \frac{|x-y|^2}{4(t-\epsilon s)}\right]}}{(t-\epsilon s)^{\frac{n}{2}}} ds dy \\
&= \frac{\epsilon}{(4\pi)^{\frac{n}{2}}} \int_{\Omega_{\eta, N}} I_{N, \epsilon}(t, x, y) dy, \tag{2.8}
\end{aligned}$$

where $I_{N, \epsilon}(t, x, y)$ is the oscillatory integral given by

$$I_{N, \epsilon}(t, x, y) = \int_0^1 e^{i\phi_{N, \epsilon}(s; t, x, y)} \chi_{N, \epsilon}(s; t, x, y) ds, \tag{2.9}$$

with the phase

$$\phi_{N, \epsilon}(s; t, x, y) = N\epsilon s - \frac{|x-y|^2}{4(t-\epsilon s)}$$

and the amplitude

$$\chi_{N, \epsilon}(s; t, x, y) = \frac{1}{(t-\epsilon s)^{\frac{n}{2}}}.$$

The first and second derivatives of the phase are given by

$$\partial_s \phi_{N, \epsilon}(s; t, x, y) = N\epsilon - \frac{|x-y|^2}{4(t-\epsilon s)^2} \epsilon, \quad \partial_s^2 \phi_{N, \epsilon}(s; t, x, y) = -\frac{|x-y|^2}{2(t-\epsilon s)^3} \epsilon^2.$$

Let $s_* \in (0, 1)$ be such that

$$\partial_s \phi_{N, \epsilon}(s_*; t, x, y) = N\epsilon - \frac{|x-y|^2}{4(t-\epsilon s_*)^2} \epsilon = 0.$$

Then

$$s_* = \frac{t}{\epsilon} - \frac{|x-y|^2}{2\epsilon\sqrt{N}}.$$

So as for s_* to lie inside $(0, 1)$, we must have that

$$2\sqrt{N}(t-\epsilon) < |x-y| < 2\sqrt{N}t \implies |x-y| \approx \sqrt{N},$$

but in this case, we would have

$$|\partial_s^2 \phi_{N,\epsilon}(s; t, x, y)| = \frac{|x - y|^2}{2(t - \epsilon s)^3} \epsilon^2 \approx N \epsilon^2.$$

This estimate of $\partial_s^2 \phi_{N,\epsilon}(s; t, x, y)$ suggests rewriting the oscillatory integral in (2.9) in such a way so that it has a phase that satisfies both conditions (i) and (ii) when $|x - y| \approx \sqrt{N}$.

Rewrite (2.9) as

$$I_{N,\epsilon}(t, x, y) = \int_0^1 e^{iN\epsilon^2 \psi_{N,\epsilon}(s; t, x, y)} \chi_{N,\epsilon}(s; t, x, y) ds,$$

where the phase now is defined by

$$\psi_{N,\epsilon}(s; t, x, y) = \frac{\phi_{N,\epsilon}(s; t, x, y)}{N\epsilon^2} = \frac{s}{\epsilon} - \frac{|x - y|^2}{4N\epsilon^2(t - \epsilon s)}.$$

Now we find $L_{I_{N,\epsilon}}$. Derivatives of the phase and amplitude that we need to control are

$$\begin{aligned} \partial_s \psi_{N,\epsilon}(s; t, x, y) &= \frac{1}{\epsilon} - \frac{|x - y|^2}{4N\epsilon(t - \epsilon s)^2}, & \partial_s^2 \psi_{N,\epsilon}(s; t, x, y) &= -\frac{|x - y|^2}{2N(t - \epsilon s)^3} \\ \partial_s^j \psi_{N,\epsilon}(s; t, x, y) &= -\frac{j! |x - y|^2}{4N[t - \epsilon s]^{(j+1)}} \epsilon^{j-2}, & j &= 3, 4, 5, \\ \partial_s \chi_{N,\epsilon}(s; t, x, y) &= \frac{n}{2}(t - \epsilon s)^{-\frac{n}{2}-1} \epsilon, & \partial_s^2 \chi_{N,\epsilon}(s; t, x, y) &= \frac{n}{2} \left(\frac{n}{2} + 1\right) (t - \epsilon s)^{-\frac{n}{2}-2} \epsilon^2. \end{aligned}$$

Let $0 < \delta < \frac{1}{2}$. We have that

$$\begin{aligned} \partial_s \psi_{N,\epsilon}(s; t, x, y) = \frac{1}{\epsilon} - \frac{|x - y|^2}{4N\epsilon(t - \epsilon s)^2} = 0 &\iff s = s_* : t - \epsilon s_* = \frac{|x - y|}{2\sqrt{N}}, \\ \delta \leq s_* \leq 1 - \delta &\iff t - (1 - \delta)\epsilon \leq \frac{|x - y|}{2\sqrt{N}} \leq t - \delta\epsilon \\ &\iff 2[t - (1 - \delta)\epsilon]\sqrt{N} \leq |x - y| \leq 2(t - \delta\epsilon)\sqrt{N}. \end{aligned}$$

Assume that

$$2[t - (1 - \delta)\epsilon]\sqrt{N} \leq |x - y| \leq 2(t - \delta\epsilon)\sqrt{N}. \quad (2.10)$$

Then

- (i) The assumption (2.10) suffices for the phase $\psi_{N,\epsilon}(s; t, x, y)$ to attain a critical point in $[\delta, 1 - \delta]$.

(ii) When (2.10) is satisfied we also have that

$$|\partial_s^2 \psi_{N,\epsilon}(s; t, x, y)| = \frac{|x - y|^2}{2N(t - \epsilon s)^3} \approx 1.$$

(iii) Moreover when x and y obey (2.10) the following estimates hold

$$|\partial_s^j \psi_{N,\epsilon}(s; t, x, y)| \approx \epsilon^{j-2}, \quad j = 3, 4, 5, \quad |\partial_s^j \chi_{N,\epsilon}(s; t, x, y)| \approx \epsilon^j, \quad j = 0, 1, 2.$$

Thus the condition (2.10) is sufficient for $(t, x, y) \in L_{I_{N,\epsilon}}$ for all $t \in [2, 3]$. It is therefore sufficient to apply Lemma 2.1.1 to the oscillatory integral $I_{N,\epsilon}(t, x, y)$.

Applying Lemma 2.1.1 to $I_{N,\epsilon}(t, x, y)$ yields to

$$I_{N,\epsilon}(t, x, y) = \left[e^{\frac{\pi}{4}i} \sqrt{\frac{2\pi}{\partial_s^2 \psi_{N,\epsilon}(s_*; t, x, y)}} \chi_{N,\epsilon}(s_*; t, x, y) \right] \frac{1}{\epsilon \sqrt{N}} e^{iN\epsilon^2 \psi_{N,\epsilon}(s_*; t, x, y)} + O\left(\frac{1}{\epsilon^2 N}\right).$$

But we shall eventually need to integrate this in y over $\Omega_{\eta,N}$ to compute the solution $u(t, x)$. To fulfill the condition (2.10) and at the same time optimize the solution norm $\|u\|_{L^q([2,3], L^r(\mathbb{R}^n))}$, by avoiding doing the integration while there is still an oscillatory factor that depends on y , we take x and y in regions where either

- $|y| \lesssim \sqrt{N}$ and $|x| \leq \frac{\eta}{\sqrt{N}}$ (The first counter example)
- or $|x| \lesssim \sqrt{N}$ and $|y| \leq \frac{\eta}{\sqrt{N}}$ (The second counter example).

2.1.1 The first counter example

Let $|x| \leq \frac{\eta}{\sqrt{N}}$ and $|y| \lesssim \sqrt{N}$. Now we describe the support of $F(s, \cdot)$, $\Omega_{\eta,N}$, by giving a more precise condition on y , and discuss the conditions sufficient to have $(t, x, y) \in L_{I_{N,\epsilon}}$.

(i) When $|x| \leq \frac{\eta}{\sqrt{N}}$, then (2.10) is satisfied **if**

$$2[t - (1 - \delta)\epsilon]\sqrt{N} + \frac{\eta}{\sqrt{N}} \leq |y| \leq 2(t - \delta\epsilon)\sqrt{N} - \frac{\eta}{\sqrt{N}}, \quad (2.11)$$

but (2.11) makes sense only if

$$\begin{aligned} 2(t - \delta\epsilon)\sqrt{N} - \frac{\eta}{\sqrt{N}} &> 2[t - (1 - \delta)\epsilon]\sqrt{N} + \frac{\eta}{\sqrt{N}} \iff \epsilon(1 - 2\delta)\sqrt{N} > \frac{\eta}{\sqrt{N}} \\ &\iff \epsilon N > \frac{\eta}{(1 - 2\delta)}. \end{aligned}$$

We will take $\delta = \frac{1}{4}$ and impose as a primary condition on N and ϵ that $\epsilon N \gg \eta$.

Condition (2.11) becomes

$$2\left(t - \frac{3}{4}\epsilon\right)\sqrt{N} + \frac{\eta}{\sqrt{N}} \leq |y| \leq 2\left(t - \frac{1}{4}\epsilon\right)\sqrt{N} - \frac{\eta}{\sqrt{N}}.$$

Now, since the region $\Omega_{\eta,N}$ has to be independent of t , and regarding the need to make the variation of the phase with y small enough, we assign

$$\Omega_{\eta,N} = \left\{y \in \mathbb{R}^n : \left||y| - 4\sqrt{N}\right| \leq \frac{2\eta}{\sqrt{N}}\right\},$$

in which case t must satisfy

$$2 + \frac{\eta}{N} + \frac{\epsilon}{4} \leq t \leq 2 - \frac{\eta}{N} + \frac{3\epsilon}{4}. \quad (2.12)$$

Notice that (2.12) makes sense when $\epsilon N \gg \eta$.

(ii) When $|x| \leq \frac{\eta}{\sqrt{N}}$ and $y \in \Omega_{\eta,N}$, we immediately have that $|x - y| \approx \sqrt{N}$ and consequently

$$|\partial_s^2 \psi_{N,\epsilon}(s; t, x, y)| = \frac{|x - y|^2}{2N(t - \epsilon s)^3} \approx 1.$$

(iii) Moreover, we have

$$|\partial_s^j \psi_{N,\epsilon}(s; t, x, y)| \approx \epsilon^{j-2}, \quad j = 3, 4, 5, \quad |\partial_s^j \chi_{N,\epsilon}(s; t, x, y)| \approx \epsilon^j, \quad j = 0, 1, 2.$$

We have so far seen that if (2.12) is satisfied, $|x| \leq \frac{\eta}{\sqrt{N}}$ and $y \in \Omega_{\eta,N}$, then $(t, x, y) \in L_{I_{N,\epsilon}}$.

Applying Lemma 2.1.1 to the oscillatory integral $I_{N,\epsilon}(t, x, y)$ now yields

$$I_{N,\epsilon}(t, x, y) = \left[e^{\frac{\pi}{4}i} \sqrt{\frac{2\pi}{\partial_s^2 \psi_{N,\epsilon}(s_*; t, x, y)}} \chi_{N,\epsilon}(s_*; t, x, y) \right] \frac{1}{\epsilon\sqrt{N}} e^{iN\epsilon^2 \psi_{N,\epsilon}(s_*; t, x, y)} + O\left(\frac{1}{\epsilon^2 N}\right).$$

Choose ϵ and N such that $\epsilon\sqrt{N} = C$, where $C \gg 1$ is a large constant and observe that by this choice we have $\psi_{N,\epsilon}(s_*; t, x, y) = \frac{t}{\epsilon^2} - \frac{|x - y|}{\epsilon^2\sqrt{N}} = \frac{Nt}{C^2} - \frac{\sqrt{N}|x - y|}{C^2}$.

We thus have

$$\begin{aligned} I_{N,\epsilon}(t, x, y) &= I_{N, \frac{C}{\sqrt{N}}}(t, x, y) \\ &= \left[e^{\frac{\pi}{4}i} \sqrt{\frac{2\pi}{\partial_s^2 \psi_{N, \frac{C}{\sqrt{N}}}(s_*; t, x, y)}} \chi_{N, \frac{C}{\sqrt{N}}}(s_*; t, x, y) \right] \frac{1}{C} e^{iNt} e^{-i\sqrt{N}|x - y|} + O\left(\frac{1}{C^2}\right). \end{aligned}$$

It is easy to verify that

$$\sqrt{N}|x - y| = 4N + O(\eta). \quad (2.13)$$

Thus the oscillatory factor $e^{-i\sqrt{N}|x-y|}$ can actually be written as

$$e^{-i\sqrt{N}|x-y|} = e^{-i[4N+O(\eta)]} = e^{-4iN}[1 + O(\eta)].$$

The solution in (2.8) is now given by

$$\begin{aligned} u(t, x) &= \frac{C}{\sqrt{N}} \int_{\Omega_{\eta, N}} I_{N, \frac{C}{\sqrt{N}}}(t, x, y) dy \\ &= \frac{\sqrt{2\pi}}{(4\pi)^{\frac{n}{2}}} e^{\frac{\pi i}{4}} \frac{e^{iN(t-4)}}{\sqrt{N}} \int_{\Omega_{\eta, N}} \frac{\chi_{N, \frac{C}{\sqrt{N}}}(s_*; t, x, y)}{\sqrt{\partial_s^2 \psi_{N, \frac{C}{\sqrt{N}}}(s_*; t, x, y)}} [1 + O(\eta)] dy + \\ &\quad + \int_{\Omega_{\eta, N}} O\left(\frac{1}{C\sqrt{N}}\right) dy. \end{aligned} \quad (2.14)$$

From (2.14) and the facts that $|\chi_{N, \frac{C}{\sqrt{N}}}(s_*; t, x, y)| \approx 1$, $|\partial_s^2 \psi_{N, \frac{C}{\sqrt{N}}}(s_*; t, x, y)| \approx 1$ and $|\Omega_{\eta, N}| \approx N^{\frac{n-2}{2}}$, it follows that, when C is large enough and η is small enough, we get the following estimate

$$|u(t, x)| \gtrsim \frac{1}{\sqrt{N}} N^{\frac{n-2}{2}}.$$

Therefore, when t varies on the interval given by (2.12) which is of size of order $\epsilon = \frac{C}{\sqrt{N}}$, the norm of the solution can be estimated as

$$\|u\|_{L_t^q L_x^r} \gtrsim C^{\frac{1}{q}} N^{\frac{n-2}{2}} N^{-\frac{1}{2q}} N^{-\frac{n}{2r}}.$$

Thus we have the ratio

$$\frac{\|u\|_{L_t^q L_x^r}}{\|F\|_{L_s^{\tilde{q}'} L_{\tilde{r}'}^{\frac{n-2}{2}}}} \gtrsim \frac{C^{\frac{1}{q}} N^{\frac{n-2}{2}} N^{-\frac{1}{2q}} N^{-\frac{n}{2r}}}{C^{\frac{1}{\tilde{q}'}} N^{-\frac{1}{2\tilde{q}'}} N^{\frac{n-2}{2\tilde{r}'}}} = C^{\frac{1}{q} + \frac{1}{\tilde{q}'} - 1} N^{-\frac{1}{2q} - \frac{1}{2\tilde{q}'} - \frac{n}{2r} + \frac{n-2}{2\tilde{r}'}}.$$

When C is fixed, then this ratio blows up as $N \rightarrow +\infty$ unless $-\frac{1}{2q} - \frac{1}{2\tilde{q}'} - \frac{n}{2r} + \frac{n-2}{2\tilde{r}'} \leq 0$. This implies the necessary condition

$$\frac{1}{q} + \frac{1}{\tilde{q}'} \geq \frac{n-2}{\tilde{r}'} - \frac{n}{r}. \quad (2.15)$$

This condition and its dual are stronger than conditions (2.4). They also are stronger than conditions (2.5) in the regions R_1 and R_2 .

Remark 2.1.1. In the light of the condition $\epsilon \sqrt{N} = C$, we had to impose on the parameters, we find that the first counter example discussed above is given by

$$F(s, y) = e^{i \frac{C^2}{\epsilon^2} s} \chi_{[0, \epsilon]}(s) \chi_{\{|y| - 4 \frac{C}{\epsilon} \leq \frac{2\eta}{C} \epsilon\}}(y)$$

And we looked at the solution where

$$2 + \frac{\eta \epsilon^2}{C^2} + \frac{\epsilon}{4} \leq t \leq 2 - \frac{\eta \epsilon^2}{C^2} + \frac{3\epsilon}{4}, \quad |x| \leq \frac{\eta \epsilon}{C}.$$

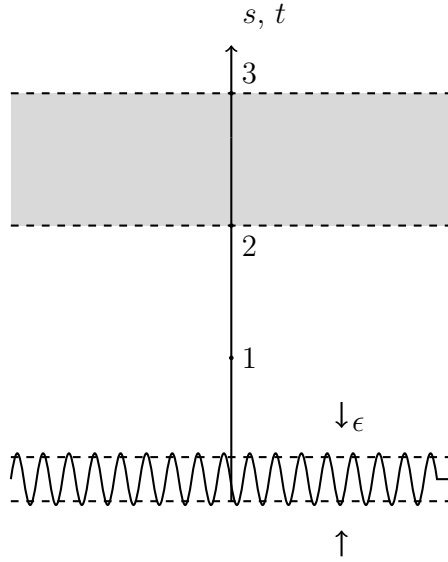
2.1.2 The second counter example

Now, that we learned how to choose ϵ , N , η and C to optimize the solution norm using the stationary phase method, we can proceed using the other choice for x and y suggested by (2.10) and sufficient to make the variation of the phase with y small enough. So, we will look at the solution when $|x| \lesssim \frac{C}{\epsilon}$ and $|y| \leq \frac{\epsilon \eta}{C}$.

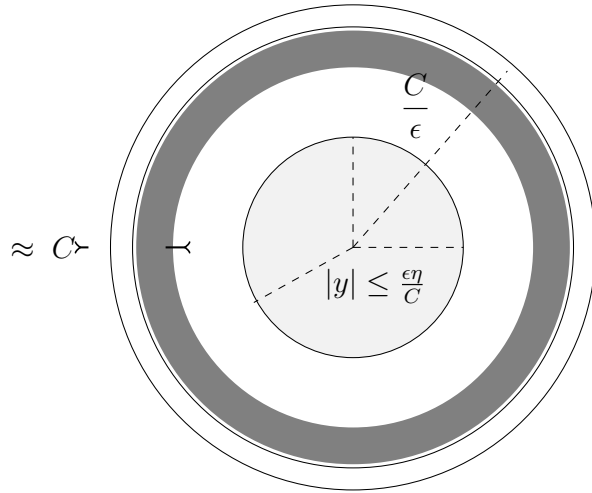
Consider the forcing term given by

$$F(s, y) = e^{i \frac{C^2}{\epsilon^2} s} \chi_{[0, \epsilon]}(s) \chi_{\Omega_{\epsilon, \eta, C}}(y)$$

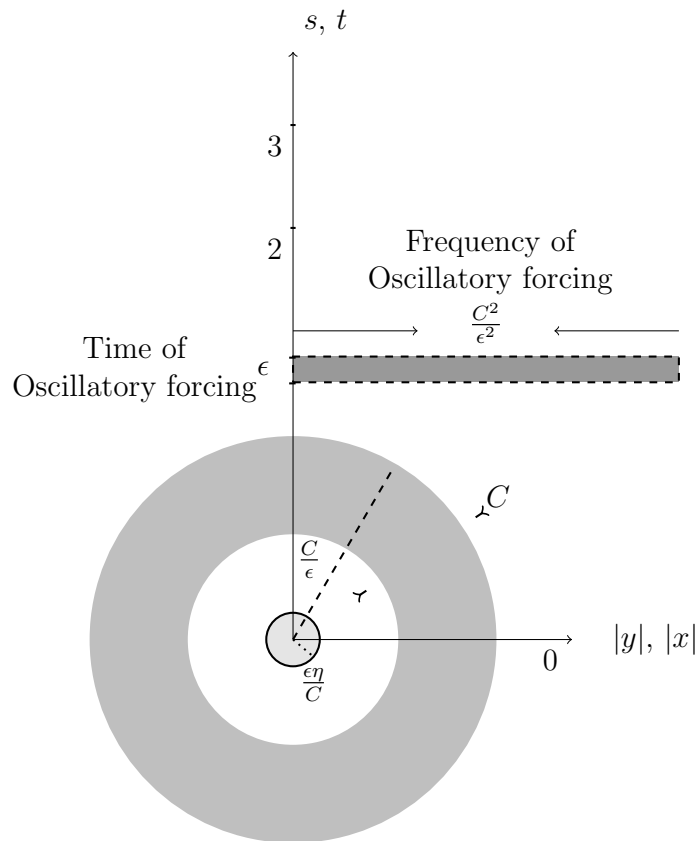
where $C \gg 1$ is a fixed large number and ϵ and η are small positive parameters such that $0 < \epsilon < \eta < 1$ and η is fixed and $\Omega_{\epsilon, \eta, C} = \{y \in \mathbb{R}^n : |y| \leq \frac{\epsilon \eta}{C}\}$.



Apply oscillatory forcing of frequency $\frac{C^2}{\epsilon^2}$ for a time period ϵ and measure the solution on the whole delayed unit time interval.



Concentrate the input mass in a ball of radius about $\frac{\epsilon}{C}$ and look at the solution in the co-centered spherical shell of radius about $\frac{C}{\epsilon}$ and thickness about C



The corresponding solution, using Fubini's theorem and then the rescaling $\frac{s}{\epsilon} \rightarrow s$ as previously can be written as follows

$$u(t, x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^\epsilon \int_{\Omega_{\epsilon, \eta, C}} \frac{e^{i\left[\frac{C^2}{\epsilon^2}s - \frac{|x-y|^2}{4(t-s)}\right]}}{(t-s)^{\frac{n}{2}}} dy ds = \frac{\epsilon}{(4\pi)^{\frac{n}{2}}} \int_{\Omega_{\epsilon, \eta, C}} I_{\epsilon, C}(t, x, y) dy, \quad (2.16)$$

where $I_{\epsilon, C}(t, x, y)$ is the oscillatory integral given by

$$I_{\epsilon, C}(t, x, y) = \int_0^1 e^{iC^2\phi_{\epsilon, C}(s; t, x, y)} \chi_{\epsilon, C}(s; t, x, y) ds,$$

with the phase $\phi_{\epsilon, C}(s; t, x, y) = \frac{s}{\epsilon} - \frac{|x-y|^2}{4C^2(t-\epsilon s)}$ and amplitude $\chi_{\epsilon, C}(s; t, x, y) = (t-\epsilon s)^{-\frac{n}{2}}$.

Derivatives of the phase and amplitude are

$$\begin{aligned} \partial_s \phi_{\epsilon, C}(s; t, x, y) &= \frac{1}{\epsilon} - \frac{\epsilon}{4C^2} \frac{|x-y|^2}{(t-\epsilon s)^2}, & \partial_s^2 \phi_{\epsilon, C}(s; t, x, y) &= -\frac{\epsilon^2}{2C^2} \frac{|x-y|^2}{(t-\epsilon s)^3} \\ \partial_s^j \phi_{\epsilon, C}(s; t, x, y) &= -\frac{j! \epsilon^j}{4C^2} \frac{|x-y|^2}{[t-\epsilon s]^{(j+1)}}, & j &= 3, 4, 5, \\ \partial_s \chi_{N, \epsilon}(s; t, x, y) &= \frac{n}{2} (t-\epsilon s)^{-\frac{n}{2}-1} \epsilon, & \partial_s^2 \chi_{N, \epsilon}(s; t, x, y) &= \frac{n}{2} \left(\frac{n}{2} + 1\right) (t-\epsilon s)^{-\frac{n}{2}-2} \epsilon^2. \end{aligned}$$

(i) Let $s = s_*$ be such that $\partial_s \phi_{\epsilon, C}(s_*; t, x, y) = 0$. Then

$$s_* = \frac{t}{\epsilon} - \frac{|x-y|}{2C}.$$

It is easy to verify that whenever $\left||x| - \left(\frac{2Ct}{\epsilon} - C\right)\right| \leq \frac{C}{2}$ and $y \in \Omega_{\epsilon, \eta, C}$ then $s_* \in \left[\frac{1}{4}, \frac{3}{4}\right]$.

(ii) Wherever $\left||x| - \left(\frac{2Ct}{\epsilon} - C\right)\right| \leq \frac{C}{2}$ and $y \in \Omega_{\epsilon, \eta, C}$ we have that $|x-y| \approx \frac{C}{\epsilon}$ and consequently

$$|\partial_s^2 \phi_{\epsilon, C}(s; t, x, y)| = \frac{|x-y|^2}{2C^2(t-\epsilon s)^3} \epsilon^2 \approx 1.$$

(iii) We also have that $|\partial_s^j \phi_{\epsilon, C}(s; t, x, y)| \approx \epsilon^{j-2}$, $j = 3, 4, 5$, and obviously

$$|\partial_s^j \chi_{\epsilon, C}(s; t, x, y)| \approx \epsilon^j, \quad j = 0, 1, 2.$$

We, observing that $\phi_{\epsilon, C}(s_* t, x, y) = \frac{t}{\epsilon^2} - \frac{|x-y|}{\epsilon C}$, apply Lemma 2.1.1 to the oscillatory integral $I_{\epsilon, C}(t, x, y)$ to get

$$I_{\epsilon, C}(t, x, y) = \left[e^{i\frac{\pi}{4}t} \sqrt{\frac{2\pi}{\partial_s^2 \phi_{\epsilon, C}(s_*; t, x, y)}} \chi_{\epsilon, C}(s_*; t, x, y) \right] \frac{1}{C^2} e^{i\frac{C^2}{2}t} e^{-i\frac{C}{\epsilon}|x-y|} + O\left(\frac{1}{C^2}\right).$$

Since we have

$$\frac{C}{\epsilon}|x-y| = \frac{C}{\epsilon}|x| + O(\eta), \quad (2.17)$$

then the oscillatory factor $e^{-i\frac{C}{\epsilon}|x-y|}$ can be written as

$$e^{-i\frac{C}{\epsilon}|x-y|} = e^{[-i\frac{C}{\epsilon}|x|+O(\eta)]} = e^{-i\frac{C}{\epsilon}|x|}[1 + O(\eta)].$$

The solution in (2.16) is now given by

$$\begin{aligned} u(t, x) &= \frac{\epsilon}{(4\pi)^{\frac{n}{2}}} \int_{\Omega_{\epsilon, \eta, C}} I_{\epsilon, C}(t, x, y) dy \\ &= \frac{\sqrt{2\pi}\epsilon}{(4\pi)^{\frac{n}{2}}} e^{\frac{\pi}{4}i} \frac{1}{C} e^{iNt} e^{-i\frac{C}{\epsilon}|x|} \int_{\Omega_{\epsilon, \eta, C}} \frac{\chi_{\epsilon, C}(s_*; t, x, y)}{\sqrt{\partial_s^2 \phi_{\epsilon, C}(s_*; t, x, y)}} [1 + O(\eta)] dy + \\ &\quad + \int_{\Omega_{\epsilon, \eta, C}} O\left(\frac{\epsilon}{C^2}\right) dy. \end{aligned} \quad (2.18)$$

From (2.18) and the facts that $|\chi_{\epsilon, C}(s_*; t, x, y)| \approx 1$ and $|\partial_s^2 \phi_{\epsilon, C}(s_*; t, x, y)| \approx 1$ and $|\Omega_{\epsilon, \eta, C}| \approx \frac{\eta^n \epsilon^n}{C^n}$. we get the following estimate

$$|u(t, x)| \gtrsim \frac{\eta^n \epsilon^{n+1}}{C^{n+1}}.$$

When $t \in [2, 3]$, the norm of the solution can therefore be estimated as

$$\|u\|_{L_t^q L_x^r} \gtrsim \frac{\eta^n \epsilon^{n+1}}{C^n} \epsilon^{-\frac{n-1}{r}} C^{\frac{n}{r}}.$$

And finally, we have the ratio

$$\frac{\|u\|_{L_t^q L_x^r}}{\|F\|_{L_s^{\tilde{q}} L_y^{\tilde{r}}}} \gtrsim \frac{\eta^n \epsilon^{n+1} C^{-n} \epsilon^{-\frac{n-1}{r}} C^{\frac{n}{r}}}{\epsilon^{\frac{1}{\tilde{q}}} \epsilon^{\frac{n}{\tilde{r}}} \eta^{\frac{n}{\tilde{r}}} C^{-\frac{n}{\tilde{r}}}} = \epsilon^{-\frac{n-1}{r} + \frac{1}{\tilde{q}} + \frac{n}{\tilde{r}}} C^{\frac{n}{r} - \frac{n}{\tilde{r}}} \eta^{\frac{n}{\tilde{r}}}.$$

When C and η are both fixed, this ratio blows up as $\epsilon \rightarrow 0$ unless $-\frac{n-1}{r} + \frac{1}{\tilde{q}} + \frac{n}{\tilde{r}} \geq 0$. This implies the necessary condition

$$\frac{1}{\tilde{q}} \geq \frac{n-1}{r} - \frac{n}{\tilde{r}}.$$

This new necessary condition and its dual one are stronger than the necessary conditions obtained in (2.15) which in their turn improve conditions (2.4) and (2.5).

2.2 Simpler counter examples and simpler proofs

Consider an inhomogeneity given by the flash forcing term

$$F(s, y) = e^{i s N^2} \chi_{\Omega_{\eta, N}}(y) \chi_{[0, \frac{\eta}{N}]}(s)$$

where

η is a fixed small number so that $0 < \eta \ll 1$,

$N \gg 1$,

$$\Omega_{\eta, N} = \{y \in \mathbb{R}^n : |y| \leq \frac{\eta}{N}\}.$$

The corresponding solution is

$$\begin{aligned} u(t, x) &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \int_{\mathbb{R}^2} e^{-i \frac{|x-y|^2}{4(t-s)}} F(s, y) (t-s)^{-\frac{n}{2}} dy ds \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^{\frac{\eta}{N}} \int_{\Omega_{\eta, N}} e^{-i \phi_N(s, y; t, x)} dy (t-s)^{-\frac{n}{2}} ds, \end{aligned} \quad (2.19)$$

with

$$\phi_N(s, y; t, x) = \frac{|x-y|^2}{4(t-s)} - sN^2.$$

The phase in the oscillatory integral (2.19) can be written in the following way

$$\begin{aligned} \phi_N(s, y; t, x) &= \frac{|x-y|^2}{4(t-s)} - sN^2 \\ &= \frac{|x|^2}{4(t-s)} - sN^2 + \frac{|x-y|^2 - |x|^2}{4(t-s)} \\ &= \frac{|x|^2}{4t} + \frac{s|x|^2}{4(t^2-ts)} - sN^2 + \frac{|x-y|^2 - |x|^2}{4(t-s)} \\ &= \frac{|x|^2}{4t} + \frac{s|x|^2 - 4(t^2-ts)sN^2}{4(t^2-ts)} + \frac{|x-y|^2 - |x|^2}{4(t-s)} \\ &= \frac{|x|^2}{4t} + \frac{s(|x|^2 - 4t^2N^2)}{4(t^2-ts)} + \frac{s^2N^2}{(t-s)} + \frac{|x-y|^2 - |x|^2}{4(t-s)}. \end{aligned}$$

Define $R_N = \{x \in \mathbb{R}^n : ||x| - 2tN| \leq 1\}$.

Now, whenever

$$x \in R_N, \quad y \in \Omega_{\eta,N}, \quad s \in [0, \frac{\eta}{N}] \quad \text{and} \quad t \in [2, 3], \quad (2.20)$$

we have that

$$\begin{aligned} -4t\eta + \frac{\eta}{N} &\leq s(|x|^2 - 4t^2N^2) \leq 4t\eta + \frac{\eta}{N}, \\ 0 &\leq s^2N^2 \leq \eta^2, \\ |x-y|^2 - |x|^2 &\leq 4t\eta + 2\frac{\eta}{N} + \frac{\eta^2}{N^2}, \\ |x-y|^2 - |x|^2 &\geq \frac{\eta^2}{N^2} - 4t\eta - 2\frac{\eta}{N}. \end{aligned}$$

The inequalities above show that under the circumstances (2.20), we have

$$|\phi_N(s, y; t, x) - \phi_N(0, 0; t, x)| = |\phi_N(s, y; t, x) - \frac{|x|^2}{4t}| < 10\eta \ll 1$$

when η is fixed so that $0 < \eta \ll \frac{1}{10}$. The variation of the phase has thus been shown to be small enough for the oscillations to die. The solution in (2.19) can then be estimated by

$$|u| \gtrsim N^{-1} |\Omega_{\eta,N}| \approx N^{-1} \eta^n N^{-n}.$$

Recall that η is fixed and look at the ratio

$$\frac{\|u\|_{L_t^q L_x^r}}{\|F\|_{L_s^{\tilde{q}'} L_y^{\tilde{r}'}}} \gtrsim \frac{N^{-1} N^{-n} N^{\frac{n-1}{r}}}{N^{-\frac{n}{\tilde{r}}} N^{-\frac{1}{\tilde{q}}}} = N^{\frac{n-1}{r} - \frac{n}{\tilde{r}} - \frac{1}{\tilde{q}}}.$$

This blows up as $N \rightarrow +\infty$ unless

$$\frac{n-1}{r} - \frac{n}{\tilde{r}} - \frac{1}{\tilde{q}} \leq 0.$$

This yields the necessary condition

$$\boxed{\frac{1}{\tilde{q}} \geq \frac{n-1}{r} - \frac{n}{\tilde{r}}}$$

We can also get the "dual" counter example using a similar method. Consider the inhomogeneity given by the flash forcing term

$$F(s, y) = e^{i\frac{|y|^2}{4(2-s)}} \chi_{\Omega_N}(y)$$

where

η is a fixed small number so that $0 < \eta \ll 1$,

$N \gg 1$,

$\Omega_N = \{y \in \mathbb{R}^n : ||y| - 2(2-s)N| \leq 1\}$.

The corresponding solution is

$$\begin{aligned} u(t, x) &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \int_{\mathbb{R}^2} e^{-i\frac{|x-y|^2}{4(t-s)}} F(s, y) (t-s)^{-\frac{n}{2}} dy ds \\ &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \int_{\Omega_N} e^{-i\phi_N(s, y; t, x)} dy (t-s)^{-\frac{n}{2}} ds, \end{aligned} \quad (2.21)$$

with

$$\phi_N(s, y; t, x) = \frac{|x-y|^2}{4(t-s)} - \frac{|y|^2}{4(2-s)}.$$

Rewrite the solution in (2.21) to be

$$u(t, x) = \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-i(t-2)N^2} \int_0^1 \int_{\Omega_N} e^{-i\psi_N(s, y; t, x)} dy (t-s)^{-\frac{n}{2}} ds,$$

where the oscillatory integral that defines the solution now has the phase

$$\psi_N(s, y; t, x) = \phi_N(s, y; t, x) + (t-2)N^2.$$

Notice that we can write

$$\begin{aligned} \frac{|x-y|^2}{4(t-s)} &= \frac{|y|^2}{4(t-s)} + \frac{|x-y|^2 - |y|^2}{4(t-s)} \\ &= \frac{|y|^2}{4[(t-2) + (2-s)]} + \frac{|x-y|^2 - |y|^2}{4(t-s)} \\ &= \frac{|y|^2}{4(2-s)} - \frac{(t-2)|y|^2}{4(2-s)[(t-2) + (2-s)]} + \frac{|x-y|^2 - |y|^2}{4(t-s)}, \end{aligned}$$

and so the phase in the oscillatory integral (2.21) can be written as

$$\begin{aligned} \phi_N(s, y; t, x) &= \frac{|x-y|^2}{4(t-s)} - \frac{|y|^2}{4(2-s)} \\ &= -\frac{(t-2)|y|^2}{4(2-s)[(t-2) + (2-s)]} + \frac{|x-y|^2 - |y|^2}{4(t-s)}, \end{aligned}$$

and thus we have

$$\begin{aligned}
\psi_N(s, y; t, x) &= (t-2)N^2 + \phi_N(s, y; t, x) = \\
&= (t-2)N^2 - \frac{(t-2)|y|^2}{4(2-s)[(t-2) + (2-s)]} + \frac{|x-y|^2 - |y|^2}{4(t-s)} \\
&= \frac{4(t-2)^2(2-s)N^2 + 4(t-2)(2-s)^2N^2 - (t-2)|y|^2}{4(2-s)[(t-2) + (2-s)]} + \frac{|x-y|^2 - |y|^2}{4(t-s)} \\
&= \frac{(t-2)^2N^2}{(t-s)} + \frac{(t-2)[4(2-s)^2N^2 - |y|^2]}{4(2-s)(t-s)} + \frac{|x-y|^2 - |y|^2}{4(t-s)}
\end{aligned}$$

Define $R_{\eta, N} = \{x \in \mathbb{R}^n : |x| \leq \frac{\eta}{N}\}$.

Now, whenever

$$x \in R_{\eta, N}, \quad y \in \Omega_N, \quad s \in [0, 1] \quad \text{and} \quad t \in [2, 2 + \frac{\eta}{N}], \quad (2.22)$$

we have that

$$\begin{aligned}
-4(2-s)t\eta + \frac{\eta}{N} &\leq (t-2)[4(2-s)^2N^2 - |y|^2] \leq 4(2-s)\eta + \frac{\eta}{N}, \\
0 &\leq (t-2)^2N^2 \leq \eta^2, \\
|x-y|^2 - |x|^2 &\leq 4(2-s)\eta + 2\frac{\eta}{N} + \frac{\eta^2}{N^2}, \\
|x-y|^2 - |x|^2 &\geq \frac{\eta^2}{N^2} - 4(2-s)\eta - 2\frac{\eta}{N}.
\end{aligned}$$

The inequalities above show that under the circumstances (2.22), we have

$$|\psi_N(s, y; t, x)| < 10\eta \ll 1 \quad \text{when } \eta \text{ is fixed so that } 0 < \eta \ll \frac{1}{10}.$$

The phase $\psi_N(s, y; t, x)$ has thus been shown to be small enough for the oscillations to die.

The solution in (2.21) can then be estimated by

$$|u| \gtrsim |\Omega_N| \approx N^{n-1}.$$

Recall that η is fixed and look at the ratio

$$\frac{\|u\|_{L_t^q L_x^r}}{\|F\|_{L_s^{q'} L_y^{r'}}} \gtrsim \frac{N^{n-1} N^{-\frac{n}{r}} N^{-\frac{1}{q}}}{N^{\frac{n-1}{r'}}} = N^{\frac{n-1}{r} - \frac{n}{r} - \frac{1}{q}}.$$

This blows up as $N \rightarrow +\infty$ unless

$$\frac{n-1}{\tilde{r}} - \frac{n}{r} - \frac{1}{q} \leq 0.$$

This implies the necessary condition

$$\boxed{\frac{1}{q} \geq \frac{n-1}{\tilde{r}} - \frac{n}{r}}.$$

2.3 Appendix: Precise contributions of the the phase and amplitude to the error term in the stationary phase approximation

Proof of Lemma 2.1.1

We give the proof of Lemma 2.1.1 in three steps. In the first step, we prove the lemma for the special case when the phase is a quadratic function of the form $\phi(s) = k(s - s_*)^2$, with $k \geq \frac{1}{2}$. Write $\chi(s) = \chi(s_*) + (s - s_*)\psi(s)$. So

$$I(N) = \int_a^b e^{iN\phi(s)} \chi(s) ds = \chi(s_*) \int_a^b e^{iNk(s-s_*)^2} ds + \int_a^b e^{iNk(s-s_*)^2} (s - s_*) \psi(s) ds. \quad (2.23)$$

For the first integral in (2.23), we have

$$\int_a^b e^{iNk(s-s_*)^2} ds = \int_{-\infty}^{+\infty} e^{iNk(s-s_*)^2} ds - \int_{-\infty}^a e^{iNk(s-s_*)^2} ds - \int_b^{+\infty} e^{iNk(s-s_*)^2} ds, \quad (2.24)$$

but

$$\int_{-\infty}^{+\infty} e^{iNk(s-s_*)^2} ds = \int_{-\infty}^{+\infty} e^{-(-i)Nks^2} ds = \sqrt{\frac{2\pi}{-iNk}} = \sqrt{\frac{2\pi i}{Nk}} = \sqrt{\frac{2\pi}{Nk}} e^{i\frac{\pi}{4}} \quad (2.25)$$

and by integration by parts, we get

$$\begin{aligned} \int_{-\infty}^a e^{iNk(s-s_*)^2} ds &= \frac{1}{2iNk} \int_{-\infty}^a \frac{1}{s - s_*} \frac{d}{ds} e^{iNk(s-s_*)^2} ds \\ &= \frac{1}{2iNk} \frac{1}{s - s_*} e^{iNk(s-s_*)^2} \Big|_{-\infty}^a - \frac{1}{2iNk} \int_{-\infty}^a e^{iNk(s-s_*)^2} \frac{d}{ds} \frac{1}{s - s_*} ds \end{aligned}$$

so that

$$\left| \int_{-\infty}^a e^{iNk(s-s_*)^2} ds \right| \leq \frac{1}{Nk} \frac{1}{s_* - a} \leq \frac{1}{Nk\delta}. \quad (2.26)$$

Similarly,

$$\begin{aligned}\int_b^{+\infty} e^{iNk(s-s_*)^2} ds &= \frac{1}{2iNk} \int_b^{+\infty} \frac{1}{s-s_*} \frac{d}{ds} e^{iNk(s-s_*)^2} ds \\ &= \frac{1}{2iNk} \frac{1}{s-s_*} e^{iNk(s-s_*)^2} \Big|_b^{+\infty} - \frac{1}{2iNk} \int_b^{+\infty} e^{iNk(s-s_*)^2} \frac{d}{ds} \frac{1}{s-s_*} ds\end{aligned}$$

so that

$$\left| \int_b^{+\infty} e^{iNk(s-s_*)^2} ds \right| \leq \frac{1}{Nk} \frac{1}{b-s_*} \leq \frac{1}{Nk\delta}. \quad (2.27)$$

Substituting from (2.25),(2.26) and (2.27) in (2.24), we get

$$\int_a^b e^{iNk(s-s_*)^2} ds = \sqrt{\frac{\pi}{Nk}} e^{i\frac{\pi}{4}} + O\left(\frac{1}{Nk\delta}\right). \quad (2.28)$$

Now, we estimate the second integral in (2.23). Again integration by parts gives

$$\begin{aligned}\int_a^b e^{iNk(s-s_*)^2} (s-s_*)\psi(s) ds &= \frac{1}{2iNk} \int_a^b \frac{d}{ds} e^{iNk(s-s_*)^2} \psi(s) ds \\ &= \frac{1}{2iNk} e^{iNk(s-s_*)^2} \psi(s) \Big|_a^b - \frac{1}{2iNk} \int_a^b e^{iNk(s-s_*)^2} \frac{d}{ds} \psi(s) ds \\ &= \frac{1}{2iNk} \left[e^{iNk(b-s_*)^2} \psi(b) - e^{iNk(a-s_*)^2} \psi(a) - \int_a^b e^{iNk(s-s_*)^2} \frac{d}{ds} \psi(s) ds \right],\end{aligned}$$

hence,

$$\left| \int_a^b e^{iNk(s-s_*)^2} (s-s_*)\psi(s) ds \right| \leq \frac{1}{2Nk} \left[|\psi(a)| + |\psi(b)| + \int_a^b \left| \frac{d}{ds} \psi(s) \right| ds \right].$$

Using the explicit formula

$$\psi(s) = \frac{\chi(s) - \chi(s_*)}{s-s_*} = \frac{1}{s-s_*} \int_{s_*}^s \chi'(s) ds,$$

we get that

$$|\psi(s)| \leq \left| \frac{1}{s-s_*} \int_{s_*}^s \chi'(s) ds \right| \leq \max |\chi'|,$$

furthermore, using the explicit formula

$$\psi'(s) = \frac{\chi'(s) - \psi(s)}{s-s_*} = \frac{\chi'(s) - \frac{\int_{s_*}^s \chi'(\rho) d\rho}{s-s_*}}{s-s_*} = \frac{\int_{s_*}^s [\chi'(s) - \chi'(\rho)] d\rho}{(s-s_*)^2} = \frac{\int_{s_*}^s \int_{\rho}^s \chi''(\sigma) d\sigma d\rho}{(s-s_*)^2},$$

and because $s_* \leq \rho \leq s$, we obtain that

$$|\psi'(s)| \leq \frac{1}{(s-s_*)^2} \left| \int_{s_*}^s \int_{\rho}^s \chi''(\sigma) d\sigma d\rho \right| \leq \frac{s-\rho}{s-s_*} \max |\chi''| \leq \max |\chi''|.$$

We deduce therefore the following estimate

$$\left| \int_a^b e^{iNk(s-s_*)^2} (s-s_*) \psi(s) ds \right| \leq \frac{1}{Nk} \left[\max |\chi'| + \frac{b-a}{2} \max |\chi''| \right], \quad (2.29)$$

in other words,

$$\left| \int_a^b e^{iNk(s-s_*)^2} (s-s_*) \psi(s) ds \right| \leq \frac{2 \max |\chi'| + (b-a) \max |\chi''|}{N \min |\phi''|}.$$

Finally, in the light of (2.28) and (2.29), (2.23) implies that

$$I(N) = \chi(s_*) \sqrt{\frac{\pi}{Nk}} e^{i\frac{\pi}{4}} + O\left(\frac{1}{Nk\delta}\right),$$

where the implicit constant in the big O -symbol depends on the quantities $b-a$ and uniform upper bounds on $|\chi|$ and $|\chi'|$ and uniform lower bound on $|\phi''|$. This concludes the proof for the special case of the quadratic phase.

The second step will show the prove the lemma for a general phase function ϕ but under the assumption that the amplitude χ has a compact support in (a, b) . Let

$$\Phi(s; \lambda) = (1-\lambda) \left[\phi(s_*) + \frac{\phi''(s_*)}{2} (s-s_*)^2 \right] + \lambda \phi(s).$$

Let $J(N; \lambda) = \int_a^b e^{iN\Phi(s; \lambda)} \chi(s) ds$. Then, applying the result in the first step, we have

$$\begin{aligned} J(N; 0) &= e^{iN\phi(s_*)} \int_a^b e^{iN\frac{\phi''(s_*)}{2}(s-s_*)^2} \chi(s) ds \\ &= e^{i\frac{\pi}{4}} \sqrt{\frac{2\pi}{N\phi''(s_*)}} \chi(s_*) e^{iN\phi(s_*)} + O\left(\frac{\max |\chi| + (b-a) \max |\chi'|}{N|\phi''(s_*)|\delta}\right). \end{aligned}$$

Since $I(N) = J(N; 1) = J(N; 0) + \int_0^1 \partial_\lambda J(N; \lambda) d\lambda$ then, in order to control $I(N)$, then it is sufficient to find uniform bounds for the derivative $\partial_\lambda J(N; \lambda)$.

To achieve this, we integrate by parts twice. Notice here that there will be no contribution

from the boundary points because χ , in this step, is assumed to have compact support in (a, b) and the fact that $\text{supp } \partial_s \chi \subset \text{supp } \chi$.

$$\begin{aligned}
\partial_\lambda J(N; \lambda) &= \partial_\lambda \int_a^b e^{iN\Phi(s; \lambda)} \chi(s) ds = \int_a^b iN \Phi_\lambda e^{iN\Phi} \chi ds = \int_a^b iN \Phi_s e^{iN\Phi} \left(\frac{\Phi_\lambda}{\Phi_s} \chi \right) ds \\
&= \int_a^b (\partial_s e^{iN\Phi}) \left(\frac{\Phi_\lambda}{\Phi_s} \chi \right) ds = - \int_a^b e^{iN\Phi} \partial_s \left(\frac{\Phi_\lambda}{\Phi_s} \chi \right) ds \\
&= - \frac{1}{iN} \int_a^b \partial_s (e^{iN\Phi}) \frac{1}{\Phi_s} \partial_s \left(\frac{\Phi_\lambda}{\Phi_s} \chi \right) ds \\
&= e^{iN\Phi} \frac{1}{\Phi_s} \left[\frac{\Phi_s \Phi_\lambda (\partial_s \chi_s) + \Phi_s (\partial_s \Phi_\lambda) \chi - \Phi_\lambda \chi (\partial_s \Phi_s)}{(\Phi_s)^2} \right] \Big|_a^b \\
&\quad - \frac{1}{iN} \int_a^b e^{iN\Phi} \partial_s \left[\frac{1}{\Phi_s} \partial_s \left(\frac{\Phi_\lambda}{\Phi_s} \chi \right) \right] ds
\end{aligned}$$

where $\Phi_\lambda(s; \lambda) = \partial_\lambda \Phi(s; \lambda)$ and $\Phi_s(s; \lambda) = \partial_s \Phi(s; \lambda)$.

It remains now to uniformly control the quantity $\left| \partial_s \left[\frac{1}{\Phi_s} \partial_s \left(\frac{\Phi_\lambda}{\Phi_s} \chi \right) \right] \right|$.

Using the integral formula for the Taylor expansion of ϕ about s_* , we have, after applying a variable change, that

$$\begin{aligned}
\Phi_\lambda &= \phi(s) - [\phi(s_*) + \frac{\phi''(s_*)}{2} (s - s_*)^2] \\
&= \int_{s_*}^s \phi'''(t) \frac{(s-t)^2}{2} dt = (s - s_*)^3 \int_0^1 \phi'''[(1-\theta)s_* + \theta s] \frac{(1-\theta)^2}{2} d\theta.
\end{aligned}$$

Let $Q(s) = \frac{\Phi_\lambda}{(s - s_*)^3}$ so that

$$Q(s) = \int_0^1 \phi'''[(1-\theta)s_* + \theta s] \frac{(1-\theta)^2}{2} d\theta. \quad (2.30)$$

From (2.30), it is easy to see that $Q \in C^2(a, b)$ whenever $\phi \in C^5(a, b)$ and that $|Q^j|$, $j = 0, 1, 2$, is controlled by uniform upper bounds on $|\phi^i|$, $i = 3, 4, 5$.

We also have that

$$\Phi_s(s; \lambda) = (1 - \lambda) \phi''(s_*) (s - s_*) + \lambda \phi'(s).$$

Let $W(s; \lambda) = \frac{\Phi_s}{s - s_*}$ so that

$$\begin{aligned}
W(s; \lambda) &= (1 - \lambda) \phi''(s_*) + \lambda \frac{\phi'(s)}{s - s_*} = (1 - \lambda) \phi''(s_*) + \lambda \frac{1}{s - s_*} \int_{s_*}^s \phi''(\sigma) d\sigma \\
&= (1 - \lambda) \phi''(s_*) + \lambda \int_0^1 \phi''[(1-\theta)s_* + \theta s] d\theta.
\end{aligned} \quad (2.31)$$

It follows from (2.31) that

$$|W| \geq \min |\phi''|.$$

Now

$$\begin{aligned} \frac{\Phi_\lambda(s)}{\Phi_s(s; \lambda)} \chi(s) &= \frac{(s - s_*)^3 Q(s)}{(s - s_*) W(s; \lambda)} \chi(s) = (s - s_*)^2 \frac{Q(s)}{W(s; \lambda)} \chi(s) = \frac{\Phi_s^2(s; \lambda)}{W^2(s; \lambda)} \frac{Q(s)}{W(s; \lambda)} \chi(s) \\ &= \Phi_s^2(s; \lambda) \frac{Q(s)}{W^3(s; \lambda)} \chi(s) = \Phi_s^2(s; \lambda) \tilde{\chi}(s; \lambda), \end{aligned}$$

where

$$\tilde{\chi}(s; \lambda) = \frac{Q(s)}{W^3(s; \lambda)} \chi(s).$$

Hence, when $\phi \in C^5$ and $\chi \in C^2$, then $\tilde{\chi} \in C^2$ and we have

$$\begin{aligned} \partial_s \tilde{\chi} &= \partial_s \left(\frac{1}{W^3} Q \chi \right) = \frac{\chi Q'}{W^3} + \frac{Q \chi'}{W^3} - \frac{3\chi Q W'}{W^4}, \\ \partial_s^2 \tilde{\chi} &= -\frac{3\chi Q W''}{W^4} + \frac{12\chi Q (W')^2}{W^5} - \frac{6\chi Q' W'}{W^4} - \frac{6Q \chi' W'}{W^4} + \frac{\chi Q''}{W^3} + \frac{2\chi' Q'}{W^3} + \frac{Q \chi''}{W^3} \end{aligned} \quad (2.32)$$

$|\partial_s^m \tilde{\chi}|$, $m = 1, 2$, is bounded uniformly by means of the uniform bounds on $\partial_s^{j+3} \phi$ and $\partial_s^j \chi$, with $j = 0, 1, 2$ as follows (we give the upper bounds up to a multiplicative numerical constant)

$$\begin{aligned} |\partial_s \tilde{\chi}| &\leq \frac{\max |\phi'''| \max |\chi|}{(\min |\phi''|)^3}, \\ |\partial_s \tilde{\chi}| &\leq \frac{(\max |\phi''''|)^2 \max |\phi^{(4)}| \max |\chi|}{(\min |\phi''|)^3 + (\min |\phi''|)^4}, \\ |\partial_s^2 \tilde{\chi}| &\leq \tilde{c}, \end{aligned}$$

where the constant \tilde{c} depends on the quantities $\max |\phi^{(j)}|$, $j = 2, 3, 4, 5$, $\max |\chi^k|$, $k = 0, 1, 2$, $\min |\phi''|$ according to (2.32).

Now, we go back to the quantity that we need to bound

$$\begin{aligned} \partial_s \left[\frac{1}{\Phi_s} \partial_s \left(\frac{\Phi_\lambda}{\Phi_s} \chi \right) \right] &= \partial_s \left[\frac{1}{\Phi_s} \partial_s (\Phi_s^2 \tilde{\chi}) \right] = \partial_s \left[2\tilde{\chi} (\partial_s^2 \Phi) + \Phi_s (\partial_s \tilde{\chi}) \right] \\ &= \partial_s \left[2\phi''(s_*) \tilde{\chi} + 2\lambda \phi''(s) \tilde{\chi} + \Phi_s (\partial_s \tilde{\chi}) \right] \\ &= 2\phi''(s_*) (\partial_s \tilde{\chi}) + 2\lambda \phi''(s) \tilde{\chi} + 2\lambda \phi''(s) \partial_s (\tilde{\chi}) + \\ &\quad (\phi''(s_*) + \lambda \phi''(s)) (\partial_s \tilde{\chi}) + (s - s_*) W(s; \lambda) (\partial_s^2 \tilde{\chi}). \end{aligned}$$

Hence, we have the estimate

$$\begin{aligned} |\partial_\lambda J(N; \lambda)| &\leq \frac{C(\max |\partial_s^{j+3} \phi|, \max |\partial_s^{j+3} \chi|, \max |\partial_s^2 \phi|, \min |\partial_s^2 \phi|, j = 0, 1, 2)}{N} \\ &= O\left(\frac{1}{N}\right). \end{aligned}$$

Finally, in the last step, we remove the assumption on χ to be compactly supported in (a, b) and estimate the contribution of the boundary to the error term.

Take a cut-off function $\alpha \in C_0^\infty((a, b))$ such that

$$\alpha(s) = \begin{cases} 1, & a + \frac{2\delta}{3} \leq s \leq b - \frac{2\delta}{3}; \\ 0, & a \leq s \leq a + \frac{\delta}{3}; \\ 0, & b - \frac{\delta}{3} \leq s \leq b. \end{cases}$$

Let $\beta(s) = 1 - \alpha(s)$. In this case, we have that

$$\beta(s) = \begin{cases} 0, & a + \frac{2\delta}{3} \leq s \leq b - \frac{2\delta}{3}; \\ 1, & a \leq s \leq a + \frac{\delta}{3}; \\ 1, & b - \frac{\delta}{3} \leq s \leq b. \end{cases}$$

Thus, the intersection of each of the supports of β and its derivative β' with $[a, b]$ are given by

$$\begin{aligned} S_\beta &= \text{supp } \beta \cap [a, b] = [a, a + \frac{2\delta}{3}] \cup [b - \frac{2\delta}{3}, b], \quad \text{and} \\ S_{\beta'} &= \text{supp } \beta'(s) \cap [a, b] = [a + \frac{\delta}{3}, a + \frac{2\delta}{3}] \cup [b - \frac{2\delta}{3}, b - \frac{\delta}{3}]. \end{aligned}$$

Obviously $S_\beta \supset S_{\beta'}$.

Write $\chi(s) = [\alpha(s) + \beta(s)]\chi(s)$, to get

$$I(N) = \int_a^b e^{iN\phi(s)} \chi(s) ds = \int_a^b e^{iN\phi(s)} \alpha(s) \chi(s) ds + \int_a^b e^{iN\phi(s)} \beta(s) \chi(s) ds.$$

The oscillatory integral $\int_a^b e^{iN\phi(s)} \alpha(s) \chi(s) ds$ is exactly of the form that we discussed in the previous step because the amplitude, for this integral $\alpha(s)\chi(s)$, is compactly supported in (a, b) , precisely, $\text{supp } \alpha(s)\chi(s) \subset \text{supp } \alpha(s) = [a + \frac{\delta}{3}, b - \frac{\delta}{3}]$.

We are left then with the second integral which we will control via integration by parts as

follows

$$\begin{aligned}
& \int_a^b e^{iN\phi(s)} \beta(s) \chi(s) ds = \int_{S_\beta} e^{iN\phi(s)} \beta(s) \chi(s) ds = \frac{1}{iN} \int_{S_\beta} \partial_s (e^{iN\phi(s)}) \frac{\beta(s) \chi(s)}{\phi'(s)} ds = \\
& = \frac{1}{iN} e^{iN\phi(s)} \frac{\beta(s) \chi(s)}{\phi'(s)} \Big|_{a+\frac{2\delta}{3}}^b + \frac{1}{iN} e^{iN\phi(s)} \frac{\beta(s) \chi(s)}{\phi'(s)} \Big|_{b-\frac{2\delta}{3}}^a - \frac{1}{iN} \int_{S_\beta} e^{iN\phi(s)} \partial_s \left[\frac{\beta(s) \chi(s)}{\phi'(s)} \right] ds \\
& = \frac{1}{iN} e^{iN\phi(a+\frac{2\delta}{3})} \frac{\beta(a+\frac{2\delta}{3}) \chi(a+\frac{2\delta}{3})}{\phi'(a+\frac{2\delta}{3})} - \frac{1}{iN} e^{iN\phi(a)} \frac{\beta(a) \chi(a)}{\phi'(a)} + \\
& \quad \frac{1}{iN} e^{iN\phi(b)} \frac{\beta(b) \chi(b)}{\phi'(b)} - \frac{1}{iN} e^{iN\phi(b-\frac{2\delta}{3})} \frac{\beta(b-\frac{2\delta}{3}) \chi(b-\frac{2\delta}{3})}{\phi'(b-\frac{2\delta}{3})} - \\
& \quad \frac{1}{iN} \int_{S_\beta} e^{iN\phi(s)} \left[\frac{\phi' \beta'(s) \chi(s) + \phi' \beta(s) \chi(s)' - \phi'' \beta(s) \chi(s)}{(\phi')^2} \right] ds \\
& = \frac{1}{iN} e^{iN\phi(b)} \frac{\chi(b)}{\phi'(b)} - \frac{1}{iN} e^{iN\phi(a)} \frac{\chi(a)}{\phi'(a)} \\
& \quad - \frac{1}{iN} \left[\int_{S_{\beta'}} e^{iN\phi(s)} \frac{\beta'(s) \chi(s)}{\phi'} ds + \int_{S_\beta} e^{iN\phi(s)} \frac{\beta(s) \chi(s)'}{\phi'} ds - \int_{S_\beta} e^{iN\phi(s)} \frac{\phi'' \beta(s) \chi(s)}{(\phi')^2} ds \right].
\end{aligned}$$

Since for all $s \in S_\beta$, we have that either $a \leq s \leq a + \frac{2\delta}{3}$ or $b - \frac{2\delta}{3} \leq s \leq b$ while we always have $a + \delta \leq s_* \leq b - \delta$, then on S_β , we always have that $|s - s_*| \geq \frac{\delta}{3}$, moreover, we get that

$$|\phi'(s)| = |\phi'(s) - \phi'(s_*)| = \left| \int_{s_*}^s \phi''(\sigma) d\sigma \right| \geq |s - s_*| \min |\phi''| \geq \frac{\delta}{3} \min |\phi''|.$$

The last computations show that

$$\left| \int_a^b e^{iN\phi(s)} \beta(s) \chi(s) ds \right| \leq c \left[\frac{\max |\chi| \max |\chi'| \max |\phi''|}{\min_{j=1,2} [(\min |\phi''| \delta)^j]} \right] \frac{1}{N},$$

where c is just a real numerical constant.

Chapter 3

Local inhomogeneous Strichartz type estimates for the Schrödinger's equation

3.1 An introduction

Let $u(t, x)$ be the fundamental solution to the Cauchy problem associated with the free linear inhomogeneous Schrödinger equation

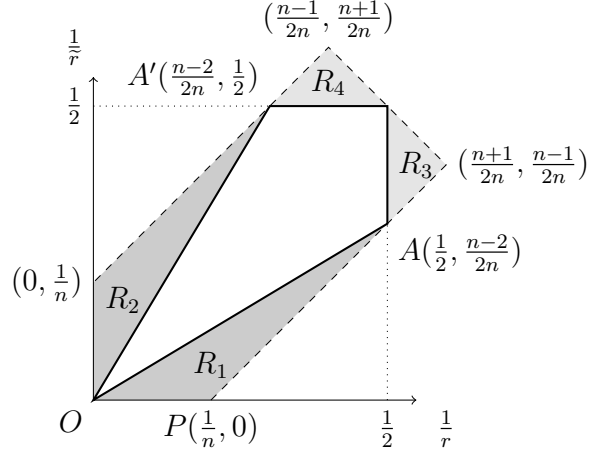
$$i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad u(0, x) = 0, \quad (3.1)$$

where the forcing term F is supported on $[0, 1] \times \mathbb{R}^n$.

Consider the following local inhomogeneous Strichartz type estimate

$$\| u \|_{L^q([2,3]; L^r(\mathbb{R}^n))} \lesssim \| F \|_{L^{\bar{q}'}([1,2]; L^{\bar{r}'}(\mathbb{R}^n))}. \quad (3.2)$$

Our main endeavour in this and the following two chapters is to provide a starting point on the way to recover some of the regions that lie in the gap between the region that represents the sufficient conditions and the one that represents the necessary conditions imposed by the counter examples given in Chapter 2. We start by looking at the point $P(\frac{1}{n}, 0)$ in the $\frac{1}{r} - \frac{1}{\bar{r}}$ plane (see the figure below).



The shadowed regions R_j represent the gap between the necessary and sufficient conditions

The necessary conditions and interpolation theory motivated this choice for the Lebesgue exponents r and \tilde{r} . As if the estimate (3.2) was proven to be satisfied for some values of q and \tilde{q} at the point P , then standard L^p interpolation techniques [2] could be employed to recover the triangle AOP . In this case, the estimate would also be valid throughout the triangle $A'OP'$ by the "duality" properties of the operator that gives the solution. These properties make all the conditions on the exponents values invariant under the symmetry $(q, r) \leftrightarrow (\tilde{q}, \tilde{r})$, (see Chapter 1 section 1.1 or [10] for more details). This is why we shall focus our interest on the estimate

$$\| u \|_{L^q([2,3], L^n(\mathbb{R}^n))} \lesssim \| F \|_{L^{\tilde{q}'}([0,1], L^1(\mathbb{R}^n))} . \quad (3.3)$$

If the estimate (3.3) was true, then so would be the estimate

$$\boxed{\| u \|_{L^q([2,3], L^n(\mathbb{R}^n))} \lesssim \| f \|_{L^{\tilde{q}'}([0,1])} \| g \|_{L^1(\mathbb{R}^n)}} \quad (3.4)$$

for inhomogeneities F is of the form

$$F(t, x) = f(t) g(x) \quad (3.5)$$

where obviously $f \in L^{\tilde{q}'}([0, 1])$ and $g \in L^1(\mathbb{R}^n)$. In this chapter we carefully choose a forcing term F that can be approximated by forcing terms of the form (3.5). Namely, we investigate

the data $F(t, x) = f(t) \delta_0(x)$ where δ_0 is the dirac delta function at the origin. We also let the Lebesgue exponents q and \tilde{q} take the extreme values given by the necessary conditions. Following two different approaches that we give separately in the next two chapters (Chapter 4 - Chapter 5), we manage to prove the estimate (3.4) for $g = \delta_0$ with a divergence of an order less than any positive ϵ . A general framework of how we get this result goes as follows. Using the Fourier transform, we get the following explicit formula for u

$$u(t, x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \int_{\mathbb{R}^n} \frac{e^{i\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{\frac{n}{2}}} F(s, y) dy ds. \quad (3.6)$$

Since the solution u is given by an oscillatory integral then it is evident that when $q = n =$ an even integer in (3.4), we can write the mixed Lebesgue norm of the solution explicitly as a multilinear form.

In both chapters 4 and 5, we use multilinear interpolation techniques to simplify the estimate of this multilinear form to an estimate involving a kernel that depends on the time variable. Since the simplified estimate, we found, was still not easy to prove, we decomposed it into several pieces and treated each piece individually. We believe that this divergence of logarithmic order of the ratio between the mixed Lebesgue norms of the solution and that of the forcing term comes from the aforementioned decomposing process. The difference between the two approaches is that in Chapter 4 we use the density of piecewise constant functions in L^p spaces to approximate the data, f , while in Chapter 5 we use the dominated convergence theorem to approximate the norm of the solution. In the current chapter, we pave the way to the proofs given in chapters 4 and 5. We justify the choice of the special data, reformulate the problem as a multilinear estimate, recall the classic Riesz-Thorin multilinear interpolation theorem and show how we apply it to the problem and summarise the strategies we pursue and the main results we obtain. We conclude this chapter with some preliminaries that will be needed in the next two chapters.

3.2 A question

As explained in Section 3.1, we look at the estimate (3.4) where a special choice of the function g is made. We justify this choice in the following section.

3.2.1 A special forcing term

We investigate the estimate (3.4) when $g = \delta_0$. That is we consider the forcing term

$$\boxed{F(t, x) = \delta_0(x) f(t)}. \quad (3.7)$$

This data is the product of the delta function δ_0 which represents a full concentration of the mass at the origin in the spatial variable and a function of time only $f \in L^{\tilde{q}}([0, 1])$ where the value of the exponent \tilde{q} is to be determined momentarily. The Dirac delta δ_0 on \mathbb{R}^n , of course, is not a function, and thus does not belong to $L^p(\mathbb{R}^n)$ for any p . However, we will sacrifice some rigor for the sake of simplicity. The main reason behind this choice of the function g is that the delta function can be viewed as the limit in the sense of distributions of a sequence of absolutely integrable functions $f_\epsilon \in L^1$ with $\|f_\epsilon\|_{L^1(\mathbb{R}^n)} = 1$. Indeed, for any test function ϕ , if we take

$$f_\epsilon = \frac{1}{\epsilon^n} f\left(\frac{x}{\epsilon}\right),$$

then by the dominated convergence theorem we have that

$$\langle f_\epsilon, \phi \rangle = \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} f\left(\frac{x}{\epsilon}\right) \phi(x) dx = \int_{\mathbb{R}^n} f(x) \phi(\epsilon x) dx \rightarrow \phi(0) = \langle \delta_0, \phi \rangle$$

as ϵ approaches zero from the right.

The fact that the δ_0 can be approximated by normalized L^1 functions means that if the estimate (3.4) holds true then so should the estimate

$$\boxed{\|u\|_{L^q([2,3], L^n(\mathbb{R}^n))} \lesssim \|f\|_{L^{\tilde{q}}([0,1])}} \quad (3.8)$$

where, using (3.6), the solution u is given in terms of the function in time only f by

$$\begin{aligned}
u(t, x) &= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{\frac{n}{2}}} f(s) g(y) dy ds \\
&= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \frac{f(s)}{(t-s)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} \delta_0(y) dy ds \\
&= \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \frac{f(s)}{(t-s)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t-s)}} ds.
\end{aligned} \tag{3.9}$$

Another motivation for considering the delta function comes from the fact that all necessary conditions for the estimate (3.2) are obtained from counter examples which involve concentration in balls or spherical shells centered at the origin. This means that concentration at the origin should be the most difficult case to deal with among data of this type. Treating this "worst" case with success would be encouraging to try to prove the more general estimate (3.4).

3.2.2 A special estimate

We look for the values of the exponents q and \tilde{q} necessary for the estimate (3.8) with u given from (3.9). First, we summarize the necessary conditions on the values of q and \tilde{q} .

The necessary conditions ([10])

$$\begin{aligned}
\frac{2}{q} &\geq \frac{n}{\tilde{r}} - \frac{n}{r}, & \frac{2}{\tilde{q}} &\geq \frac{n}{r} - \frac{n}{\tilde{r}} \\
\frac{2}{q} &\geq \frac{n-2}{r} - \frac{n}{\tilde{r}}, & \frac{2}{\tilde{q}} &\geq \frac{n-2}{\tilde{r}} - \frac{n}{r}.
\end{aligned} \tag{3.10}$$

The new necessary conditions (see Chapter 2)

$$\frac{1}{\tilde{q}} \geq \frac{n-1}{r} - \frac{n}{\tilde{r}}, \quad \frac{1}{q} \geq \frac{n-1}{\tilde{r}} - \frac{n}{r}. \tag{3.11}$$

When $(r, \tilde{r}) = (n, \infty)$, the necessary conditions above imply the following restriction on \tilde{q}

$$\frac{1}{\tilde{q}} \geq 1 - \frac{1}{n}, \quad \text{which is equivalent to} \quad \tilde{q}' \geq n.$$

While the strongest necessary condition on the values of the exponent q is

$$\frac{1}{q} \geq \frac{1}{2} - \frac{1}{n}, \quad \text{which is equivalent to} \quad q \geq \frac{2n}{n-2}.$$

We shall take the extreme values

$$(q, \tilde{q}') = \left(\frac{2n}{n-2}, n \right).$$

That is we shall consider the estimate

$$\boxed{\| u \|_{L^{\frac{2n}{n-2}}([2,3], L^n(\mathbb{R}^n))} \lesssim \| f \|_{L^n([0,1])}}. \quad (3.12)$$

The trick here is to choose

$$\frac{2n}{n-2} = n = \text{an even integer.}$$

Luckily enough, this is possible only when $n = 4$. This particular choice of the dimension has the technical importance of enabling us to substitute the estimate of the norm $\| \cdot \|_{L_t^{\frac{2n}{n-2}} L_x^n}$ of the solution by the estimate of a multilinear form. As we can in this case use the explicit expression (3.9) for the solution to write

$$\begin{aligned} \| u \|_{L^4([2,3]; L^4(\mathbb{R}^4))}^4 &= \int_2^3 \int_{\mathbb{R}^4} |u(t, x)|^4 dx dt \\ &= \int_2^3 \int_{\mathbb{R}^4} \bar{u}(t, x) u(t, x) \bar{u}(t, x) u(t, x) dx dt \\ &= \frac{1}{2\pi^2} \int_2^3 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{iP(t, s_1, s_2, s_3, s_4)|x|^2}}{(t-s_1)^2(t-s_2)^2(t-s_3)^2(t-s_4)^2} \\ &\quad f(s_1) f(s_2) f(s_3) f(s_4) ds_1 ds_2 ds_3 ds_4 dx dt \end{aligned} \quad (3.13)$$

with

$$P(t, s_1, s_2, s_3, s_4) = \frac{-1}{t-s_1} + \frac{1}{t-s_2} + \frac{-1}{t-s_3} + \frac{1}{t-s_4}.$$

which is an integral form with an oscillatory kernel that that depends on both variables x and t . Clearly, integration in x and in t is a must to get the decay due to the oscillation.

This reasoning finally leads us to consider the local inhomogeneous estimate

$$\boxed{\| u \|_{L^4([2,3] \times \mathbb{R}^4)} \lesssim \| f \|_{L^4([0,1])}}. \quad (3.14)$$

3.2.3 The question

Finally, we summarize and simplify the problem we are going to solve throughout the rest of this thesis in the form of a question that we give here. First, we take into account the following observations on the formula (3.13) and the estimate (3.14). Using the change of variables

$$t - 2 \longrightarrow \tau, \quad 1 - s \longrightarrow \sigma$$

in (3.13) we immediately see that

$$\begin{aligned} & \| u(t, x) \|_{L^4([2,3]; L^4(\mathbb{R}^4))}^4 = \| u(\tau, x) \|_{L^4([0,1]; L^4(\mathbb{R}^4))}^4 = \\ &= \frac{1}{2\pi^2} \int_0^1 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{iA(\tau, \sigma_1, \sigma_2, \sigma_3, \sigma_4)|x|^2}}{(1 + \tau + \sigma_1)^2 (1 + \tau + \sigma_2)^2 (1 + \tau + \sigma_3)^2 (1 + \tau + \sigma_4)^2} \\ & \quad f(1 - \sigma_1) f(1 - \sigma_2) f(1 - \sigma_3) f(1 - \sigma_4) d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_4 dx dt \end{aligned} \quad (3.15)$$

where

$$A(\tau, \sigma_1, \sigma_2, \sigma_3, \sigma_4) = P(2 + \tau, 1 - \sigma_1, 1 - \sigma_2, 1 - \sigma_3, 1 - \sigma_4) = \sum_{l=1}^4 \frac{(-1)^l}{1 + \tau + \sigma_l}.$$

Moreover, and because $\|f(1 - s)\|_{L^4([0,1])} = \|f(s)\|_{L^4([0,1])}$, the estimate (3.14) becomes

$$\| u(t + 2, x) \|_{L^4([0,1] \times \mathbb{R}^4)} \lesssim \| f \|_{L^4([0,1])} \quad (3.16)$$

Now, consider the quadrilinear form $T : L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \rightarrow \mathbb{C}$ given by

$$\begin{aligned} T(f_1, f_2, f_3, f_4) &= \int_0^1 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{iA(t, s_1, s_2, s_3, s_4)|x|^2}}{(1 + t + s_1)^2 (1 + t + s_2)^2 (1 + t + s_3)^2 (1 + t + s_4)^2} \\ & \quad f_1(1 - s_1) f_2(1 - s_2) f_3(1 - s_3) f_4(1 - s_4) ds_1 ds_2 ds_3 ds_4 dx dt. \end{aligned}$$

Then, it follows from (3.15) that

$$\| u(t + 2, x) \|_{L^4([0,1] \times \mathbb{R}^4)} = L(f),$$

where

$$L(f) = T(f, f, f, f).$$

Remark 3.2.1. It is worth noticing that it is enough to prove the estimate (3.14) for f real. This is clear since if f is a complex valued function with real part f_R and imaginary part f_{Im} so that $f = f_R + \imath f_{Im}$, then, by (3.9), we have that

$$u(t, x) = u_R(t, x) + \imath u_{Im}(t, x)$$

where

$$u_R(t, x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \frac{f_R(s)}{(t-s)^{\frac{n}{2}}} e^{\imath \frac{|x|^2}{4(t-s)}} ds, \quad u_{Im}(t, x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \frac{f_{Im}(s)}{(t-s)^{\frac{n}{2}}} e^{\imath \frac{|x|^2}{4(t-s)}} ds.$$

Once the estimates,

$$\|u_R\|_{L^4([2,3] \times \mathbb{R}^n)} \lesssim \|f_R\|_{L^4([0,1])}, \quad \|u_{Im}\|_{L^4([2,3] \times \mathbb{R}^n)} \lesssim \|f_{Im}\|_{L^4([0,1])},$$

are proven, we get

$$\begin{aligned} \|u\|_{L^4([2,3] \times \mathbb{R}^n)} &\leq \|u_R\|_{L^4([2,3] \times \mathbb{R}^n)} + \|u_{Im}\|_{L^4([2,3] \times \mathbb{R}^n)} \\ &\lesssim \|f_R\|_{L^4([0,1])} + \|f_{Im}\|_{L^4([0,1])} \\ &\lesssim \|f\|_{L^4([0,1])}. \end{aligned}$$

We are ready now to state the main problem we shall work on.

Question: Given a wave $u = u(t, x)$ defined by the oscillatory integral

$$\boxed{u(t, x) = \frac{1}{(4\pi)^2} \int_0^1 \frac{f(s)}{(t-s)^2} e^{\imath \frac{|x|^2}{4(t-s)}} ds} \quad (3.17)$$

where $f \in L^4([0, 1])$ is a real valued function. Prove the following estimate

$$\boxed{\|u\|_{L^4([2,3] \times \mathbb{R}^4)} \lesssim \|f\|_{L^4([0,1])}.} \quad (3.18)$$

A quadrilinear estimate question:

Given the quadrilinear form $T : L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \rightarrow \mathbb{C}$ defined

by

$$\boxed{T(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{\imath A(t, s_1, s_2, s_3, s_4) |x|^2}}{(1+t+s_1)^2 (1+t+s_2)^2 (1+t+s_3)^2 (1+t+s_4)^2} f_1(1-s_1) f_2(1-s_2) f_3(1-s_3) f_4(1-s_4) ds_1 ds_2 ds_3 ds_4 dx dt} \quad (3.19)$$

Prove the following estimate

$$\boxed{|T(f_1, f_2, f_3, f_4)| \lesssim \|f_1\|_{L^4([0,1])}^4 \|f_2\|_{L^4([0,1])}^4 \|f_3\|_{L^4([0,1])}^4 \|f_4\|_{L^4([0,1])}^4.} \quad (3.20)$$

Since

$$T(f, f, f, f) = \| u \|_{L^4([2,3] \times \mathbb{R}^4)}.$$

then the quadrilinear estimate (3.20) implies the estimate (3.18).

3.2.4 A strategy to give the answer

Let us sketch a general scheme that highlights the steps we shall follow in chapters 4-5 to address the question posed in Section 3.2.3. The main idea is to prove the quadrilinear estimate (3.20). To benefit from the oscillations, we need to do the integrations in (3.19). The problem is that we cannot do the integration in x because the integrand is not absolutely convergent and thus we can not change the order of integration. Notice that if we start by integrating the oscillatory kernel $e^{iA(t,s_1,s_2,s_3,s_4)|x|^2}$ in x , we get

$$\int_{\mathbb{R}^4} e^{iA(t,s_1,s_2,s_3,s_4)|x|^2} dx = \frac{c}{A^2(t, s_1, s_2, s_3, s_4)},$$

for some constant c . We shall see in Section 3.4.2 that the function $t \mapsto A(t, s_1, s_2, s_3, s_4)$ can vanish for some values of s . This way we would face a divergent integral in t . We overcame this difficulty in two different ways. The first way is to approximate the data f by piecewise constant functions of the form $\sum_{k=1}^N c_k \chi_{[\frac{k-1}{N}, \frac{k}{N}]}$. Notice from (3.17) that for data of the form $\chi_{[a,b]}$, $0 \leq a < b \leq 1$, we can integrate explicitly in the variable s and get

$$u(t, x) = \frac{e^{\frac{i|x|^2}{t-b}} - e^{\frac{i|x|^2}{t-a}}}{i|x|^2}.$$

So, when $f = \chi_{[a,b]}$, we have that $|u(t, x)| \lesssim \frac{1}{1+|x|^2}$. This enables us to integrate in x . The second way is to approximate the norm of the solution by $\|u\|_{L^4([2,3] \times \mathbb{R}^4)} = \lim_{\epsilon \rightarrow 0^+} \|u e^{-\epsilon|x|^2}\|_{L^4([2,3] \times \mathbb{R}^4)}$. This will result in an absolutely integrable kernel in the quadrilinear form (3.19). We will then be allowed to change order of integration and obtain the decay due integrating in x .

1. Rewrite the quadrilinear form (3.19) as a quadrilinear form whose kernel is an integral in the time variable.

- In Chapter 4, we use piecewise constant functions to approximate the function f . This helps us integrate explicitly in the variables s_i and in the spatial variable x so that we obtain a quadrilinear form with a kernel given by an integral in the time variable. We will discover that if $f_l(s) = \sum_{k_l=1}^N c_{k_l}^l \chi_{[\frac{k_l-1}{N}, \frac{k_l}{N}]}(s)$, $l = 1, 2, 3, 4$, then we have that $T(f_1, f_2, f_3, f_4) = S_N(c^1, c^2, c^3, c^4)$ where

$$S_N(c^1, c^2, c^3, c^4) = \sum_{k_1, k_2, k_3, k_4=1}^N c_{k_1}^1 c_{k_2}^2 c_{k_3}^3 c_{k_4}^4 J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right)$$

where

$$J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) = \sum_{\gamma} \sigma_{\gamma} \int_0^1 A_{\gamma}^2\left(t, \frac{k}{N}\right) \log |A_{\gamma}\left(t, \frac{k}{N}\right)| dt,$$

$$A_{\gamma}\left(t, \frac{k}{N}\right) = A\left(t, \frac{k-\gamma}{N}\right) = \sum_{l=1}^4 \frac{(-1)^l}{1+t+\frac{k_l}{N}-\frac{\gamma_l}{N}},$$

$$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \Gamma = \{0, 1\}^4, \quad k = (k_1, k_2, k_3, k_4) \in \{1, \dots, N\}^4,$$

$$\sigma_{\gamma} = (-1)^{(\gamma_1+\gamma_2+\gamma_3+\gamma_4)},$$

$$A(t, k) = A(t, k_1, k_2, k_3, k_4) = \sum_{l=1}^4 \frac{(-1)^l}{1+t+k_l}.$$

- In Chapter 5, we do the integration in x . To be able to do this, we use the dominated convergence theorem to approximate the quadrilinear form (3.19) by a quadrilinear integral form S_{ϵ} in which interchanging the order of integration is allowed. We shall actually write $T(f_1, f_2, f_3, f_4) = \lim_{\epsilon \rightarrow 0^+} S_{\epsilon}(f_1, f_2, f_3, f_4)$ where

$$S_{\epsilon}(f_1, f_2, f_3, f_4) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 K_{\epsilon}(s_1, s_2, s_3, s_4) f_1(1-s_1) f_2(1-s_2) f_3(1-s_3) f_4(1-s_4) ds_1 ds_2 ds_3 ds_4.$$

with the kernel $K_\epsilon(s_1, s_2, s_3, s_4)$ given from

$$\begin{aligned} K_\epsilon(s_1, s_2, s_3, s_4) &= \int_0^1 H_\epsilon(t, s_1, s_2, s_3, s_4) dt, \\ H_\epsilon(t, s_1, s_2, s_3, s_4) &= \frac{1}{\prod_{l=1}^4 (1+t+s_l)^2} \int_{\mathbb{R}^4} e^{(-\epsilon+tA(t,s_1,s_2,s_3,s_4))|x|^2} dx \\ &= \frac{1}{\prod_{l=1}^4 (1+t+s_l)^2} \frac{\epsilon^2 - A^2(t, s_1, s_2, s_3, s_4)}{(\epsilon^2 + A^2(t, s_1, s_2, s_3, s_4))^2}. \end{aligned}$$

2. Reformulate the problem to become a problem of estimating the quadrilinear form obtained in 1. That is we consider the estimates

$$\boxed{|S_N(c^1, c^2, c^3, c^4)| \lesssim \frac{1}{N} \|c^1\|_{L^4([0,1])} \|c^2\|_{L^4([0,1])} \|c^3\|_{L^4([0,1])} \|c^4\|_{L^4([0,1])}}, \quad (3.21)$$

$$\boxed{|S_\epsilon(f_1, f_2, f_3, f_4)| \lesssim \|f_1\|_{L^4([0,1])} \|f_2\|_{L^4([0,1])} \|f_3\|_{L^4([0,1])} \|f_4\|_{L^4([0,1])}} \quad (3.22)$$

in chapters 4 and 5 respectively.

3. Using the multilinear interpolation technique, it suffices to prove the estimates

$$\sup_{k_{i_1} \quad k_{i_2}, k_{i_3}, k_{i_4}=1}^N |J_{\frac{1}{N}}(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N})| \lesssim \frac{1}{N}, \quad (3.23)$$

$$\sup_{s_i \in [0,1]} \int \int \int_{[0,1]^3} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim 1. \quad (3.24)$$

so as to prove the estimates (3.21) and (3.22) respectively.

4. Decompose the estimate (3.23) suggested by the multilinear interpolation by splitting it into a sum of estimates each of which is obtained from restricting the estimate (3.23) on a certain region. The same procedure of course will be performed on the estimate (3.24).
5. Estimate the kernels $J_{\frac{1}{N}}$ and K_ϵ on the regions considered in the step 4. There will be some easy regions where there is no need to integrate and some other more difficult regions where the singular function that defines the kernel oscillates and must be integrated.

6. Prove the estimates obtained in 4 that came from decomposition. (we get a divergence of log type in this step)
7. Interpolate and get the estimate (3.20) with a divergence of logarithmic order.

3.3 Preliminaries

Although the preliminary results that we give here will seem unmotivated, we decided to dedicate this section to the computations and proofs of these preliminaries. This helps us avoid giving unnecessarily lengthy proofs in the next two chapters. However, and for the sake of convenience, we will briefly state the main result we use whenever we need it. In the next subsection we recall some classic interpolation theorems and prepare to apply them to our problem.

3.3.1 Classical interpolation theory and its application to the problem

We recall the classic Reisz-Thorin interpolation theorem and its multilinear version. Let (U, μ) be a measure space. Denote by $L^p(U, d\mu)$ the Lebesgue-space of (all equivalence classes of) scalar-valued μ -measurable functions f on U such that

$$\|f\|_{L^p(U)} = \left(\int_U |f|^p d\mu \right)^{\frac{1}{p}}$$

is finite. Here $1 \leq p < \infty$. In the limiting case, $p = \infty$, L^p consists of all μ -measurable and bounded functions. Then we write

$$\|f\|_{L^\infty(U)} = \operatorname{ess\,sup}_U |f|.$$

Let T be a linear mapping from $L^p(U, d\mu)$ to $L^q(V, d\nu)$. We shall write

$$T : L^p \rightarrow L^q$$

if in addition T is bounded, i.e. if

$$M = \sup_{f \neq 0} \frac{\|Tf\|_{L^q}}{\|f\|_{L^p}}$$

is finite. The number M is of course the norm of the bounded linear mapping T .

Theorem 3.3.1. (*The Riesz-Thorin interpolation theorem*) [2]. Assume that $p_0 \neq p_1$, $q_0 \neq q_1$ and that

$$T : L^{p_0}(U, d\mu) \rightarrow L^{q_0}(V, d\nu)$$

with norm M_0 , and that

$$T : L^{p_1}(U, d\mu) \rightarrow L^{q_1}(V, d\nu)$$

with norm M_1 . Then

$$T : L^p(U, d\mu) \rightarrow L^q(V, d\nu)$$

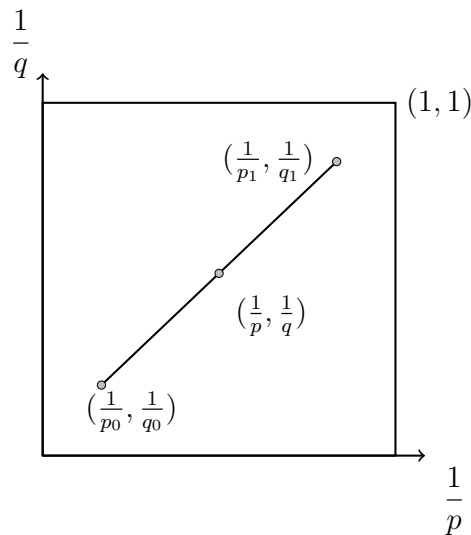
with norm

$$M \leq M_0^{1-\theta} M_1^\theta. \tag{3.25}$$

provided that $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{3.26}$$

Note that (3.25) means that M is logarithmically convex, i.e. $\log M$ is convex.



Note also the geometrical meaning of (3.26). The points $(\frac{1}{p}, \frac{1}{q})$ described by (3.26) are the points on the line segment between $(\frac{1}{p_0}, \frac{1}{q_0})$ and $(\frac{1}{p_1}, \frac{1}{q_1})$. (Obviously one should think of L^p as a "function" of $\frac{1}{p}$ rather than of p .)

For an elementary proof of the Riesz-Thorin interpolation (or convexity) theorem given by Thorin the reader may consult [2]. Now, we state the multilinear version of the interpolation theorem.

Theorem 3.3.2. (Multilinear Riesz-Thorin Interpolation Theorem)[2]

Let

$$T : L^{p_1} \times L^{p_2} \times \dots \times L^{p_m} \rightarrow \mathbb{C}$$

be a bounded multilinear functional such that

$$|T(f_1, f_2, \dots, f_m)| \leq M_1 \prod_{j=1}^m \|f_j\|_{L^{p_j}} .$$

Assume also that

$$T : L^{q_1} \times L^{q_2} \times \dots \times L^{q_m} \rightarrow \mathbb{C}$$

is a bounded multilinear functional such that

$$|T(f_1, f_2, \dots, f_m)| \leq M_2 \prod_{j=1}^m \|f_j\|_{L^{q_j}} .$$

Then

$$T : L^{r_1} \times L^{r_2} \times \dots \times L^{r_m} \rightarrow \mathbb{C}$$

is a bounded linear functional that satisfies

$$|T(f_1, f_2, \dots, f_m)| \leq M_0 \prod_{j=1}^m \|f_j\|_{L^{r_j}}$$

where

$$\frac{1}{r_j} = \frac{\theta}{p_j} + \frac{1-\theta}{q_j}, \quad j = 1, 2, \dots, m, \quad M_0 \leq M_1^\theta M_2^{1-\theta}, \quad 0 \leq \theta \leq 1.$$

3.3.1.1 Application to the problem (A quadrilinear interpolation result)

The following consequence of Theorem 3.3.1 will prove very useful. Again we employ the same notations of Theorem 3.3.1.

Theorem 3.3.3. *Let $k = 1, 2, 3, 4$, and consider the quadrilinear operator*

$$T : L^{p_1^k}([0, 1]) \times L^{p_2^k}([0, 1]) \times L^{p_3^k}([0, 1]) \times L^{p_4^k}([0, 1]) \rightarrow \mathbb{C},$$

where

$$\begin{aligned} (p_1^1, p_2^1, p_3^1, p_4^1) &= (1, \infty, \infty, \infty), \\ (p_1^2, p_2^2, p_3^2, p_4^2) &= (\infty, 1, \infty, \infty), \\ (p_1^3, p_2^3, p_3^3, p_4^3) &= (\infty, \infty, 1, \infty), \\ (p_1^4, p_2^4, p_3^4, p_4^4) &= (\infty, \infty, \infty, 1). \end{aligned}$$

Suppose that

$$|T(f_1, f_2, f_3, f_4)| \lesssim \prod_{j=1}^4 \|f_j\|_{L^{p_j^k}}, \quad k = 1, 2, 3, 4.$$

Then we have that

$$T : L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \rightarrow \mathbb{C}$$

is a bounded quadrilinear operator that satisfies

$$|T(f_1, f_2, f_3, f_4)| \lesssim \prod_{j=1}^4 \|f_j\|_{L^4}. \quad (3.27)$$

Moreover, if the quadrilinear form T is given by the integral

$$T(f) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 H(x_1, x_2, x_3, x_4) f(x_1) f(x_2) f(x_3) f(x_4) dx_1 dx_2 dx_3 dx_4.$$

Then it is enough, for (3.27) to be satisfied, to have that

$$\sup_{x_i} \int_0^1 \int_0^1 \int_0^1 |H(x_1, x_2, x_3, x_4)| dx_j dx_k dx_l \lesssim 1, \quad (3.28)$$

for all the permutations (i, j, k, l) of the integers $\{1, 2, 3, 4\}$.

The following interpolation result is a direct consequence of Theorem 3.3.3. It provides an estimate of a quadrilinear form defined on the finite-dimensional Banach space of all finite sequences.

Theorem 3.3.4. *Suppose that $T : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{C}$ is the quadrilinear form given by*

$$T(c^1, c^2, c^3, c^4) = \sum_{k_1, k_2, k_3, k_4=1}^N M_{k_1, k_2, k_3, k_4} c_{k_1}^1 c_{k_2}^2 c_{k_3}^3 c_{k_4}^4,$$

$c^j = (c_1^j, \dots, c_N^j)$, $j = 1, 2, 3, 4$. Let (i_1, i_2, i_3, i_4) be a permutation of the integers $\{1, 2, 3, 4\}$.

$$\text{If} \quad \sup_{k_{i_1} \dots k_{i_4}=1}^N |M_{k_1, k_2, k_3, k_4}| \lesssim 1.$$

$$\text{Then} \quad |T(c^1, c^2, c^3, c^4)| \lesssim \prod_{j=1}^4 \|c^j\|_{l^{i_j}}^4.$$

3.3.2 Approximation of L^p functions by Piecewise constant functions

Lemma 3.3.5. *Let $V_N(0, 1) = \{f : [0, 1] \rightarrow \mathbb{R} \mid f(s) = \sum_{m=1}^N c_m \chi_{[\frac{m-1}{N}, \frac{m}{N}]}(s), c_m \in \mathbb{R}, m = 1, \dots, N\}$. $V_N(0, 1)$ is a finite dimensional linear space of piecewise constant functions on $[0, 1]$ with $\dim V_N(0, 1) = N$. Then $\cup_{N \geq 1} V_N(0, 1)$ is dense in $L^p([0, 1])$, $1 \leq p < +\infty$ in the L^p -norm, that is, $\overline{\cup_{N \geq 1} V_N(0, 1)} = L^p([0, 1])$. Furthermore, given any $f \in L^p([0, 1])$, there is a sequence of functions $f_N \in V_N(0, 1)$ such that*

$$\lim_{N \rightarrow \infty} \|f - f_N\|_{L^p([0,1])} = 0.$$

Proof. Since the space $C([0, 1])$ of all continuous functions on $[0, 1]$ is dense in $L^p([0, 1])$ then for all $f \in L^p([0, 1])$ and any however small ϵ there is a continuous function $g \in C([0, 1])$ such that

$$\|f - g\|_{L^p(0,1)} < \frac{\epsilon}{2}. \quad (3.29)$$

Let $g_N \in V_N(0, 1)$ be the piecewise constant function given by

$$g_N(x) = N \sum_{m=1}^N \left(\int_{\frac{m-1}{N}}^{\frac{m}{N}} g(y) dy \right) \chi_{[\frac{m-1}{N}, \frac{m}{N}]}(x).$$

Notice that the constants of g_N are the mean values of the function on the intervals $[\frac{m-1}{N}, \frac{m}{N}]$. We have

$$\begin{aligned} & \|g - g_N\|_{L^p([0,1])}^p \\ &= \int_0^1 \left| g(x) - N \sum_{m=1}^N \left(\int_{\frac{m-1}{N}}^{\frac{m}{N}} g(y) dy \right) \chi_{[\frac{m-1}{N}, \frac{m}{N}]}(x) \right|^p dx \\ &= \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left| g(x) - N \left(\int_0^{\frac{1}{N}} g(y) dy \right) \chi_{[0, \frac{1}{N}]}(x) - \dots - N \left(\int_{\frac{N-1}{N}}^1 g(y) dy \right) \chi_{[\frac{N-1}{N}, 1]}(x) \right|^p dx \\ &= \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left| g(x) - N \left(\int_{\frac{k-1}{N}}^{\frac{k}{N}} g(y) dy \right) \right|^p dx \\ &\leq \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left[N \left(\int_{\frac{k-1}{N}}^{\frac{k}{N}} |g(x) - g(y)| dy \right) \right]^p dx \end{aligned} \quad (3.30)$$

$$(3.31)$$

Since g is continuous on the compact set $[0, 1]$ then it is also uniformly continuous there. That is, given any $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ so that $|g(x) - g(y)| < \epsilon$ for all $x, y \in [0, 1]$ with $|x - y| < \delta_\epsilon$. Given any $\epsilon > 0$, there is a large enough $N_\epsilon > \frac{1}{\delta_\epsilon}$ such that for all $x, y \in [\frac{k-1}{N}, \frac{k}{N}]$, $|x - y| < \frac{1}{N_\epsilon} < \delta_\epsilon$. Thus, by uniform continuity of g , we get

$$|g(x) - g(x_m)| < \frac{\epsilon}{2}.$$

Using this in (3.30), it follows that

$$\|g - g_N\|_{L^p([0,1])}^p < \frac{\epsilon}{2}. \quad (3.32)$$

Combining (3.29) and (3.32), we get that for every $\epsilon > 0$, there is N_ϵ such that for all $N > N_\epsilon$,

$$\begin{aligned} \|f - g_N\|_{L^p([0,1])} &\leq \|f - g\|_{L^p([0,1])} + \|g - g_N\|_{L^p([0,1])} \\ &< \epsilon. \end{aligned}$$

That is

$$\lim_{N \rightarrow \infty} \|f - g_N\|_{L^p(0,1)} = 0.$$

This concludes the proof. \square

3.3.3 Singularity-related Calculus

Lemma 3.3.6. *Let $A, \phi \in C^2([a, b])$ and assume that $A(t_*) = 0$ at a unique $t_* \in]a, b[$ so that $A(t) = Q(t)(t - t_*)$, $Q(t) \neq 0, \forall t$. Assume moreover that A' never vanishes on $[a, b]$. Then*

$$P.V. \int_a^b \frac{\phi(t)}{A(t)} dt = - \int_a^b \log |A(t)| \partial_t \left(\frac{\phi(t)}{\partial_t A(t)} \right) dt + \frac{\phi(b)}{\partial_t A(b)} \log |A(b)| - \frac{\phi(a)}{\partial_t A(a)} \log |A(a)|.$$

Proof. Integrating by parts, we get

$$\begin{aligned} P.V. \int_a^b \frac{\phi(t)}{A(t)} dt &= \lim_{\delta \rightarrow 0^+} \left(\int_a^{t_* - \delta} + \int_{t_* + \delta}^b \right) \frac{\phi(t)}{\partial_t A(t)} \frac{\partial_t A(t)}{A(t)} dt \\ &= \lim_{\delta \rightarrow 0^+} \left(\int_a^{t_* - \delta} + \int_{t_* + \delta}^b \right) \frac{\phi(t)}{\partial_t A(t)} d(\log |A(t)|) \\ &= - \lim_{\delta \rightarrow 0^+} \left[\frac{\phi(t_* + \delta)}{\partial_t A(t_* + \delta)} \log |Q(t_* + \delta)| - \frac{\phi(t_* - \delta)}{\partial_t A(t_* - \delta)} \log |Q(t_* - \delta)| \right] + \\ &\quad - \lim_{\delta \rightarrow 0^+} \left[\frac{\phi(t_* + \delta)}{\partial_t A(t_* + \delta)} - \frac{\phi(t_* - \delta)}{\partial_t A(t_* - \delta)} \right] \log \delta + \frac{\phi(b)}{\partial_t A(b)} \log |A(b)| - \frac{\phi(a)}{\partial_t A(a)} \log |A(a)| + \\ &\quad - \lim_{\delta \rightarrow 0^+} \left(\int_a^{t_* - \delta} + \int_{t_* + \delta}^b \right) \log |A(t)| \partial_t \left(\frac{\phi(t)}{\partial_t A(t)} \right) dt \\ &= - \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \left[\frac{\phi(t_* + \delta)}{\partial_t A(t_* + \delta)} - \frac{\phi(t_* - \delta)}{\partial_t A(t_* - \delta)} \right] \lim_{\delta \rightarrow 0^+} \delta \log \delta + \frac{\phi(b)}{\partial_t A(b)} \log |A(b)| - \\ &\quad \frac{\phi(a)}{\partial_t A(a)} \log |A(a)| - \lim_{\delta \rightarrow 0^+} \left(\int_a^{t_* - \delta} + \int_{t_* + \delta}^b \right) \log |A(t)| \partial_t \left(\frac{\phi(t)}{\partial_t A(t)} \right) dt \\ &= - \int_a^b \log |A(t)| \partial_t \left(\frac{\phi(t)}{\partial_t A(t)} \right) dt + \frac{\phi(b)}{\partial_t A(b)} \log |A(b)| - \frac{\phi(a)}{\partial_t A(a)} \log |A(a)| \end{aligned}$$

where we used the continuity of the function $\frac{\phi}{\partial_t A} \log |Q|$ and the fact that the function $\frac{\phi}{\partial_t A}$ is continuously differentiable. \square

Lemma 3.3.7. *Let $A, \phi \in C^1$. Assume that the map $t \mapsto A(t, s)$ has a unique zero, $t_*(s) \in]a, b[$, so that $A(t, s) = Q(t, s)(t - t_*(s))$ and $Q(t, s) \neq 0$, for t and s . Then*

$$\partial_s \int_a^b \log |A(t, s)| \phi(t, s) dt = P.V. \int_a^b \frac{\partial_s A(t, s)}{A(t, s)} \phi(t, s) dt + \int_a^b \log |A(t, s)| \partial_s \phi(t, s) dt.$$

Proof.

$$\begin{aligned} \int_a^b \log |A(t, s)| \phi(t, s) dt &= \int_a^b \phi(t, s) \log |t - t_*(s)| dt + \int_a^b \phi(t, s) \log |Q(t, s)| dt \\ &= \int_{a-t_*(s)}^{b-t_*(s)} \phi(x + t_*(s), s) \log |x| dx + \int_a^b \phi(t, s) \log |Q(t, s)| dt. \end{aligned} \quad (3.33)$$

Since Q never vanishes,

$$\partial_s \int_a^b \log |Q(t, s)| \phi(t, s) dt = \int_a^b \frac{\partial_s Q(t, s)}{Q(t, s)} \phi(t, s) dt + \int_a^b \partial_s \phi(t, s) \log |Q(t, s)| dt. \quad (3.34)$$

The function Q never vanishing implies that $\partial_t A(t_*(s), s) \neq 0$ for all s . And since

$$\frac{d}{ds} t_*(s) = -\frac{\partial_s A(t_*(s), s)}{\partial_t A(t_*(s), s)},$$

then $t_* \in C^1$. By this and the assumption that $\phi \in C^1([a, b])$ we have

$$\begin{aligned} &\partial_s \int_{a-t_*(s)}^{b-t_*(s)} \phi(x + t_*(s), s) \log |x| dx \\ &= \partial_s t_*(s) \int_{a-t_*(s)}^{b-t_*(s)} [\partial_t \phi(x + t_*, s)] \log |x| dx + \int_{a-t_*(s)}^{b-t_*(s)} [\partial_s \phi(x + t_*, s)] \log |x| dx - \\ &\quad [\partial_s t_*(s)] \phi(b, s) \log |b - t_*(s)| + [\partial_s t_*(s)] \phi(a, s) \log |a - t_*(s)| \\ &= \partial_s t_*(s) \int_a^b \log |t - t_*(s)| \partial_t \phi(t, s) dt + \int_a^b \log |t - t_*(s)| \partial_s \phi(t, s) dt - \\ &\quad [\partial_s t_*(s)] \phi(b, s) \log |b - t_*(s)| + [\partial_s t_*(s)] \phi(a, s) \log |a - t_*(s)|. \end{aligned} \quad (3.35)$$

By integration by parts, we have

$$\begin{aligned} \int_a^b \log |t - t_*(s)| \partial_t \phi(t, s) dt &= -P.V. \int_a^b \frac{\phi(t, s)}{t - t_*(s)} dt + \phi(b, s) \log |b - t_*(s)| - \\ &\quad \phi(a, s) \log |a - t_*(s)| \end{aligned}$$

using this and (3.35), we get that

$$\begin{aligned} \partial_s \int_{a-t_*(s)}^{b-t_*(s)} \phi(x + t_*(s), s) \log |x| dx &= -P.V. \int_a^b \frac{\phi(t, s) \partial_s t_*(s)}{t - t_*(s)} dt + \\ &\quad \int_a^b \partial_s \phi(t, s) \log |t - t_*(s)| dt. \end{aligned} \quad (3.36)$$

Finally, from (3.33), (3.34) and (3.36), we obtain

$$\begin{aligned} \partial_s \int_a^b \log |A(t, s)| \phi(t, s) dt &= P.V. \int_a^b \left[\frac{-\partial_s t_*(s)}{t - t_*(s)} + \frac{\partial_s Q(t, s)}{Q(t, s)} \right] \phi(t, s) dt + \\ &\quad \int_a^b [\log |Q(t, s)| + \log |t - t_*(s)|] \partial_s \phi(t, s) dt \\ &= P.V. \int_a^b \frac{\partial_s A(t, s)}{A(t, s)} \phi(t, s) dt + \int_a^b \log |A(t, s)| \partial_s \phi(t, s) dt. \end{aligned}$$

□

The following corollary is a direct consequence of lemmas 3.3.6 and 3.3.7.

Corollary 3.3.8. *Let A and ϕ be as in Lemma 3.3.7. Then*

$$\begin{aligned} \partial_s \int_a^b \log |A(t, s)| \phi(t, s) dt &= \int_a^b \log |A(t, s)| \left[\partial_s \phi(t, s) - \partial_t \left(\frac{\phi(t, s) \partial_s A(t, s)}{\partial_t A(t, s)} \right) \right] dt + \\ &\quad + \frac{\phi(b, s) \partial_s A(b, s)}{\partial_t A(b, s)} \log |A(b, s)| - \frac{\phi(a, s) \partial_s A(a, s)}{\partial_t A(a, s)} \log |A(a, s)| \end{aligned}$$

3.4 On the function $A(t, s_1, s_2, s_3, s_4)$

3.4.1 Zeros of the function $t \mapsto A(t, s_1, s_2, s_3, s_4)$

It will be evident in both chapters 4 and 5 that we need to learn all about the zeros of the map $t \mapsto A(t, s)$ for all $t \in [0, 1]$ where

$$A(t, s) = A(t, s_1, s_2, s_3, s_4) = \frac{-1}{1+t+s_1} + \frac{1}{1+t+s_2} + \frac{-1}{1+t+s_3} + \frac{1}{1+t+s_4}.$$

. This will determine the smoothness and integrability properties of the kernels that appear in the multilinear forms there.

We will also estimate the function $A(t, s_1, s_2, s_3, s_4)$ for all different values of the variables s_j . This also is going to be very useful when we estimate the previously mentioned kernels. As a matter of fact we obtain uniform estimates for A except on a certain "small" region where the function changes its sign. Here is where the estimates of the roots and derivatives of A enter the game. First of all we show that at most one of the zeros of this map $t \mapsto A(t, \cdot)$ can lie inside $[0, 1]$.

Lemma 3.4.1. *The mapping $t \mapsto A(t, s)$ has **at most** one zero in $[0, 1]$ for all $s \in [0, 1]^4$.*

Proof. Now consider the functions

$$\tilde{A}(t, s) = A(t - 1, s) = \sum_{l=1}^4 \frac{(-1)^{(l)}}{t + s_l}.$$

This can be rewritten as

$$\tilde{A}(t, s) = \frac{f(t, s)}{\prod_{l=1}^4 (t + s_l)}$$

where

$$\begin{aligned} f(t, s) &= a(s)t^2 + b(s)t + c(s), \\ a(s) &= s_1 + s_3 - (s_2 + s_4), \\ b(s) &= 2(s_1s_3 - s_2s_4), \\ c(s) &= s_1s_3(s_2 + s_4) - s_2s_4(s_1 + s_3). \end{aligned}$$

Obviously the function $t \mapsto A(t, s)$ has a real zero in $[0, 1]$ if and only if the function $t \mapsto \tilde{A}(t, s)$ has a real zero in $[1, 2]$. The latter occurs if and only if at most one of the two roots $t_1(s)$ and $t_2(s)$ of the quadratic polynomial $t \mapsto f(t, s)$ lies in $[1, 2]$ where

$$t_j(s) = \frac{-b(s) + (-1)^j \sqrt{\Delta(s)}}{2a(s)}, \quad j = 1, 2$$

with $\Delta(s) = b^2(s) - 4a(s)c(s)$.

If $a(s) = 0$, then $t \mapsto f(t, s)$ has only one root, namely, $-\frac{c(s)}{b(s)}$. So, we assume that $a(s) \neq 0$. By contradiction assume that we have both roots are in $[1, 2]$, i.e.,

$$\begin{aligned} 1 \leq t_1(s) &= \frac{-b(s) - \sqrt{\Delta(s)}}{2a(s)} \leq 2, \\ 1 \leq t_2(s) &= \frac{-b(s) + \sqrt{\Delta(s)}}{2a(s)} \leq 2. \end{aligned}$$

Then only one of the following four possibilities can happen:

- (i) $a(s) < 0, \quad b(s) \leq 0$
 - (ii) $a(s) < 0, \quad b(s) \geq 0$
 - (iii) $a(s) > 0, \quad b(s) \leq 0$
 - (iv) $a(s) > 0, \quad b(s) \geq 0$
- (i) $a(s) < 0, \quad b(s) \leq 0$.

In this case we have that $t_2(s) < 0$ which is a contradiction to the assumption.

(ii) $a(s) < 0, \quad b(s) \geq 0$

$$a(s) < 0 \quad \implies s_2 + s_4 < s_1 + s_3, \quad (3.37)$$

$$b(s) \geq 0 \quad \implies s_1 s_3 \leq s_2 s_4, \quad (3.38)$$

(3.37) together with (3.38) imply that $c(s) \geq 0$ which means that $t_1(s)t_2(s) = \frac{c(s)}{a(s)} \geq 0$ which contradicts the assumption that the two roots lie in $[1, 2]$.

(iii) $a(s) > 0, \quad b(s) \leq 0,$

This case is similar to case (ii) since

$$a(s) > 0 \quad \implies s_1 + s_3 > s_2 + s_4,$$

$$b(s) \leq 0 \quad \implies s_2 s_4 \geq s_1 s_3,$$

then $c(s) \leq 0$ and $t_1(s)t_2(s) = \frac{c(s)}{a(s)} < 0$ which is a contradiction.

(iv) $a(s) > 0, \quad b(s) \geq 0$

In this case we have that $t_1(s) < 0$ which is a contradiction to the assumption.

□

Let $s = (s_1, s_2, s_3, s_4)$. The function $A(t, s) = A(t, s_1, s_2, s_3, s_4)$ can be rewritten as

$$A(t, s) = \frac{s_1 - s_2}{(1+t+s_1)(1+t+s_1)} + \frac{s_3 - s_4}{(1+t+s_3)(1+t+s_4)} \quad (3.39)$$

$$= \frac{s_3 - s_2}{(1+t+s_2)(1+t+s_3)} + \frac{s_1 - s_4}{(1+t+s_1)(1+t+s_4)}. \quad (3.40)$$

Now, considering all the possible values of s_j , we are in one of the following three situations

$$(I) \quad (s_1 - s_2)(s_3 - s_4) > 0$$

$$(II) \quad (s_3 - s_2)(s_1 - s_4) > 0$$

$$(III) \quad (s_1 - s_2)(s_3 - s_4) < 0 \quad \text{and} \quad (s_3 - s_2)(s_1 - s_4) < 0.$$

It is clear by (3.39) that $t \mapsto A(t, s)$ does not change sign on $[0, 1]$ for all s in the region (I).

Similarly looking at (3.40) we see that $t \mapsto A(t, s)$ has no zeros for all s in the region (II).

This restricts the possibility of attaining real zeros solely to the region (III). We summarize this in the following lemma.

Lemma 3.4.2. *The mapping $t \mapsto A(t, s)$ has **at most** one real zero in $[0, 1]$ only when*

$$(s_1 - s_2)(s_3 - s_4) < 0 \quad \text{and} \quad (s_3 - s_2)(s_1 - s_4) < 0.$$

3.4.2 Estimates for the function $A(t, s_1, s_2, s_3, s_4)$

The following lemma gives estimates for $A(t, s)$ on the regions I and (II) . These estimates follow directly from (3.39) and (3.40).

Lemma 3.4.3.

$$|A(t, s)| \approx \begin{cases} \max \{|s_1 - s_2|, |s_3 - s_4|\} & \text{if } (s_1 - s_2)(s_3 - s_4) > 0, \\ \max \{|s_1 - s_4|, |s_2 - s_3|\} & \text{if } (s_3 - s_2)(s_1 - s_4) > 0. \end{cases}$$

The figures (Figure (1)- Figure (2)) below illustrate the estimates given in Lemma 3.4.3.

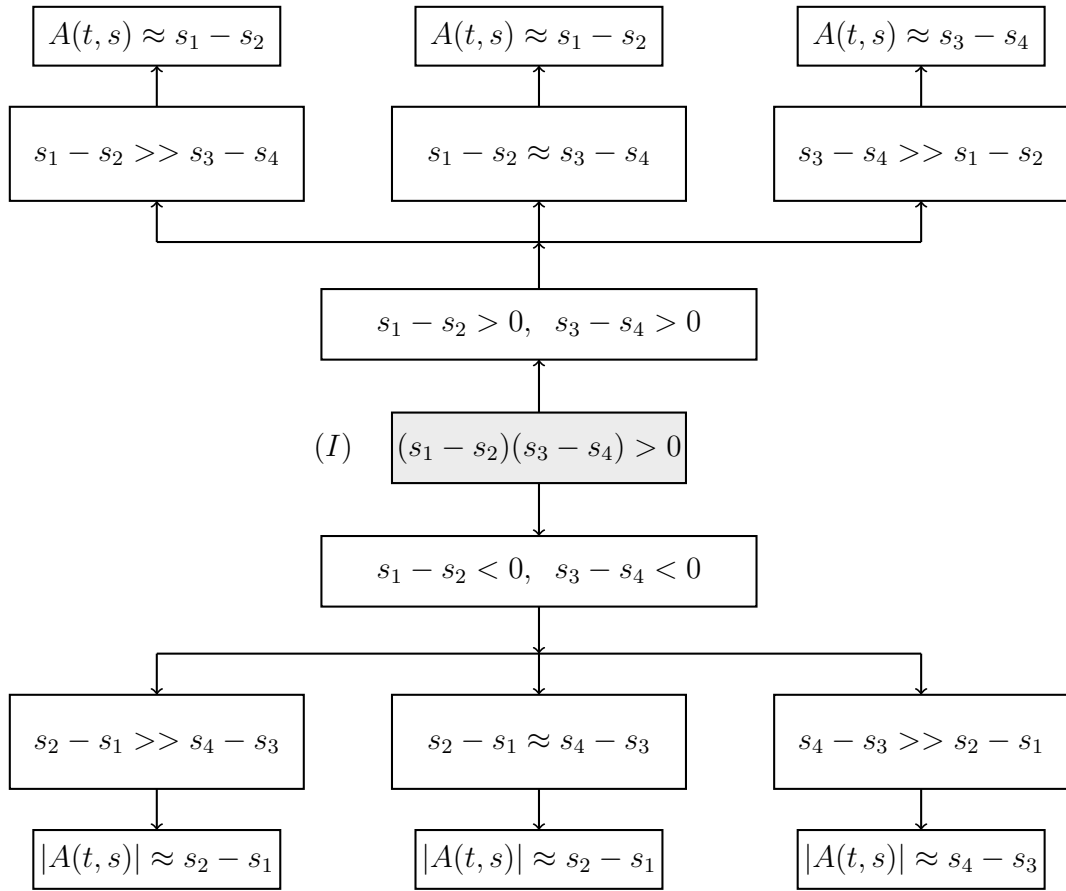


Figure (1): Estimates for $A(t, s)$ in the region (I)

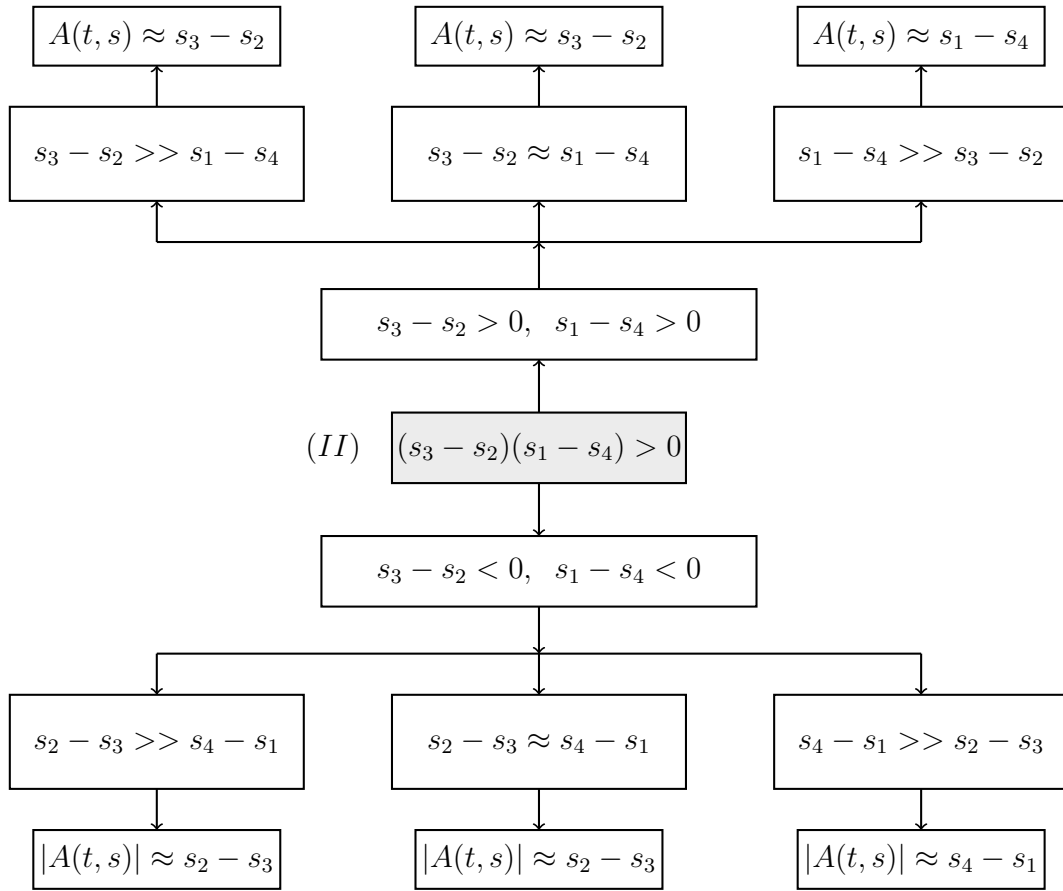


Figure (2): Estimates for $A(t, s)$ in the region (II)

Case (III) occurs only if $s_1, s_2, s_3,$ and s_4 come arranged in one of the following eight ways shown in the table below

	$s_1 - s_2$	$s_3 - s_4$	$s_3 - s_2$	$s_1 - s_4$
$s_2 < s_1 < s_3 < s_4$	+	-	+	-
$s_2 < s_3 < s_1 < s_4$	+	-	+	-
$s_4 < s_1 < s_3 < s_2$	-	+	-	+
$s_4 < s_3 < s_1 < s_2$	-	+	-	+
$s_1 < s_2 < s_4 < s_3$	-	+	+	-
$s_1 < s_4 < s_2 < s_3$	-	+	+	-
$s_3 < s_2 < s_4 < s_1$	+	-	-	+
$s_3 < s_4 < s_2 < s_1$	+	-	-	+

To make these arrangements easier to understand we show them in the following picture.

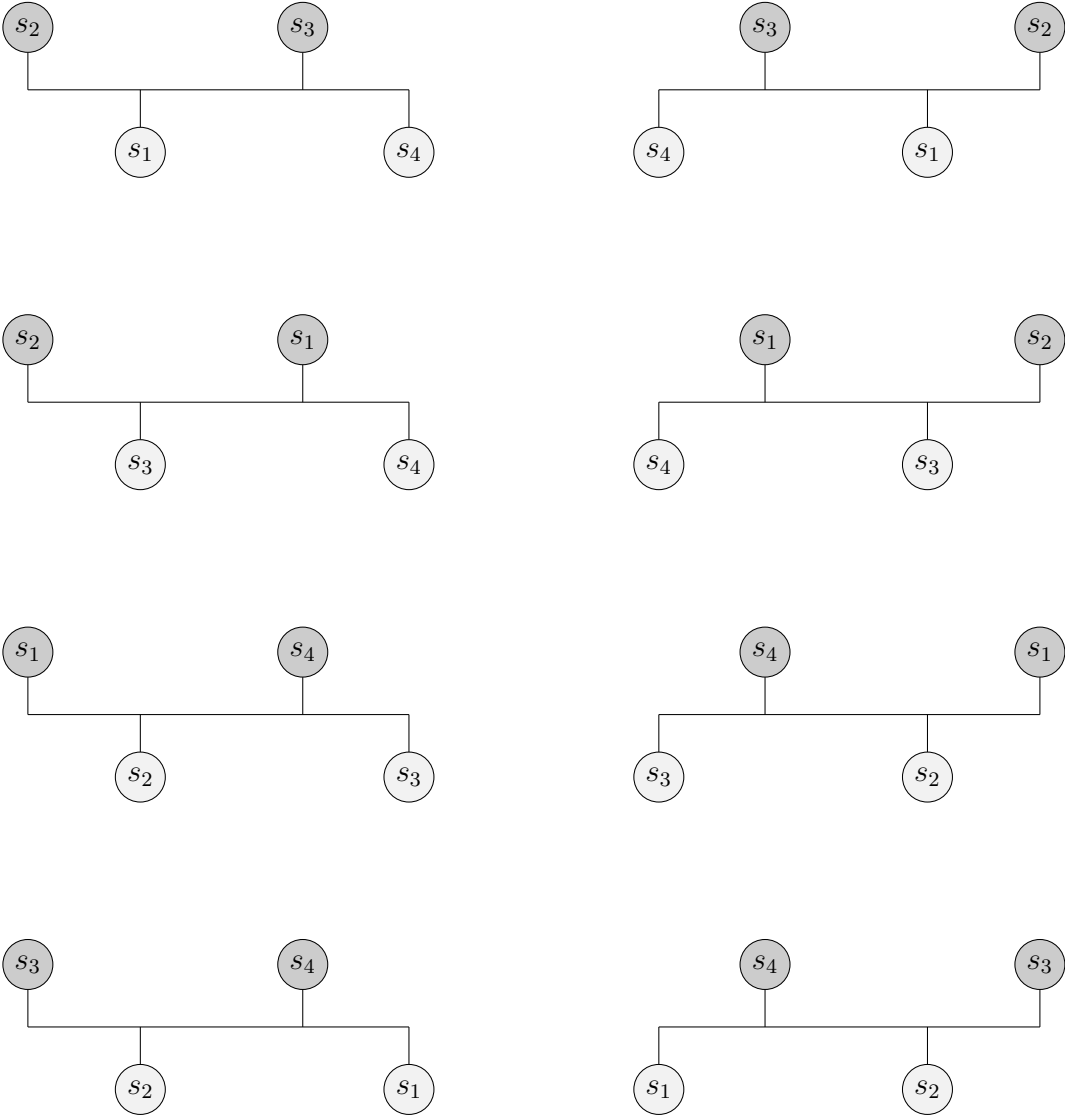


Figure (3): All possible arrangements of s_1, s_2, s_3, s_4 for the region (III).

Now, notice that we can rewrite $A(t, s)$ in one of the following formulas

$$A(t, s) = \frac{s_1 - s_2}{(1+t+s_1)(1+t+s_1)} - \frac{s_4 - s_3}{(1+t+s_3)(1+t+s_4)} \quad (3.41)$$

$$= \frac{s_3 - s_4}{(1+t+s_3)(1+t+s_4)} - \frac{s_2 - s_1}{(1+t+s_1)(1+t+s_1)} \quad (3.42)$$

$$= \frac{s_3 - s_2}{(1+t+s_2)(1+t+s_3)} - \frac{s_4 - s_1}{(1+t+s_1)(1+t+s_4)} \quad (3.43)$$

$$= \frac{s_1 - s_4}{(1+t+s_1)(1+t+s_4)} - \frac{s_2 - s_3}{(1+t+s_2)(1+t+s_3)}. \quad (3.44)$$

A careful look at the picture above and the formulas (3.41)-(3.44) shows a kind of symmetry that makes studying any of these arrangements equivalent to studying any other arrangement. Indeed, we have that $A(t, s)$ enjoys the following symmetry properties

$$\begin{aligned} A(t, s_1, s_2, s_3, s_4) &= A(t, s_3, s_2, s_1, s_4) = A(t, s_1, s_4, s_3, s_2) = A(t, s_3, s_4, s_1, s_2) \\ &= -A(t, s_2, s_1, s_4, s_3) = -A(t, s_2, s_3, s_4, s_1) = -A(t, s_4, s_1, s_2, s_3) = -A(t, s_4, s_3, s_2, s_1). \end{aligned}$$

Thus, we can freely choose one arrangement and stick to it and any argument corresponding to this arrangement can then be repeated in the same way to all other possible cases. We can therefore proceed without loss of rigor choosing for instance the arrangement

$$\boxed{s_2 < s_1 < s_3 < s_4}.$$

Next, fix s so that $0 \leq s_2 < s_1 < s_3 < s_4 \leq 1$ and let

$$\mu = s_1 - s_2,$$

$$\nu = s_3 - s_2,$$

$$\tau = 1 + t + s_2,$$

$$\alpha = s_1 - s_2 + s_3 - s_4.$$

In this case, we have that

$$A(t, s) = \frac{g(\tau)}{\tau(\tau + \mu)(\tau + \nu)(\tau + \mu + \nu - \alpha)}, \quad (3.45)$$

where

$$g(\tau) = \alpha\tau^2 + 2\mu\nu\tau + \mu\nu(\mu + \nu - \alpha).$$

Since whenever $t \in [0, 1]$ we have that $1 + t + s_l \in [1, 3]$, for all $l = 1, 2, 3, 4$, and since essentially

$$\tau(\tau + \mu)(\tau + \mu)(\tau + \mu + \nu - \alpha) = \prod_{l=1}^4 (1 + t + s_l),$$

then

$$\tau(\tau + \mu)(\tau + \mu)(\tau + \mu + \nu - \alpha) \approx 1.$$

Therefore and from (3.45), we learn that

$$|A(t, s)| \approx |g(\tau)|.$$

That is

$$|A(t, s)| \approx |\alpha\tau^2 + 2\mu\nu\tau + \mu\nu(\mu + \nu - \alpha)|. \quad (3.46)$$

At this point, since $\mu > 0$, $\nu > 0$ and $\tau \approx 1$, we can continue investigating the behavior of $A(t, s)$ considering α . We shall distinguish two different cases according to the sign of the quantity α as follows:

$$(III1) \quad \alpha \geq 0$$

$$(III2) \quad \alpha < 0$$

3.4.3 (III1): $\alpha \geq 0$

In this case and since

$$\mu > 0, \quad \nu > 0, \quad \tau \in [1, 3] \quad \text{and} \quad \mu + \nu - \alpha = s_4 - s_2 > 0,$$

then

$$g(\tau) > 0, \quad \text{and consequently} \quad A(t, s) > 0.$$

Moreover, the facts that

$$\tau \approx 1 \quad \text{and} \quad 0 < \mu + \nu - \alpha \lesssim 1$$

imply that

$$A(t, s) \approx \alpha + \mu\nu.$$

We can estimate $A(t, s)$ according to the relation between $\mu\nu$ and α as follows

$$(III1i) \quad \text{If } \boxed{\alpha \gg \mu\nu} \quad \text{then} \quad \boxed{A(t, s) \approx \alpha}$$

$$(III1ii) \quad \text{If } \boxed{\mu\nu \gtrsim \alpha} \quad \text{then} \quad \boxed{A(t, s) \approx \mu\nu}$$

3.4.4 (III2): $\alpha < 0$

The estimate (3.46) yields the following three subcases of the case (III2).

$$(III2i) \quad \text{When } \boxed{|\alpha| \gg \mu\nu} \quad \text{then} \quad \boxed{|A(t, s)| \approx |\alpha|}$$

$$(III2ii) \quad \text{When } \boxed{\mu\nu \gg |\alpha|} \quad \text{then} \quad \boxed{A(t, s) \approx \mu\nu}$$

$$(III2iii) \quad \text{When } \boxed{|\alpha| \approx \mu\nu}$$

To complete the picture, it remains to estimate $A(t, s)$ when $\alpha < 0$ and $|\alpha| \approx \mu\nu$, that is the subcase (III2iii).

Recall from (3.46), that

$$|A(t, s)| \approx |g(\tau)|, \quad g(\tau) = \alpha\tau^2 + 2\mu\nu\tau + \mu\nu(\mu + \nu - \alpha).$$

Whenever $\alpha < 0$, the quadratic polynomial $g(\tau)$ may change its sign meaning that it may attain one of or both its real zeros in $[0, 1]$. Technically, by Lemma 3.4.2, g can have at most one zero in $[0, 1]$. Thus, it is plausible here to estimate $g(\tau)$ and hence $A(t, s)$ by referring to its real roots. It is easy to verify that in the region (III), g can attain only real roots.

We can write

$$g(\tau) = \alpha (\tau - \tau_-(s)) (\tau - \tau_+(s)) = \alpha (t - t_-(s)) (t - t_*(s)), \quad (3.47)$$

where

$$\begin{aligned}\tau_+(s) &= \frac{\mu\nu}{-\alpha} \left[1 + \sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} \right], \\ \tau_-(s) &= \frac{\mu\nu}{-\alpha} \left[1 - \sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} \right], \\ t_*(s) &= \tau_+(s) - 1 - s_2, \\ t_-(s) &= \tau_-(s) - 1 - s_2.\end{aligned}$$

Since $\alpha < 0$, then $\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right) > 1$ and hence we always have that $\boxed{\tau_-(s) < 0}$.

Furthermore, we have

$$\tau_-(s) = \frac{\mu\nu}{\alpha} \left[\sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} - 1 \right] = -\frac{\mu + \nu - \alpha}{\sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right) + 1}}.$$

But, because $s_2 < s_1 < s_3 < s_4$,

$$\begin{aligned}\mu + \nu - \alpha &= s_4 - s_2 > 0, \\ 1 + \sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} &\in \left(\frac{s_4 - s_2}{s_3 - s_2}, \frac{s_4 - s_2}{s_1 - s_2}\right).\end{aligned}$$

Thus implies that

$$-\tau_-(s) = \frac{\mu + \nu - \alpha}{\sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right) + 1}} \in [s_1 - s_2, s_3 - s_2] = [\mu, \nu]. \quad (3.48)$$

Thus

$$\tau - \tau_-(s) \in [1 + t + s_1, 1 + t + s_3] \subset (1, 3),$$

that is

$$\boxed{\tau - \tau_-(s) \approx 1}$$

Hence, we can rewrite (3.47) as

$$|g(\tau)| \approx -\alpha |\tau - \tau_+(s)| = -\alpha |t - t_*(s)|,$$

and consequently

$$|A(t, s)| \approx -\alpha |t - t_*(s)|.$$

Hence

(III2iii) When $\boxed{|\alpha| \approx \mu\nu}$ then $\boxed{|A(t, s)| \approx -\alpha |t - t_*(s)|}$

Figure (4) below summarizes the estimates of $A(t, s)$ in all subcases of the case (III), when $s_2 < s_1 < s_3 < s_4$, that we discussed above.

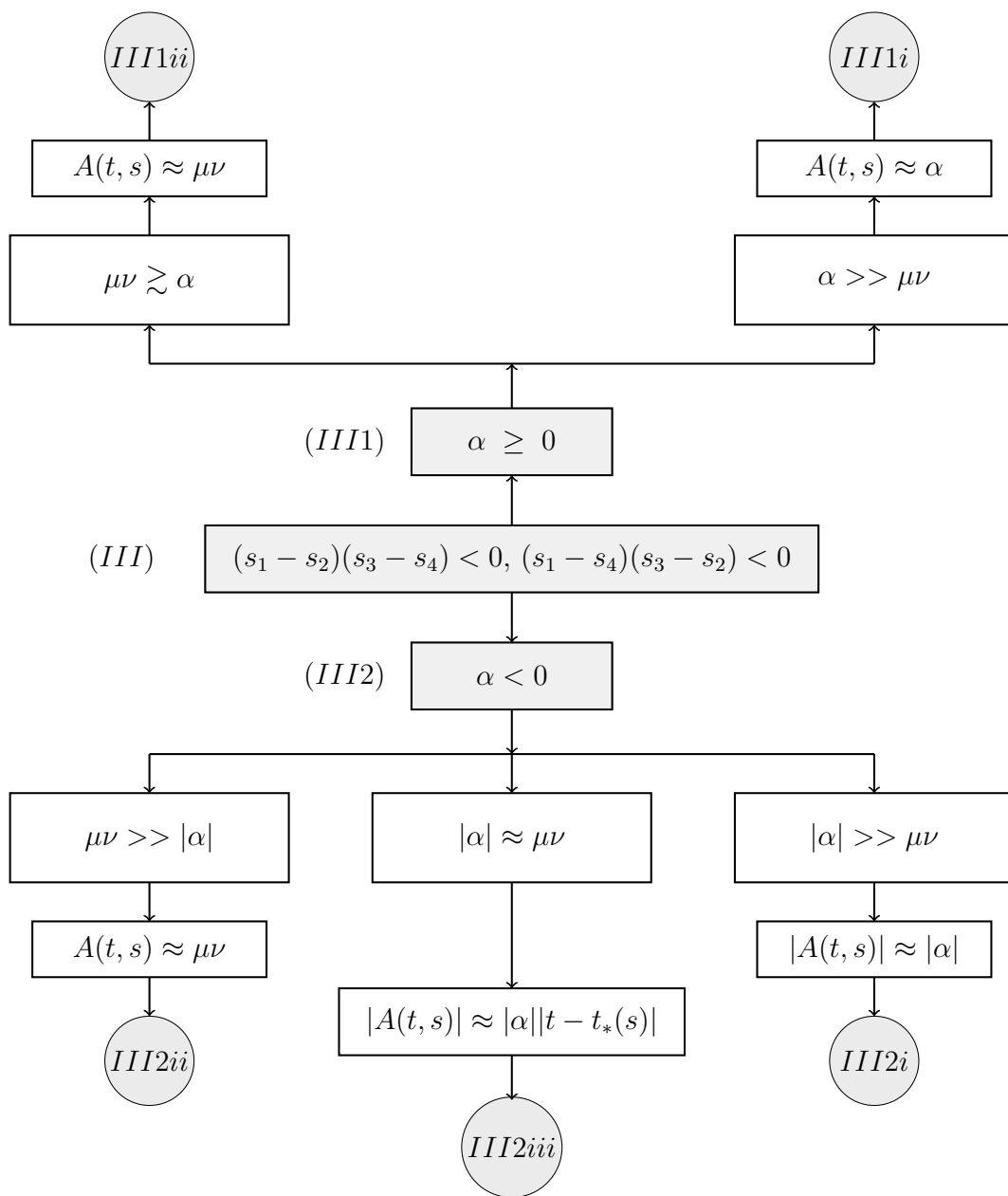


Figure (4): Estimates for $A(t, s)$ when $s_2 < s_1 < s_3 < s_4$

In particular, we have proven the following estimate

Lemma 3.4.4. *Let $s_{i_1} < s_{i_2} < s_{i_3} < s_{i_4}$ where*

$$(i_1, i_2, i_3, i_4) \in \{(1, 2, 4, 3), (1, 4, 2, 3), (3, 2, 4, 1), (3, 4, 2, 1), \\ (2, 1, 3, 4), (2, 3, 1, 4), (4, 1, 3, 2), (4, 3, 1, 2)\}.$$

And assume that $\tilde{\alpha} \approx \tilde{\mu}\tilde{\nu}$ where

$$\tilde{\alpha} = s_{i_1} - s_{i_2} + s_{i_3} - s_{i_4}, \quad \tilde{\mu} = s_{i_2} - s_{i_1}, \quad \tilde{\nu} = s_{i_3} - s_{i_4}.$$

Then

$$|A(t, s)| \approx -\tilde{\alpha}|t - \tilde{t}_*(s)|$$

where $\tilde{t}_(s)$ is the positive zero of $t \mapsto A(t, s)$.*

3.4.5 Estimates for the derivatives of A

Recall that

$$\alpha = s_1 - s_2 + s_3 - s_4, \quad \mu = s_1 - s_2, \quad \nu = s_3 - s_2.$$

We shall prove the following estimates

Lemma 3.4.5. *If*

$$s_2 < s_1 < s_3 < s_4, \quad \alpha < 0, \quad -\alpha \approx \mu\nu, \quad s_3 - s_2 \approx s_4 - s_1,$$

then the derivatives $\partial_t A(t, s)$, $\partial_{tt} A(t, s)$ and $\partial_{ttt} A(t, s)$ satisfy the following uniform estimates whenever $|t - t_(s)| \ll 1$ where $t_*(s)$ is such that $A(t_*(s), s) = 0$.*

$$\boxed{\begin{aligned} -\partial_t A(t, s) &\approx \mu\nu \\ \partial_{tt} A(t, s) &\approx \mu\nu \\ -\partial_{ttt} A(t, s) &\approx \mu\nu \end{aligned}} \tag{3.49}$$

In particular, when $t_(s) > \mu\nu$, we have*

$$\boxed{\begin{aligned} -\partial_t A(t, s) &\approx \mu\nu \quad \text{whenever } t \in (t_*(s) - \mu\nu, t_*(s) + \mu\nu), \\ -\partial_t A(1, s) &\approx \mu\nu \quad \text{whenever } |1 - t_*(s)| \ll 1, \\ -\partial_t A(0, s) &\approx \mu\nu, \end{aligned}} \tag{3.50}$$

where $t_(s)$ is such that $A(t_*(s), s) = 0$.*

Proof. Let $a(t, s)$, $b(t, s)$, $c(t, s)$ and $d(t, s)$ be the functions $\frac{1}{1+t+s_l}$, $l = 1, 2, 3, 4$, respectively. We then have

$$\begin{aligned} A(t, s) &= -a + b - c + d, & \partial_t^j A(t, s) &= (-1)^{j+1} j! (-a^{j+1} + b^{j+1} - c^{j+1} + d^{j+1}), \\ b - a &= \frac{s_1 - s_2}{(1+t+s_1)(1+t+s_2)} \approx \mu, \\ b - c &= \frac{s_3 - s_2}{(1+t+s_2)(1+t+s_3)} \approx \nu, \\ a - d &= \frac{s_4 - s_1}{(1+t+s_1)(1+t+s_4)} \approx \nu. \end{aligned}$$

Now since

$$\begin{aligned} a^2 - b^2 + c^2 - d^2 &= (a-b)(a+b) + (c-d)(c+d) \\ &= [(a-b)(a+b) - (a-b)(c+d)] + [(a-b)(c+d) + (c-d)(c+d)] \\ &= (a-b)[(a-d) + (b-c)] + (c+d)(a-b+c-d) \\ &= -(b-a)[(a-d) + (b-c)] - (c+d)(-a+b-c+d) \end{aligned}$$

then

$$-\partial_t A(t, s) = \xi(t, s) \mu \nu + \eta(t, s) A(t, s). \quad (3.51)$$

where

$$\begin{aligned} \xi(t, s) &= [(1+t+s_1)(1+t+s_4) \frac{s_4 - s_1}{s_3 - s_2} + (1+t+s_2)(1+t+s_3)] \\ &\quad a^2(t, s) b^2(t, s) c(t, s) d(t, s) \approx 1, \end{aligned} \quad (3.52)$$

$$\eta(t, s) = (2 + 2t + s_3 + s_4) c(t, s) d(t, s) \approx 1. \quad (3.53)$$

Recall from (3.45) that

$$A(t, s) = \frac{\alpha \tau^2 + 2\mu\nu\tau + \mu\nu(\mu + \nu - \alpha)}{\tau(\tau + \mu)(\tau + \nu)(\tau + \mu + \nu - \alpha)},$$

where

$$\tau = \tau(t, s) = 1 + t + s_2.$$

Since $A(t, s) = 0$ if and only if $t = t_*(s)$, then we have

$$\alpha \tau^2(t_*(s), s) + 2\mu\nu\tau(t_*(s), s) + \mu\nu[\mu + \nu - \alpha] = 0$$

and we get that $A(t, s) = 0$ if and only if

$$\alpha = -\frac{2\tau(t_*(s), s) + \mu + \nu}{\tau^2(t_*(s), s) - \mu\nu} \mu\nu.$$

Thus we can write

$$\begin{aligned} A(t, s) &= \alpha [t - t_*(s)] [t - t_-(s)] a(t, s) b(t, s) c(t, s) d(t, s) = \\ &= - \frac{2\tau(t_*(s), s) + \mu + \nu}{\tau^2(t_*(s), s) - \mu\nu} \mu\nu [t - t_*(s)] [t - t_-(s)] a(t, s) b(t, s) c(t, s) d(t, s). \end{aligned} \quad (3.54)$$

Substituting from (3.52), (3.53) and (3.54) in the identity (3.51) we obtain

$$\frac{-\partial_t A(t, s)}{a(t, s) b(t, s) c(t, s) d(t, s)} = [f(t, s) - (t - t_*(s)) e(t, s)] \mu\nu \quad (3.55)$$

where

$$\begin{aligned} e(t, s) &= (2 + 2t + s_3 + s_4) \frac{2\tau(t_*(s), s) + \mu + \nu}{\tau^2(t_*(s), s) - \mu\nu} [t - t_-(s)] c(t, s) d(t, s), \\ f(t, s) &= [2(1 + t)^2 + (s_1 + s_2 + s_3 + s_4)(1 + t) + s_1 s_4 + s_2 s_3] a(t, s) b(t, s). \end{aligned}$$

Because of the estimates (see Section 3.4.4)

$$\begin{aligned} t - t_-(s) &\approx 1, \\ \tau(t_*(s), s) + \mu + \nu &\approx 1, \\ \tau^2(t_*(s), s) - \mu\nu &\approx 1, \\ t_*(s) &< 1, \\ 1 + t + s_l &\approx 1, \\ a(t, s) \approx b(t, s) \approx c(t, s) \approx d(t, s) &\approx 1, \end{aligned}$$

we can see that

$$e(t, s) \approx 1, \quad f(t, s) \approx 1.$$

Thus whenever $|t - t_*(s)| \ll 1$, we have that

$$e(t, s) |t - t_*(s)| \ll 1$$

and consequently

$$f(t, s) - (t - t_*(s)) e(t, s) \approx 1.$$

It follows then from (3.55) that for all t such that $|t - t_*(s)| \ll 1$, we have the estimate

$$-\partial_t A(t, s) \approx \mu\nu,$$

which proves the first (5.38) and the first two estimates in (5.39) for $\partial_t A(t, s)$.

When $t = 0$, we get from (3.55) that

$$\frac{-\partial_t A(0, s)}{a(0, s) b(0, s) c(0, s) d(0, s)} = [f(0, s) + t_*(s) e(0, s)] \mu\nu \approx \mu\nu, \quad (3.56)$$

as

$$f(0, s) + t_*(s) e(0, s) \approx 1.$$

This shows the last estimate in (5.39).

Now we return to (5.38) and prove the second estimate which is a uniform estimate for $\partial_{tt}A(t, s)$ and the third one which is a uniform estimate for $\partial_{ttt}A(t, s)$ in a neighborhood small enough of $t_*(s)$. Similarly as before we can write

$$\begin{aligned} \partial_{tt}A(t, s) &= -2(a^3 - b^3 + c^3 - d^3) \\ &= -2(a - b)(a^2 + ab + b^2) - 2(c - d)(c^2 + cd + d^2) \\ &= -2(a - b)(a^2 + ab + b^2) + 2(a - b)(c^2 + cd + d^2) + \\ &\quad - 2(a - b)(c^2 + cd + d^2) - 2(c - d)(c^2 + cd + d^2) \\ &= -2(a - b)(a^2 - d^2 + b^2 - c^2 + ab - cd) - 2(c^2 + cd + d^2)(a - b + c - d) \\ &= -2(a - b)[(a - d)(a + d) + (b - c)(b + c) + a(b - c) + c(a - d)] + \\ &\quad - 2(c^2 + cd + d^2)(a - b + c - d) \\ &= 2(b - a)[(a - d)(a + c + d) + (b - c)(a + b + c)] + 2(c^2 + cd + d^2)(-a + b - c + d). \end{aligned}$$

This interprets to the following identity

$$\partial_{tt}A(t, s) = 2\mu\nu\tilde{e}(t, s) + 2\tilde{f}(t, s)A(t, s) \quad (3.57)$$

where

$$\begin{aligned} \tilde{e}(t, s) &= a^2(t, s)b(t, s)d(t, s)\frac{s_4 - s_1}{s_3 - s_2}[a(t, s) + c(t, s) + d(t, s)] + \\ &\quad + a(t, s)b^2(t, s)c(t, s)[a(t, s) + b(t, s) + c(t, s)], \\ \tilde{f}(t, s) &= c^2(t, s) + c(t, s)d(t, s) + d^2(t, s). \end{aligned}$$

Since each of the functions

$$a(t, s), b(t, s), c(t, s), d(t, s) \approx 1.$$

Then we clearly have that both functions

$$\tilde{e}(t, s), \tilde{f}(t, s) \approx 1.$$

Therefore and because of the estimate

$$|A(t, s)| \approx \mu\nu|t - t_*(s)|$$

it we infer from (3.58) that

$$\partial_{tt}A(t, s) \approx \mu\nu, \quad (3.58)$$

as long as $|t - t_*(s)| \ll 1$. Analogously, we may write

$$\begin{aligned}
\partial_{ttt}A(t, s) &= 6(a^4 - b^4 + c^4 - d^4) \\
&= 6(a - b)(a + b)(a^2 + b^2) + 6(c - d)(c + d)(c^2 + d^2) \\
&= 6(a - b)(a + b)(a^2 + b^2) - 6(a - b)(c + d)(c^2 + d^2) + \\
&\quad + 6(a - b)(c + d)(c^2 + d^2) + 6(c - d)(c + d)(c^2 + d^2) \\
&= 6(a - b)(a^3 - d^3 + b^3 - c^3 + a^2b - cd^2 + ab^2 - c^2d) + \\
&\quad + 6(c + d)(c^2 + d^2)(a - b + c - d) \\
&= 6(a - b)[(a - d)(a^2 + ad + d^2) + (b - c)(b^2 + bc + c^2) + \\
&\quad + (a^2b - bd^2) + (bd^2 - cd^2) + (ab^2 - ac^2) + (ac^2 - c^2d)] + \\
&\quad + 6(c + d)(c^2 + d^2)(a - b + c - d) \\
&= -6(b - a)(a - d)[a^2 + ad + c^2 + d^2 + b(a + d)] + \\
&\quad - 6(b - a)(b - c)[b^2 + bc + c^2 + d^2 + a(b + c)] + \\
&\quad - 6(c + d)(c^2 + d^2)(-a + b - c + d).
\end{aligned}$$

The simplified equality above yields that

$$\partial_{ttt}A(t, s) = -6\mu\nu\tilde{e}(t, s) + -6\tilde{f}(t, s)A(t, s), \quad (3.59)$$

where

$$\begin{aligned}
\tilde{e}(t, s) &= a^2(t, s)b(t, s)d(t, s)\frac{s_4 - s_1}{s_3 - s_2} \\
&\quad [a^2(t, s) + a(t, s)d(t, s) + c^2(t, s) + d^2(t, s) + b(t, s)(a(t, s) + d(t, s))] + \\
&\quad a(t, s)b^2(t, s)c(t, s) \\
&\quad [b^2(t, s) + b(t, s)c(t, s) + c^2(t, s) + d^2(t, s) + a(t, s)(b(t, s) + c(t, s))], \\
\tilde{f}(t, s) &= [c(t, s) + d(t, s)][c^2(t, s) + d^2(t, s)].
\end{aligned}$$

Again we can easily see the estimates

$$\tilde{e}(t, s) \approx 1, \quad \tilde{f}(t, s) \approx 1.$$

Plugging these estimates together with the estimate $A(t, s) \approx \mu\nu|t - t_*(s)|$ into (3.59), we deduce that

$$-\partial_{ttt}A(t, s) \approx \mu\nu$$

whenever $|t - t_*(s)| \ll 1$. □

Chapter 4

A quadrilinear estimate (I)

In this chapter, we work on answering the question raised in Chapter 3 Section 5. We prove the estimate

$$\boxed{\| u \|_{L^4([2,3] \times \mathbb{R}^4)} \lesssim (\log N)^{\frac{3}{4}} \| f \|_{L^4([0,1])}} \quad (4.1)$$

where f is a piecewise constant function given by

$$f(t) = \sum_{k=1}^N c_k \chi_{[\frac{k-1}{N}, \frac{k}{N}]}(t), \quad c_k \in \mathbb{C}. \quad (4.2)$$

Using a standard argument, we justified this approximation of an L^4 function by a piecewise constant function in Chapter 3 Section 3.3.2. Recall from Remark 3.2.1 that assuming that the constants c_k are real adds no further restrictions on the function f and the estimate because proving the estimate (5.60) for f real is equivalent to proving it for the more general case when f is complex valued.

In the light of the equation

$$u(t, x) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^1 \frac{f(s)}{(t-s)^{\frac{n}{2}}} e^{i\frac{|x|^2}{4(t-s)}} ds$$

we can foresee how this piecewise constant function makes explicit integration in the variable s achievable. This is useful for the purpose of proving the estimate (5.60) because it enables us to recover the cancellations due to the existence of the oscillatory factor $e^{i\frac{|x|^2}{4(t-s)}}$ in the integral that defines u . Using the change of variables $t-2 \rightarrow t$ and $1-s \rightarrow s$ we have that

$$u(t+2, x) = \frac{1}{(4\pi)^2} \int_0^1 \frac{e^{i\frac{|x|^2}{4(1+t+s)}}}{(1+t+s)^2} f(1-s) ds. \quad (4.3)$$

Later in Section 4.1, we show that

$$\| u(t+2, x) \|_{L^4([0,1] \times \mathbb{R}^4)}^4 = \sum_{k_1, k_2, k_3, k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right), \quad (4.4)$$

where

$$\begin{aligned} J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) &= \sum_{\gamma \in \Gamma} \sigma_\gamma \int_0^1 A_\gamma^2 \left(t, \frac{k}{N} \right) \log |A_\gamma \left(t, \frac{k}{N} \right)| dt, \\ A_\gamma \left(t, \frac{k}{N} \right) &= A \left(t, \frac{k - \gamma}{N} \right) = \sum_{l=1}^4 \frac{(-1)^l}{1 + t + \frac{k_l}{N} - \frac{\gamma_l}{N}}, \\ k &= (k_1, k_2, k_3, k_4), \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4), \quad \sigma_\gamma = (-1)^{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4}, \quad \Gamma = \{0, 1\}^4. \end{aligned}$$

We shall see in the process of deriving (4.4) the power of the cancellations due to the oscillation and interaction between the waves. These cancellations appear explicitly the moment we begin writing $\| u \|_{L^4([0,1] \times \mathbb{R}^4)}^4$ in this multilinear form. The next step then is to estimate the quadrilinear form in (4.4) and show that it is controlled by $\| f \|_{L^4([0,1])}^4 = \| c \|_{l^4}^4 / N$. Here is where we play with the interpolation theory tools (see Theorem 3.3.4 in Section 3.3.1.1 that allows us to merely consider the estimate

$$\sup_{k_{i_1} \quad k_{i_2}, k_{i_3}, k_{i_4}=1}^N |J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right)| \lesssim \frac{1}{N}. \quad (4.5)$$

instead of (5.60). The estimate (4.5), as it is, seemed difficult to prove. Thus, and because of the properties of the function A that has direct impact on the kernel $J_{\frac{1}{N}}$, we divide the region where the parameters that define the function A live in a certain way into smaller regions and prove restricted versions of the estimate (4.5). Each of these restricted estimates is nothing more than the estimate (4.5) restricted to each of those subregions. As expected, we find that the properties and hence the estimates of the function A and consequently the kernel $J_{\frac{1}{N}}$ differ from one subregion to the other. This is how we end up with a number of different simplified estimates that we need to prove so as to prove the estimate (4.5). This, we believe, is one of the reasons we get a divergence of order less than any $\epsilon > 0$. For more interesting and deeper details we refer the reader to Section 4.4. We finally prove the

following estimate

$$\sum_{k_1, k_2, k_3, k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \lesssim \frac{(\log N)^3}{N} \|f\|_{L^4([0,1])}^4.$$

4.1 An explicit quadrilinear form for $\|u(t, x)\|_{L^4([0,1] \times \mathbb{R}^n)}$ when $f = \sum_{k=1}^N c_k \chi_{[\frac{k-1}{N}, \frac{k}{N}]}$

Here we compute and simplify a quadrilinear integral form for the norm $\|u(t, x)\|_{L^4([0,1] \times \mathbb{R}^n)}$ for the solution u given by (4.3) that corresponds to the piecewise constant data

$$f(s) = \sum_{k=1}^N c_k \chi_{[\frac{k-1}{N}, \frac{k}{N}]}(s).$$

From (4.3), and taking into account that $\|u(t+2, x)\|_{L^4([0,1] \times \mathbb{R}^4)}^4 = \|u(t, x)\|_{L^4([2,3] \times \mathbb{R}^4)}^4$, we have that

$$u(t, x) = \frac{1}{(4\pi)^2} \int_0^1 \frac{e^{i\frac{|x|^2}{4(1+t+s)}}}{(1+t+s)^2} f(s) ds = \frac{1}{(4\pi)^2} \sum_{k=1}^N c_k u_{[\frac{k-1}{N}, \frac{k}{N}]}(t, x),$$

where, for each $k = 1, \dots, N$,

$$u_k(t, x) = u_{[\frac{k-1}{N}, \frac{k}{N}]}(t, x) = \int_{\frac{k-1}{N}}^{\frac{k}{N}} \frac{e^{i\frac{|x|^2}{4(1+t+s)}}}{(1+t+s)^2} ds = \frac{e^{i\frac{|x|^2}{4(1+t+\frac{k-1}{N})}} - e^{i\frac{|x|^2}{4(1+t+\frac{k}{N})}}}{i|x|^2}. \quad (4.6)$$

Now, we proceed to compute explicitly the L^4 -norm in (3.16) of $u(t, x)$. In the sequel, we will violate some unimportant multiplicative numerical constants.

$$\begin{aligned} |u(t, x)|^4 &= \bar{u}(t, x) u(t, x) \bar{u}(t, x) u(t, x) \\ &= \sum_{k_1=1}^N \sum_{k_2=1}^N \sum_{k_3=1}^N \sum_{k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} \bar{u}_{k_1}(t, x) u_{k_2}(t, x) \bar{u}_{k_3}(t, x) u_{k_4}(t, x). \end{aligned}$$

The following lemma emphasizes the cancellations due to the interaction between the waves $u(t, x)$ in (4.6).

Lemma 4.1.1. *Let*

$$u_k(t, x) = \frac{e^{i\frac{|x|^2}{4(1+t+\frac{k-1}{N})}} - e^{i\frac{|x|^2}{4(1+t+\frac{k}{N})}}}{i|x|^2}.$$

Then

$$\begin{aligned} & \bar{u}_{k_1}(t, x)u_{k_2}(t, x)\bar{u}_{k_3}(t, x)u_{k_4}(t, x) \\ &= \frac{1}{|x|^8} \prod_{l=1}^4 \left(e^{(-1)^{l_i} \frac{|x|^2}{4(1+t+\frac{k_l-1}{N})}} - e^{(-1)^{l_i} \frac{|x|^2}{4(1+t+\frac{k_l}{N})}} \right) = \frac{1}{|x|^8} \sum_{\gamma} \sigma_{\gamma} e^{iA(t, \frac{k-\gamma}{N})|x|^2}, \end{aligned}$$

where

$$\begin{aligned} \gamma &= (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \Gamma = \{0, 1\}^4, \quad k = (k_1, k_2, k_3, k_4) \in \{1, \dots, N\}^4, \\ \sigma_{\gamma} &= (-1)^{(\gamma_1+\gamma_2+\gamma_3+\gamma_4)}, \quad A(t, \frac{k}{N}) = A(t, \frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}) = \sum_{l=1}^4 \frac{(-1)^{l_i}}{1+t+k_l}. \end{aligned}$$

Now, using Lemma 4.1.1, we have

$$\begin{aligned} \|u\|_{L^4([0,1] \times \mathbb{R}^4)}^4 &= \int_0^1 \int_{\mathbb{R}^4} |u(t, x)|^4 dx dt \\ &= \sum_{k_1=1}^N \sum_{k_2=1}^N \sum_{k_3=1}^N \sum_{k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} \int_0^1 \int_{\mathbb{R}^4} \Re \left[\bar{u}_{k_1}(t, x)u_{k_2}(t, x)\bar{u}_{k_3}(t, x)u_{k_4}(t, x) \right] dx dt \\ &= \sum_{k_1=1}^N \sum_{k_2=1}^N \sum_{k_3=1}^N \sum_{k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} \int_0^1 \int_{\mathbb{R}^4} \sum_{\gamma} \sigma_{\gamma} \Re \left[\frac{e^{iA(t, \frac{k-\gamma}{N})|x|^2}}{|x|^8} \right] dx dt. \end{aligned}$$

Let us begin by computing the integral in x . We exploit the fact that the integral is radial and use polar coordinates. Notice here that we will overlook the constant giving the volume of the unit ball in \mathbb{R}^4 . For all γ and k let

$$A_{\gamma}(t, \frac{k}{N}) = A(t, \frac{k-\gamma}{N}).$$

Using polar coordinates we get

$$\begin{aligned} \int_{\mathbb{R}^4} \sum_{\gamma} \sigma_{\gamma} \Re \left[\frac{e^{iA(t, \frac{k-\gamma}{N})|x|^2}}{|x|^8} \right] dx &= \int_0^{\infty} \sum_{\gamma} \sigma_{\gamma} \Re \left[\frac{e^{iA(t, \frac{k-\gamma}{N})\rho^2}}{\rho^5} \right] d\rho \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^{\infty} \sum_{\gamma} \sigma_{\gamma} \frac{\cos [|A_{\gamma}(t, \frac{k}{N})|s]}{s^3} ds = \lim_{\epsilon \rightarrow 0^+} \sum_{\gamma} \sigma_{\gamma} q_{\gamma}(t, k, \epsilon), \end{aligned} \tag{4.7}$$

where

$$q_{\gamma}(t, k, \epsilon) = \int_{\epsilon}^{\infty} \frac{\cos [|A_{\gamma}(t, \frac{k}{N})|s]}{s^3} ds.$$

Consider the integral

$$I_\epsilon(\alpha) = \int_\epsilon^\infty \frac{\cos(\alpha s)}{s^3} ds, \quad \alpha \geq 0.$$

If $\alpha = 0$, then

$$I_\epsilon(\alpha) = \int_\epsilon^\infty \frac{1}{s^3} ds = \frac{1}{2\epsilon^2}.$$

If $\alpha > 0$, then

$$I_\epsilon(\alpha) = \alpha^2 \int_{\epsilon\alpha}^\infty \frac{\cos s}{s^3} ds = \alpha^2 \int_1^\infty \frac{\cos s}{s^3} ds + \alpha^2 \int_{\epsilon\alpha}^1 \frac{\cos s}{s^3} ds,$$

and using the Taylor's expansion of the cosine function, we get

$$\begin{aligned} \int_{\epsilon\alpha}^1 \frac{\cos s}{s^3} ds &= \int_{\epsilon\alpha}^1 \frac{1 - \frac{1}{2}s^2 + s^4\psi(s)}{s^3} ds = \frac{1}{2s^2} \Big|_1^{\epsilon\alpha} + \frac{1}{2} \log s \Big|_1^{\epsilon\alpha} + \int_{\epsilon\alpha}^1 s\psi(s) ds \\ &= \frac{1}{2\epsilon^2\alpha^2} - \frac{1}{2} + \frac{1}{2} \log \alpha + \frac{1}{2} \log \epsilon + \int_0^1 s\psi(s) ds - \int_0^{\epsilon\alpha} s\psi(s) ds, \end{aligned}$$

for some $\psi \in C([0, 1])$. Thus

$$\begin{aligned} I_\epsilon(\alpha) &= \alpha^2 \left(\int_1^\infty \frac{\cos s}{s^3} ds - \frac{1}{2} + \frac{1}{2} \log \epsilon + \int_0^1 s\psi(s) ds \right) + \frac{1}{2\epsilon^2} + \frac{1}{2} \alpha^2 \log \alpha + \\ &\quad - \alpha^2 \int_0^{\epsilon\alpha} s\psi(s) ds \\ &= \alpha^2 \left(c + \frac{1}{2} \log \epsilon \right) + \frac{1}{2\epsilon^2} + \frac{1}{2} \alpha^2 \log \alpha - \alpha^2 \int_0^{\epsilon\alpha} s\psi(s) ds, \end{aligned}$$

where

$$c = \int_1^\infty \frac{\cos s}{s^3} ds - \frac{1}{2} + \int_0^1 s\psi(s) ds$$

is a real constant. Therefore, we have

$$q_\gamma(t, k, \epsilon) = A_\gamma^2(t, k) \left(c + \frac{1}{2} \log \epsilon \right) + \frac{1}{2\epsilon^2} + \frac{1}{2} A_\gamma^2(t, k) \log |A_\gamma(t, k)| - A_\gamma^2(t, k) \int_0^{\epsilon |A_\gamma(t, k)|} s\psi(s) ds.$$

Now, substituting in (4.7), we obtain that

$$\begin{aligned}
& \int_{\mathbb{R}^4} \Re \left[\frac{e^{iA(t, \frac{k-\gamma}{N})|x|^2}}{|x|^8} \right] dx = \lim_{\epsilon \rightarrow 0^+} \sum_{\gamma} \sigma_{\gamma} q_{\gamma}(t, k, \epsilon) \\
&= \lim_{\epsilon \rightarrow 0^+} \sum_{\gamma} \sigma_{\gamma} \left[A_{\gamma}^2(t, \frac{k}{N}) \left(c + \frac{1}{2} \log \epsilon \right) + \frac{1}{2\epsilon^2} + \frac{1}{2} A_{\gamma}^2(t, \frac{k}{N}) \log |A_{\gamma}(t, \frac{k}{N})| + \right. \\
&\quad \left. - A_{\gamma}^2(t, \frac{k}{N}) \int_0^{\epsilon |A_{\gamma}(t, \frac{k}{N})|} s \psi(s) ds \right] \\
&= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\epsilon^2} \sum_{\gamma} \sigma_{\gamma} + \lim_{\epsilon \rightarrow 0^+} \left(c + \frac{1}{2} \log \epsilon \right) \sum_{\gamma} \sigma_{\gamma} A_{\gamma}^2(t, \frac{k}{N}) + \\
&\quad + \sum_{\gamma} \sigma_{\gamma} \frac{1}{2} A_{\gamma}^2(t, \frac{k}{N}) \log |A_{\gamma}(t, \frac{k}{N})| - \sum_{\gamma} \sigma_{\gamma} A_{\gamma}^2(t, \frac{k}{N}) \lim_{\epsilon \rightarrow 0^+} \int_0^{\epsilon |A_{\gamma}(t, \frac{k}{N})|} s \psi(s) ds.
\end{aligned}$$

Lemma 4.1.2 (Cancelations (1)). *The following identities (cancelations) hold for any t*

$$\sum_{\gamma} \sigma_{\gamma} (A_{\gamma}(t, \frac{k}{N}))^m = 0, \quad m = 0, 1, 2, 3.$$

Proof. Recall that

$$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \Gamma = \{0, 1\}^4, \quad \sigma_{\gamma} = (-1)^{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4}$$

and notice that

$$\sum_{\gamma \in \Gamma} \sigma_{\gamma} \phi(\gamma_1, \gamma_2, \gamma_3) = \sum_{(\gamma_1, \gamma_2, \gamma_3) \in \{0, 1\}^3} (-1)^{\gamma_1 + \gamma_2 + \gamma_3} \phi(\gamma_1, \gamma_2, \gamma_3) \sum_{\gamma_4 \in \{0, 1\}} (-1)^{\gamma_4} = 0$$

for any function, ϕ that does not depend on γ_4 . Thus, by symmetry, we have

$$\sum_{\gamma \in \Gamma} \sigma_{\gamma} \phi(\gamma_i, \gamma_j, \gamma_k) = 0,$$

for all $(i, k, j) \in \{1, 2, 3, 4\}$. Also the function

$$A_{\gamma}(t, \frac{k}{N}) = -\frac{1}{1+t+\frac{k_1}{N}-\frac{\gamma_1}{N}} + \frac{1}{1+t+\frac{k_2}{N}-\frac{\gamma_2}{N}} - \frac{1}{1+t+\frac{k_3}{N}-\frac{\gamma_3}{N}} + \frac{1}{1+t+\frac{k_4}{N}-\frac{\gamma_4}{N}}$$

has the property that its powers up to the third power contains at most three of the indices γ_i . \square

Now, the cancelations $\sum_{\gamma} \sigma_{\gamma} = \sum_{\gamma} \sigma_{\gamma} A_{\gamma}^2(t, \frac{k}{N}) = 0$, given in Lemma 4.1.2, imply the following

$$\lim_{\epsilon \rightarrow 0^+} \left(c + \frac{1}{2} \log \epsilon \right) \sum_{\gamma \in \Gamma} \sigma_{\gamma} A_{\gamma}^2(t, \frac{k}{N}) = 0, \quad \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\epsilon^2} \sum_{\gamma \in \Gamma} \sigma_{\gamma} = 0.$$

and clearly

$$\sum_{\gamma} \sigma_{\gamma} A_{\gamma}^2(t, \frac{k}{N}) \lim_{\epsilon \rightarrow 0^+} \int_0^{\epsilon |A_{\gamma}(t, \frac{k}{N})|} s \psi(s) ds = 0.$$

Therefore

$$\int_{\mathbb{R}^4} \sum_{\gamma} \sigma_{\gamma} \Re \left[\frac{e^{iA(t, \frac{k-\gamma}{N})|x|^2}}{|x|^8} \right] dx = \frac{1}{2} \sum_{\gamma \in \Gamma} \sigma_{\gamma} A_{\gamma}^2(t, \frac{k}{N}) \log |A_{\gamma}(t, \frac{k}{N})|. \quad (4.8)$$

The norm now becomes the quadrilinear form given by

$$\| u \|_{L^4([0,1] \times \mathbb{R}^4)}^4 = \sum_{k_1, k_2, k_3, k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}}(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}) \quad (4.9)$$

with

$$J_{\frac{1}{N}}(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}) = \sum_{\gamma} \sigma_{\gamma} \int_0^1 A_{\gamma}^2(t, \frac{k}{N}) \log |A_{\gamma}(t, \frac{k}{N})| dt.$$

In the context of deriving the formula (4.9), we obtained Lemma 4.1.1 and the identity (4.8).

Doing that, we have actually proved the following Lemma

Lemma 4.1.3.

$$J_{\epsilon}(s_1, s_2, s_3, s_4) = \int_0^1 G_{\epsilon}(t, s) dt,$$

where

$$\begin{aligned} G_{\epsilon}(t, s) &= \int_0^1 \int_{\mathbb{R}^4} \bar{u}_{[s_1-\epsilon, s_1]}(t, x) u_{[s_2-\epsilon, s_2]}(t, x) \bar{u}_{[s_3-\epsilon, s_3]}(t, x) u_{[s_4-\epsilon, s_4]}(t, x) dx dt \\ &= \sum_{\gamma \in \Gamma} \sigma_{\gamma} f(A(t, s - \epsilon\gamma)), \quad f(x) = x^2 \log |x|, \quad x \in \mathbb{R}, \end{aligned}$$

where we use the notations

$$\begin{aligned} N &\gg 1, \quad k_l = 1, \dots, N, \quad s_l = \frac{k_l}{N}, \quad l = 1, 2, 3, 4, \\ \epsilon &= \frac{1}{N}, \quad \gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \Gamma = \{0, 1\}^4, \quad \sigma_{\gamma} = (-1)^{(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)}, \\ s &= (s_1, s_2, s_3, s_4) \in [0, 1]^4. \end{aligned}$$

4.2 Reformulation of the problem

Using the theory of interpolation between bounded linear operators we will reformulate the estimate (3.16) that we need to prove and replace it by a further simpler estimate (see the estimate (4.11) below). Let us take a look at the terms of the inequality (3.16). In Lemma 4.2.1 below, we compute $\|f\|_{L^4([0,1])}$.

Lemma 4.2.1. *Let f be a piecewise constant function on $[0, 1]$ given by $f(x) = \sum_{k=1}^N c_k \chi_{[\frac{k-1}{N}, \frac{k}{N}]}(x)$. Then*

$$\|f\|_{L^p([0,1])} = \frac{1}{N^{\frac{1}{p}}} \left(\sum_{k=1}^N |c_k|^p \right)^{\frac{1}{p}} = \frac{\|c\|_{l^p}}{N^{\frac{1}{p}}}, \quad c = (c_1, \dots, c_N), \quad 1 \leq p < \infty.$$

Proof.

$$\begin{aligned} \|f\|_{L^p([0,1])} &= \int_0^1 |f(s)|^p ds = \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} |f(s)|^p ds = \sum_{k=1}^N \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left| \sum_{m=1}^N c_m \chi_{[\frac{k-1}{N}, \frac{k}{N}]}(s) \right|^p ds = \\ &= \left[\sum_{k=1}^N |c_k|^p \int_{\frac{k-1}{N}}^{\frac{k}{N}} ds \right]^{\frac{1}{p}} = \frac{1}{N^{\frac{1}{p}}} \left(\sum_{k=1}^N |c_k|^p \right)^{\frac{1}{p}} = \frac{1}{N^{\frac{1}{p}}} \|c\|_{l^p}. \end{aligned}$$

□

Having computed an explicit formula for $\|u\|_{L^4([0,1] \times \mathbb{R}^4)}$ in (4.9) and applying Lemma 4.2.1 to the data, we can reformulate the estimate (3.16) that we are considering here to the following estimate

$$\boxed{\sum_{k_1, k_2, k_3, k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \lesssim \frac{1}{N} \|c\|_{l^4}^4}. \quad (4.10)$$

To prove (4.10), it is enough, by Theorem 3.3.4, to prove the estimate (4.11) below.

$$\sup_{k_{i_1}, k_{i_2}, k_{i_3}, k_{i_4}=1}^N \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \lesssim \frac{1}{N}. \quad (4.11)$$

4.3 Some preliminary estimates

It is clear now from the updated question with which we concluded Section 4.1 and the estimates discussed in Section 4.2 that all we have to do to prove the estimate (5.60) is to

show the estimate

$$\boxed{\sup_{k_{i_1}, k_{i_2}, k_{i_3}, k_{i_4}=1} \sum_{k_1, k_2, k_3, k_4=1}^N \left| J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \right|} \lesssim \frac{1}{N}. \quad (4.12)$$

We therefore have to estimate the kernel $J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right)$.

Recall from Lemma 4.1.3 that

$$J_\epsilon(s) = \sum_{\gamma \in \Gamma} \sigma_\gamma \int_0^1 f(A(t, s - \epsilon\gamma)) dt, \quad f(x) = x^2 \log|x|, \quad x \in \mathbb{R}.$$

In this section we find a global estimate for $J_\epsilon(s)$. Unfortunately, this estimate is not enough to fulfill the estimate (4.12). Notice that the pointwise behavior of the kernel $J_\epsilon(s_1, s_2, s_3, s_4)$ on the unit hypercube $[0, 1]^4$ depends on the corresponding pointwise behavior of the function $A(\cdot, s_1, s_2, s_3, s_4)$ there. Motivated by these facts, we divide the hypercube $[0, 1]^4$ into smaller regions. Then, we estimate $J_\epsilon(s)$ and prove the estimate (4.12) on each of these subregions. see Section 4.4 below for the details. Before doing all this we need to prepare some preliminary estimates.

4.3.1 A trivial estimate for $J_\epsilon(s_1, s_2, s_3, s_4)$

We have seen that the solution is a superposition of waves of the form

$$u_{[s-\epsilon, s]} = \int_{s-\epsilon}^s \frac{e^{i\frac{|x|^2}{1+t+s}}}{(1+t+s)^2} ds = \frac{e^{i\frac{|x|^2}{4(1+t+s-\epsilon)}} - e^{i\frac{|x|^2}{4(1+t+s)}}}{i|x|^2}.$$

One trivial estimate for $u_{[s-\epsilon, s]}$ is

$$|u_{[s-\epsilon, s]}| \leq \frac{2}{|x|^2}. \quad (4.13)$$

It can on the other hand be estimated by

$$|u_{[s-\epsilon, s]}| = \frac{|e^{i\frac{|x|^2}{4(1+t+s-\epsilon)}} - e^{i\frac{|x|^2}{4(1+t+s)}}|}{|x|^2} = \frac{|e^{\frac{i}{4(1+t+s-\epsilon)(1+t+s)} \epsilon|x|^2} - 1|}{|x|^2} = O(\epsilon). \quad (4.14)$$

Estimates (4.13) and (4.14) together imply that

$$|u_{[s-\epsilon, s]}| \lesssim \min \left\{ \frac{1}{|x|^2}, \epsilon \right\}.$$

Hence, we have the following trivial estimate

$$\int_{\mathbb{R}^4} |u_{[s-\epsilon, s]}(t, x)|^4 dx \lesssim \int_{\mathbb{R}^4} (\min \{ \frac{1}{|x|^2}, \epsilon \})^4 dx \approx \int_{|x| < \frac{1}{\sqrt{\epsilon}}} \epsilon^4 dx + \int_{|x| > \frac{1}{\sqrt{\epsilon}}} \frac{1}{|x|^8} dx \approx \epsilon^2. \quad (4.15)$$

By the generalized Hölder's inequality, we get

$$\begin{aligned} |J_\epsilon(s_1, s_2, s_3, s_4)| &\leq \int_0^1 \int_{\mathbb{R}^4} \bar{u}_{[s_1-\epsilon, s_1]}(t, x) u_{[s_2-\epsilon, s_2]}(t, x) \bar{u}_{[s_3-\epsilon, s_3]}(t, x) u_{[s_4-\epsilon, s_4]}(t, x) dx dt \\ &\leq \|u_{[s_1-\epsilon, s_1]}\|_{L^4} \|u_{[s_2-\epsilon, s_2]}\|_{L^4} \|u_{[s_3-\epsilon, s_3]}\|_{L^4} \|u_{[s_4-\epsilon, s_4]}\|_{L^4} \lesssim \epsilon^2. \end{aligned} \quad (4.16)$$

4.3.2 A control for $G_\epsilon(t, s)$ when $|A(t, s)| \gg \epsilon$

We shall show that whenever $|A(t, s)| \gg \epsilon$, $f(A(t, s - \epsilon\gamma))$ is smooth and $|G_\epsilon(t, s)| \lesssim \frac{\epsilon^4}{A^2(t, s)}$. We start with noticing some cancelations properties. We may write

$$A(t, s - \epsilon\gamma) = \sum_{l=1}^4 \frac{(-1)^l}{1+t+s_l - \epsilon\gamma_l} = \sum_{l=1}^4 \frac{(-1)^l}{1+t+s_l} + \sum_{l=1}^4 \frac{(-1)^l \epsilon\gamma_l}{(1+t+s_l)(1+t+s_l - \epsilon\gamma_l)}$$

Hence

$$A(t, s - \epsilon\gamma) = A(t, s) + \epsilon B_\gamma(t, s, \epsilon), \quad (4.17)$$

where

$$B_\gamma(t, s, \epsilon) = \sum_{l=1}^4 \frac{(-1)^l \gamma_l}{(1+t+s_l)(1+t+s_l - \epsilon\gamma_l)}.$$

Lemma 4.3.1 (Cancelations (2)).

$$\sum_{\gamma \in \Gamma} \sigma_\gamma (B_\gamma(t, s, \epsilon))^m = 0, \quad m = 1, 2, 3.$$

Proof. The proof is a direct consequence of Lemma 4.1.2 and (4.17). \square

Lemma 4.3.2.

$$|G_\epsilon(t, s)| \lesssim \frac{\epsilon^4}{A^2(t, s)} \quad (4.18)$$

whenever $|A(t, s)| \gg \epsilon$ uniformly in t .

Proof. Recall that for all $\gamma \in \Gamma$, we have

$$A(t, s - \epsilon\gamma) = A(t, s) + \epsilon B_\gamma(t, s, \epsilon).$$

And we have

$$|B_\gamma(t, s, \epsilon)| = \left| \sum_{l=1}^4 \frac{(-1)^l \gamma_l}{(1+t+s_l)(1+t+s_l-\epsilon\gamma_l)} \right| \leq \sum_{l=1}^4 \frac{1}{(1+t+s_l)(1+t+s_l-\epsilon\gamma_l)} \lesssim 1$$

Consequently,

$$\epsilon B_\gamma(t, s, \epsilon) = O(\epsilon).$$

Thus, whenever $|A(t, s)| \gg \epsilon$, one gets that

$$|A(t, s)| \gg \epsilon |B_\gamma(t, s, \epsilon)|.$$

This shows that $f(A(t, s - \epsilon\gamma))$ is a smooth function for small enough ϵ and hence we can Taylor expand it about $A(t, s)$ as follows

$$\begin{aligned} f(A(t, s - \epsilon\gamma)) &= f(A(t, s) + \epsilon B_\gamma(t, s, \epsilon)) \\ &= f(A(t, s)) + f'(A(t, s))\epsilon B_\gamma(t, s, \epsilon) + \frac{1}{2}f''(A(t, s))\epsilon^2 B_\gamma^2(t, s, \epsilon) + \\ &\quad + \frac{1}{6}f'''(A(t, s))\epsilon^3 B_\gamma^3(t, s, \epsilon) + \frac{1}{24}f^{(4)}(A(t, s) + \tilde{\epsilon} B_\gamma(t, s, \tilde{\epsilon}))\epsilon^4 B_\gamma^4(t, s, \epsilon), \end{aligned}$$

for some $0 < \tilde{\epsilon} < \epsilon$. Now, notice that $f^4(a) = \frac{-2}{a^2}$. We have

$$\begin{aligned} G_\epsilon(t, s) &= \sum_{\gamma \in \Gamma} \sigma_\gamma f(A(t, s - \epsilon\gamma)) \\ &= \sum_{\gamma \in \Gamma} \sigma_\gamma f(A(t, s)) + f'(A(t, s))\epsilon \sum_{\gamma \in \Gamma} \sigma_\gamma B_\gamma(t, s, \epsilon) + \frac{1}{2}f''(A(t, s))\epsilon^2 \sum_{\gamma \in \Gamma} \sigma_\gamma B_\gamma^2(t, s, \epsilon) + \\ &\quad + \frac{1}{6}f'''(A(t, s))\epsilon^3 \sum_{\gamma \in \Gamma} \sigma_\gamma B_\gamma^3(t, s, \epsilon) + \frac{\epsilon^4}{24} \sum_{\gamma \in \Gamma} \sigma_\gamma f^{(4)}(A(t, s) + \tilde{\epsilon} B_\gamma(t, s, \tilde{\epsilon})) B_\gamma^4(t, s, \epsilon) \\ &= -\frac{\epsilon^4}{12} \sum_{\gamma \in \Gamma} \sigma_\gamma \frac{B_\gamma^4(t, s, \epsilon)}{[A(t, s) + \tilde{\epsilon} B_\gamma(t, s, \tilde{\epsilon})]^2}, \end{aligned}$$

by the cancelations given in Lemma 4.3.1. Since $|A(t, s)| \gg \epsilon$ and $\tilde{\epsilon} B_\gamma(t, s, \tilde{\epsilon}) = O(\epsilon)$ then

$$|A(t, s) + \tilde{\epsilon} B_\gamma(t, s, \tilde{\epsilon})| \approx |A(t, s)|$$

whence we have the following estimate

$$|G_\epsilon(t, s)| \lesssim \frac{\epsilon^4}{A^2(t, s)}.$$

□

4.4 Estimates for the kernel $J_\epsilon(s)$ and proofs of the decomposed simplified estimates

We have seen that it is difficult to verify the estimate (4.12) in one step. Thus we divide the unit hypercube $[0, 1]^4$ into smaller subregions analogous to those subregions on which we studied and estimated the function $A(t, s)$ in Chapter 3 Section 3.4. We shall treat each of these subregions uniquely and estimate $J_\epsilon(s_1, s_2, s_3, s_4)$ there. Except for the case (III2iii) on which the function $t \mapsto A(t, s)$ may change sign on $[0, 1]$ (see Section 4.4.6 below), we shall see that in all of these cases (subregions), we have a uniform estimate for $A(t, s)$ that is independent of t . This makes estimating $J_\epsilon(s)$ and hence verifying (4.12) relatively easy because there is no need to integrate $G_\epsilon(t, s)$ w.r.t the time t to estimate $J_\epsilon(s)$. As a matter of fact, we shall look at each of the aforementioned regions when

$$|A(t, s)| \lesssim \epsilon$$

in which case it suffices to use the global estimate (4.16) for $J_\epsilon(s)$ and when

$$|A(t, s)| \gg \epsilon$$

in which case we shall use the estimate (4.18) that we obtained in Section 4.3.2 and get the estimate

$$|J_\epsilon(s)| \lesssim |G_\epsilon(\cdot, s)| \lesssim \frac{\epsilon^4}{A^2(\cdot, s)}$$

as in these cases $A(t, s)$ and consequently $G_\epsilon(t, s)$ have uniform estimates independent of t . The following figure summarizes the proof steps in all cases except the critical case.

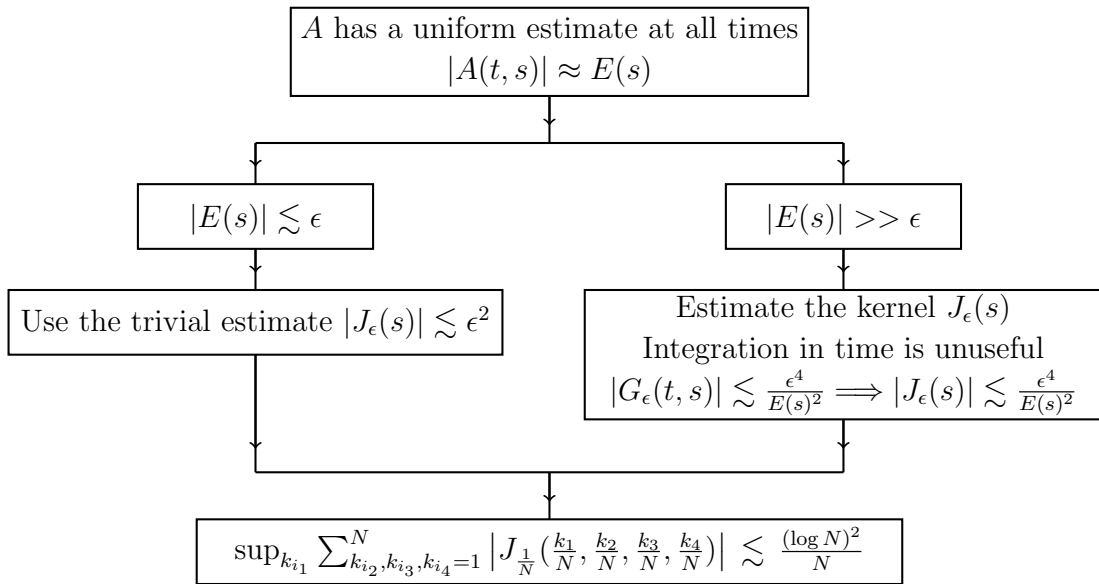


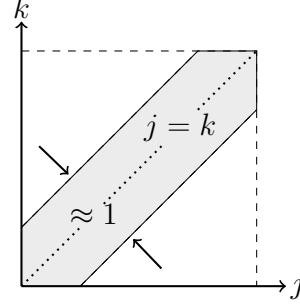
Figure (5): The process of proving the estimate (??) when $A(t, s)$ has a uniform estimate (for all s except for the subcase *III3iii* and the cases symmetric to it)

4.4.1 The trivial estimate for $J_\epsilon(s_1, s_2, s_3, s_4)$ is enough when

$$|s_1 - s_2| \lesssim \epsilon, |s_3 - s_4| \lesssim \epsilon \text{ or } |s_3 - s_2| \lesssim \epsilon, |s_1 - s_4| \lesssim \epsilon$$

Lemma 4.4.1.

$$\sum_{j,k|j-k|\lesssim 1} c_j c_k \lesssim \sum_j c_j^2 \approx \|c\|_{l^2}^2$$



The sum is taken over the shadowed region where $|j - k| \lesssim 1$

When

$$|s_1 - s_2| \lesssim \epsilon \quad \text{and} \quad |s_3 - s_4| \lesssim \epsilon,$$

we have that

$$|k_1 - k_2| \lesssim 1 \quad \text{and} \quad |k_3 - k_4| \lesssim 1.$$

In this case we use the estimate (4.16), that is

$$|J_{\frac{1}{N}}(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N})| \lesssim \frac{1}{N^2}.$$

Using this estimate and applying Lemma 4.4.1 followed by Cauchy Schwartz inequality, we deduce that

$$\begin{aligned} & \sum_{\substack{k_1, k_2, k_3, k_4 \\ |k_1 - k_2| \lesssim 1, \\ |k_3 - k_4| \lesssim 1}} c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}}(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}) \lesssim \frac{1}{N^2} \sum_{\substack{k_1, k_2, k_3, k_4 \\ |k_1 - k_2| \lesssim 1, \\ |k_3 - k_4| \lesssim 1}} |c_{k_1}| |c_{k_2}| |c_{k_3}| |c_{k_4}| \\ & \lesssim \frac{1}{N^2} \sum_{\substack{k_1, k_2 \\ |k_1 - k_2| \lesssim 1}} |c_{k_1}| |c_{k_2}| \sum_{\substack{k_3, k_4 \\ |k_3 - k_4| \lesssim 1}} |c_{k_3}| |c_{k_4}| \lesssim \frac{1}{N^2} \left(\sum_{k=1}^N c_k^2 \right)^2 \\ & \lesssim \frac{1}{N^2} \left[\left(\sum_{k=1}^N c_k^4 \right)^{\frac{1}{2}} \sqrt{N} \right]^2 \lesssim \frac{1}{N} \|c\|_{l^4}^4. \end{aligned} \tag{4.19}$$

The case when

$$|k_3 - k_2| \lesssim 1 \quad \text{and} \quad |k_1 - k_4| \lesssim 1$$

can also be done in the same exact manner and we obtain

$$\begin{aligned} & \sum_{\substack{k_1, k_2, k_3, k_4 \\ |k_1 - k_4| \lesssim 1, \\ |k_3 - k_2| \lesssim 1}} c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \\ & \lesssim \frac{1}{N^2} \sum_{\substack{k_1, k_4 \\ |k_1 - k_4| \lesssim 1}} |c_{k_1}| |c_{k_4}| \sum_{\substack{k_2, k_3 \\ |k_2 - k_3| \lesssim 1}} |c_{k_2}| |c_{k_3}| \lesssim \frac{1}{N} \|c\|_{l^4}^4. \end{aligned} \quad (4.20)$$

Now we move on to the cases (regions for $s \in [0, 1]^4$) where the trivial estimate is not enough and we need to do some more work.

4.4.2 Case (I)

$$(I) \quad \boxed{(s_1 - s_2)(s_3 - s_4) > 0}$$

This case subcategorizes into the following two subcases

$$(I1) \quad (s_1 - s_2) > 0 \quad \text{and} \quad (s_3 - s_4) > 0$$

$$(I2) \quad (s_1 - s_2) < 0 \quad \text{and} \quad (s_3 - s_4) < 0$$

The subcase (I1) can be subdivided into three subsubcases as follows

$$(I1i) \quad s_1 - s_2 \gg s_3 - s_4 > 0, \quad s_1 - s_2 \gg \epsilon$$

$$(I1ii) \quad s_3 - s_4 \gg s_1 - s_2 > 0, \quad s_3 - s_4 \gg \epsilon$$

$$(I1iii) \quad s_1 - s_2 \approx s_3 - s_4 \gg \epsilon$$

Similarly, the subcase (I2) can be subdivided into three subsubcases as follows

$$(I2i) \quad s_2 - s_1 \gg s_4 - s_3 > 0, \quad s_2 - s_1 \gg \epsilon$$

$$(I2ii) \quad s_4 - s_3 \gg s_2 - s_1 > 0, \quad s_4 - s_3 \gg \epsilon$$

$$(I2iii) \quad s_2 - s_1 \approx s_4 - s_3 \gg \epsilon$$

Now we do the estimate for each of the cases above.

$$(I1i) \quad s_1 - s_2 \gg s_3 - s_4 > 0, \quad s_1 - s_2 \gg \epsilon.$$

By (3.39), and since $1 + t + s_l \approx 1$, for all $l = 1, 2, 3, 4$, we have that

$$A(t, s) \approx s_1 - s_2.$$

Using this and (4.18), we get

$$G_\epsilon(t, s) \lesssim \frac{\epsilon^4}{(s_1 - s_2)^2}, \quad \text{uniformly in } t.$$

It follows from here that

$$J_\epsilon(s) \lesssim \frac{\epsilon^4}{(s_1 - s_2)^2}.$$

That is

$$J_{\frac{1}{N}}\left(\frac{k}{N}\right) \lesssim \frac{1}{N^2} \frac{1}{(k_1 - k_2)^2}.$$

Now we prove Lemma 4.4.2 below. The estimate provided by this lemma implies the estimate (4.12) in this subregion with a loss of $\log N$ by the interpolation Theorem 3.3.4.

Lemma 4.4.2.

$$\sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ k_1 - k_2 \gg k_3 - k_4 \\ k_1 - k_2 \gtrsim 1}} \left| J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \right| \lesssim \frac{\log N}{N}.$$

Proof. The estimates (4.21), (4.22), (4.23) and (4.24) proven below together give the proof of Lemma 4.4.2.

$$\begin{aligned} & \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ k_1 - k_2 \gg k_3 - k_4 \\ k_1 - k_2 \gtrsim 1}} \left| J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \right| \\ & \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{k_2, k_3} \frac{1}{|k_1 - k_2|^2} \sum_{\substack{k_4 \\ k_1 - k_2 \gg k_3 - k_4}} 1 \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{k_2} \frac{1}{|k_1 - k_2|} \sum_{k_3} 1 \\ & \lesssim \frac{1}{N} \sup_{k_1} \sum_{k_2} \frac{1}{|k_1 - k_2|} \lesssim \frac{\log N}{N}. \end{aligned} \tag{4.21}$$

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_2} \sum_{\substack{k_1, k_3, k_4, \\ k_1 - k_2 \gg k_3 - k_4}} \frac{1}{|k_1 - k_2|^2} \\
\lesssim & \frac{1}{N^2} \sup_{k_2} \sum_{k_1, k_3} \frac{1}{|k_1 - k_2|^2} \sum_{\substack{k_4 \\ k_1 - k_2 \gg k_3 - k_4}} 1 \\
\lesssim & \frac{1}{N^2} \sup_{k_2} \sum_{k_1} \frac{1}{|k_1 - k_2|} \sum_{k_3} 1 \\
\lesssim & \frac{1}{N} \sup_{k_2} \sum_{k_1} \frac{1}{|k_1 - k_2|} \lesssim \frac{\log N}{N}. \tag{4.22}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_3} \sum_{\substack{k_1, k_2, k_4, \\ k_1 - k_2 \gg k_3 - k_4}} \frac{1}{|k_1 - k_2|^2} \\
\lesssim & \frac{1}{N^2} \sup_{k_3} \sum_{k_1, k_2} \frac{1}{|k_1 - k_2|^2} \sum_{\substack{k_4 \\ k_1 - k_2 \gg k_3 - k_4}} 1 \\
\lesssim & \frac{1}{N^2} \sup_{k_3} \sum_{k_1} \sum_{k_2} \frac{1}{|k_1 - k_2|} \\
\lesssim & \frac{\log N}{N^2} \sup_{k_3} \sum_{k_1} 1 \lesssim \frac{\log N}{N}. \tag{4.23}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3, \\ k_1 - k_2 \gg k_3 - k_4}} \frac{1}{|k_1 - k_2|^2} \\
\lesssim & \frac{1}{N^2} \sup_{k_4} \sum_{k_1, k_2} \frac{1}{|k_1 - k_2|^2} \sum_{\substack{k_3 \\ k_1 - k_2 \gg k_3 - k_4}} 1 \\
\lesssim & \frac{1}{N^2} \sup_{k_4} \sum_{k_1} \sum_{k_2} \frac{1}{|k_1 - k_2|} \\
\lesssim & \frac{\log N}{N^2} \sup_{k_4} \sum_{k_1} 1 \lesssim \frac{\log N}{N}. \tag{4.24}
\end{aligned}$$

□

Now we go on and look at the next subcase (subregion) and prove the estimate (4.12) there.

$$(I1ii) \quad s_3 - s_4 \gg s_1 - s_2 > 0, \quad s_3 - s_4 \gg \epsilon.$$

This case is symmetrically similar to the case above, (I1i). Indeed, looking at (3.39), we find that

$$A(t, s) \approx s_3 - s_4.$$

And using (4.18), we get

$$|G_\epsilon(t, s)| \lesssim \frac{\epsilon^4}{(s_3 - s_4)^2} \quad \text{uniformly in } t.$$

From which we immediately get the estimate

$$|J_\epsilon(s)| \lesssim \frac{\epsilon^4}{(s_3 - s_4)^2}.$$

$$(I1iii) \quad s_1 - s_2 \approx s_3 - s_4 \gg \epsilon.$$

This case can obviously be treated in as exactly the same way as in either the case (I1i) or the case (I1ii). The cases (I2i), (I2ii) and (I2iii) are symmetric to the cases (I1i), (I1ii) and (I1iii), respectively.

4.4.3 Cases (II)

$$(II) \quad \boxed{(s_3 - s_2)(s_1 - s_4) > 0}$$

The symmetry between the formulas (3.39) and (3.40) for $A(t, s)$ makes estimating $J_\epsilon(s)$ in case (II) equivalent to estimating it in case (I). Indeed, all one has to do is replace $s_1 - s_2$ by $s_3 - s_2$ and $s_3 - s_4$ by $s_1 - s_4$ or replace $s_1 - s_2$ by $s_1 - s_4$ and $s_3 - s_4$ by $s_3 - s_2$.

The following simple figures (Figure (6)-Figure (8)) is a helpful way to see the symmetry used in the previous argument as they illustrate and summarize all the cases and their subcases studied so far and the estimates of $A(t, s)$, $G_\epsilon(t, s)$ and $J_\epsilon(s)$ in each of these subcases. Notice also that up to this moment, all the estimates for $G_\epsilon(t, s)$ are uniform in time. This is why

integrating $G_\epsilon(t, s)$ in time in these cases to estimate $J_\epsilon(s)$ insignificant.

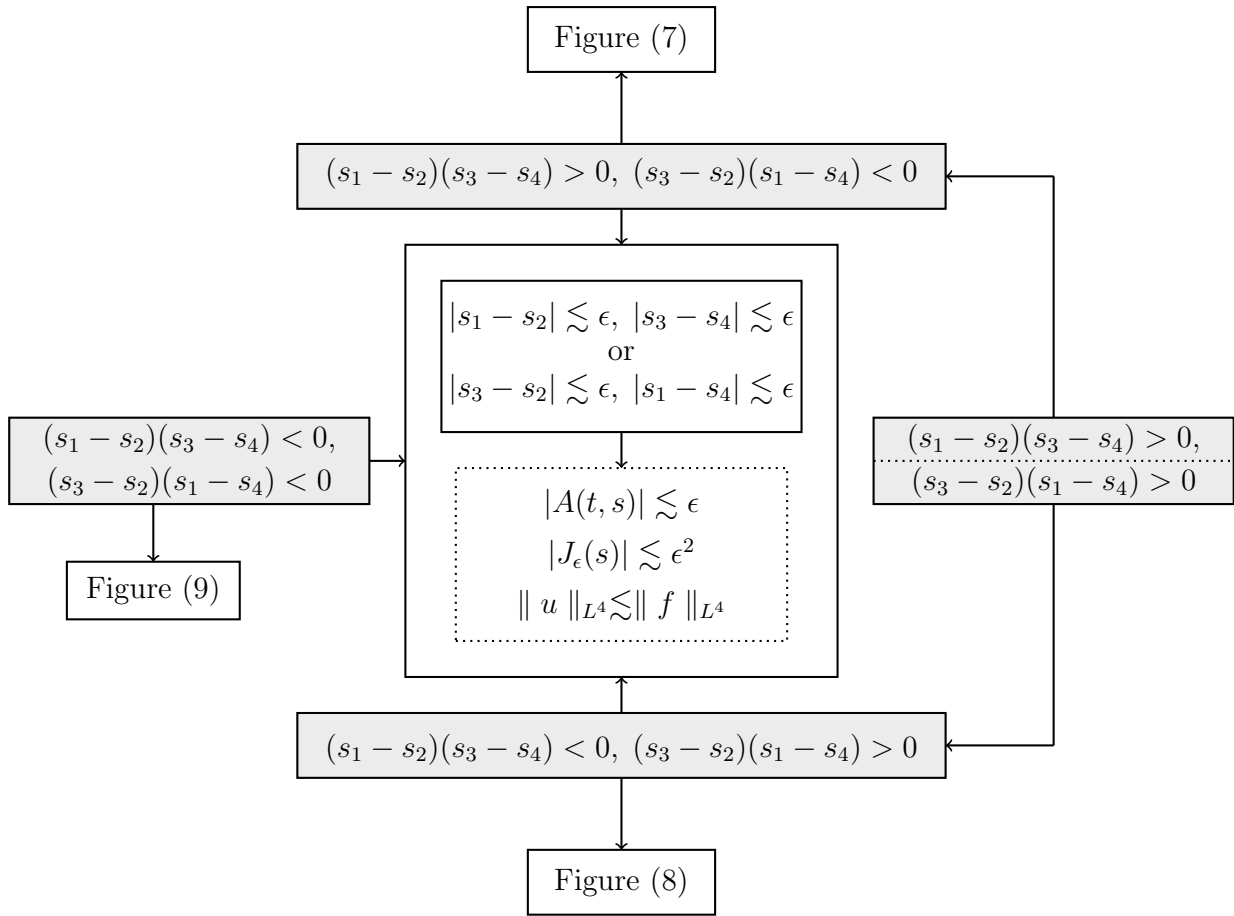


Figure (6)

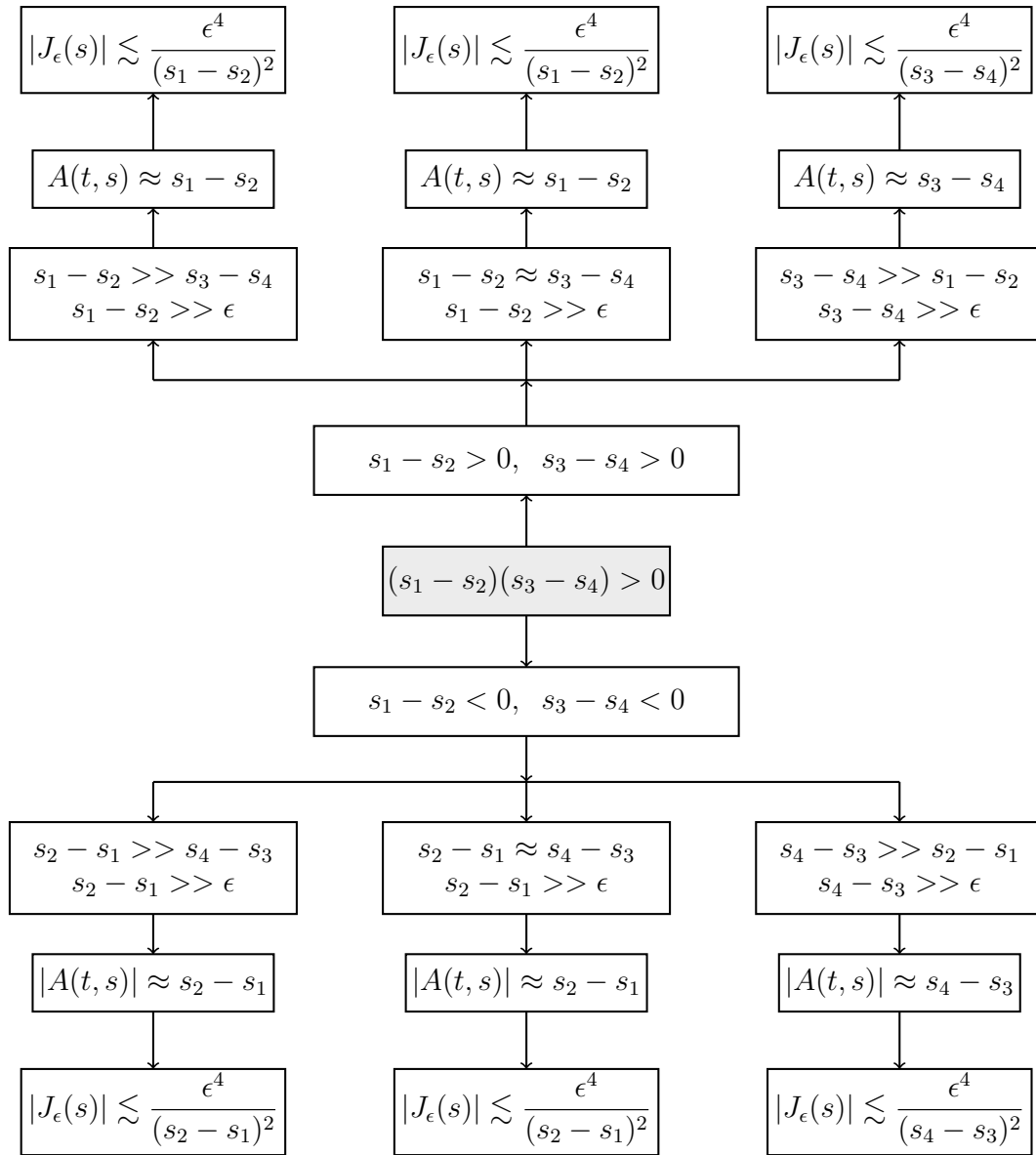


Figure (7): Estimates for the kernel $J_\epsilon(s)$ in the case (I)

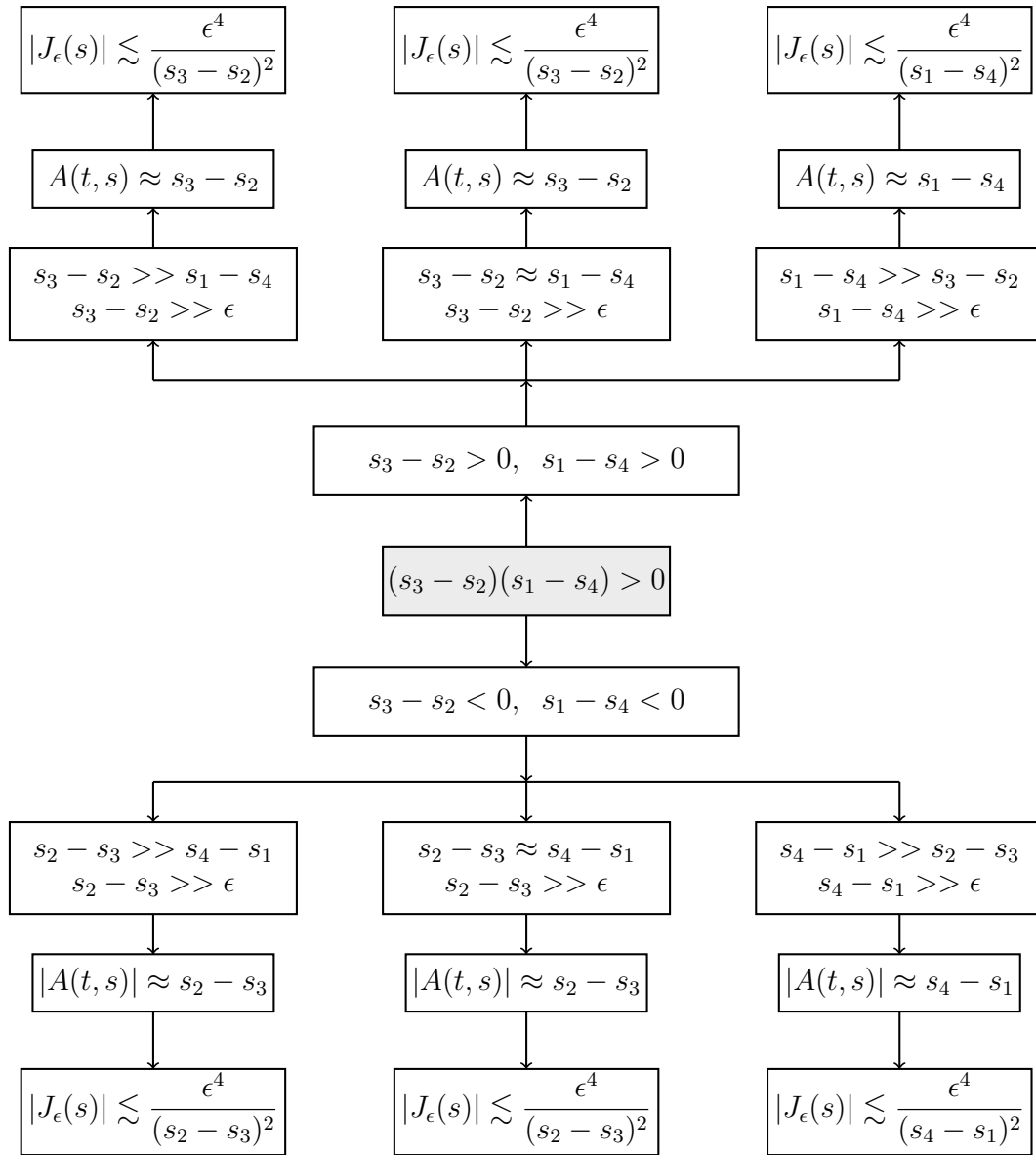


Figure (8): Estimates for the kernel $J_\epsilon(s)$ in the case (II)

Remark 4.4.1. Notice that in all the subcases of the cases (I) and (II) studied above we have that $|A(t, s)| \gg \epsilon$. This is crucial because otherwise we would not be entitled to use the control (4.18) for $G_\epsilon(t, s)$ as we actually did. As a matter of fact when

$$\begin{aligned} s_{j_1} - s_{j_2} > s_{j_3} - s_{j_4} > 0, \\ (j_1, j_2, j_3, j_4) \in \{(1, 2, 3, 4), (2, 1, 4, 3), (3, 4, 1, 2), (4, 3, 2, 1), (3, 2, 1, 4), (2, 3, 4, 1), \\ (1, 4, 3, 2), (4, 1, 2, 3)\}, \end{aligned}$$

we have that either $s_{j_1} - s_{j_2} \gg \epsilon$, which is what we assumed in the argument above, or we could have that $s_{j_1} - s_{j_2} \lesssim \epsilon$. But in the latter case, we would also have that $s_{j_3} - s_{j_4} \lesssim \epsilon$. This clearly means that $|s_{j_1} - s_{j_2}| \lesssim \epsilon$, $|s_{j_3} - s_{j_4}| \lesssim \epsilon$ which puts us in the situation identical to the one we encountered in 4.4.1 where we applied the trivial estimate (4.16) for $J_\epsilon(s)$ and obtained the estimate in (4.19) which is the estimate (4.12) restricted to that subregion.

By the help of Remark 4.4.1, we can summarize all the estimates obtained for both the cases (I) and (II) in the following lemma.

Lemma 4.4.3.

$$\sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_{i_1} - k_{i_2} \gg k_{i_3} - k_{i_4} \\ k_{i_1} - k_{i_2} \gtrsim 1}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \lesssim \frac{\log N}{N}$$

for all

$$\begin{aligned} (i_1, i_2, i_3, i_4) \in \{(1, 2, 3, 4), (2, 1, 4, 3), (3, 4, 1, 2), (4, 3, 2, 1), (3, 2, 1, 4), (2, 3, 4, 1), \\ (1, 4, 3, 2), (4, 1, 2, 3)\} \end{aligned}$$

4.4.4 Case (III)

$$(III) \quad \boxed{(s_1 - s_2)(s_3 - s_4) < 0 \quad \text{and} \quad (s_3 - s_2)(s_1 - s_4) < 0}$$

This is the last case to do to complete the process of estimating the norm. This case stands as the most critical case as it contains- among other easier subcases free from the singularity - the subcases for which the function $A(t, s)$ has a real zero inside $[0, 1]$ which creates a singularity in the function

$$G_\epsilon(t, s) = \sum_{\gamma} \sigma_\gamma f(A(t, s - \epsilon\gamma)) = \sum_{\gamma} \sigma_\gamma A^2(t, s - \epsilon\gamma) \log |A(t, s - \epsilon\gamma)|.$$

(see 4.4.6 below for the subcase where $A(t, s)$ may have a zero in $[0, 1]$). Although the integration in t smoothes out this singularity, it is still a challenge to estimate $J_\epsilon(s)$ as we shall see. Obtaining such an estimate in this case in one step is very difficult. That is why we had to split the estimate here into further simpler but rather long arguments. We actually prove the estimate (4.12) here with a logarithmic divergence. Since we have explicit functions, it was very natural to investigate, through numerical plotting, these functions how the cancelations coming from integration in time in this case work. The function $t \mapsto A(t, s)$ is a rational function which behaves as a quadratic polynomial when $0 \leq t \leq 1$. Fortunately it has at most one zero in $[0, 1]$. More luckily, this happens only on a region of "small" measure in the hypercube $[0, 1]^4$. What we did is divide this most critical case to essentially two subcases each of which is treated differently. The first subcase is when the zero, t_* , of $t \mapsto A(t, s)$ satisfies $|t_*| \lesssim \epsilon$ or $|1 - t_*| \lesssim \epsilon$. In this subcase, instead of integrating over the whole interval $[0, 1]$, we localize the zero of $A(t, s)$ in a "small" enough interval where we use the trivial estimate (4.16). This way, not only does $f(A(t, s - \epsilon\gamma))$ restore its smoothness outside that interval, but we also have $|A(t, s)| \gg \epsilon$ there and we may use again the estimate (4.18) for $G_\epsilon(t, s)$. The second subcase is when $|t_*| \gg \epsilon$ or $|1 - t_*| \gg \epsilon$. This is when we had to show the smoothing effect of integration in time on the function $f(A(t, s - \epsilon\gamma))$. We then managed to find a decent estimate for $J_\epsilon(s)$ in this case. This estimate will give us the desired result however with a divergence of logarithmic type. What we would like to emphasize here is that **in this most critical case where $A(t, s)$ has a zero in $[0, 1]$ and $G_\epsilon(t, s)$ oscillates there, the cancelations that come from integration in time was extremely important** and can not be done without. Any naive attempts to estimate the norm without performing this step of integration and taking for instance absolute values fails immediately.

Now we go through the details. As we did in cases (I) and (II), we will continue verifying (4.12) in case (III) by dividing it into simpler subcases in a way parallel to that used in section 3.4.2 to estimate $A(t, s)$. Recall that we had two subregions of the region (III):

$$(III1) \quad \alpha(s) \geq 0$$

$$(III2) \quad \alpha(s) < 0$$

We begin with the subregion (III1). But before proceeding, we recall how conditions on α , μ , ν and ϵ translate into conditions on $k = (k_1, k_2, k_3, k_4)$.

α	$\frac{k_1 - k_2 + k_3 - k_4}{N}$
μ	$\frac{k_1 - k_2}{N}$
ν	$\frac{k_3 - k_2}{N}$
ϵ	$\frac{1}{N}$

$$4.4.5 \quad (III1) \quad \alpha(s) > 0$$

When

$$(III1i) \quad \boxed{\alpha \gg \mu\nu, \quad A(t, s) \approx \alpha}$$

The argument for the case (i) will depend on the size of α compared to that of ϵ . That is we will treat two different subcases as follows

$$(III1ia) \quad \alpha \lesssim \epsilon$$

$$(III1ib) \quad \alpha \gg \epsilon$$

First look at

$$(III1ia) \quad \boxed{\mu\nu \ll \alpha \lesssim \epsilon}$$

In this subcase, we will simply use the trivial estimate (4.16). And to show the validity of (4.12) for this subregion we need to estimate

$$\sup_{k_1} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \ll N(k_1 - k_2 + k_3 - k_4)}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} 1. \quad (4.25)$$

We have the estimate

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{k_2} \sum_{\substack{k_3 \\ k_3 - k_2 \lesssim \frac{N}{k_1 - k_2}}} \sum_{\substack{k_4 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{k_2} \sum_{\substack{k_3 \\ k_3 - k_2 \lesssim \frac{N}{k_1 - k_2}}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_1} \sum_{k_2} \frac{1}{k_1 - k_2} \lesssim \frac{\log N}{N}. \tag{4.26}
\end{aligned}$$

We also have that

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_2} \sum_{\substack{k_1, k_3, k_4 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_2} \sum_{k_1} \sum_{\substack{k_3 \\ k_3 - k_2 \lesssim \frac{N}{k_1 - k_2}}} \sum_{\substack{k_4 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_2} \sum_{k_1} \sum_{\substack{k_3 \\ k_3 - k_2 \lesssim \frac{N}{k_1 - k_2}}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_2} \sum_{k_1} \frac{1}{k_1 - k_2} \lesssim \frac{\log N}{N}. \tag{4.27}
\end{aligned}$$

And because $k_1 - k_2 + k_3 - k_4 \geq 0$, we have that

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_3} \sum_{\substack{k_1, k_2, k_4, \\ k_1 - k_2 \geq k_4 - k_3, \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_3} \sum_{k_4} \sum_{\substack{k_2 \\ k_3 - k_2 \lesssim \frac{N}{k_4 - k_3}}} \sum_{\substack{k_1 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_3} \sum_{k_4} \sum_{\substack{k_2 \\ k_3 - k_2 \lesssim \frac{N}{k_4 - k_3}}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_3} \sum_{k_4} \frac{1}{k_4 - k_3} \lesssim \frac{\log N}{N} \tag{4.28}
\end{aligned}$$

and that

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3, \\ k_1 - k_2 \geq k_4 - k_3, \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_4} \sum_{k_3} \sum_{\substack{k_2 \\ k_3 - k_2 \lesssim \frac{N}{k_4 - k_3}}} \sum_{\substack{k_1 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_4} \sum_{k_3} \sum_{\substack{k_2 \\ k_3 - k_2 \lesssim \frac{N}{k_4 - k_3}}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_4} \sum_{k_3} \frac{1}{k_4 - k_3} \lesssim \frac{\log N}{N}. \tag{4.29}
\end{aligned}$$

It thus follows from the estimates (4.26) - (4.29) that the quadrilinear form in (4.25) is estimated by

$$\sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \ll N(k_1 - k_2 + k_3 - k_4) \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \lesssim \frac{\log N}{N}. \tag{4.30}$$

which completes the work for this subcase. Secondly, consider the subcase when

$$(III1ib) \quad \boxed{\alpha \gg \mu\nu, \alpha \gg \epsilon}$$

In this subregion, we have that $A(t, s) \approx \alpha \gg \epsilon$. Thus, we can use the estimate (4.18) and get

$$|G_\epsilon(t, s)| \lesssim \frac{\epsilon^4}{\alpha^2},$$

uniformly in t . So

$$|J_\epsilon(s)| \lesssim \frac{\epsilon^4}{(s_1 - s_2 + s_3 - s_4)^2}.$$

We end up then with the task of estimating the quantity given in (4.31) below.

$$\begin{aligned}
& \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ k_1 - k_2 + k_3 - k_4 \gg 1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_4 - k_3)(k_3 - k_2)}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \\
& \lesssim \frac{1}{N^2} \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ k_1 - k_2 + k_3 - k_4 \gg 1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_4 - k_3)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2}. \tag{4.31}
\end{aligned}$$

We shall show that (4.31) is bounded by $\frac{(\log N)^2}{N}$. We have that

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ k_1 - k_2 + k_3 - k_4 \gg 1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_1 - k_2)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{k_2, k_3} \sum_{\substack{k_4 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_1 - k_2)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N} \sup_{k_1} \sum_{k_2, k_3} \frac{1}{(k_1 - k_2)(k_3 - k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_1} \sum_{k_2} \frac{1}{k_1 - k_2} \sum_{k_3} \frac{1}{k_3 - k_2} \lesssim \frac{(\log N)^2}{N}. \tag{4.32}
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_2} \sum_{\substack{k_1, k_3, k_4 \\ k_1 - k_2 + k_3 - k_4 \gg 1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_1 - k_2)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N^2} \sup_{k_2} \sum_{k_1, k_3} \sum_{\substack{k_4 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_1 - k_2)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N} \sup_{k_2} \sum_{k_1, k_3} \frac{1}{(k_1 - k_2)(k_3 - k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_2} \sum_{k_1} \frac{1}{k_1 - k_2} \sum_{k_3} \frac{1}{k_3 - k_2} \lesssim \frac{(\log N)^2}{N}. \tag{4.33}
\end{aligned}$$

And

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_3} \sum_{\substack{k_1, k_2, k_4 \\ k_1 - k_2 + k_3 - k_4 \gg 1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_1 - k_2)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N^2} \sup_{k_3} \sum_{k_1, k_2} \sum_{\substack{k_4 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_1 - k_2)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N} \sup_{k_3} \sum_{k_1, k_2} \frac{1}{(k_1 - k_2)(k_3 - k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_3} \sum_{k_2} \frac{1}{k_3 - k_2} \sum_{k_1} \frac{1}{k_1 - k_2} \lesssim \frac{(\log N)^2}{N}. \tag{4.34}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1 - k_2 \geq k_4 - k_3 \\ k_1 - k_2 + k_3 - k_4 \gg 1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_1 - k_2)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N^2} \sup_{k_4} \sum_{k_2, k_3} \sum_{\substack{k_1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_4 - k_3)(k_3 - k_2)}} \frac{1}{(k_1 - k_2 + k_3 - k_4)^2} \\
& \lesssim \frac{1}{N} \sup_{k_4} \sum_{k_2, k_3} \frac{1}{(k_4 - k_3)(k_3 - k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_4} \sum_{k_3} \frac{1}{k_4 - k_3} \sum_{k_2} \frac{1}{k_3 - k_2} \lesssim \frac{(\log N)^2}{N}. \tag{4.35}
\end{aligned}$$

The estimates (4.32)-(4.35) together estimate the quantity (4.31) by

$$\sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ k_1 - k_2 + k_3 - k_4 \gg 1 \\ (k_1 - k_2 + k_3 - k_4)N \gg (k_4 - k_3)(k_3 - k_2)}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \lesssim \frac{(\log N)^2}{N}. \tag{4.36}$$

$$(III1ii) \quad \boxed{\mu\nu \gtrsim \alpha, \quad A(t, s) \approx \mu\nu}$$

Similarly, we need to differentiate between two subcases related to the two possibilities for the size of $\mu\nu$ compared to that of ϵ . Namely

$$(III1iia) \quad \mu\nu \lesssim \epsilon$$

$$(III1iib) \quad \mu\nu \gg \epsilon$$

$$(III1ia) \quad \boxed{\epsilon \gtrsim \mu\nu \gtrsim \alpha}$$

Using the trivial estimate for $J_\epsilon(s)$ in this subregion, the estimate (4.30) directly gives that

$$\begin{aligned} & \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \gtrsim N(k_1 - k_2 + k_3 - k_4) \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \\ & \lesssim \frac{1}{N^2} \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 + k_3 - k_4 \lesssim 1 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} 1 \lesssim \frac{\log N}{N}. \end{aligned} \quad (4.37)$$

This is it then for this subcase and we move on to the next subregion where $A(t, s) \approx \mu\nu \gg \epsilon$.

$$(III1ib) \quad \boxed{\mu\nu \gtrsim \alpha, \quad \mu\nu \gg \epsilon}$$

Here, we have $A(t, s) \approx \mu\nu \gg \epsilon$. Therefore, we can use the estimate (4.18) for $G_\epsilon(t, s)$ to obtain that

$$|J_\epsilon(s)| \lesssim \int_0^1 G_\epsilon(t, s) dt \lesssim \int_0^1 \frac{\epsilon^4}{A^2(t, s)} dt \lesssim \frac{\epsilon^4}{\mu^2 \nu^2}.$$

Once again, we shall prove- with a loss of logarithmic order- the estimate (4.12) after restricting the operator to the subregion (III1ib). We shall actually prove that

$$\begin{aligned} & \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ (k_1 - k_2)(k_3 - k_2) \gtrsim N(k_1 - k_2 + k_3 - k_4) \\ (k_1 - k_2)(k_3 - k_2) \gg N}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \\ & \lesssim \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \geq k_4 - k_3 \\ (k_1 - k_2)(k_3 - k_2) \gtrsim N(k_1 - k_2 + k_3 - k_4) \\ (k_1 - k_2)(k_3 - k_2) \gg N}} \frac{1}{(k_1 - k_2)^2 (k_3 - k_2)^2} \lesssim \frac{(\log N)^2}{N}. \end{aligned} \quad (4.38)$$

The estimate (4.38) follows from the estimates (4.39)-(4.42) below.

$$\begin{aligned}
& \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ (k_1-k_2)(k_3-k_2) \gtrsim N(k_1-k_2+k_3-k_4) \\ (k_1-k_2)(k_3-k_2) \gg N}} \frac{1}{(k_1-k_2)^2(k_3-k_2)^2} \\
& \lesssim \sup_{k_1} \sum_{k_2, k_3} \frac{1}{(k_1-k_2)^2(k_3-k_2)^2} \sum_{\substack{k_4 \\ (k_1-k_2)(k_3-k_2) \gtrsim N(k_1-k_2+k_3-k_4)}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_1} \sum_{k_2, k_3} \frac{1}{(k_1-k_2)(k_3-k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_1} \sum_{k_2} \frac{1}{k_1-k_2} \sum_{k_3} \frac{1}{k_3-k_2} \lesssim \frac{(\log N)^2}{N}. \tag{4.39}
\end{aligned}$$

Also

$$\begin{aligned}
& \sup_{k_2} \sum_{\substack{k_1, k_3, k_4 \\ (k_1-k_2)(k_3-k_2) \gtrsim N(k_1-k_2+k_3-k_4) \\ (k_1-k_2)(k_3-k_2) \gg N}} \frac{1}{(k_1-k_2)^2(k_3-k_2)^2} \\
& \lesssim \sup_{k_2} \sum_{k_1, k_3} \frac{1}{(k_1-k_2)^2(k_3-k_2)^2} \sum_{\substack{k_4 \\ (k_1-k_2)(k_3-k_2) \gtrsim N(k_1-k_2+k_3-k_4)}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_2} \sum_{k_1, k_3} \frac{1}{(k_1-k_2)(k_3-k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_2} \sum_{k_1} \frac{1}{k_1-k_2} \sum_{k_3} \frac{1}{k_3-k_2} \lesssim \frac{(\log N)^2}{N}. \tag{4.40}
\end{aligned}$$

Similarly

$$\begin{aligned}
& \sup_{k_3} \sum_{\substack{k_1, k_2, k_4 \\ (k_1-k_2)(k_3-k_2) \gtrsim N(k_1-k_2+k_3-k_4) \\ (k_1-k_2)(k_3-k_2) \gg N}} \frac{1}{(k_1-k_2)^2(k_3-k_2)^2} \\
& \lesssim \sup_{k_3} \sum_{k_1, k_2} \frac{1}{(k_1-k_2)^2(k_3-k_2)^2} \sum_{\substack{k_4 \\ (k_1-k_2)(k_3-k_2) \gtrsim N(k_1-k_2+k_3-k_4)}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_3} \sum_{k_1, k_2} \frac{1}{(k_1-k_2)(k_3-k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_3} \sum_{k_2} \frac{1}{k_3-k_2} \sum_{k_1} \frac{1}{k_1-k_2} \lesssim \frac{(\log N)^2}{N}. \tag{4.41}
\end{aligned}$$

And it only remains to prove the estimate (4.42) below.

$$\sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1 - k_2 \geq k_4 - k_3 \\ (k_1 - k_2)(k_3 - k_2) \gtrsim N(k_1 - k_2 + k_3 - k_4) \\ (k_1 - k_2)(k_3 - k_2) \gg N}} \frac{1}{(k_1 - k_2)^2 (k_3 - k_2)^2} \lesssim \frac{(\log N)^2}{N}. \quad (4.42)$$

It is difficult to prove the estimate (4.42) as it is. We will actually prove a weaker version of this estimate, namely

$$\sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1 - k_2 \approx k_4 - k_3 \\ (k_1 - k_2)(k_3 - k_2) \gtrsim N(k_1 - k_2 + k_3 - k_4) \\ (k_1 - k_2)(k_3 - k_2) \gg N}} \frac{1}{(k_1 - k_2)^2 (k_3 - k_2)^2} \lesssim \frac{(\log N)^2}{N}. \quad (4.43)$$

Lemma 4.4.4 below justifies replacing the estimate (4.42) by (4.43).

Lemma 4.4.4.

$$\left| \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ |k_1 - k_2| \gg |k_4 - k_3| \text{ or } |k_3 - k_4| \gg |k_1 - k_2|}} J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \right| \lesssim \frac{\log N}{N}. \quad (4.44)$$

Proof. Notice that when $\epsilon \gtrsim |s_1 - s_2| \gg |s_3 - s_4|$, then by (4.19), we have nothing more to do. So, let $|s_1 - s_2| \gg |s_3 - s_4|$ and $|s_1 - s_2| \gg \epsilon$. We know from (3.39) that $|A(t, s)| \gg \epsilon$ in this case. We can therefore use the estimate (4.18) and get that

$$|J_\epsilon(s)| \lesssim \frac{\epsilon^4}{(s_1 - s_2)^2}.$$

Thus

$$\left| \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ |k_1 - k_2| \gg |k_4 - k_3| \\ |k_1 - k_2| \gg 1}} J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \right| \lesssim \frac{1}{N^2} \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ |k_1 - k_2| \gg |k_4 - k_3| \\ |k_1 - k_2| \gg 1}} \frac{1}{(k_1 - k_2)^2} \lesssim \frac{\log N}{N}$$

by the proof of Lemma 4.4.2. This completes the proof of the Lemma 4.4.4. \square

We go back to showing the estimate (4.43) which together with the assertion of Lemma 4.4.4 proves the estimate (4.42) which in its turn together with the estimates (4.39)-(4.41)

prove (4.38).

$$\begin{aligned}
& \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1 - k_2 \approx k_4 - k_3 \\ (k_1 - k_2)(k_3 - k_2) \gtrsim N(k_1 - k_2 + k_3 - k_4) \\ (k_1 - k_2)(k_3 - k_2) \gg N}} \frac{1}{(k_1 - k_2)^2 (k_3 - k_2)^2} \\
& \lesssim \sup_{k_4} \sum_{k_2, k_3} \frac{1}{(k_4 - k_3)^2 (k_3 - k_2)^2} \sum_{\substack{k_1 \\ (k_4 - k_3)(k_3 - k_2) \gtrsim N(k_1 - k_2 + k_3 - k_4)}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_4} \sum_{k_2, k_3} \frac{1}{(k_4 - k_3)(k_3 - k_2)} \\
& \lesssim \frac{1}{N} \sup_{k_4} \sum_{k_3} \frac{1}{k_4 - k_3} \sum_{k_2} \frac{1}{k_3 - k_2} \lesssim \frac{(\log N)^2}{N}.
\end{aligned}$$

This concludes the estimate for the case (III1), $\alpha \geq 0$. We move on to the case (III2), $\alpha < 0$. The following corollary will be useful later on.

Corollary 4.4.5.

$$\left| \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ |k_1 - k_2| \gg 1 \\ |k_3 - k_2| \gg |k_1 - k_4| \text{ or } |k_3 - k_2| \ll |k_1 - k_4|}} J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \lesssim \frac{\log N}{N}.$$

Proof. The proof is obvious by the assertion of Lemma 4.4.4 and the symmetry properties of the kernel $J_{\frac{1}{N}}$ that come from the symmetry properties of the functions $A_\gamma(t, \frac{k}{N})$. (recall the two formulas (3.39) and (3.40) in Chapter 3 Section 3.4). \square

4.4.6 (III2) $\alpha(s) < 0$

We begin studying this case by the following remark

Remark 4.4.2. Notice that the two cases (III1i) and (III1ii) that constitute the study done in Section 4.4.5 can be slightly generalized with minor modifications to the following two respectively corresponding cases

$$\begin{aligned}
(III1i') & \quad |\alpha| \gg \mu\nu \\
(III1ii') & \quad \mu\nu \gtrsim |\alpha|
\end{aligned}$$

in which we have the following respective estimates for $A(t, s)$

$$\begin{aligned}
(i') & \quad |A(t, s)| \approx |\alpha| \\
(ii') & \quad |A(t, s)| \approx \mu\nu
\end{aligned}$$

Remark 4.4.2 leaves us with a more restricted subcase of the case $\alpha < 0$, that is when

$$\boxed{\alpha < 0, \quad |\alpha| \approx \mu\nu}$$

In Lemma 4.4.6 below, we prove the estimate (4.12) with a logarithmic divergence in the particular subcase when $\alpha < 0$, $|\alpha| \approx \mu\nu \lesssim \epsilon$ using the trivial estimate (4.16). This will restrict the case $\alpha < 0$, $|\alpha| \approx \mu\nu$ even more to when

$$\boxed{\alpha < 0, \quad |\alpha| \approx \mu\nu \gg \epsilon}$$

But first let us show Lemma 4.4.6.

Lemma 4.4.6.

$$\sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |k_1 - k_2 + k_3 - k_4| \lesssim 1}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \lesssim \frac{\log N}{N}.$$

Proof. We have the following four estimates. First

$$\begin{aligned} & \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |k_1 - k_2 + k_3 - k_4| \lesssim 1}} \left| J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \right| \\ & \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |k_1 - k_2 + k_3 - k_4| \lesssim 1}} 1 \\ & \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{k_2} \sum_{\substack{k_3 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} \sum_{\substack{k_4 \\ |k_1 - k_2 + k_3 - k_4| \lesssim 1}} 1 \\ & \lesssim \frac{1}{N^2} \sup_{k_1} \sum_{k_2} \sum_{\substack{k_3 \\ (k_1 - k_2)(k_3 - k_2) \lesssim N}} 1 \\ & \lesssim \frac{1}{N} \sup_{k_1} \sum_{k_2} \frac{1}{k_1 - k_2} \lesssim \frac{\log N}{N}. \end{aligned} \tag{4.45}$$

Secondly, we have the estimate

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_2} \sum_{\substack{k_1, k_3, k_4 \\ (k_1-k_2)(k_3-k_2) \approx N |k_1-k_2+k_3-k_4| \\ |k_1-k_2+k_3-k_4| \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_2} \sum_{k_1} \sum_{\substack{k_3 \\ (k_1-k_2)(k_3-k_2) \lesssim N}} \sum_{\substack{k_4 \\ |k_1-k_2+k_3-k_4| \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_2} \sum_{k_1} \sum_{\substack{k_3 \\ (k_1-k_2)(k_3-k_2) \lesssim N}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_2} \sum_{k_1} \frac{1}{k_1 - k_2} \lesssim \frac{\log N}{N}. \tag{4.46}
\end{aligned}$$

Similarly

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_3} \sum_{\substack{k_1, k_2, k_4 \\ (k_1-k_2)(k_3-k_2) \approx N |k_1-k_2+k_3-k_4| \\ |k_1-k_2+k_3-k_4| \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_3} \sum_{k_2} \sum_{\substack{k_1 \\ (k_1-k_2)(k_3-k_2) \lesssim N}} \sum_{\substack{k_4 \\ |k_1-k_2+k_3-k_4| \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_3} \sum_{k_2} \sum_{\substack{k_1 \\ (k_1-k_2)(k_3-k_2) \lesssim N}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_3} \sum_{k_2} \frac{1}{k_3 - k_2} \lesssim \frac{\log N}{N}. \tag{4.47}
\end{aligned}$$

According to the assertion of Lemma 4.4.4, it suffices, to complete the proof of Lemma 4.4.6, to show the estimate

$$\frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1-k_2 \approx k_4-k_3 \\ (k_1-k_2)(k_3-k_2) \approx N |k_1-k_2+k_3-k_4| \\ |k_1-k_2+k_3-k_4| \lesssim 1}} 1 \lesssim \frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ (k_4-k_3)(k_3-k_2) \ll N \\ |k_1-k_2+k_3-k_4| \lesssim 1}} 1 \lesssim \frac{\log N}{N} \tag{4.48}$$

instead of

$$\frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ (k_1-k_2)(k_3-k_2) \approx N |k_1-k_2+k_3-k_4| \\ |k_1-k_2+k_3-k_4| \lesssim 1}} 1 \lesssim \frac{\log N}{N}.$$

We have

$$\begin{aligned}
& \frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1 - k_2 \approx k_4 - k_3 \\ (k_1 - k_2)(k_3 - k_2) \ll N \\ |k_1 - k_2 + k_3 - k_4| \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_4} \sum_{\substack{k_2, k_3 \\ (k_4 - k_3)(k_3 - k_2) \ll N}} \sum_{\substack{k_1 \\ |k_1 - k_2 + k_3 - k_4| \lesssim 1}} 1 \\
& \lesssim \frac{1}{N^2} \sup_{k_4} \sum_{k_3} \sum_{\substack{k_2 \\ (k_4 - k_3)(k_3 - k_2) \ll N}} 1 \\
& \lesssim \frac{1}{N} \sup_{k_4} \sum_{k_3} \frac{1}{k_4 - k_3} \lesssim \frac{\log N}{N}.
\end{aligned}$$

The proof of Lemma 4.4.6 follows immediately then from the estimates (4.45)-(4.48) and Lemma 4.4.4. \square

Let us summarize what we have achieved so far. In Remark 4.4.2, we have seen that it is redundant to deal with both the cases when $|\alpha| \gg \mu\nu$ and $\mu\nu \gg |\alpha|$. This left us with the case $|\alpha| \approx \mu\nu$. Theorem 3.3.4 together with Lemma 4.4.6 give us the desired estimate with a logarithmic divergence for the case $|\alpha| \approx \mu\nu \lesssim \epsilon$. We are now ready to look at the only case remaining, that is when

$$\boxed{\alpha < 0, \quad |\alpha| \approx \mu\nu \gg \epsilon}$$

According to Lemma 4.4.4 and its Corollary 4.4.5, the restrictions we are actually left with in terms of μ, ν , and α and the corresponding ones in terms of $k = (k_1, k_2, k_3, k_4)$ are given by the table below

α, μ, ν	k
$\alpha < 0$	$k_1 - k_2 < k_4 - k_3$
$ \alpha \approx \mu\nu$	$k_2 - k_1 + k_4 - k_3 \approx \mu\nu N$
$ \alpha \gg \epsilon$	$k_2 - k_1 + k_4 - k_3 \gg 1$
$\mu\nu \gg \epsilon$	$(k_1 - k_2)(k_3 - k_2) \gg N$
$k_4 - k_3 \approx \mu N$	$k_1 - k_2 \approx k_4 - k_3$
$k_4 - k_1 \approx \nu N$	$k_3 - k_2 \approx k_4 - k_1$

(4.49)

Recall that we estimated

$$|A(t, s)| \approx -\alpha |t - t_*|. \quad (4.50)$$

The estimate (4.50) puts us at crossroads. Before we continue as before to estimate the norm given by the formula (4.9), we assume that $|t_*| \lesssim |1 - t_*|$. Computations will show that there is no loss of generality in making this choice. Any way, we have that

$$|t_*| \lesssim 1.$$

This follows directly from the estimates (4.71) and (4.83) shown below. We shall distinguish the following two different cases that we study separately:

$$(1*) \quad |t_*| \lesssim \epsilon$$

$$(2*) \quad |t_*| \gg \epsilon$$

First, we consider the subcase

$$(1*) \quad \boxed{|t_*| \lesssim \epsilon}$$

In this case, to estimate $J_\epsilon(s) = \int_0^1 G_\epsilon(t, s) dt$, we need to actually perform the integration in t to capture the cancelation coming from the change of the sign of the function $G_\epsilon(t, s) = \sum_{\gamma \in \Gamma} \sigma_\gamma f(A(t, s - \epsilon\gamma))$ with $f(a) = a^2 \log |a|$, on $[0, 1]$. However, estimating $J_\epsilon(s)$ by doing the integral over the whole interval involves a number of difficulties that we shall face in the subcase (2*). Since, the integrand is an explicit function, we found it very useful, in order to develop deeper intuition about the behavior of $G_\epsilon(t, s)$ over the interval $[0, 1]$, to see concrete pictures of how this function actually behaves for the different of values of s_1, s_2, s_3 and s_4 that tie $A(t, s)$ with the estimate (4.50) and what happens to its profile as $\epsilon \rightarrow 0^+$. To avoid interrupting the proof of the estimate, we prefer to put these in chapter 5. This gives us the opportunity to compare between the kernels of the quadrilinear forms, the one we are estimating here and the one that appears in Chapter 5. Now, what we shall do is divide the

interval $[0, 1]$ into two subintervals, I_1, I_2 as follows

$$I_1 = \{t \in [0, 1] : |t - t_*| \lesssim \frac{\epsilon}{\mu\nu}\},$$

$$I_2 = [0, 1] - I_1, \quad \text{i. e.,} \quad I_2 = \{t \in [0, 1] : |t - t_*| \gg \frac{\epsilon}{\mu\nu}\}.$$

On the interval I_1 , we merely use the trivial estimate (4.15) for $\int_{\mathbb{R}^4} |u_{[s-\epsilon, s]}(t, x)|^4 dx$. While on the interval I_2 we have by (4.50) that

$$|A(t, s)| \approx -\alpha |t - t_*| \gg -\alpha \frac{\epsilon}{\mu\nu}.$$

And since $-\alpha \approx \mu\nu$ then

$$|A(t, s)| \gg \epsilon, \quad \text{when } t \in I_2.$$

This allows us to use the estimate (4.18) for $G_\epsilon(t, s)$ and get

$$|G_\epsilon(t, s)| \lesssim \frac{\epsilon^4}{A^2(t, s)} \approx \frac{\epsilon^4}{\mu^2\nu^2(t - t_*)^2}, \quad t \in I_2.$$

Finally, we integrate the trivial estimate (4.15) on the interval I_1 and integrate $G_\epsilon(t, s)$ over I_2 to estimate $J_\epsilon(s)$ on $[0, 1]$ in this case.

$$\begin{aligned} |J_\epsilon(s)| &\lesssim \epsilon^2 \int_{I_1} dt + \int_{I_2} |G_\epsilon(t, s)| dt \\ &\lesssim \epsilon^2 \int_{|t-t_*| \lesssim \frac{\epsilon}{\mu\nu}} dt + \frac{\epsilon^4}{\mu^2\nu^2} \int_{|t-t_*| \gg \frac{\epsilon}{\mu\nu}} \frac{dt}{(t - t_*)^2} \\ &\lesssim \frac{\epsilon^3}{\mu\nu}. \end{aligned} \tag{4.51}$$

From (4.9), the estimate (4.51) and dyadically decomposing the quadrilinear form, we know that we have to estimate the quantity

$$\begin{aligned} &\sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \\ k_3 - k_2 \approx k_4 - k_1 \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |t_* (\frac{k}{N})| \lesssim \frac{1}{N}}} c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}} \left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N} \right) \\ &\lesssim \frac{1}{N^3} \sum_{\substack{\mu: \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu: \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |t_* (\frac{k}{N})| \lesssim \frac{1}{N}}} |c_{k_1}| |c_{k_2}| |c_{k_3}| |c_{k_4}|. \end{aligned} \tag{4.52}$$

Let us see what restriction does the condition $|t_*(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N})| \lesssim \frac{1}{N}$ put on $k = (k_1, k_2, k_3, k_4)$.

Since $A(t_*(s), s) = 0$. Then, for any $l = 1, 2, 3, 4$, we have

$$\partial_{s_l}[A(t_*(s), s)] = \partial_t A(t_*(s), s) \partial_{s_l} t_*(s) + \partial_{s_l} A(t_*(s), s) = 0.$$

That is

$$\partial_{s_l} t_*(s) = -\frac{\partial_{s_l} A(t_*(s), s)}{\partial_t A(t_*(s), s)}. \quad (4.53)$$

But, we have the following estimate

$$|\partial_{s_l} A(t_*(s), s)| = \left| \partial_{s_l} \sum_{l=1}^4 \frac{(-1)^l}{1+t+s_l} \right| = \frac{1}{(1+t+s_l)^2} \approx 1, \quad l = 1, 2, 3, 4. \quad (4.54)$$

Moreover, we have the following estimate previously obtained in Lemma 3.4.5 in Chapter 3 Section 3.4.5.

$$-\partial_t A(t_*(s), s) \approx \mu\nu \quad (4.55)$$

where $t_*(s)$ is such that $A(t_*(s), s) = 0$. Using (4.54) and (4.55) in (4.53) we get

$$|\partial_{s_l} t_*(s)| \approx \frac{1}{\mu\nu}. \quad (4.56)$$

Now let $I_{s_l} \subset [0, 1]$ be the interval on which $|t_*(s)| \lesssim \epsilon$. Then, by the mean value theorem and the estimate (4.56), we must have that

$$|I_{s_l}| \lesssim \mu\nu\epsilon.$$

That is whenever $|t_*(\frac{k}{N})| \lesssim \frac{1}{N}$, then

$$|k_l - k_l^*| \lesssim \mu\nu, \quad \text{for some } k_l^*, \quad l = 1, 2, 3, 4, \quad (4.57)$$

where k_l depends on the other variables k_j , $j \neq l$. From the condition (4.57) and the estimate (4.52), and to prove the estimate (4.10) in this case, we need to prove the estimate (4.58) below.

$$\frac{1}{N^3} \sum_{\substack{\mu; \text{ dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu; \text{ dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |t_*(\frac{k}{N})| \lesssim \frac{1}{N} (\Leftrightarrow |k_l - k_l^*| \lesssim \mu\nu)}} |c_{k_1}| |c_{k_2}| |c_{k_3}| |c_{k_4}| \lesssim \frac{1}{N} \|c\|_{l^4}^4. \quad (4.58)$$

We accomplish this using the result of the interpolation method in Theorem 3.3.4. For this purpose we merely have to prove the following estimate

$$\frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sup_{k_{i_1}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |k_l - k_l^*| \lesssim \mu \nu, l \neq i_1}} 1 \lesssim \frac{1}{N}.$$

The latter comes out using the estimates (4.59) - (4.62) that we show below. We have the estimate

$$\begin{aligned} & \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |k_4 - k_4^*| \lesssim \mu \nu}} 1 \\ & \lesssim \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sup_{k_1} \sum_{\substack{k_2, k_3 \\ k_1 - k_2 \approx \mu N \\ k_3 - k_2 \approx \nu N}} \sum_{\substack{k_4 \\ |k_4 - k_4^*| \lesssim \mu \nu}} 1 \\ & \lesssim \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} 1 \sup_{k_1} \sum_{\substack{k_2 \\ k_1 - k_2 \approx \mu N}} \sum_{\substack{k_3 \\ k_3 - k_2 \approx \nu N}} 1 \\ & \lesssim \frac{1}{N^2} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \nu \sup_{k_1} \sum_{\substack{k_2 \\ k_1 - k_2 \approx \mu N}} 1 \\ & \lesssim \frac{1}{N} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \mu \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \nu \lesssim \frac{1}{N}. \end{aligned} \tag{4.59}$$

Also,

$$\begin{aligned}
& \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sup_{k_2} \sum_{\substack{k_1, k_3 \\ k_1 - k_2 \approx \mu N \\ k_3 - k_2 \approx \nu N}} \sum_{\substack{k_4 \\ |k_4 - k_4^*| \lesssim \mu \nu}} 1 \\
& \lesssim \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} 1 \sup_{k_2} \sum_{\substack{k_1 \\ k_1 - k_2 \approx \mu N}} \sum_{\substack{k_3 \\ k_3 - k_2 \approx \nu N}} 1 \\
& \lesssim \frac{1}{N^2} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \nu \sup_{k_2} \sum_{\substack{k_1 \\ k_1 - k_2 \approx \mu N}} 1 \\
& \lesssim \frac{1}{N} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \mu \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \nu \lesssim \frac{1}{N}.
\end{aligned} \tag{4.60}$$

Similarly

$$\begin{aligned}
& \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \sup_{k_3} \sum_{\substack{k_1, k_2 \\ k_1 - k_2 \approx \mu N \\ k_3 - k_2 \approx \nu N}} 1 \\
& \lesssim \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \sup_{k_3} \sum_{\substack{k_2 \\ k_3 - k_2 \approx \nu N}} \sum_{\substack{k_1 \\ k_1 - k_2 \approx \mu N}} 1 \\
& \lesssim \frac{1}{N^2} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \mu \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} 1 \sup_{k_3} \sum_{\substack{k_2 \\ k_3 - k_2 \approx \nu N}} 1 \\
& \lesssim \frac{1}{N} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \mu \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \nu \lesssim \frac{1}{N}.
\end{aligned} \tag{4.61}$$

Lastly

$$\begin{aligned}
& \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ (k_1 - k_2)(k_3 - k_2) \approx N |k_1 - k_2 + k_3 - k_4| \\ |k_1 - k_1^*| \lesssim \mu \nu}} 1 \\
& \lesssim \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu} \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu} \sup_{k_4} \sum_{\substack{k_2, k_3 \\ k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx \nu N}} \sum_{\substack{k_1 \\ |k_1 - k_1^*| \lesssim \mu \nu}} 1 \\
& \lesssim \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} 1 \sup_{k_4} \sum_{\substack{k_2, k_3 \\ k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx \nu N}} 1 \\
& \lesssim \frac{1}{N^3} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} 1 \sup_{k_4} \sum_{\substack{k_3 \\ k_4 - k_3 \approx \mu N}} \sum_{\substack{k_2 \\ k_3 - k_2 \approx \nu N}} 1 \\
& \lesssim \frac{1}{N^2} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \nu \sup_{k_4} \sum_{\substack{k_3 \\ k_4 - k_3 \approx \mu N}} 1 \\
& \lesssim \frac{1}{N} \sum_{\substack{\mu; \text{dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \mu \sum_{\substack{\nu; \text{dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \nu \lesssim \frac{1}{N}. \tag{4.62}
\end{aligned}$$

This completes the argument for the subcase (1*), $\boxed{|t_*| \lesssim \epsilon}$. Observe that the condition

$$|\alpha| \approx \mu \nu \quad \text{or} \quad N|k_1 - k_2 + k_3 - k_4| \approx (k_1 - k_2)(k_3 - k_2)$$

may seem to have added no useful restriction in doing the estimates (4.59) - (4.62). But it was already used to estimate the kernel $J_{\frac{1}{N}}$ in (4.51). Finally, it remains to prove the estimate (4.12) and consequently (4.10) for the subcase

$$(2*) \quad \boxed{|t_*| \gg \epsilon}$$

We shall prove the desired estimate in this subcase with a logarithmic loss.

We start with proving the following key lemma

Lemma 4.4.7. *If*

$$\begin{aligned}
& \alpha < 0, \quad -\alpha \approx \mu \nu, \\
& s_1 - s_2 \approx s_4 - s_3 \approx \mu, \\
& s_3 - s_2 \approx s_4 - s_1 \approx \nu, \\
& |t_*(s)| < |1 - t_*(s)|, \quad |t_*| \gg \epsilon,
\end{aligned}$$

then

$$|J_\epsilon(s)| \lesssim \frac{\epsilon^4}{\mu^2 \nu^2} \frac{1}{|t_*(s)|}.$$

Proof. Recall that $f(a) = a^2 \log |a|$. To make the computations clear we introduce the function

$$H_\gamma(\epsilon; s) = F(s - \epsilon\gamma) = \int_0^1 f(A(t, s - \epsilon\gamma)) dt.$$

We have seen in Corollary 3.3.8 that, by the local integrability of the function $x \rightarrow \log |x|$, F is a smooth function. We can then Taylor expand F around $\epsilon = 0$ as follows

$$H_\gamma(\epsilon; s) = H_\gamma(0; s) + H'_\gamma(0; s)\epsilon + \frac{1}{2}H''_\gamma(0; s)\epsilon^2 + \frac{1}{6}H_\gamma^{(3)}(0; s)\epsilon^3 + \frac{1}{24}H_\gamma^{(4)}(\tilde{\epsilon}; s)\epsilon^4 \quad (4.63)$$

for some $\tilde{\epsilon} \in (0, \epsilon)$.

Lemma 4.4.8. (*Cancellations*) For any multiindex $\alpha \in \{\alpha \in \mathbb{Z}_+^4 : |\alpha| \leq 4\} - \{(1, 1, 1, 1)\}$

$$\sum_\gamma \sigma_\gamma \gamma^\alpha = \sum_\gamma \sigma_\gamma \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \gamma_3^{\alpha_3} \gamma_4^{\alpha_4} = 0.$$

Now, considering the fact that, for all $m = 1, 2, 3, 4$, we have

$$\sum_\gamma \sigma_\gamma H_\gamma^{(m)}(0; s) = \sum_{\alpha \in \mathbb{Z}_+^4 : |\alpha|=m} \frac{1}{\alpha!} \frac{\partial^\alpha F(s)}{\partial s_1^{\alpha_1} \partial s_2^{\alpha_2} \partial s_3^{\alpha_3} \partial s_4^{\alpha_4}} \left[\sum_\gamma \sigma_\gamma \gamma_1^{\alpha_1} \gamma_2^{\alpha_2} \gamma_3^{\alpha_3} \gamma_4^{\alpha_4} \right],$$

and the cancelations given by Lemma 4.4.8, we deduce that

$$\sum_\gamma \sigma_\gamma H_\gamma^{(m)}(0; s) = 0, \quad m = 1, 2, 3.$$

Moreover, and since $\sum_\gamma \sigma_\gamma \gamma_1 \gamma_2 \gamma_3 \gamma_4 = 1$, we have

$$\sum_\gamma \sigma_\gamma H_\gamma^{(4)}(0; s) = \frac{\partial^4 F(s)}{\partial s_1 \partial s_2 \partial s_3 \partial s_4}.$$

By the smoothness of F , we have

$$\begin{aligned} \partial_{s_1} F(s) &= \int_0^1 f'(A(t, s)) \partial_{s_1} A(t, s) dt, \\ \partial_{s_1} \partial_{s_2} F(s) &= \int_0^1 f''(A(t, s)) \partial_{s_1} A(t, s) \partial_{s_2} A(t, s) dt \\ &= 2 \int_0^1 \log |A(t, s)| \partial_{s_1} A(t, s) \partial_{s_2} A(t, s) dt + 3 \int_0^1 \partial_{s_1} A(t, s) \partial_{s_2} A(t, s) dt. \end{aligned}$$

Differentiating w.r.t s_3 according to Lemma 3.3.7 we get

$$\begin{aligned}\partial_{s_1}\partial_{s_2}\partial_{s_3}F(s) &= 2P.V. \int_0^1 \frac{\partial_{s_1}A(t,s)\partial_{s_2}A(t,s)\partial_{s_3}A(t,s)}{A(t,s)}dt \\ &= \frac{-2}{\alpha(s)} P.V. \int_0^1 \frac{1+t+s_4}{(t-t_*(s))(t-t_-(s))[1+t+s_1][1+t+s_2][1+t+s_3]}dt.\end{aligned}$$

This is because $\partial_{s_1}A(t,s)\partial_{s_2}A(t,s)$ is independent of s_3 . Recall that

$$\alpha(s) = s_1 - s_2 + s_3 - s_4.$$

For the sake of simplicity, we shall use the following notations in what follows

$$\begin{aligned}q_m(s_1, s_2, s_3, s_4) &= 1 + t_-(s) + s_m, & m = 1, 2, 3, 4, \\ r_m(s_1, s_2, s_3, s_4) &= 1 + t_*(s) + s_m, & m = 1, 2, 3, 4, \\ \beta(s) &= r_1(s)r_2(s)r_3(s)r_4(s)\alpha(s)[t_*(s) - t_-(s)], \\ \gamma(s) &= q_1(s)q_2(s)q_3(s)q_4(s)\alpha(s)[t_-(s) - t_*(s)].\end{aligned}$$

Calculating the integral explicitly by decomposing it into its partial fractions, we get

$$\begin{aligned}\partial_{s_1}\partial_{s_2}\partial_{s_3}F(s) &= -2a_*(s)P.V. \int_0^1 \frac{1}{t-t_*(s)}dt - 2a_-(s) \int_0^1 \frac{1}{t-t_-(s)}dt - 2 \sum_{l=1}^3 a_l(s) \int_0^1 \frac{1}{1+t+s_l}dt \\ &= -2a_*(s) \log \frac{|1-t_*(s)|}{|t_*(s)|} - 2a_-(s) \log \frac{|1-t_-(s)|}{|t_-(s)|} - 2 \sum_{l=1}^3 a_l(s) \log \frac{2+s_l}{1+s_l},\end{aligned}\quad (4.64)$$

where

$$\begin{aligned}a_1(s_1, s_2, s_3, s_4) &= \frac{s_4 - s_1}{\alpha(s)q_1(s)r_1(s)(s_2 - s_1)(s_3 - s_1)}, \\ a_2(s_1, s_2, s_3, s_4) &= \frac{s_4 - s_2}{\alpha(s)q_2(s)r_2(s)(s_1 - s_2)(s_3 - s_2)}, \\ a_3(s_1, s_2, s_3, s_4) &= \frac{s_4 - s_3}{\alpha(s)q_3(s)r_3(s)(s_1 - s_3)(s_2 - s_3)}, \\ a_*(s_1, s_2, s_3, s_4) &= \frac{1}{\alpha(s)[t_*(s) - t_-(s)]} \frac{r_4(s)}{r_1(s)r_2(s)r_3(s)} = \frac{r_4^2(s)}{\beta(s)}, \\ a_-(s_1, s_2, s_3, s_4) &= \frac{1}{\alpha(s)[t_-(s) - t_*(s)]} \frac{q_4(s)}{q_1(s)q_2(s)q_3(s)} = \frac{q_4^2(s)}{\gamma(s)}.\end{aligned}$$

Now, the derivative w.r.t. s_4 takes the form

$$\begin{aligned}\partial_{s_1}\partial_{s_2}\partial_{s_3}\partial_{s_4}F(s) &= -2 \log \frac{1-t_*(s)}{t_*(s)} \partial_{s_4}a_*(s) - 2 \log \frac{|1-t_-(s)|}{|t_-(s)|} \partial_{s_4}a_-(s) + \\ &+ 2a_*(s) \frac{\partial_{s_4}t_*(s)}{t_*(s)[1-t_*(s)]} + 2a_-(s) \frac{\partial_{s_4}t_-(s)}{t_-(s)[1-t_-(s)]} - 2\partial_{s_4} \left(\sum_{l=1}^3 a_l(s) \log \frac{2+s_l}{1+s_l} \right).\end{aligned}$$

We shall show that $\sum_{l=1}^3 a_l(s) \log \frac{2+s_l}{1+s_l}$ does not depend on s_3 . We have

$$\alpha(s)r_l(s)q_l(s) = (-1)^l \prod_{m \neq l} (s_m - s_l), \quad \forall l \in \{1, 2, 3, 4\}. \quad (4.65)$$

The identities (4.65) imply that

$$\begin{aligned} a_1(s_1, s_2, s_3, s_4) &= \frac{s_4 - s_1}{\alpha(s)q_1(s)r_1(s)(s_2 - s_1)(s_3 - s_1)} = \frac{-1}{(s_2 - s_1)^2(s_3 - s_1)^2}, \\ a_2(s_1, s_2, s_3, s_4) &= \frac{s_4 - s_2}{\alpha(s)q_2(s)r_2(s)(s_1 - s_2)(s_3 - s_2)} = \frac{1}{(s_1 - s_2)^2(s_3 - s_2)^2}, \\ a_3(s_1, s_2, s_3, s_4) &= \frac{s_4 - s_3}{\alpha(s)q_3(s)r_3(s)(s_1 - s_3)(s_2 - s_3)} = \frac{-1}{(s_1 - s_3)^2(s_2 - s_3)^2}. \end{aligned}$$

and hence

$$\partial_{s_4} \left[\sum_{l=1}^3 a_l(s_1, s_2, s_3, s_4) \log \frac{2+s_l}{1+s_l} \right] = 0.$$

This shows that the last three terms in (4.64) actually vanish when differentiated w.r.t s_4 . We now look at the derivative of the first two terms.

$$\partial_{s_4} \left[a_*(s) \log \frac{|1 - t_*(s)|}{|t_*(s)|} \right] = \left(\log \frac{|1 - t_*(s)|}{|t_*(s)|} \right) \partial_{s_4} a_*(s) + a_*(s) \left[\partial_{s_4} \log \frac{|1 - t_*(s)|}{|t_*(s)|} \right],$$

and

$$\begin{aligned} \partial_{s_4} a_*(s) &= \partial_{s_4} \frac{r_4^2(s)}{\beta(s)} = \frac{2r_4(s)\partial_{s_4}r_4(s)}{\beta(s)} - \frac{r_4^2(s)\partial_{s_4}\beta(s)}{\beta^2(s)} \\ &= \frac{r_4(s)}{\beta^2(s)} [2\beta(s)\partial_{s_4}r_4(s) - r_4(s)\partial_{s_4}\beta(s)] \\ &= \frac{r_4(s)}{\beta^2(s)} [2\beta(s) + 2\beta(s)\partial_{s_4}t_*(s) - r_4(s)\partial_{s_4}\beta(s)]. \end{aligned} \quad (4.66)$$

Now

$$\begin{aligned} \partial_{s_4}\beta(s) &= \beta(s) \left[\frac{-1}{\alpha(s)} + \frac{1}{r_4(s)} + \left(\frac{1}{r_1(s)} + \frac{1}{r_2(s)} + \frac{1}{r_3(s)} + \frac{1}{r_4(s)} \right) \partial_{s_4}t_*(s) \right. \\ &\quad \left. + \frac{\partial_{s_4}t_*(s) - \partial_{s_4}t_-(s)}{t_*(s) - t_-(s)} \right]. \end{aligned} \quad (4.67)$$

At this point, we have to do some estimates. The reason we insist on doing precise computations here is to make sure that we are getting the best possible estimate for $\partial_{s_1}\partial_{s_2}\partial_{s_3}\partial_{s_4}F(s_1, s_2, s_3, s_4)$.

Let

$$\tilde{\mu} = s_4 - s_3, \quad \tilde{\nu} = s_4 - s_1,$$

and recall that

$$\begin{aligned}
t_*(s) &= -1 + \frac{s_1 s_3 - s_2 s_4}{-\alpha(s)} + \frac{\sqrt{\Delta(s)}}{-\alpha(s)}, \\
t_-(s) &= -1 + \frac{s_1 s_3 - s_2 s_4}{-\alpha(s)} + \frac{\sqrt{\Delta(s)}}{\alpha(s)}, \\
\Delta(s) &= (s_1 - s_2)(s_3 - s_2)(s_4 - s_3)(s_4 - s_1) = \mu \nu \tilde{\mu} \tilde{\nu}, \\
\alpha(s) &= s_1 - s_2 + s_3 - s_4.
\end{aligned} \tag{4.68}$$

We have that

$$\begin{aligned}
r_1(s) &= \frac{\mu \tilde{\nu} + \sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}}{-\alpha(s)} \approx 1 \\
r_2(s) &= \frac{\mu \nu + \sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}}{-\alpha(s)} \approx 1 \\
r_3(s) &= \frac{\tilde{\mu} \nu + \sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}}{-\alpha(s)} \approx 1 \\
r_4(s) &= \frac{\tilde{\mu} \tilde{\nu} + \sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}}{-\alpha(s)} \approx 1
\end{aligned} \tag{4.69}$$

And since by the hypotheses of the Lemma 4.4.7.

$$t_*(s) - t_-(s) = \frac{2\sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}}{-\alpha}, \quad \mu \approx \tilde{\mu}, \quad \nu \approx \tilde{\nu} \quad \text{and} \quad \mu \nu \approx -\alpha \tag{4.70}$$

then

$$t_*(s) - t_-(s) \approx 1. \tag{4.71}$$

By (4.69) and (4.71), we have

$$|\beta(s)| \approx |\alpha(s)| \approx \mu \nu. \tag{4.72}$$

To estimate $\partial_{s_4} a_*(s)$, it remains to estimate $\partial_{s_4} t_*(s)$ and $\partial_{s_4} t_-(s)$. Taking the derivatives of $t_*(s)$ and $t_-(s)$ w.r.t s_4 , we get the following estimates

$$-\partial_{s_4} t_*(s) = \frac{\mu \nu (2\sqrt{\mu \tilde{\mu} \nu \tilde{\nu}} + \mu \tilde{\nu} + \tilde{\mu} \nu)}{2\alpha^2(s) \sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}} \approx \frac{1}{\mu \nu}, \tag{4.73}$$

$$\begin{aligned}
\partial_{s_4} t_-(s) &= \frac{\mu \nu (2\sqrt{\mu \tilde{\mu} \nu \tilde{\nu}} - \mu \tilde{\nu} - \tilde{\mu} \nu)}{2\alpha^2(s) \sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}} \\
&= \frac{\mu \nu (\mu - \nu)^2}{2\sqrt{\mu \tilde{\mu} \nu \tilde{\nu}} (2\sqrt{\mu \tilde{\mu} \nu \tilde{\nu}} + \mu \tilde{\nu} + \tilde{\mu} \nu)} \approx \frac{(\nu - \mu)^2}{\mu \nu}.
\end{aligned} \tag{4.74}$$

Actually,

$$\partial_{s_4} t_-(s) - \partial_{s_4} t_*(s) = \frac{\mu \nu (\mu \tilde{\nu} + \tilde{\mu} \nu)}{\alpha^2(s) \sqrt{\mu \tilde{\mu} \nu \tilde{\nu}}} \approx \frac{1}{\mu \nu}. \tag{4.75}$$

Using the estimates (4.69), (4.71), (4.75) and (4.67), we get the estimate

$$|\partial_{s_4}\beta(s)| \lesssim 1. \quad (4.76)$$

When $|t_*(s)| < |1 - t_*(s)|$, then $\frac{|1 - t_*(s)|}{|t_*(s)|} > 1$ and we have

$$\log \frac{|1 - t_*(s)|}{|t_*(s)|} \lesssim \frac{1}{|t_*(s)|}. \quad (4.77)$$

From (4.69), (4.72), (4.73), (4.76), (4.77), and (4.66)

$$\left| \log \frac{|1 - t_*(s)|}{|t_*(s)|} \partial_{s_4} a_*(s) \right| \lesssim \frac{1}{\mu^2 \nu^2 |t_*(s)|}. \quad (4.78)$$

To finish estimating $\partial_{s_4} \left[a_*(s) \log \frac{|1 - t_*(s)|}{|t_*(s)|} \right]$, it remains to look at

$$a_*(s) \partial_{s_4} \log \frac{|1 - t_*(s)|}{|t_*(s)|} = -\frac{a_*(s) \partial_{s_4} t_*(s)}{t_*(s)[1 - t_*(s)]} = -\frac{r^2(s)}{\beta(s)} \frac{\partial_{s_4} t_*(s)}{t_*(s)[1 - t_*(s)]}.$$

Using the estimates (4.72) and (4.73), we get the estimate

$$\left| a_*(s) \partial_{s_4} \log \frac{|1 - t_*(s)|}{|t_*(s)|} \right| \lesssim \frac{1}{\mu^2 \nu^2 |t_*(s)|}. \quad (4.79)$$

The last term we need to estimate to complete estimating $\sum_\gamma \sigma_\gamma H_\gamma^{(4)}(0; s)$ is

$$\partial_{s_4} \left[a_-(s) \log \frac{1 - t_-(s)}{|t_-(s)|} \right] = a_-(s) \partial_{s_4} \left(\log \frac{1 - t_-(s)}{|t_-(s)|} \right) + \log \frac{1 - t_-(s)}{|t_-(s)|} \partial_{s_4} a_-(s),$$

where

$$\begin{aligned} a_-(s_1, s_2, s_3, s_4) &= \frac{1}{\alpha(s)[t_-(s) - t_*(s)]} \frac{q_4(s)}{q_1(s)q_2(s)q_3(s)} = \frac{q_4^2(s)}{\gamma(s)}, \\ \gamma(s) &= q_1(s)q_2(s)q_3(s)q_4(s)\alpha(s)[t_-(s) - t_*(s)], \\ q_m(s) &= 1 + s_m + t_-(s), \quad m = 1, 2, 3, 4. \end{aligned}$$

We have

$$\begin{aligned} a_-(s) \partial_{s_4} \left(\log \frac{1 - t_-(s)}{|t_-(s)|} \right) &= a_-(s) \partial_{s_4} \left(\log \frac{1 - t_-(s)}{-t_-(s)} \right) \\ &= -a_-(s) \frac{\partial_{s_4} t_-(s)}{t_-(s)[1 - t_-(s)]} \end{aligned}$$

We begin with estimating the term $a_-(s)\partial_{s_4} \log \frac{1-t_-(s)}{|t_-(s)|}$. Let us first estimate the quantities $q_m(s)$, $m = 1, 2, 3, 4$.

$$\begin{aligned}
-q_1(s) &= \frac{\mu\tilde{\nu} - \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\alpha(s)} = \frac{\mu\tilde{\nu}(\nu - \mu)}{\mu\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} \approx \nu - \mu \\
-q_2(s) &= \frac{\mu\nu - \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\alpha(s)} = \frac{\mu\nu(\tilde{\mu} + \nu)}{\mu\nu + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} \approx \mu + \nu \\
q_3(s) &= \frac{\tilde{\mu}\nu - \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\alpha(s)} = \frac{\tilde{\mu}\nu(\nu - \mu)}{\tilde{\mu}\nu + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} \approx \nu - \mu \\
q_4(s) &= \frac{\tilde{\mu}\tilde{\nu} - \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\alpha(s)} = \frac{\tilde{\mu}\tilde{\nu}(\tilde{\mu} + \nu)}{\tilde{\mu}\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} \approx \mu + \nu
\end{aligned} \tag{4.80}$$

The estimates (4.80) yield that

$$\frac{q_4(s)}{q_1(s)q_2(s)q_3(s)} \approx \frac{1}{(\nu - \mu)^2}. \tag{4.81}$$

Look at $\frac{1}{|t_-(s)|(1-t_-(s))}$. Recall that

$$t_- = \tau_- - 1 - s_2, \quad \tau_- = \frac{\mu\nu}{-\alpha} \left[1 - \sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} \right] < 0, \quad \text{because } \alpha < 0$$

And that

$$\tau_- = \frac{\mu\nu}{\alpha} \left[\sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} - 1 \right] = -\frac{\mu + \nu - \alpha}{\sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} + 1},$$

whereas

$$\mu + \nu - \alpha = s_4 - s_2 > 0, \quad 1 + \sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} \in \left(\frac{s_4 - s_2}{s_3 - s_2}, \frac{s_4 - s_2}{s_1 - s_2} \right).$$

This implies that

$$-\tau_- = \frac{\mu + \nu - \alpha}{\sqrt{\left(1 - \frac{\alpha}{\mu}\right)\left(1 - \frac{\alpha}{\nu}\right)} + 1} \in (s_1 - s_2, s_3 - s_2) = (\mu, \nu),$$

hence

$$t_-(s) = -1 - s_2 + \tau_- \in (-1 - s_3, -1 - s_1). \tag{4.82}$$

Consequently

$$\boxed{|t_-(s)| \approx 1, \quad 1 - t_-(s) \approx 1} \tag{4.83}$$

Also

$$\frac{t_-(s) - 1}{t_-(s)} \in \left(1 + \frac{1}{1 + s_3}, 1 + \frac{1}{1 + s_1}\right) \subset \left(\frac{3}{2}, 2\right). \quad (4.84)$$

Thus

$$\frac{|1 - t_-(s)|}{|t_-(s)|} = \frac{t_-(s) - 1}{t_-(s)} \approx 1. \quad (4.85)$$

Therefore, by (4.70), (4.71), (4.74), (4.81) and (4.83), we get that

$$\left| a_-(s) \partial_{s_4} \log \frac{|1 - t_-(s)|}{|t_-(s)|} \right| \approx \frac{1}{\mu^2 \nu^2}. \quad (4.86)$$

Now, we turn to estimating the term $\log \frac{|1 - t_-(s)|}{|t_-(s)|} \partial_{s_4} a_-(s)$. In what follows, we compute and estimate $\partial_{s_4} a_-(s)$.

$$\begin{aligned} \partial_{s_4} a_-(s) &= \partial_{s_4} \frac{q_4^2(s)}{\gamma(s)} \\ &= \frac{2q_4(s) \partial_{s_4} q_4(s)}{\gamma(s)} - \frac{q_4^2(s) \partial_{s_4} \gamma(s)}{\gamma^2(s)} \\ &= \frac{q_4(s)}{\gamma^2(s)} [2\gamma(s) \partial_{s_4} q_4(s) - q_4(s) \partial_{s_4} \gamma(s)] \\ &= \frac{q_4(s)}{\gamma^2(s)} [2\gamma(s) + 2\gamma(s) \partial_{s_4} t_-(s) - q_4(s) \partial_{s_4} \gamma(s)]. \end{aligned} \quad (4.87)$$

We have

$$\begin{aligned} \partial_{s_4} \gamma(s) &= \gamma(s) \left[\frac{-1}{\alpha(s)} + \frac{1}{q_4(s)} + \left(\frac{1}{q_1(s)} + \frac{1}{q_2(s)} + \frac{1}{q_3(s)} + \frac{1}{q_4(s)} \right) \partial_{s_4} t_-(s) \right. \\ &\quad \left. + \frac{\partial_{s_4} t_-(s) - \partial_{s_4} t_*(s)}{t_-(s) - t_*(s)} \right]. \end{aligned} \quad (4.88)$$

Substituting from (4.88) in (4.87), we get

$$\begin{aligned} \partial_{s_4} a_-(s) &= \frac{q_4(s)}{\gamma(s)} \left[1 + 2\partial_{s_4} t_-(s) + \frac{q_4(s)}{\alpha(s)} - \frac{q_4(s) [\partial_{s_4} t_-(s) - \partial_{s_4} t_*(s)]}{t_-(s) - t_*(s)} \right. \\ &\quad \left. - \left(\frac{1}{q_1(s)} + \frac{1}{q_2(s)} + \frac{1}{q_3(s)} + \frac{1}{q_4(s)} \right) q_4(s) \partial_{s_4} t_-(s) \right] \\ &= \frac{1}{-\alpha(s) [t_*(s) - t_-(s)] q_1(s) q_2(s) q_3(s)} \\ &\quad \left[1 + \frac{q_4(s)}{\alpha(s)} + \frac{q_4(s) [\partial_{s_4} t_-(s) - \partial_{s_4} t_*(s)]}{t_*(s) - t_-(s)} \right. \\ &\quad \left. + \left(\frac{1}{q_4(s)} - \frac{1}{q_1(s)} - \frac{1}{q_2(s)} - \frac{1}{q_3(s)} \right) q_4(s) \partial_{s_4} t_-(s) \right] \end{aligned} \quad (4.89)$$

By (4.70), (4.75) and (4.80) we have

$$\begin{aligned}
& 1 + \frac{q_4(s)}{\alpha(s)} + \frac{q_4(s)[\partial_{s_4} t_-(s) - \partial_{s_4} t_*(s)]}{t_*(s) - t_-(s)} \\
&= 1 + \frac{\tilde{\mu}\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\tilde{\mu}\tilde{\nu}(\tilde{\mu} + \nu)} \left[\frac{1}{\alpha(s)} + \frac{-\alpha(s)}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} \frac{\mu\nu(\mu\tilde{\nu} + \tilde{\mu}\nu)}{\alpha^2(s)\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} \right] \\
&= 1 + \frac{\tilde{\mu}\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\tilde{\mu}\tilde{\nu}(\tilde{\mu} + \nu)} \left[\frac{1}{\alpha(s)} - \frac{1}{2\alpha(s)} \frac{\mu\tilde{\nu} + \tilde{\mu}\nu}{\tilde{\mu}\tilde{\nu}} \right] \\
&= 1 - \frac{(\tilde{\mu} + \tilde{\nu})(\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} - \tilde{\mu}\tilde{\nu})}{2\alpha(s)\tilde{\mu}\tilde{\nu}} \\
&= \frac{2\alpha(s)\tilde{\mu}\tilde{\nu} - (\tilde{\mu} + \tilde{\nu})\tilde{\mu}\tilde{\nu} - (\tilde{\mu} + \tilde{\nu})\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{2\alpha(s)\tilde{\mu}\tilde{\nu}} \tag{4.90}
\end{aligned}$$

$$= - \frac{(\tilde{\mu} + \nu)(\nu - \mu)^2}{2[(\tilde{\mu} + \tilde{\nu})\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \tilde{\mu}\tilde{\nu}(\mu + \nu)]}. \tag{4.91}$$

It also follows from (4.80) that

$$\begin{aligned}
& \frac{1}{q_4(s)} - \frac{1}{q_1(s)} - \frac{1}{q_2(s)} - \frac{1}{q_3(s)} \\
&= \frac{\tilde{\mu}\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\tilde{\mu}\tilde{\nu}(\tilde{\mu} + \nu)} + \frac{\mu\nu + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\mu\nu(\tilde{\mu} + \nu)} + \frac{\mu\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\mu\tilde{\nu}(\nu - \mu)} - \frac{\tilde{\mu}\nu + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\tilde{\mu}\nu(\nu - \mu)} \\
&= \frac{\tilde{\mu}\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\tilde{\mu}\tilde{\nu}(\tilde{\mu} + \nu)} + \frac{\mu\nu + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\mu\nu(\tilde{\mu} + \nu)} + \frac{1}{\nu - \mu} \frac{(\tilde{\mu}\nu - \mu\tilde{\nu})\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\mu\tilde{\mu}\nu\mu} \\
&= \frac{\tilde{\mu}\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\tilde{\mu}\tilde{\nu}(\tilde{\mu} + \nu)} + \frac{\mu\nu + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\mu\nu(\tilde{\mu} + \nu)} + \frac{-\alpha(s)\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}}{\mu\tilde{\mu}\nu\mu} \\
&= \frac{1}{\mu\tilde{\mu}\nu\mu(\tilde{\mu} + \nu)} [2\mu\nu\tilde{\mu}\tilde{\nu} + (\tilde{\mu}\tilde{\nu} + \tilde{\mu}^2 - \tilde{\mu}\mu + \tilde{\mu}\nu)\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}] \\
&= \frac{1}{\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(\tilde{\mu} + \nu)} [2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \tilde{\mu}\tilde{\nu} + \tilde{\mu}^2 - \tilde{\mu}\mu + \tilde{\mu}\nu] \\
&= \frac{2(\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \tilde{\mu}\tilde{\nu})}{\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(\tilde{\mu} + \nu)}. \tag{4.92}
\end{aligned}$$

Using (4.74), (4.80) and (4.92), we have

$$\begin{aligned}
& \left(\frac{1}{q_4(s)} - \frac{1}{q_1(s)} - \frac{1}{q_2(s)} - \frac{1}{q_3(s)} \right) q_4(s) \partial_{s_4} t_-(s) \\
&= \frac{2(\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \tilde{\mu}\tilde{\nu})}{\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(\tilde{\mu} + \nu)} \frac{\tilde{\mu}\tilde{\nu}(\tilde{\mu} + \nu)}{\tilde{\mu}\tilde{\nu} + \sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} \frac{\mu\nu(\nu - \mu)^2}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\nu} + \tilde{\mu}\nu)} \\
&= \frac{(\nu - \mu)^2}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\nu} + \tilde{\mu}\nu}. \tag{4.93}
\end{aligned}$$

Combining (4.91) and (4.93) we have

$$\begin{aligned}
& 1 + \frac{q_4(s)}{\alpha(s)} + \frac{q_4(s)[\partial_{s_4} t_-(s) - \partial_{s_4} t_*(s)]}{t_*(s) - t_-(s)} + \left(\frac{1}{q_4(s)} - \frac{1}{q_1(s)} - \frac{1}{q_2(s)} - \frac{1}{q_3(s)} \right) q_4(s) \partial_{s_4} t_-(s) \\
&= \frac{(\nu - \mu)^2}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\nu} + \tilde{\mu}\nu} - \frac{(\tilde{\mu} + \nu)(\nu - \mu)^2}{2[(\tilde{\mu} + \tilde{\nu})\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \tilde{\mu}\tilde{\nu}(\mu + \nu)]} \\
&= \frac{[2(\tilde{\nu} - \nu)\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + 2\tilde{\mu}\tilde{\nu}(\mu + \nu) - (\tilde{\mu} + \nu)(\mu\tilde{\nu} + \tilde{\mu}\nu)](\nu - \mu)^2}{2[2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\nu} + \tilde{\mu}\nu][(\tilde{\mu} + \tilde{\nu})\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \tilde{\mu}\tilde{\nu}(\mu + \nu)]} \\
&= \frac{-\alpha[2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\mu} + \nu\tilde{\nu}](\nu - \mu)^2}{2[2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\nu} + \tilde{\mu}\nu][(\tilde{\mu} + \tilde{\nu})\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \tilde{\mu}\tilde{\nu}(\mu + \nu)]} \\
&\approx \frac{\nu^2}{\mu + \nu} \frac{(\nu - \mu)^2}{\mu\nu} \approx \frac{(\nu - \mu)^2}{\mu}. \tag{4.94}
\end{aligned}$$

Lastly, in the light of (4.89), the estimates (4.71), (4.80), and (4.94) imply the following estimate for $\partial_{s_4} a_-(s)$

$$|\partial_{s_4} a_-(s)| \approx \frac{\nu}{\mu\nu(\mu + \nu)(\nu - \mu)^2} \frac{(\nu - \mu)^2}{\mu\nu} \approx \frac{1}{\mu^2\nu^2}. \tag{4.95}$$

From (4.84) and (4.85), we have

$$\log \frac{|1 - t_-(s)|}{|t_-(s)|} = \log \frac{t_-(s) - 1}{t_-(s)} \approx 1. \tag{4.96}$$

And (4.95) and (4.96) together give the following estimate for $\log \frac{|1 - t_-(s)|}{|t_-(s)|} \partial_{s_4} a_-(s)$

$$\left| \log \frac{|1 - t_-(s)|}{|t_-(s)|} \partial_{s_4} a_-(s) \right| \approx \frac{1}{\mu^2\nu^2(\mu + \nu)}. \tag{4.97}$$

Finally, in regard of both estimates (4.86) and (4.97), we have

$$\left| \partial_{s_4} \left(a_-(s) \log \frac{|1 - t_-(s)|}{|t_-(s)|} \right) \right| \lesssim \frac{1}{\mu^2\nu^2(\mu + \nu)}. \tag{4.98}$$

We have seen how using the cancellations in Lemma 4.4.8 in (4.63) implied

$$\sum_{\gamma} \sigma_{\gamma} H_{\gamma}(\epsilon; s) = \frac{1}{24} \sum_{\gamma} \sigma_{\gamma} H_{\gamma}^{(4)}(\tilde{\epsilon}; s) \epsilon^4.$$

The estimates (4.78), (4.79) and (4.98) give the estimate

$$\left| \sum_{\gamma} \sigma_{\gamma} H_{\gamma}^{(4)}(0; s) \right| = |\partial_{s_1} \partial_{s_2} \partial_{s_3} \partial_{s_4} F(s_1, s_2, s_3, s_4)| \lesssim \frac{1}{\mu^2\nu^2|t_*(s)|}.$$

Since

$$|J_\epsilon(s)| = \left| \sum_\gamma \sigma_\gamma H_\gamma(\epsilon; s) \right| = \frac{1}{24} \left| \sum_\gamma \sigma_\gamma H_\gamma^{(4)}(\tilde{\epsilon}; s) \right|$$

then, for the assertion of Lemma 4.4.7 to follow, we actually have to show that

$$\left| \sum_\gamma \sigma_\gamma H_\gamma^{(4)}(\tilde{\epsilon}; s) \right| \lesssim \frac{1}{\mu^2 \nu^2 |t_*(s)|}. \quad (4.99)$$

uniformly in $0 \leq \tilde{\epsilon} \leq \epsilon$. First let

$$A_j(t, s - \epsilon\gamma) = \frac{d^j}{d\epsilon^j} A(t, s - \epsilon\gamma) = j! \sum_{l=1}^4 \frac{(-1)^l \gamma_l^j}{(1+t+s_l-\epsilon\gamma_l)^{j+1}}, \quad j = 1, \dots, 4.$$

Now, let us compute $H_\gamma^{(4)}(\epsilon; s)$. Notice that the function f is locally continuously differentiable.

$$\begin{aligned} H_\gamma''(\epsilon; s) &= \int_0^1 f''(A(t, s - \epsilon\gamma)) A_1^2(t, s - \epsilon\gamma) dt + \int_0^1 f'(A(t, s - \epsilon\gamma)) A_2(t, s - \epsilon\gamma) dt \\ &= 2 \int_0^1 \log(|A(t, s - \epsilon\gamma)|) A_1^2(t, s - \epsilon\gamma) dt + 3 \int_0^1 A_1^2(t, s - \epsilon\gamma) dt + \\ &\quad + \int_0^1 f'(A(t, s - \epsilon\gamma)) A_2(t, s - \epsilon\gamma) dt. \end{aligned}$$

Since

$$Q_\gamma(t, s, \epsilon) = \frac{A(t, s - \epsilon\gamma)}{t - t_*(s - \epsilon\gamma)} = \frac{\alpha(s - \epsilon\gamma)(t - t_-(s - \epsilon\gamma))}{\prod_{l=1}^4 (1 + t + s_l - \epsilon\gamma_l)}$$

is a smooth function that satisfies $|Q_\gamma(t, s, \epsilon)| \approx \mu\nu \gg \epsilon$, then we are entitled to employ Lemma 3.3.7 to compute $H_\gamma'''(\epsilon; s)$. We shall occasionally denote $A(t, s - \epsilon\gamma)$ by A and

$A_j(t, s - \epsilon\gamma)$ by A_j , $j = 1, \dots, 4$ and so forth.

$$\begin{aligned}
H_\gamma'''(\epsilon; s) &= 2P.V. \int_0^1 \frac{A_1^3}{A} dt + 4 \int_0^1 \log(|A|) A_1 A_2 dt + \int_0^1 f''(A) A_1 A_2 dt + \\
&+ \int_0^1 f'(A) A_3 dt + 6 \int_0^1 A_1 A_2 dt \\
&= 2P.V. \int_0^1 \frac{A_1^3}{A} dt + 6 \int_0^1 \log(|A|) A_1 A_2 dt + \int_0^1 f'(A) A_3 dt + 9 \int_0^1 A_1 A_2 dt \\
&= 2P.V. \int_0^1 \frac{\phi_\gamma(t, s, \epsilon)}{t - t_*(s - \epsilon\gamma)} dt + 6 \int_0^1 \log(|A|) A_1 A_2 dt + \int_0^1 f'(A) A_3 dt + 9 \int_0^1 A_1 A_2 dt \\
&= 2P.V. \int_0^1 \frac{\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)}{t - t_*(s - \epsilon\gamma)} dt + 2 \int_0^1 \psi_\gamma(t, s, \epsilon) dt + 6 \int_0^1 \log(|A|) A_1 A_2 dt + \\
&+ \int_0^1 f'(A) A_3 dt + 9 \int_0^1 A_1 A_2 dt \\
&= 2\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \log \frac{1 - t_*(s - \epsilon\gamma)}{|t_*(s - \epsilon\gamma)|} + 2 \int_0^1 \psi_\gamma(t, s, \epsilon) dt + 6 \int_0^1 \log(|A|) A_1 A_2 dt + \\
&+ \int_0^1 f'(A) A_3 dt + 9 \int_0^1 A_1 A_2 dt
\end{aligned}$$

where ϕ_γ and ψ_γ are the smooth functions given by

$$\phi_\gamma(t, s, \epsilon) = \frac{A_1^3(t, s - \epsilon\gamma)}{Q_\gamma(t, s, \epsilon)}, \quad \psi_\gamma(t, s, \epsilon) = \frac{\phi_\gamma(t, s, \epsilon) - \phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)}{t - t_*(s - \epsilon\gamma)}.$$

Proceeding with the derivatives, we have

$$\begin{aligned}
H_\gamma^{(4)}(\epsilon; s) &= 2 \log \frac{1 - t_*(s - \epsilon\gamma)}{|t_*(s - \epsilon\gamma)|} \frac{d}{d\epsilon} \phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) - \frac{2\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \frac{d}{d\epsilon} t_*(s - \epsilon\gamma)}{t_*(s - \epsilon\gamma)(1 - t_*(s - \epsilon\gamma))} + \\
&+ 2 \int_0^1 \frac{d}{d\epsilon} \psi_\gamma(t, s, \epsilon) dt + 6P.V. \int_0^1 \frac{\chi_\gamma(t, s, \epsilon)}{t - t_*(s - \epsilon\gamma)} dt + \int_0^1 f''(A) A_1 A_3 dt + \\
&+ \int_0^1 f'(A) A_4 dt + 9 \int_0^1 (A_2^2 + A_1 A_3) dt \\
&= 2 \log \frac{1 - t_*(s - \epsilon\gamma)}{|t_*(s - \epsilon\gamma)|} \frac{d}{d\epsilon} \phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) - \frac{2\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \frac{d}{d\epsilon} t_*(s - \epsilon\gamma)}{t_*(s - \epsilon\gamma)(1 - t_*(s - \epsilon\gamma))} + \\
&+ 2 \int_0^1 \frac{d}{d\epsilon} \psi_\gamma(t, s, \epsilon) dt + 6\chi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \log \frac{1 - t_*(s - \epsilon\gamma)}{|t_*(s - \epsilon\gamma)|} + 6 \int_0^1 \omega_\gamma(t, s, \epsilon) dt + \\
&+ 2 \int_0^1 \log(|A|) A_1 A_3 dt + \int_0^1 f'(A) A_4 dt + 9 \int_0^1 (A_2^2 + \frac{4}{3} A_1 A_3) dt \tag{4.100}
\end{aligned}$$

where χ_γ and ω_γ are the smooth functions given by

$$\chi_\gamma(t, s, \epsilon) = \frac{A_1^2(t, s - \epsilon\gamma) A_2(t, s - \epsilon\gamma)}{Q_\gamma(t, s, \epsilon)}, \quad \omega_\gamma(t, s, \epsilon) = \frac{\chi_\gamma(t, s, \epsilon) - \chi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)}{t - t_*(s - \epsilon\gamma)}.$$

Since the cancelations coming from summing over γ as demonstrated in Lemma 4.4.8 are no longer useful. Then what is actually left to do now is estimate $|H_\gamma^{(4)}(\epsilon; s)|$ and show that it has an estimate consistent with (4.99). We shall prove that $H_\gamma^{(4)}(\epsilon; s)$ enjoys the following estimate

$$|H_\gamma^{(4)}(\epsilon, s)| \lesssim \frac{1}{\mu^2 \nu^2 t_*(s)}. \quad (4.101)$$

This follows from (4.100) and a careful look at (4.102) below that summarizes the estimates of the terms that appear in it.

$$\begin{aligned} |\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)| &\lesssim \frac{1}{\mu\nu} \\ \left| \frac{d}{d\epsilon} t_*(s - \epsilon\gamma) \right| &\lesssim \sum_{l=1}^4 |\partial_{s_l} t_*(s)| \lesssim \frac{1}{\mu\nu} \\ \left| \frac{d}{d\epsilon} \phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \right| &\lesssim \frac{1}{\mu^2 \nu^2} \\ \left| \frac{d}{d\epsilon} \psi_\gamma(t, s, \epsilon) \right| &\lesssim \frac{1}{\mu^2 \nu^2} \\ |\chi_\gamma(t, s, \epsilon)| &\lesssim \frac{1}{\mu\nu} \\ |\omega_\gamma(t, s, \epsilon)| &\lesssim \frac{1}{\mu\nu} \\ \left| \int_0^1 \log(|A(t, s - \epsilon\gamma)|) A_1(t, s - \epsilon\gamma) A_3(t, s - \epsilon\gamma) dt \right| &\lesssim \frac{1}{\mu\nu} \\ \left| \int_0^1 f'(A(t, s - \epsilon\gamma)) A_4(t, s - \epsilon\gamma) dt \right| &\lesssim 1 \\ \left| \int_0^1 (A_2^2(t, s - \epsilon\gamma) + \frac{4}{3} A_1(t, s - \epsilon\gamma) A_3(t, s - \epsilon\gamma)) dt \right| &\lesssim 1 \end{aligned} \quad (4.102)$$

So as to prove the estimates in (4.102), we do some preliminary estimates first. We begin with the obvious observation that

$$|A_j(t, s - \epsilon\gamma)| \lesssim 1, \quad (4.103)$$

$$|\partial_t A_j(t, s - \epsilon\gamma)| = \left| (j+1)! \sum_{l=1}^4 \frac{(-1)^{l+1} \gamma_l^j}{(1+t+s_l-\epsilon\gamma_l)^{j+2}} \right| \lesssim 1 \quad (4.104)$$

uniformly in $t \in [0, 1]$ for all $j = 1, 2, 3, 4$. We have

$$\frac{d}{d\epsilon} t_*(s - \epsilon\gamma) = - \sum_{l=1}^4 \gamma_l \partial_{s_l} t_*(s)$$

Since $A(t_*(s), s) = 0$. Then

$$\partial_{s_l} A(t_*(s), s) = \partial_t A(t_*(s), s) \partial_{s_l} t_*(s) + \partial_{s_l} A(t_*(s), s) = 0, \quad l = 1, 2, 3, 4.$$

Hence, applying Lemma 3.4.5, we get

$$|\partial_{s_l} t_*(s)| = \frac{|\partial_{s_l} A(t_*(s), s)|}{|\partial_t A(t_*(s), s)|} \approx \frac{1}{\mu\nu}, \quad l = 1, 2, 3, 4.$$

We thus have the following estimate

$$\left| \frac{d}{d\epsilon} t_*(s - \epsilon\gamma) \right| \lesssim \sum_{l=1}^4 |\partial_{s_l} t_*(s)| \lesssim \frac{1}{\mu\nu}. \quad (4.105)$$

To estimate $\frac{d}{d\epsilon} t_-(s - \epsilon\gamma)$, it suffices to notice that

$$\begin{aligned} \partial_{s_1} t_-(s) &= \frac{\tilde{\mu}\nu(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} - \mu\nu - \tilde{\mu}\tilde{\nu})}{2\alpha^2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} = -\frac{\tilde{\mu}\nu(\tilde{\mu} + \nu)^2}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\nu + \tilde{\mu}\tilde{\nu})} \\ \partial_{s_2} t_-(s) &= -\frac{\tilde{\mu}\tilde{\nu}(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} - \mu\tilde{\nu} - \tilde{\mu}\nu)}{2\alpha^2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} = \frac{\tilde{\mu}\tilde{\nu}(\nu - \mu)^2}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\nu} + \tilde{\mu}\nu)} \\ \partial_{s_3} t_-(s) &= \frac{\mu\tilde{\nu}(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} - \mu\nu - \tilde{\mu}\tilde{\nu})}{2\alpha^2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} = -\frac{\mu\tilde{\nu}(\tilde{\mu} + \nu)^2}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\nu + \tilde{\mu}\tilde{\nu})} \\ \partial_{s_4} t_-(s) &= -\frac{\mu\nu(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} - \mu\tilde{\nu} - \tilde{\mu}\nu)}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}} = \frac{\mu\nu(\mu - \nu)^2}{2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}}(2\sqrt{\mu\tilde{\mu}\nu\tilde{\nu}} + \mu\tilde{\nu} + \tilde{\mu}\nu)} \end{aligned}$$

So

$$|\partial_{s_1} t_-(s)| \approx |\partial_{s_3} t_-(s)| \approx \frac{(\mu + \nu)^2}{\mu\nu}, \quad |\partial_{s_2} t_-(s)| \approx |\partial_{s_4} t_-(s)| \approx \frac{(\nu - \mu)^2}{\mu\nu},$$

and we have

$$\left| \frac{d}{d\epsilon} t_-(s - \epsilon\gamma) \right| \lesssim \sum_{l=1}^4 |\partial_{s_l} t_-(s)| \lesssim \frac{\nu^2}{\mu\nu} \lesssim \frac{1}{\mu\nu}, \quad (4.106)$$

because, $0 < \mu < \nu \lesssim 1$. Moreover

$$|\alpha(s - \epsilon\gamma)| = |\alpha(s) - \epsilon\alpha(\gamma)| \lesssim |\alpha(s)| + \epsilon \approx |\alpha(s)| \approx \mu\nu, \quad (4.107)$$

whenever $|\alpha(s)| \gg \epsilon$.

Remark 4.4.3. The smoothness of the maps $\epsilon \mapsto t_*(s - \epsilon\gamma)$ and $\epsilon \mapsto t_-(s - \epsilon\gamma)$ is a consequence of (4.68) and the facts that $\alpha(s - \epsilon\gamma) > 0$ and that $\mu \approx \tilde{\mu} \gg \epsilon$ and $\nu \approx \tilde{\nu} \gg \epsilon$ which imply that

$$\begin{aligned} \Delta(s - \epsilon\gamma) &= (s_1 - s_2 - \epsilon(\gamma_1 - \gamma_2))(s_3 - s_2 - \epsilon(\gamma_3 - \gamma_2))(s_4 - s_3 - \epsilon(\gamma_4 - \gamma_3)) \\ &\quad (s_4 - s_1 - \epsilon(\gamma_4 - \gamma_1)) \\ &= (\mu - \epsilon(\gamma_1 - \gamma_2))(\nu - \epsilon(\gamma_3 - \gamma_2))(\tilde{\mu} - \epsilon(\gamma_4 - \gamma_3))(\tilde{\nu} - \epsilon(\gamma_4 - \gamma_1)) \\ &= \mu\tilde{\mu}\nu\tilde{\nu} = \Delta(s) + O(\epsilon\mu\nu). \end{aligned} \quad (4.108)$$

It also follows from (4.68) and the estimates (4.107) and (4.108) that

$$t_*(s - \epsilon\gamma) - t_-(s - \epsilon\gamma) = \frac{2\Delta(s - \epsilon\gamma)}{-\alpha(s - \epsilon\gamma)} = \frac{\Delta(s) + O(\epsilon\mu\nu)}{-\alpha(s) + O(\epsilon)} \approx \frac{\Delta(s)}{-\alpha(s)} \approx 1 \quad (4.109)$$

when $\mu\nu \gg \epsilon$. Because of (4.82), we have that

$$t - t_-(s) \in (1 + s_1, 2 + s_3) \quad (4.110)$$

and, as a result of Remark 4.4.3 and the estimate (4.106), we have

$$t - t_-(s - \epsilon\gamma) = t - t_-(s) + O\left(\frac{\epsilon}{\mu\nu}\right) \approx 1 \quad (4.111)$$

when $\mu\nu \gg \epsilon$. This, together with the estimate (4.107) for $\alpha(s - \epsilon\gamma)$ mean that $Q_\gamma(t, s, \epsilon)$ has the following estimate for all $t \in [0, 1]$,

$$|Q_\gamma(t, s, \epsilon)| \approx \mu\nu. \quad (4.112)$$

Now, we have

$$\begin{aligned} \frac{d}{d\epsilon} Q_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) &= \frac{d}{d\epsilon} \frac{\alpha(s - \epsilon\gamma)[t_*(s - \epsilon\gamma) - t_-(s - \epsilon\gamma)]}{\prod_{l=1}^4 (1 + t_*(s - \epsilon\gamma) + s_l - \epsilon\gamma_l)} \\ &= \frac{-\alpha(\gamma)[t_*(s - \epsilon\gamma) - t_-(s - \epsilon\gamma)]}{\prod_{l=1}^4 (1 + t_*(s - \epsilon\gamma) + s_l - \epsilon\gamma_l)} + \frac{\alpha(s - \epsilon\gamma)\left[\frac{d}{d\epsilon}t_*(s - \epsilon\gamma) - \frac{d}{d\epsilon}t_-(s - \epsilon\gamma)\right]}{\prod_{l=1}^4 (1 + t_*(s - \epsilon\gamma) + s_l - \epsilon\gamma_l)} + \\ &\quad + \sum_{l=1}^4 \frac{\alpha(s - \epsilon\gamma)[t_*(s - \epsilon\gamma) - t_-(s - \epsilon\gamma)]\left[\frac{d}{d\epsilon}t_*(s - \epsilon\gamma) - \gamma_l\right]}{(1 + t_*(s - \epsilon\gamma) + s_l - \epsilon\gamma_l) \prod_{l=1}^4 (1 + t_*(s - \epsilon\gamma) + s_l - \epsilon\gamma_l)} \\ &= Q_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \left(\frac{-\alpha(\gamma)}{\alpha(s - \epsilon\gamma)} + \frac{\frac{d}{d\epsilon}t_*(s - \epsilon\gamma) - \frac{d}{d\epsilon}t_-(s - \epsilon\gamma)}{t_*(s - \epsilon\gamma) - t_-(s - \epsilon\gamma)} + \right. \\ &\quad \left. + \sum_{l=1}^4 \frac{\frac{d}{d\epsilon}t_*(s - \epsilon\gamma) - \gamma_l}{1 + t_*(s - \epsilon\gamma) + s_l - \epsilon\gamma_l} \right) \end{aligned}$$

which, regarding the estimates (4.105), (4.106), (4.107), (4.109) and (4.112), gives the following estimate for $\frac{d}{d\epsilon}Q_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)$.

$$\left| \frac{d}{d\epsilon} Q_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \right| \lesssim \mu\nu + 1 \approx 1. \quad (4.113)$$

We move on to estimating $\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)$ and $\frac{d}{d\epsilon}\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)$. Considering the estimates (4.103) and (4.112), we easily get the estimate

$$|\phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)| = \frac{|A_1^3(t, s - \epsilon\gamma)|}{|Q_\gamma(t, s, \epsilon)|} \lesssim \frac{1}{\mu\nu}. \quad (4.114)$$

On the other hand

$$\begin{aligned}
\frac{d}{d\epsilon}\phi_\gamma(t_*(s-\epsilon\gamma), s, \epsilon) &= \frac{d}{d\epsilon} \frac{A_1^3(t_*(s-\epsilon\gamma), s-\epsilon\gamma)}{Q_\gamma(t_*(s-\epsilon\gamma), s, \epsilon)} \\
&= \frac{3A_1^2(t_*(s-\epsilon\gamma), s-\epsilon\gamma)}{Q_\gamma(t_*(s-\epsilon\gamma), s, \epsilon)} \left[\partial_t A_1(t_*(s-\epsilon\gamma), s-\epsilon\gamma) \frac{d}{d\epsilon} t_*(s-\epsilon\gamma) + A_2(t_*(s-\epsilon\gamma), s-\epsilon\gamma) \right] - \\
&\quad + \frac{A_1^3(t_*(s-\epsilon\gamma), s-\epsilon\gamma)}{Q_\gamma^2(t_*(s-\epsilon\gamma), s, \epsilon)} \frac{d}{d\epsilon} Q_\gamma(t_*(s-\epsilon\gamma), s, \epsilon).
\end{aligned}$$

Once again, the estimates (4.103), (4.104), (4.112) and (4.113) imply that

$$\left| \frac{d}{d\epsilon} \phi_\gamma(t_*(s-\epsilon\gamma), s, \epsilon) \right| \lesssim \frac{1}{\mu^2 \nu^2}. \quad (4.115)$$

Before estimating $\frac{d}{d\epsilon} \psi_\gamma(t, s, \epsilon)$, it is better to simplify the function $\psi_\gamma(t, s, \epsilon)$ and show that it is continuously differentiable. We have the difference

$$\begin{aligned}
\phi_\gamma(t, s, \epsilon) - \phi_\gamma(t_*(s-\epsilon\gamma), s, \epsilon) &= \frac{A_1^3(t, s-\epsilon\gamma)}{Q_\gamma(t, s, \epsilon)} - \frac{A_1^3(t_*(s-\epsilon\gamma), s-\epsilon\gamma)}{Q_\gamma(t_*(s-\epsilon\gamma), s, \epsilon)} \\
&= \frac{A_1^3(t, s-\epsilon\gamma)}{Q_\gamma(t, s, \epsilon)} - \frac{A_1^3(t_*, s-\epsilon\gamma)}{Q_\gamma(t, s, \epsilon)} + \frac{A_1^3(t_*, s-\epsilon\gamma)}{Q_\gamma(t, s, \epsilon)} - \frac{A_1^3(t_*, s-\epsilon\gamma)}{Q_\gamma(t_*, s, \epsilon)} \\
&= \frac{1}{Q_\gamma(t, s, \epsilon)} [A_1^3(t, s-\epsilon\gamma) - A_1^3(t_*, s-\epsilon\gamma)] + \left(\frac{1}{Q_\gamma(t, s, \epsilon)} - \frac{1}{Q_\gamma(t_*, s, \epsilon)} \right) A_1^3(t_*, s-\epsilon\gamma) \\
&= \frac{A_1(t) - A_1(t_*)}{Q_\gamma(t)} [A_1^2(t) + A_1(t)A_1(t_*) + A_1^2(t_*)] + \frac{Q_\gamma(t_*) - Q_\gamma(t)}{Q_\gamma(t)Q_\gamma(t_*)} A_1^3(t_*). \quad (4.116)
\end{aligned}$$

But, we have that

$$A_1(t) - A_1(t_*) = -(t - t_*) \sum_{l=1}^4 \frac{(-1)^l \gamma_l (2 + 2s_l + t + t_* - 2\epsilon\gamma_l)}{(1 + t + s_l - \epsilon\gamma_l)^2 (1 + t_* + s_l - \epsilon\gamma_l)^2}. \quad (4.117)$$

Furthermore, we have

$$\begin{aligned}
\frac{Q_\gamma(t) - Q_\gamma(t_*)}{\alpha(s - \epsilon\gamma)} &= \frac{\prod_{l=1}^4(1+t+s_l - \epsilon\gamma_l)}{t - t_-(s_l - \epsilon\gamma_l)} - \frac{\prod_{l=1}^4(1+t_*(s_l - \epsilon\gamma_l) + s_l - \epsilon\gamma_l)}{t_*(s_l - \epsilon\gamma_l) - t_-(s_l - \epsilon\gamma_l)} = \\
&= \frac{1+t+s_1 - \epsilon\gamma_1 - (1+t_*+s_1 - \epsilon\gamma_1)}{t - t_-} \prod_{l=2}^4(1+t+s_l - \epsilon\gamma_l) + \\
&\quad + (1+t_*+s_1 - \epsilon\gamma_1) \frac{1+t+s_2 - \epsilon\gamma_2 - (1+t_*+s_2 - \epsilon\gamma_2)}{t - t_-} \prod_{l=3}^4(1+t+s_l - \epsilon\gamma_l) + \\
&\quad + \prod_{l=1}^2(1+t_*+s_l - \epsilon\gamma_l) \frac{1+t+s_3 - \epsilon\gamma_3 - (1+t_*+s_3 - \epsilon\gamma_3)}{t - t_-} (1+t+s_4 - \epsilon\gamma_4) + \\
&\quad + \prod_{l=1}^3(1+t_*+s_l - \epsilon\gamma_l) \frac{1+t+s_4 - \epsilon\gamma_4 - (1+t_*+s_4 - \epsilon\gamma_4)}{t - t_-} + \\
&\quad + \prod_{l=1}^4(1+t_*+s_l - \epsilon\gamma_l) \left(\frac{1}{t - t_-} - \frac{1}{t_* - t_-} \right) = \frac{t - t_*}{t - t_-} \tilde{\psi}_\gamma(t, s, \epsilon),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\psi}_\gamma(t, s, \epsilon) &= \prod_{l=2}^4(1+t+s_l - \epsilon\gamma_l) + (1+t_*+s_1 - \epsilon\gamma_1) \prod_{l=3}^4(1+t+s_l - \epsilon\gamma_l) + \\
&\quad + \prod_{l=1}^2(1+t_*+s_l - \epsilon\gamma_l)(1+t+s_4 - \epsilon\gamma_4) + \prod_{l=1}^3(1+t_*+s_l - \epsilon\gamma_l) - \frac{\prod_{l=1}^4(1+t_*+s_l - \epsilon\gamma_l)}{t_* - t_-}.
\end{aligned}$$

This leads to the following identity

$$\frac{Q_\gamma(t_*) - Q_\gamma(t)}{Q_\gamma(t)Q_\gamma(t_*)} = -(t - t_*) \frac{\alpha(s - \epsilon\gamma)}{t - t_-} \frac{\tilde{\psi}_\gamma(t, s, \epsilon)}{Q_\gamma(t)Q_\gamma(t_*)}. \quad (4.118)$$

Thus, in the light of (4.117) and (4.118), it follows from (4.116) that

$$\begin{aligned}
\psi_\gamma(t, s, \epsilon) &= \frac{\phi_\gamma(t, s, \epsilon) - \phi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)}{t - t_*(s - \epsilon\gamma)} \\
&= \frac{1}{Q_\gamma(t)} [A_1^2(t) + A_1(t)A_1(t_*) + A_1^2(t_*)] \sum_{l=1}^4 \frac{(-1)^{l+1}\gamma_l(2 + 2s_l + t + t_* - 2\epsilon\gamma_l)}{(1+t+s_l - \epsilon\gamma_l)^2(1+t_*+s_l - \epsilon\gamma_l)^2} + \\
&\quad - \frac{\alpha(s - \epsilon\gamma)}{t - t_-} \frac{\tilde{\psi}_\gamma(t, s, \epsilon)}{Q_\gamma(t)Q_\gamma(t_*)} A_1^3(t_*).
\end{aligned}$$

Differentiating w.r.t. ϵ we obtain

$$\begin{aligned}
\frac{d}{d\epsilon}\psi_\gamma(t, s, \epsilon) &= \frac{-1}{Q_\gamma^2(t, s - \epsilon\gamma)} \frac{dQ_\gamma(t, s - \epsilon\gamma)}{d\epsilon} [A_1^2(t) + A_1(t)A_1(t_*) + A_1^2(t_*)] \\
&\quad \sum_{l=1}^4 \frac{(-1)^{l+1}\gamma_l(2 + 2s_l + t + t_* - 2\epsilon\gamma_l)}{(1 + t + s_l - \epsilon\gamma_l)^2(1 + t_* + s_l - \epsilon\gamma_l)^2} + \frac{1}{Q_\gamma(t, s - \epsilon\gamma)} \\
&\quad [2A_1(t)A_2(t) + A_1(t)A_2(t_*) + 3A_1(t_*)\partial_t A_1(t_*) \frac{d}{d\epsilon} t_*(s - \epsilon\gamma) + 2A_1(t_*)A_2(t_*)] \\
&\quad \sum_{l=1}^4 \frac{(-1)^{l+1}\gamma_l(2 + 2s_l + t + t_* - 2\epsilon\gamma_l)}{(1 + t + s_l - \epsilon\gamma_l)^2(1 + t_* + s_l - \epsilon\gamma_l)^2} + \frac{1}{Q_\gamma(t)} [A_1^2(t) + A_1(t)A_1(t_*) + A_1^2(t_*)] \\
&\quad \left(\sum_{l=1}^4 \frac{2(-1)^l \gamma_l^2 (2 + 2s_l + t + t_* - 2\epsilon\gamma_l)^2}{(1 + t + s_l - \epsilon\gamma_l)^3 (1 + t_* + s_l - \epsilon\gamma_l)^3} + \right. \\
&\quad \left. + \sum_{l=1}^4 \frac{2(-1)^l \gamma_l^2 (2 + 2s_l + t + t_* - 2\epsilon\gamma_l)}{(1 + t + s_l - \epsilon\gamma_l)^2 (1 + t_* + s_l - \epsilon\gamma_l)^3} \frac{d}{d\epsilon} t_*(s - \epsilon\gamma) + \right. \\
&\quad \left. + \sum_{l=1}^4 \frac{(-1)^{l+1}\gamma_l \left(\frac{d}{d\epsilon} t_*(s - \epsilon\gamma) - 2\gamma_l \right)}{(1 + t + s_l - \epsilon\gamma_l)^2 (1 + t_* + s_l - \epsilon\gamma_l)^2} \right) + \\
&\quad + \frac{\alpha(\gamma)}{t - t_-} \frac{\tilde{\psi}_\gamma(t, s, \epsilon)}{Q_\gamma(t)Q_\gamma(t_*)} A_1^3(t_*) + \frac{\alpha(s - \epsilon\gamma)}{(t - t_-)^2} \frac{\tilde{\psi}_\gamma(t, s, \epsilon)}{Q_\gamma(t)Q_\gamma(t_*)} A_1^3(t_*) \frac{d}{d\epsilon} t_*(s - \epsilon\gamma) + \\
&\quad + \frac{\alpha(s - \epsilon\gamma)}{t - t_-} \frac{\tilde{\psi}_\gamma(t, s, \epsilon)}{Q_\gamma^2(t)Q_\gamma^2(t_*)} A_1^3(t_*) [Q_\gamma(t_*) \frac{dQ_\gamma(t, s, \epsilon)}{d\epsilon} + Q_\gamma(t) \frac{dQ_\gamma(t_*(s), s, \epsilon)}{d\epsilon}] + \\
&\quad - 3 \frac{\alpha(s - \epsilon\gamma)}{t - t_-} \frac{\tilde{\psi}_\gamma(t, s, \epsilon)}{Q_\gamma(t)Q_\gamma(t_*)} A_1^2(t_*) (\partial_t A_1(t_*) \frac{dt_*(s - \epsilon\gamma)}{d\epsilon} + A_2(t_*)) + \\
&\quad - \frac{\alpha(s - \epsilon\gamma)}{t - t_-} \frac{1}{Q_\gamma(t)Q_\gamma(t_*)} A_1^3(t_*) \frac{d\tilde{\psi}_\gamma(t, s, \epsilon)}{d\epsilon}.
\end{aligned}$$

Now, since $\frac{d}{d\epsilon}\tilde{\psi}_\gamma(t, s, \epsilon)$ is linear in $\frac{d}{d\epsilon}t_*(s - \epsilon\gamma)$ with coefficients that are ≈ 1 , then it follows by (4.105), that

$$\left| \frac{d}{d\epsilon}\tilde{\psi}_\gamma(t, s, \epsilon) \right| \lesssim \frac{1}{\mu\nu}.$$

In view of this and the estimates (4.103) - (4.107), it is easy to deduce that

$$\left| \frac{d}{d\epsilon}\psi_\gamma(t, s, \epsilon) \right| \lesssim \frac{1}{\mu^2\nu^2}. \tag{4.119}$$

Now, we turn to estimating $\chi_\gamma(t, s, \epsilon)$. By both the estimates (4.103) and (4.112), it immediately follows that

$$|\chi_\gamma(t, s, \epsilon)| = \left| \frac{A_1^2(t, s - \epsilon\gamma)A_2(t, s - \epsilon\gamma)}{Q_\gamma(t, s, \epsilon)} \right| \lesssim \frac{1}{\mu\nu}. \tag{4.120}$$

To estimate $\omega_\gamma(t, s, \epsilon)$, notice that

$$\begin{aligned}
& \chi_\gamma(t, s, \epsilon) - \chi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon) \\
&= \frac{A_1^2(t, s - \epsilon\gamma)A_2(t, s - \epsilon\gamma)}{Q_\gamma(t, s, \epsilon)} - \frac{A_1^2(t_*(s - \epsilon\gamma), s - \epsilon\gamma)A_2(t_*(s - \epsilon\gamma), s - \epsilon\gamma)}{Q_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)} \\
&= \frac{A_1^2(t)A_2(t)}{Q_\gamma(t)} - \frac{A_1^2(t_*)A_2(t)}{Q_\gamma(t)} + \frac{A_1^2(t_*)A_2(t)}{Q_\gamma(t)} - \frac{A_1^2(t_*)A_2(t_*)}{Q_\gamma(t)} + \frac{A_1^2(t_*)A_2(t_*)}{Q_\gamma(t)} - \frac{A_1^2(t_*)A_2(t_*)}{Q_\gamma(t_*)} \\
&= \frac{A_2(t)}{Q_\gamma(t)} [A_1^2(t) - A_1^2(t_*)] + \frac{A_1^2(t_*)}{Q_\gamma(t)} [A_2(t) - A_2(t_*)] + A_1^2(t_*)A_2(t_*) \left[\frac{1}{Q_\gamma(t)} - \frac{1}{Q_\gamma(t_*)} \right] \\
&= \frac{A_2(t)}{Q_\gamma(t)} [A_1(t) - A_1(t_*)] [A_1(t) + A_1(t_*)] + \frac{A_1^2(t_*)}{Q_\gamma(t)} [A_2(t) - A_2(t_*)] + \\
&\quad + A_1^2(t_*)A_2(t_*) \frac{Q_\gamma(t_*) - Q_\gamma(t)}{Q_\gamma(t)Q_\gamma(t_*)}. \tag{4.121}
\end{aligned}$$

But

$$\begin{aligned}
A_2(t) - A_2(t_*) &= \sum_{l=1}^4 (-1)^l \gamma_l^2 \left[\frac{1}{(1+t+s_l-\epsilon\gamma_l)^3} - \frac{1}{(1+t_*+s_l-\epsilon\gamma_l)^3} \right] \\
&= -(t-t_*) \sum_{l=1}^4 \frac{(-1)^l \gamma_l^2 \tilde{A}_\gamma(t, s, \epsilon)}{(1+t+s_l-\epsilon\gamma_l)^3 (1+t_*+s_l-\epsilon\gamma_l)^3}, \tag{4.122}
\end{aligned}$$

with

$$\tilde{A}_\gamma(t, s, \epsilon) = (1+t+s_l-\epsilon\gamma_l)^2 + (1+t+s_l-\epsilon\gamma_l)(1+t_*+s_l-\epsilon\gamma_l) + (1+t_*+s_l-\epsilon\gamma_l)^2.$$

It follows then from (4.117), (4.118), (4.121) and (4.122) that

$$\begin{aligned}
\omega_\gamma(t, s, \epsilon) &= \frac{\chi_\gamma(t, s, \epsilon) - \chi_\gamma(t_*(s - \epsilon\gamma), s, \epsilon)}{t - t_*} \\
&= \frac{A_2(t)}{Q_\gamma(t)} [A_1(t) + A_1(t_*)] \sum_{l=1}^4 \frac{(-1)^{l+1} \gamma_l (2 + 2s_l + t + t_* - 2\epsilon\gamma_l)}{(1+t+s_l-\epsilon\gamma_l)^2 (1+t_*+s_l-\epsilon\gamma_l)^2} + \\
&\quad + \frac{A_1^2(t_*)}{Q_\gamma(t)} \sum_{l=1}^4 \frac{(-1)^{l+1} \gamma_l^2 \tilde{A}_\gamma(t, s, \epsilon)}{(1+t+s_l-\epsilon\gamma_l)^3 (1+t_*+s_l-\epsilon\gamma_l)^3} - A_1^2(t_*)A_2(t_*) \frac{\alpha(s - \epsilon\gamma)}{t - t_*} \frac{\tilde{\psi}_\gamma(t, s, \epsilon)}{Q_\gamma(t)Q_\gamma(t_*)}.
\end{aligned}$$

From here and since $\tilde{A}_\gamma(t, s, \epsilon) \approx 1$ and $\tilde{\psi}_\gamma(t, s, \epsilon) \lesssim 1$ and because of the estimates (4.103), (4.107), (4.111) and (4.112), we have that

$$|\omega_\gamma(t, s, \epsilon)| \lesssim \frac{1}{\mu\nu}. \tag{4.123}$$

In addition, because of the estimates (4.103), we have

$$\begin{aligned}
& \left| \int_0^1 \log(|A(t, s - \epsilon\gamma)|) A_1(t, s - \epsilon\gamma) A_3(t, s - \epsilon\gamma) dt \right| \\
& \leq \left| \int_0^1 \log(|t - t_*(s - \epsilon\gamma)|) A_1(t, s - \epsilon\gamma) A_3(t, s - \epsilon\gamma) dt \right| + \\
& \quad + \left| \int_0^1 \log(|Q_\gamma(t, s, \epsilon)|) A_1(t, s - \epsilon\gamma) A_3(t, s - \epsilon\gamma) dt \right| \\
& \lesssim \int_0^1 |\log(|t - t_*(s - \epsilon\gamma)|)| dt + \int_0^1 |\log(|Q_\gamma(t, s, \epsilon)|)| dt.
\end{aligned}$$

But

$$\begin{aligned}
& \int_0^1 |\log(|t - t_*(s - \epsilon\gamma)|)| dt = - \int_0^1 \log(|t - t_*(s - \epsilon\gamma)|) dt \\
& = 1 - (1 - t_*(s - \epsilon\gamma)) \log|1 - t_*(s - \epsilon\gamma)| - t_*(s - \epsilon\gamma) \log|t_*(s - \epsilon\gamma)| \approx 1,
\end{aligned}$$

whenever $t_*(s - \epsilon\gamma) \in (0, 1)$, and, by (4.110), we have

$$\begin{aligned}
& \int_0^1 |\log(|Q_\gamma(t, s, \epsilon)|)| dt \\
& \leq \int_0^1 |\log(|\alpha(s - \epsilon\gamma)|)| dt + \int_0^1 \log(t - t_-(s - \epsilon\gamma)) dt + \sum_{l=1}^4 \int_0^1 |\log(1 + t + s_l - \epsilon\gamma_l)| dt \\
& \lesssim \frac{1}{|\alpha(s - \epsilon\gamma)|} \approx \frac{1}{\mu\nu}.
\end{aligned}$$

Thus

$$\left| \int_0^1 \log(|A(t, s - \epsilon\gamma)|) A_1(t, s - \epsilon\gamma) A_3(t, s - \epsilon\gamma) dt \right| \lesssim \frac{1}{\mu\nu}. \quad (4.124)$$

Since the functions $t \mapsto A_j(t, s - \epsilon\gamma)$, $j = 1, \dots, 4$, and f' are continuous, then the following estimates hold

$$\left| \int_0^1 f'(A(t, s - \epsilon\gamma)) A_4(t, s - \epsilon\gamma) dt \right| \lesssim 1, \quad (4.125)$$

$$\left| \int_0^1 (A_2^2(t, s - \epsilon\gamma) + \frac{4}{3} A_1(t, s - \epsilon\gamma) A_3(t, s - \epsilon\gamma)) dt \right| \lesssim 1. \quad (4.126)$$

Finally, the estimates (4.114), (4.115), (4.119), (4.120), (4.123), (4.124), (4.125), (4.126) imply (4.102) and hence (4.101) when $\mu\nu \gg \epsilon$. This completes the proof of Lemma 4.4.7. \square

The following remark will be used in proving the estimate (4.12) for the case

$$(2^*) \quad \boxed{\alpha < 0, \quad |\alpha| \approx \mu\nu \gg \epsilon, \quad |t_*(s)| \gg \epsilon}$$

with the additional restrictions

$$\boxed{s_1 - s_2 \approx s_4 - s_3 \approx \mu, \quad s_3 - s_2 \approx s_4 - s_1 \approx \nu}$$

Remark 4.4.4. For $l \in \{1, 2, 3, 4\}$, fix s_j , $j \neq l$ and define $I_{s_l} = I_{s_l}(s_j)$ to be the subinterval of the interval $[0, 1]$ on which $|t_*(s)| \approx \lambda$, where $\lambda \lesssim 1$ is a dyadic number. Then, by the mean value theorem and the estimate (4.56), we get

$$|I_{s_l}| \approx \lambda \mu \nu.$$

Thus, whenever $|t_*(\frac{k}{N})| \approx \lambda$, then

$$|k_l - k_l^{**}| \lesssim \lambda \mu \nu N, \quad \text{for some } k_l^{**}, \quad l = 1, 2, 3, 4,$$

where k_l^{**} depends on k_j , $j \neq l$.

In the light of formula 4.9, Remark 4.4.4 and Lemma 4.4.7 lead to considering proving the estimate (4.127).

$$\begin{aligned} & \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |t_*(\frac{k}{N})| \gg \frac{1}{N}}} J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \\ & \lesssim \frac{1}{N^4} \sum_{\substack{\lambda; \text{ dyadic} \\ \frac{1}{N} \lesssim \lambda \lesssim 1}} \frac{1}{\lambda} \sum_{\substack{\mu; \text{ dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu^2} \sum_{\substack{\nu; \text{ dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu^2} \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |k_l - k_l^{**}| \lesssim \lambda \mu \nu N}} |c_{k_1}| |c_{k_2}| |c_{k_3}| |c_{k_4}| \lesssim \frac{1}{N} \|c\|_{l^4}. \end{aligned} \tag{4.127}$$

We can accomplish this with a loss of logarithmic order again using the interpolation result presented in Theorem 3.3.4. All we have to do is prove the following lemma necessary for the interpolation step.

Lemma 4.4.9.

$$\frac{1}{N^4} \sum_{\substack{\lambda; \text{ dyadic} \\ \frac{1}{N} \lesssim \lambda \lesssim 1}} \frac{1}{\lambda} \sum_{\substack{\mu; \text{ dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} \frac{1}{\mu^2} \sum_{\substack{\nu; \text{ dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} \frac{1}{\nu^2} \sup_{k_{i_2}} \sum_{\substack{k_{i_2}, k_{i_3}, k_{i_4} \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |k_l - k_l^{**}| \lesssim \lambda \mu \nu N}} 1 \lesssim \frac{(\log N)^3}{N}$$

Proof. We shall show that

$$\sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |k_l - k_l^{**}| \lesssim \lambda \mu \nu N}} 1 \lesssim \lambda \mu^2 \nu^2 N^3. \tag{4.128}$$

This will suffice to prove Lemma 4.4.9, considering that

$$\sum_{\substack{\lambda; \text{ dyadic} \\ \frac{1}{N} \lesssim \lambda \lesssim 1}} 1 \sum_{\substack{\mu; \text{ dyadic} \\ \frac{1}{N} \lesssim \mu \lesssim 1}} 1 \sum_{\substack{\nu; \text{ dyadic} \\ \frac{1}{N} \lesssim \nu \lesssim 1}} 1 \approx (\log N)^3.$$

The estimate (4.128) in its turn follows from the estimates (4.129)-(4.132) below.

$$\begin{aligned} & \sup_{k_1} \sum_{\substack{k_2, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |k_4 - k_4^{**}| \lesssim \lambda \mu \nu N}} 1 \\ & \lesssim \sup_{k_1} \sum_{\substack{k_2 \\ k_1 - k_2 \approx \mu N}} \sum_{\substack{k_3 \\ k_3 - k_2 \approx \nu N}} \sum_{\substack{k_4 \\ |k_4 - k_4^{**}| \lesssim \lambda \mu \nu N}} 1 \\ & \lesssim \lambda \mu \nu N \sup_{k_1} \sum_{\substack{k_2 \\ k_1 - k_2 \approx \mu N}} \sum_{\substack{k_3 \\ k_3 - k_2 \approx \nu N}} 1 \\ & \lesssim \lambda \mu \nu^2 N^2 \sup_{k_1} \sum_{\substack{k_2 \\ k_1 - k_2 \approx \mu N}} 1 \\ & \lesssim \lambda \mu^2 \nu^2 N^3. \end{aligned} \tag{4.129}$$

Similarly we have that

$$\begin{aligned} & \sup_{k_2} \sum_{\substack{k_1, k_3, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |k_4 - k_4^{**}| \lesssim \lambda \mu \nu N}} 1 \\ & \lesssim \sup_{k_2} \sum_{\substack{k_1 \\ k_1 - k_2 \approx \mu N}} \sum_{\substack{k_3 \\ k_3 - k_2 \approx \nu N}} \sum_{\substack{k_4 \\ |k_4 - k_4^{**}| \lesssim \lambda \mu \nu N}} 1 \\ & \lesssim \lambda \mu^2 \nu^2 N^3. \end{aligned} \tag{4.130}$$

and symmetrically we deduce that

$$\begin{aligned} & \sup_{k_3} \sum_{\substack{k_1, k_2, k_4 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |k_4 - k_4^{**}| \lesssim \lambda \mu \nu N}} 1 \\ & \lesssim \sup_{k_3} \sum_{\substack{k_2 \\ k_3 - k_2 \approx \nu N}} \sum_{\substack{k_1 \\ k_1 - k_2 \approx \mu N}} \sum_{\substack{k_4 \\ |k_4 - k_4^{**}| \lesssim \lambda \mu \nu N}} 1 \\ & \lesssim \lambda \mu^2 \nu^2 N^3 \end{aligned} \tag{4.131}$$

and finally

$$\begin{aligned}
& \sup_{k_4} \sum_{\substack{k_1, k_2, k_3 \\ k_1 - k_2 \approx k_4 - k_3 \approx \mu N \\ k_3 - k_2 \approx k_4 - k_1 \approx \nu N \\ |k_4 - k_4^*| \lesssim \lambda \mu \nu N}} 1 \\
& \lesssim \sup_{k_4} \sum_{\substack{k_1 \\ k_4 - k_1 \approx \nu N}} \sum_{\substack{k_2 \\ k_1 - k_2 \approx \mu N}} \sum_{\substack{k_3 \\ |k_3 - k_3^*| \lesssim \lambda \mu \nu N}} 1 \\
& \lesssim \lambda \mu^2 \nu^2 N^3.
\end{aligned} \tag{4.132}$$

□

This completes the study of the subcase (2*). Thus we have also completed proving the estimate (4.12) on the region (III). By the machinery of the interpolation introduced in Theorem 3.3.4 we have actually proved the estimate (4.10) here too.

4.5 An inhomogeneous Strichartz estimate for the special inhomogeneity

Finally Lemma 4.4.3, the interpolation Theorem (3.3.4) applied to the estimates (4.19), (4.20), (4.30), (4.36), (4.37), (4.38) and the estimates (4.58) and (4.127) all together prove the following theorem

Theorem 4.5.1.

$$\boxed{\sum_{k_1, k_2, k_3, k_4=1}^N c_{k_1} c_{k_2} c_{k_3} c_{k_4} J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) \lesssim \frac{(\log N)^3}{N} \|c\|_{l^4}^4}. \tag{4.133}$$

where

$$\begin{aligned}
J_{\frac{1}{N}}\left(\frac{k_1}{N}, \frac{k_2}{N}, \frac{k_3}{N}, \frac{k_4}{N}\right) &= \sum_{\gamma} \sigma_{\gamma} \int_0^1 A^2\left(t, \frac{k - \gamma}{N}\right) \log |A\left(t, \frac{k - \gamma}{N}\right)| dt, \\
\gamma &= (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \Gamma = \{0, 1\}^4, \quad k = (k_1, k_2, k_3, k_4), \quad \sigma_{\gamma} = (-1)^{(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)}, \\
A(t, s) &= A(t, s_1, s_2, s_3, s_4) = \sum_{l=1}^4 \frac{(-1)^l}{(1 + t + s_l)^2}.
\end{aligned}$$

The following theorem is a direct consequence of Theorem 4.5.1.

Theorem 4.5.2. *Let*

$$u(t, x) = \int_0^1 \frac{e^{\frac{i|x|^2}{1+t+\sigma}}}{(1+t+\sigma)^2} f(\sigma) d\sigma.$$

This is the fundamental solution of the Cauchy problem

$$i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad u(0, x) = 0,$$

where the forcing term F is given by

$$F(t, x) = f(t) \delta_0(x), \quad f = \sum_{k=1}^N c_k \chi_{[\frac{k-1}{N}, \frac{k}{N}]}$$

and f is supported on $[0, 1]$. Then we have the following estimate

$$\frac{\|u\|_{L^4([2,3] \times \mathbb{R}^4)}}{\|f\|_{L^4([0,1])}} \lesssim (\log N)^{\frac{3}{4}}. \tag{4.134}$$

Chapter 5

A quadrilinear estimate (II)

In this chapter we study the estimate problem given in Chapter 3 Section 5. That is we investigate the estimate

$$\| u \|_{L^4([2,3] \times \mathbb{R}^4)} \lesssim \| f \|_{L^4([0,1])} \quad (5.1)$$

where

$$u(t, x) = \frac{1}{(4\pi)^2} \int_0^1 \frac{e^{i\frac{|x|^2}{4(1+t+s)}}}{(1+t+s)^2} f(s) ds. \quad (5.2)$$

As we have seen in chapter 3 Section one way to prove the estimate (5.1) is to prove the quadrilinear estimate

$$|T(f_1, f_2, f_3, f_4)| \lesssim \| f_1 \|_{L^4([0,1])} \| f_2 \|_{L^4([0,1])} \| f_3 \|_{L^4([0,1])} \| f_4 \|_{L^4([0,1])}$$

where $T : L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \rightarrow \mathbb{C}$ is the quadrilinear form defined by

$$T(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{iA(t, s_1, s_2, s_3, s_4)|x|^2}}{\prod_{l=1}^4 (1+t+s_l)^2} f_1(s_1) f_2(s_2) f_3(s_3) f_4(s_4) ds_1 ds_2 ds_3 ds_4 dx dt. \quad (5.3)$$

Recall that one of the bases on which we chose the Lebesgue exponents values in the estimate (5.1) was to be able to look at a multilinear estimate. The kernel in the quadrilinear form (5.3) contains an oscillatory factor that oscillates with the variation in the variables t , x and s_j . To find a descent estimate for T , we need to integrate in as many of these variables as possible to get the decay due to this oscillation. We can not integrate in any of the variables

s_j because we have no regularity assumptions on the functions f_j . Therefore, we decided to integrate in x or t . Unfortunately, the integral (5.3) is not absolutely convergent. Thus, we can not interchange the order of integration. From here came the idea of throwing in a gaussian $e^{-\epsilon|x|^2}$ with $\epsilon > 0$ in the integrand of (5.3) and considering the quadrilinear form

$$T_\epsilon(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{(-\epsilon + \iota A(t, s_1, s_2, s_3, s_4))|x|^2}}{\prod_{l=1}^4 (1 + t + s_l)^2} f_1(s_1) f_2(s_2) f_3(s_3) f_4(s_4) ds_1 ds_2 ds_3 ds_4 dx dt \quad (5.4)$$

in which changing the order of integration is allowed. Notice that the dominated convergence theorem implies that

$$T(f_1, f_2, f_3, f_4) = \lim_{\epsilon \rightarrow 0^+} T_\epsilon(f_1, f_2, f_3, f_4).$$

Thus our quadrilinear form (5.3) is approximated by $T_\epsilon(f_1, f_2, f_3, f_4)$. In the following lemma we show how integrating explicitly in x helps us rewrite the kernel of the quadrilinear form $T_\epsilon(f_1, f_2, f_3, f_4)$ as an integral in the time variable t .

Lemma 5.0.3.

$$T_\epsilon(f_1, f_2, f_3, f_4) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 K_\epsilon(s_1, s_2, s_3, s_4) f(s_1) f(s_2) f(s_3) f(s_4) ds_1 ds_2 ds_3 ds_4 \quad (5.5)$$

where

$$\begin{aligned} K_\epsilon(s_1, s_2, s_3, s_4) &= \int_0^1 H_\epsilon(t, s_1, s_2, s_3, s_4) dt, \\ H_\epsilon(t, s_1, s_2, s_3, s_4) &= \frac{\epsilon^2 - A^2(t, s_1, s_2, s_3, s_4)}{[\epsilon^2 + A^2(t, s_1, s_2, s_3, s_4)]^2} B(t, s_1, s_2, s_3, s_4), \\ A(t, s_1, s_2, s_3, s_4) &= \sum_{l=1}^4 \frac{(-1)^l}{1 + t + s_l}, \quad B(t, s_1, s_2, s_3, s_4) = \prod_{l=1}^4 (1 + t + s_l)^{-2}. \end{aligned}$$

Proof. Applying Fubini's theorem we have

$$\begin{aligned} T_\epsilon(f_1, f_2, f_3, f_4) &= \int_0^1 \int_{\mathbb{R}^4} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{e^{(-\epsilon + \iota A(t, s_1, s_2, s_3, s_4))|x|^2}}{\prod_{l=1}^4 (1 + t + s_l)^2} f_1(s_1) f_2(s_2) f_3(s_3) f_4(s_4) ds_1 ds_2 ds_3 ds_4 dx dt = \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 f_1(s_1) f_2(s_2) f_3(s_3) f_4(s_4) \\ &\quad \int_0^1 \prod_{l=1}^4 (1 + t + s_l)^{-2} \left(\int_{\mathbb{R}^4} e^{(-\epsilon + \iota A(t, s_1, s_2, s_3, s_4))|x|^2} dx \right) dt ds_1 ds_2 ds_3 ds_4. \end{aligned}$$

The integral

$$\begin{aligned}
& \int_{\mathbb{R}^4} e^{-[\epsilon - \imath A(t, s_1, s_2, s_3, s_4)]|x|^2} dx = \int_{\mathbb{R}^4} e^{-[\epsilon - \imath A(t, s_1, s_2, s_3, s_4)] \sum_{i=1}^4 x_i^2} dx = \\
& = \int_{\mathbb{R}^4} \prod_{l=1}^4 e^{-[\epsilon - \imath A(t, s_1, s_2, s_3, s_4)] x_l^2} dx = \prod_{l=1}^4 \int_{-\infty}^{+\infty} e^{-[\epsilon - \imath A(t, s_1, s_2, s_3, s_4)] x_l^2} dx_l = \\
& = \frac{\pi^2}{(\epsilon - \imath A(t, s_1, s_2, s_3, s_4))^2}. \tag{5.6}
\end{aligned}$$

We have seen that $T_\epsilon(f_1, f_2, f_3, f_4)$ approximates the quadrilinear form $T_\epsilon(f_1, f_2, f_3, f_4)$. Recall from Chapter 3 Section 3.2.3 that $\|u(t, x)\|_{L^4([2,3] \times \mathbb{R}^4)} = T(f, f, f, f)$. Recall also from Remark 3.2.1 that the function f can be assumed to be realvalued. Therefore, we are interested in the real part of the integral (5.6) which is

$$\Re \int_{\mathbb{R}^4} e^{-[\epsilon - \imath A(t, s_1, s_2, s_3, s_4)]|x|^2} dx = \frac{\epsilon^2 - A^2(t, s_1, s_2, s_3, s_4)}{[\epsilon^2 + A^2(t, s_1, s_2, s_3, s_4)]^2}.$$

This concludes the proof of Lemma 5.0.3. □

We shall show the following theorem

Theorem 5.0.4. *Let $T_\epsilon : L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \times L^4([0, 1]) \rightarrow \mathbb{C}$ be the quadrilinear form given in Lemma 5.0.3. Then*

$$|T_\epsilon(f_1, f_2, f_3, f_4)| \lesssim |\log \epsilon|^2 \|f_1\|_{L^4([0,1])} \|f_2\|_{L^4([0,1])} \|f_3\|_{L^4([0,1])} \|f_4\|_{L^4([0,1])}.$$

It is convenient to summarize the idea of the proof of Theorem 5.0.4 before going into all the details. In Chapter 3 Section 3.4.2, we split the unit hypercube $[0, 1]^4$ to which $s = (s_1, s_2, s_3, s_4)$ belongs into different regions and estimated $A(t, s)$ on each of these

subregions. We recall and summarize those estimates in the following table

The region in $[0, 1]^4$	Estimate of $A(t, s)$
R_I	$ A(t, s) \approx \begin{cases} s_1 - s_2, & \text{if } s_1 - s_2 > s_3 - s_4 > 0 \\ s_3 - s_4, & \text{if } s_3 - s_4 > s_1 - s_2 > 0 \\ s_2 - s_1, & \text{if } s_2 - s_1 > s_4 - s_3 > 0 \\ s_4 - s_3, & \text{if } s_4 - s_3 > s_2 - s_1 > 0 \end{cases}$
R_{II}	$ A(t, s) \approx \begin{cases} s_1 - s_4, & \text{if } s_1 - s_4 > s_3 - s_2 > 0 \\ s_3 - s_2, & \text{if } s_3 - s_2 > s_1 - s_4 > 0 \\ s_4 - s_1, & \text{if } s_4 - s_1 > s_2 - s_3 > 0 \\ s_2 - s_3, & \text{if } s_2 - s_3 > s_4 - s_1 > 0 \end{cases}$
R_{III1i}, R_{III2i}	$A(t, s) \approx \alpha $
R_{III1ii}, R_{III2ii}	$A(t, s) \approx \mu\nu$
$R_{III2iii}$	$ A(t, s) \approx \alpha t - t_*(s) $
R_{IIISYM}	see Section 5.6

where

$$\begin{aligned}
\alpha &= s_1 - s_2 + s_3 - s_4, \quad \mu = s_1 - s_2, \quad \nu = s_3 - s_2, \\
R_I &= \{s \in [0, 1]^4 : (s_1 - s_2)(s_3 - s_4) > 0\}, \\
R_{II} &= \{s \in [0, 1]^4 : (s_3 - s_2)(s_1 - s_4) > 0\}, \\
R_{III} &= \{s \in [0, 1]^4 : (s_1 - s_2)(s_3 - s_4) < 0 \quad \text{and} \quad (s_3 - s_2)(s_1 - s_4) < 0, \\
&\quad s_2 < s_1 < s_3 < s_4\}, \\
&= R_{III1i} \cup R_{III1\bar{i}} \cup R_{III2i} \cup R_{III2\bar{i}} \cup R_{III2\bar{i}\bar{i}}, \\
R_{III1i} &= \{s \in R_{III} : \alpha \geq 0, \quad \alpha \gg \mu\nu\}, \\
R_{III1\bar{i}} &= \{s \in R_{III} : \alpha \geq 0, \quad \alpha \lesssim \mu\nu\}, \\
R_{III2i} &= \{s \in R_{III} : \alpha < 0, \quad |\alpha| \gg \mu\nu\}, \\
R_{III2\bar{i}} &= \{s \in R_{III} : \alpha < 0, \quad |\alpha| \ll \mu\nu\}, \\
R_{III2\bar{i}\bar{i}} &= \{s \in R_{III} : \alpha < 0, \quad |\alpha| \approx \mu\nu\} \\
R_{IIISYM} &= \{s \in [0, 1]^4 : (s_1 - s_2)(s_3 - s_4) < 0 \quad \text{and} \quad (s_3 - s_2)(s_1 - s_4) < 0\} - R_{III}.
\end{aligned}$$

What we are going to do now is estimate the kernel $K_\epsilon(s_1, s_2, s_3, s_4)$ in each of these subregions by the help of these estimates for $A(t, s)$. Then we will verify the condition

$$\sup_{s_i \in [0, 1]} \int \int \int_{[0, 1]^3} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim (|\log \epsilon|)^3, \quad (5.7)$$

where (i, j, k, l) are all the permutations of the integers $\{1, 2, 3, 4\}$, for each of these subregions and hence for the entire hypercube $[0, 1]^4$. Actually, on each of the above mentioned subregions except for the "critical" subregion ($R_{III2\bar{i}\bar{i}}$) and its symmetric subregions R_{IIISYM} where upon $A(t, s)$ may change its sign on $t \in [0, 1]$, we have estimates independent of t for $H_\epsilon(t, s)$. First of all, we have the trivial estimate

$$|H_\epsilon(t, s)| \lesssim \frac{1}{\epsilon^2 + A^2(t, s)} \lesssim \frac{1}{\epsilon^2}. \quad (5.8)$$

This yields the following global estimate for the kernel $k_\epsilon(s)$.

$$|K_\epsilon(s)| \lesssim \frac{1}{\epsilon^2}. \quad (5.9)$$

Whenever $|A(t, s)|$ has a uniform estimate for all t and s such that $|A(t, s)| \lesssim \epsilon$ we are going to use the estimate (5.9). If $|A(t, s)| \gg \epsilon$ we are going to use the estimate

$$|H_\epsilon(t, s)| \approx \frac{1}{A^2(t, s)}, \quad (5.10)$$

for all $t \in [0, 1]$. Observe, from the estimates (5.8) and (5.10), that as long as $t \mapsto A(t, s)$ does not vanish, which is the case for all $s \in [0, 1]^4 - R_{III2iii} \cup R_{IIISYM}$, the uniform estimates of the function $A(t, s)$ for all $t \in [0, 1]$ makes integration in the time t of $H_\epsilon(t, s)$ to estimate the kernel $K_\epsilon(s)$ pointless. In the critical region ($R_{III2iii}$), where $A(t, s)$ may attain at most one zero inside $[0, 1]$ (Lemma 3.4.2), estimating $K_\epsilon(s)$ becomes more difficult because integration in time of $H_\epsilon(t, s)$ becomes inevitable. We shall discuss this case in detail in Section 5.5.

Once condition (5.7) is verified, we are entitled to use the interpolation result introduced in Theorem 3.3.3 and obtain the estimate of Theorem 5.0.4. The following diagram outlines

the proof of the estimate (5.7) everywhere except for the critical region $R_{III3iii}$.

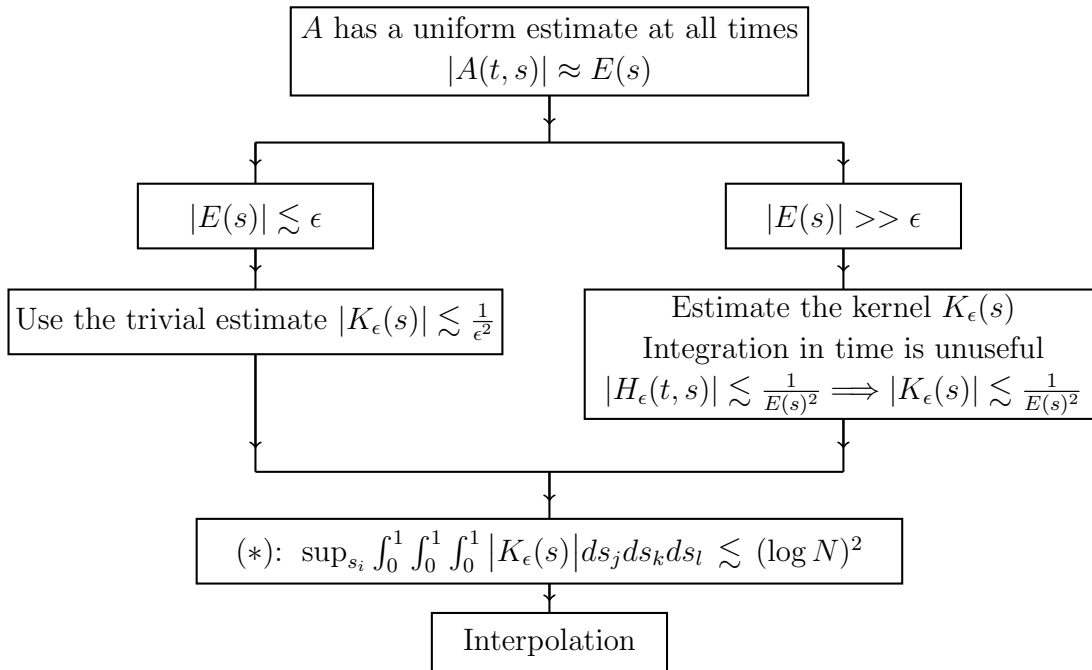


Figure (9): The process of proving the estimate (*) for all $s \in [0, 1]^4 - R_{III2iii} \cup R_{IIISYM}$

We begin with proving (5.0.4) in the regions R_I and R_{II} .

5.1 In $R_I \cup R_{II}$

5.1.1 When $s \in R_I$ and $|A(t, s)| \lesssim \epsilon$

Fix $s \in R_I$ so that $s_1 - s_2 > s_3 - s_4 > 0$. In this case we have that $A(t, s) \approx s_1 - s_2$.

Assume moreover that $s_1 - s_2 \lesssim \epsilon$ so that $|A(t, s)| \lesssim \epsilon$. Then we also have $|s_3 - s_4| \lesssim \epsilon$.

Using the trivial estimate (5.9),

$$K_\epsilon(s_1, s_2, s_3, s_4) \lesssim \frac{1}{\epsilon^2},$$

we get

$$\begin{aligned} & \sup_{s_1} \int \int \int_{\epsilon \gtrsim s_1 - s_2 > s_3 - s_4 > 0} K_\epsilon(s_1, s_2, s_3, s_4) ds_2 ds_3 ds_4 \\ & \approx \frac{1}{\epsilon^2} \sup_{s_1} \int \int \int_{\substack{|s_1 - s_2| \lesssim \epsilon, \\ |s_3 - s_4| \lesssim \epsilon}} ds_2 ds_3 ds_4 \\ & \approx \frac{1}{\epsilon^2} \sup_{s_1} \int_{|s_1 - s_2| \lesssim \epsilon} ds_2 \int ds_3 \int_{|s_3 - s_4| \lesssim \epsilon} ds_4 \lesssim 1. \end{aligned}$$

We can see from the symmetry that

$$\sup_{s_i} \int \int \int_{\epsilon \gtrsim s_1 - s_2 > s_3 - s_4 > 0} |K_\epsilon|(s_1, s_2, s_3, s_4) ds_j ds_k ds_l \lesssim 1. \quad (5.11)$$

The rest of the cases in s that constitute the region R_I follow in a similar way thanks to the symmetry.

5.1.2 When $s \in R_I$ and $|A(t, s)| \gg \epsilon$

Take $s \in R_I$ such that $s_1 - s_2 > s_3 - s_4 > 0$ and $s_1 - s_2 > \epsilon$. Then $A(t, s) \gg \epsilon$ because $A(t, s) \approx s_1 - s_2$, in this case. By the estimate (5.10), it follows that

$$|K_\epsilon(s_1, s_2, s_3, s_4)| \approx \approx \frac{1}{(s_1 - s_2)^2}.$$

We can therefore deduce the estimates (5.12) below.

$$\begin{aligned}
& \sup_{s_1} \int \int \int_{\substack{s_1-s_2 > s_3-s_4 \\ s_1-s_2 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_2 ds_3 ds_4 \\
& \approx \sup_{s_1} \int \int \int_{\substack{s_1-s_2 > s_3-s_4 \\ s_1-s_2 \gg \epsilon}} \frac{ds_2}{(s_1-s_2)^2} ds_3 ds_4 \\
& \approx \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \frac{ds_2}{(s_1-s_2)^2} \int ds_3 \int_{s_1-s_2 > s_3-s_4} ds_4 \\
& \approx \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \frac{ds_2}{s_1-s_2} \approx |\log \epsilon|. \tag{5.12}
\end{aligned}$$

And since

$$\begin{aligned}
& \sup_{s_2} \int \int \int_{\substack{s_1-s_2 > s_3-s_4 \\ s_1-s_2 \gg \epsilon}} \frac{ds_1}{(s_1-s_2)^2} ds_3 ds_4 \\
& \approx \sup_{s_2} \int_{s_1-s_2 \gg \epsilon} \frac{ds_1}{(s_1-s_2)^2} \int ds_3 \int_{s_1-s_2 > s_3-s_4} ds_4 \\
& \approx \sup_{s_2} \int_{s_1-s_2 \gg \epsilon} \frac{ds_1}{s_1-s_2} \approx |\log \epsilon|. \tag{5.13}
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{s_3} \int \int \int_{\substack{s_1-s_2 > s_3-s_4 \\ s_1-s_2 \gg \epsilon}} \frac{ds_1 ds_2}{(s_1-s_2)^2} ds_4 \\
& \approx \sup_{s_3} \int ds_1 \int_{s_1-s_2 \gg \epsilon} \frac{ds_2}{(s_1-s_2)^2} \int_{s_1-s_2 > s_3-s_4} ds_4 \\
& \approx \sup_{s_3} \int ds_1 \int_{s_1-s_2 \gg \epsilon} \frac{ds_2}{s_1-s_2} \approx |\log \epsilon|. \tag{5.14}
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{s_4} \int \int \int_{\substack{s_1-s_2 > s_3-s_4 \\ s_1-s_2 \gg \epsilon}} \frac{ds_1 ds_2}{(s_1-s_2)^2} ds_3 \\
& \approx \sup_{s_4} \int_{s_1-s_2 \gg \epsilon} \frac{ds_2}{(s_1-s_2)^2} \int_{s_1-s_2 > s_3-s_4} ds_3 \\
& \approx \sup_{s_4} \int ds_1 \int_{s_1-s_2 \gg \epsilon} \frac{ds_2}{s_1-s_2} \approx |\log \epsilon|. \tag{5.15}
\end{aligned}$$

Then we have by the estimates (5.12)-(5.15) that

$$\sup_{s_i} \int \int \int_{\substack{s_1-s_2 > s_3-s_4 \\ s_1-s_2 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|. \tag{5.16}$$

Again, the rest of the cases in s that form the region R_I can be treated in an analogous way to this case.

It follows from (5.11), (5.16) and the symmetric cases that

$$\sup_{s_i} \int \int \int_{R_I} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|. \quad (5.17)$$

5.1.3 When $s \in R_{II}$

If we repeat the arguments in 5.1.1 and 5.1.2 after replacing

$$s_1 - s_2 \text{ by } s_1 - s_4, \quad s_3 - s_4 \text{ by } s_3 - s_2,$$

or replacing

$$s_1 - s_2 \text{ by } s_3 - s_2, \quad s_3 - s_4 \text{ by } s_1 - s_4,$$

we will directly get that

$$\sup_{s_i} \int \int \int_{R_{II}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|. \quad (5.18)$$

5.2 In R_{III}

Before continuing the proof of Theorem 5.0.4 and proceeding to cover the remaining sub-regions, it is worthwhile to take into account the following remark that will be employed occasionally.

Remark 5.2.1. Notice that for all $s \in R_{III}$, if $s_1 - s_2 > s_4 - s_3$ so that $A(t, s) \approx s_1 - s_2$ or $s_4 - s_3 > s_1 - s_2$ so that $A(t, s) \approx s_4 - s_3$ or if $s_3 - s_2 > s_4 - s_1$ so that $A(t, s) \approx s_3 - s_2$ or $s_4 - s_1 > s_3 - s_2$ so that $A(t, s) \approx s_4 - s_1$, then, following the same steps as in 5.1, we get the following estimates

$$\sup_{s_i} \int \int \int_{\substack{s_1 - s_2 > s_4 - s_3 \\ \text{or } s_4 - s_3 > s_1 - s_2}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|.$$

and

$$\sup_{s_i} \int \int \int_{\substack{s_3 - s_2 > s_4 - s_1 \\ \text{or } s_4 - s_1 > s_3 - s_2}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|.$$

Therefore, from now on, we shall always assume that that for all $s \in R_{III}$, we have that

$$\begin{aligned} s_1 - s_2 &\approx s_4 - s_3, \\ s_3 - s_2 &\approx s_4 - s_1. \end{aligned}$$

Not only that but we can also assume that

$$\boxed{\begin{aligned} s_1 - s_2 &\approx s_4 - s_3 \gg \epsilon \\ s_3 - s_2 &\approx s_4 - s_1 \gg \epsilon \end{aligned}}$$

since otherwise we can easily conclude the estimate in this case

$$\sup_{s_i} \int \int \int_{\substack{s_1 - s_2 \approx s_4 - s_3 \lesssim \epsilon \\ \text{or } s_3 - s_2 \approx s_4 - s_1 \lesssim \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim 1.$$

exactly as we did in 5.1.1.

5.3 In $R_{III1i} \cup R_{III2i}$

Since on both subregions R_{III1i} and R_{III2i} , we have that

$$|\alpha| \gg \mu\nu$$

and that

$$|A(t, s)| \approx |\alpha|,$$

then we will feel free to treat them in a unified manner.

5.3.1 When $s \in R_{III1i} \cup R_{III2i}$ and $|A(t, s)| \lesssim \epsilon$

When $|\alpha| = |s_1 - s_2 + s_3 - s_4| \lesssim \epsilon$, that is when $|A(t, s)| \lesssim \epsilon$, we use the trivial estimate

$$|K_\epsilon(s_1, s_2, s_3, s_4)| \lesssim \frac{1}{\epsilon^2}.$$

We have in this case that

$$\mu\nu = (s_1 - s_2)(s_3 - s_2) \ll \epsilon.$$

Hence, we have

$$\begin{aligned}
& \sup_{s_1} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \lesssim \epsilon, \\ (s_1-s_2)(s_3-s_2) \ll \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_2 ds_3 ds_4 \\
& \lesssim \frac{1}{\epsilon^2} \sup_{s_1} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \lesssim \epsilon, \\ (s_1-s_2)(s_3-s_2) \ll \epsilon, \\ s_1-s_2 \gg \epsilon}} ds_2 ds_3 ds_4 \\
& \approx \frac{1}{\epsilon^2} \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \int_{s_3-s_2 \ll \frac{\epsilon}{s_1-s_2}} \int_{|s_1-s_2+s_3-s_4| \lesssim \epsilon} ds_4 ds_3 ds_2 \\
& \lesssim \frac{1}{\epsilon} \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \int_{s_3-s_2 \ll \frac{\epsilon}{s_1-s_2}} ds_3 ds_2 \\
& \lesssim \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \frac{ds_2}{s_1-s_2} \lesssim |\log \epsilon|. \tag{5.19}
\end{aligned}$$

We also have

$$\begin{aligned}
& \frac{1}{\epsilon^2} \sup_{s_2} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \lesssim \epsilon, \\ (s_1-s_2)(s_3-s_2) \ll \epsilon, \\ s_1-s_2 \gg \epsilon}} ds_1 ds_3 ds_4 \\
& \approx \frac{1}{\epsilon^2} \sup_{s_2} \int_{s_1-s_2 \gg \epsilon} \int_{s_3-s_2 \ll \frac{\epsilon}{s_1-s_2}} \int_{|s_1-s_2+s_3-s_4| \lesssim \epsilon} ds_4 ds_3 ds_1 \\
& \lesssim \frac{1}{\epsilon} \sup_{s_2} \int_{s_1-s_2 \gg \epsilon} \int_{s_3-s_2 \ll \frac{\epsilon}{s_1-s_2}} ds_3 ds_1 \lesssim \sup_{s_2} \int_{s_1-s_2 \gg \epsilon} \frac{ds_1}{s_1-s_2} \lesssim |\log \epsilon|. \tag{5.20}
\end{aligned}$$

and similarly

$$\begin{aligned}
& \frac{1}{\epsilon^2} \sup_{s_3} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \lesssim \epsilon, \\ (s_1-s_2)(s_3-s_2) \ll \epsilon, \\ s_3-s_2 \gg \epsilon}} ds_1 ds_2 ds_4 \\
& \approx \frac{1}{\epsilon^2} \sup_{s_3} \int_{s_3-s_2 \gg \epsilon} \int_{s_1-s_2 \ll \frac{\epsilon}{s_3-s_2}} \int_{|s_1-s_2+s_3-s_4| \lesssim \epsilon} ds_4 ds_1 ds_2 \\
& \lesssim \frac{1}{\epsilon} \sup_{s_3} \int_{s_3-s_2 \gg \epsilon} \int_{s_1-s_2 \ll \frac{\epsilon}{s_3-s_2}} ds_1 ds_2 \lesssim \sup_{s_3} \int_{s_3-s_2 \gg \epsilon} \frac{ds_2}{s_3-s_2} \lesssim |\log \epsilon|. \tag{5.21}
\end{aligned}$$

Moreover, Remark 5.2.1 allows us to look at the weaker the estimate

$$\begin{aligned}
& \sup_{s_4} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \lesssim \epsilon, \\ (s_1-s_2)(s_3-s_2) \ll \epsilon, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_1 ds_2 ds_3 \\
& \lesssim \frac{1}{\epsilon^2} \sup_{s_4} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \lesssim \epsilon, \\ (s_4-s_3)(s_3-s_2) \ll \epsilon, \\ s_4-s_3 \gg \epsilon}} ds_1 ds_2 ds_3 \\
& \approx \frac{1}{\epsilon^2} \sup_{s_4} \int_{s_4-s_3 \gg \epsilon} \int_{s_3-s_2 \ll \frac{\epsilon}{s_4-s_3}} \int_{|s_1-s_2+s_3-s_4| \lesssim \epsilon} ds_1 ds_2 ds_3 \\
& \lesssim \frac{1}{\epsilon} \sup_{s_4} \int_{s_4-s_3 \gg \epsilon} \int_{s_3-s_2 \ll \frac{\epsilon}{s_4-s_3}} ds_2 ds_3 \lesssim \sup_{s_4} \int_{s_4-s_3 \gg \epsilon} \frac{ds_3}{s_4-s_3} \lesssim |\log \epsilon|. \quad (5.22)
\end{aligned}$$

Hence, by the estimates (5.19)-(5.22), we get the following estimate

$$\begin{aligned}
& \sup_{s_i} \int \int \int_{\substack{s \in R_{III1i} \cup R_{III2i}, \\ |A(t,s)| \lesssim \epsilon, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \\
& \approx \sup_{s_i} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \lesssim \epsilon, \\ (s_1-s_2)(s_3-s_2) \ll \epsilon, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|. \quad (5.23)
\end{aligned}$$

5.3.2 When $s \in R_{III1i} \cup R_{III2i}$ and $|A(t, s)| \gg \epsilon$

If $|\alpha| \gg \epsilon$ then $|A(t, s)| \gg \epsilon$ because $|A(t, s)| \approx |\alpha|$ whenever $s \in R_{III1i} \cup R_{III2i}$. Thus, in this case we have that

$$|K_\epsilon(s_1, s_2, s_3, s_4)| \approx \frac{1}{\alpha^2(s)}.$$

This leads to the following estimate

$$\begin{aligned}
& \sup_{s_1} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \gg \epsilon, \\ |s_1-s_2+s_3-s_4| \gg (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_2 ds_3 ds_4 \\
& \lesssim \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \int_{s_3-s_2 \gg \epsilon} \int_{|s_1-s_2+s_3-s_4| \gg (s_1-s_2)(s_3-s_2)} \frac{ds_4}{|s_1-s_2+s_3-s_4|^2} ds_3 ds_2 \\
& \lesssim \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \frac{1}{s_1-s_2} \int_{s_3-s_2 \gg \epsilon} \frac{ds_3}{s_3-s_2} ds_2 \approx |\log \epsilon|^2. \quad (5.24)
\end{aligned}$$

We also have that

$$\begin{aligned} & \sup_{s_2} \int \int_{\substack{s_1-s_2 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \int_{|s_1-s_2+s_3-s_4| \gg (s_1-s_2)(s_3-s_2)} \frac{ds_4}{|s_1-s_2+s_3-s_4|^2} ds_3 ds_1 \\ & \lesssim \sup_{s_2} \int_{s_1-s_2 \gg \epsilon} \frac{1}{s_1-s_2} \int_{s_3-s_2 \gg \epsilon} \frac{ds_3}{s_3-s_2} ds_1 \approx |\log \epsilon|^2. \end{aligned} \quad (5.25)$$

and similarly

$$\begin{aligned} & \sup_{s_3} \int \int_{\substack{s_1-s_2 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \int_{|s_1-s_2+s_3-s_4| \gg (s_1-s_2)(s_3-s_2)} \frac{ds_4}{|s_1-s_2+s_3-s_4|^2} ds_1 ds_2 \\ & \lesssim \sup_{s_3} \int_{s_3-s_2 \gg \epsilon} \frac{1}{s_3-s_2} \int_{s_1-s_2 \gg \epsilon} \frac{ds_1}{s_1-s_2} ds_2 \approx |\log \epsilon|^2. \end{aligned} \quad (5.26)$$

We have in addition the estimate

$$\begin{aligned} & \sup_{s_4} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \gg \epsilon, \\ |s_1-s_2+s_3-s_4| \gg (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_1 ds_2 ds_3 \\ & \lesssim \sup_{s_4} \int \int_{\substack{s_4-s_3 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \int_{|s_1-s_2+s_3-s_4| \gg (s_4-s_3)(s_3-s_2)} \frac{ds_1}{|s_1-s_2+s_3-s_4|^2} ds_2 ds_3 \\ & \lesssim \sup_{s_4} \int_{s_4-s_3 \gg \epsilon} \frac{1}{s_4-s_3} \int_{s_3-s_2 \gg \epsilon} \frac{ds_2}{s_3-s_2} ds_3 \approx |\log \epsilon|^2. \end{aligned} \quad (5.27)$$

Using the estimates (5.24)-(5.27) and Remark 5.2.1 we obtain the estimate

$$\sup_{s_i} \int \int \int_{\substack{s \in R_{III1i} \cup R_{III2i}, \\ |A(t,s)| \gg \epsilon, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|^2. \quad (5.28)$$

The estimates (5.23) and (5.28) together imply that

$$\sup_{s_i} \int \int \int_{\substack{s \in R_{III1i} \cup R_{III2i}, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|^2. \quad (5.29)$$

5.4 In $R_{III1ii} \cup R_{III2ii}$

We shall handle these two subregions in a unified manner analogous to that in which we previously treated the regions R_{III1i} and R_{III2i} . This is because for all $s \in R_{III1ii} \cup R_{III2ii}$, we have that

$$A(t, s) \approx \mu\nu$$

and that

$$\mu\nu \gtrsim |\alpha|. \quad (5.30)$$

Now, we proceed as we did in (5.1) and (5.3). We investigate the validity of the condition (5.7) when

$$\mu\nu \lesssim \epsilon$$

so that

$$|A(t, s)| \lesssim \epsilon$$

in which case we make use of the trivial estimate (5.9),

$$K_\epsilon(s) \approx \frac{1}{\epsilon^2},$$

and when

$$\mu\nu \gg \epsilon,$$

so that

$$|A(t, s)| \gg \epsilon,$$

where we shall employ the estimate

$$|K_\epsilon(s)| \approx \frac{1}{\mu^2\nu^2}$$

taking into account the restriction (5.30) in both cases.

5.4.1 When $s \in R_{III1ii} \cup R_{III2ii}$ and $|A(t, s)| \lesssim \epsilon$

In this case we find that

$$\begin{aligned} & \sup_{s_i} \int \int \int_{\substack{s \in R_{III1ii} \cup R_{III2ii}, \\ |A(t,s)| \lesssim \epsilon, \\ s_1 - s_2 \approx s_4 - s_3 \gg \epsilon, \\ s_3 - s_2 \approx s_4 - s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \\ & \lesssim \sup_{s_i} \int \int \int_{\substack{|s_1 - s_2 + s_3 - s_4| \lesssim (s_1 - s_2)(s_3 - s_2) \lesssim \epsilon, \\ s_1 - s_2 \approx s_4 - s_3 \gg \epsilon, \\ s_3 - s_2 \approx s_4 - s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon| \end{aligned} \quad (5.31)$$

by the estimate (5.23).

5.4.2 When $s \in R_{III1ii} \cup R_{III2ii}$ and $|A(t, s)| \gg \epsilon$

Here, we have

$$\begin{aligned}
& \sup_{s_1} \int \int \int_{\substack{(s_1-s_2)(s_3-s_2) \gg \epsilon, \\ |s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_2 ds_3 ds_4 \\
& \lesssim \sup_{s_1} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \frac{ds_2 ds_3 ds_4}{\mu^2(s) \nu^2(s)} \\
& \approx \sup_{s_1} \int \int_{\substack{s_1-s_2 \gg \epsilon \\ s_3-s_2 \gg \epsilon}} \frac{1}{(s_1-s_2)^2 (s_3-s_2)^2} \int_{|s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2)} ds_4 ds_3 ds_2 \\
& \lesssim \sup_{s_1} \int_{s_1-s_2 \gg \epsilon} \frac{1}{s_1-s_2} \int_{s_3-s_2 \gg \epsilon} \frac{ds_3}{s_3-s_2} ds_2 \lesssim |\log \epsilon|^2. \tag{5.32}
\end{aligned}$$

We in addition have the estimates

$$\begin{aligned}
& \sup_{s_2} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \frac{ds_1 ds_3 ds_4}{\mu^2(s) \nu^2(s)} \\
& \approx \sup_{s_2} \int \int_{\substack{s_1-s_2 \gg \epsilon \\ s_3-s_2 \gg \epsilon}} \frac{1}{(s_1-s_2)^2 (s_3-s_2)^2} \int_{|s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2)} ds_4 ds_3 ds_1 \\
& \lesssim \sup_{s_2} \int_{s_1-s_2 \gg \epsilon} \frac{1}{s_1-s_2} \int_{s_3-s_2 \gg \epsilon} \frac{ds_3}{s_3-s_2} ds_1 \lesssim |\log \epsilon|^2. \tag{5.33}
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{s_3} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \frac{ds_1 ds_2 ds_4}{\mu^2(s) \nu^2(s)} \\
& \approx \sup_{s_3} \int \int_{\substack{s_1-s_2 \gg \epsilon \\ s_3-s_2 \gg \epsilon}} \frac{1}{(s_1-s_2)^2 (s_3-s_2)^2} \int_{|s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2)} ds_4 ds_1 ds_2 \\
& \lesssim \sup_{s_3} \int_{s_3-s_2 \gg \epsilon} \frac{1}{s_3-s_2} \int_{s_1-s_2 \gg \epsilon} \frac{ds_1}{s_1-s_2} ds_2 \lesssim |\log \epsilon|^2. \tag{5.34}
\end{aligned}$$

Also, with the help of the result in Remark 5.2.1, we have the estimate

$$\begin{aligned}
& \sup_{s_4} \int \int \int_{\substack{(s_1-s_2)(s_3-s_2) \gg \epsilon, \\ |s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_1 ds_2 ds_3 \\
& \lesssim \sup_{s_4} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \ll (s_1-s_2)(s_3-s_2), \\ s_1-s_2 \approx s_4-s_3, \\ s_4-s_3 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \frac{ds_1 ds_2 ds_3}{(s_1-s_2)^2 (s_3-s_2)^2} \\
& \approx \sup_{s_4} \int \int_{\substack{s_4-s_3 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \int_{|s_1-s_2+s_3-s_4| \ll (s_4-s_3)(s_3-s_2)} \frac{ds_1 ds_2 ds_3}{(s_4-s_3)^2 (s_3-s_2)^2} \\
& \approx \sup_{s_4} \int \int_{\substack{s_4-s_3 \gg \epsilon, \\ s_3-s_2 \gg \epsilon}} \frac{1}{(s_4-s_3)^2 (s_3-s_2)^2} \int_{|s_1-s_2+s_3-s_4| \ll (s_4-s_3)(s_3-s_2)} ds_1 ds_2 ds_3 \\
& \lesssim \sup_{s_4} \int_{s_4-s_3 \gg \epsilon} \frac{1}{s_4-s_3} \int_{s_3-s_2 \gg \epsilon} \frac{ds_2}{s_3-s_2} ds_3 \lesssim |\log \epsilon|^2. \tag{5.35}
\end{aligned}$$

It follows from the estimates (5.32)-(5.35) then that

$$\sup_{s_i} \int \int \int_{\substack{s \in R_{III1ii} \cup R_{III2ii}, \\ |A(t,s)| \gg \epsilon, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|^2 \tag{5.36}$$

which together with the estimate (5.31) imply that

$$\sup_{s_i} \int \int \int_{\substack{s \in R_{III1ii} \cup R_{III2ii}, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim |\log \epsilon|^2 \tag{5.37}$$

5.5 In $R_{III2iii}$

Recall that we must verify condition (5.7) on the unit hypercube $[0, 1]^4$ in order to apply the interpolation Theorem 3.3.3 and get the desired estimate in Theorem 5.0.4. All the previous subregions enjoyed the privilege of the existence of uniform estimates for $H_\epsilon(t, s_1, s_2, s_3, s_4)$ that are independent of t . This made proving (5.7) on each of these regions relatively easy because there was no need to integrate in time to estimate $K_\epsilon(s_1, s_2, s_3, s_4)$. The subregion $R_{III2iii}$, where

$$\alpha < 0, \quad |\alpha| \approx \mu\nu \quad \text{and} \quad |A(t, s)| \approx \mu\nu |t - t_*(s)|,$$

is the most delicate region to deal with for this purpose. This is because whenever $s = (s_1, s_2, s_3, s_4) \in R_{III2iii}$, there is the possibility that $A(t, s)$ attains a zero inside $[0, 1]$ and we

no longer have a uniform estimate for $A(t, s)$ for all $t \in [0, 1]$ as it changes sign there now. As promised in Chapter 4 Section 4.4.6, we give numerical examples that show the importance of integration in the time variable of the functions $G_\epsilon(t, s)$ and $H_\epsilon(t, s)$ to estimate the kernels $J_\epsilon(s)$ and $K_\epsilon(s)$, respectively. Recall that

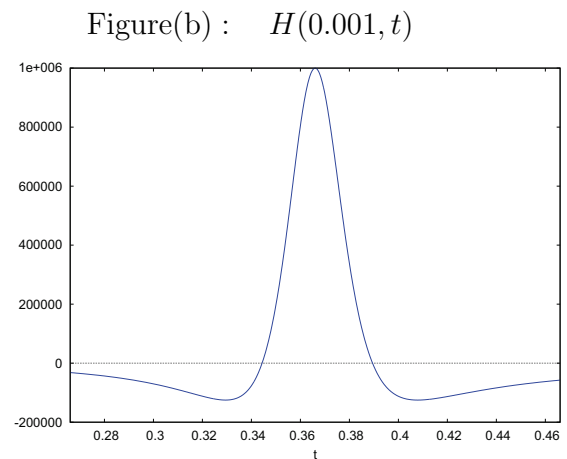
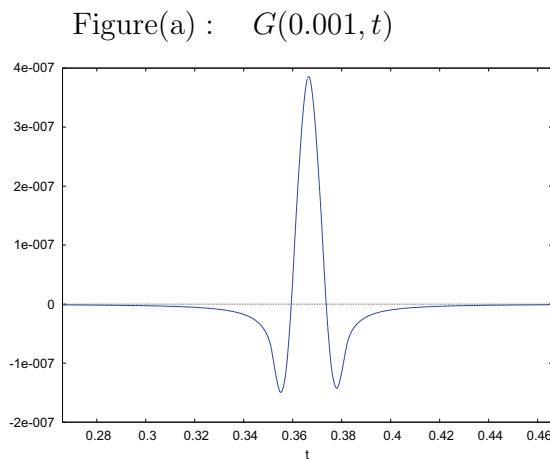
$$G_\epsilon(t, s) = \sum_{\gamma} \sigma_{\gamma} A^2(t, s - \epsilon\gamma) \log |A(t, s - \epsilon\gamma)|, \quad \sigma_{\gamma} = (-1)^{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4},$$

$$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \{0, 1\}^4, \quad H_\epsilon(t, s) = \frac{\epsilon^2 - A^2(t, s)}{(\epsilon^2 + A^2(t, s))^2}.$$

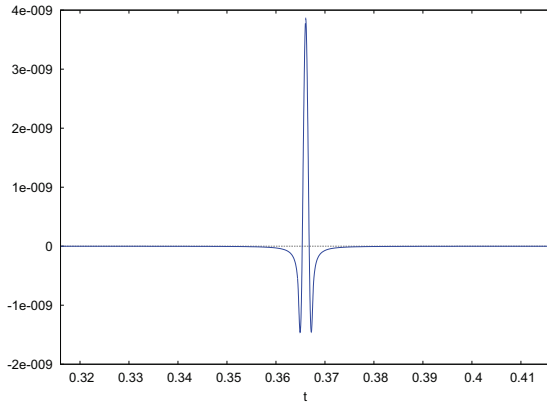
As expected, the functions $G_\epsilon(t, s)$ and $H_\epsilon(t, s)$ have identical profiles. Both change sign on the interval $[0, 1]$ when $s \in R_{III2iii}$, where

$$R_{III3iii} = \{s \in [0, 1]^4 : s_2 < s_1 < s_3 < s_4, \alpha < 0, |\alpha| \approx \mu\nu\}.$$

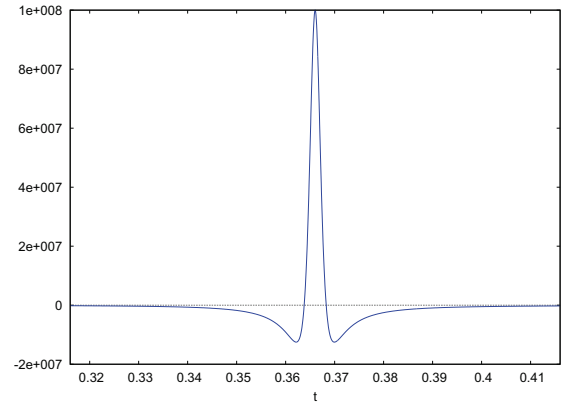
Indeed, both smooth functions have their positive peaks right near $t_*(s)$, the unique zero of $A(t, s)$ inside $[0, 1]$, and change sign once in a smooth way to become negative ever after achieving their minima in a neighborhood of $t_*(s)$ of radius about $\frac{\epsilon}{\mu\nu}$ so that they are almost symmetric around $t = t_*(s)$. In figures (a) - (f) below, $G(\epsilon, t)$, $H(\epsilon, t)$ denote the functions $G(\epsilon, t, 0.25, 0, 0.5, 1)$, $H(\epsilon, t, 0.25, 0, 0.5, 1)$, respectively.



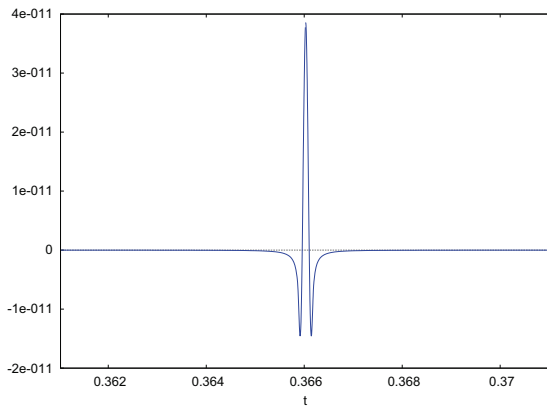
Figure(c) : $G(0.0001, t)$



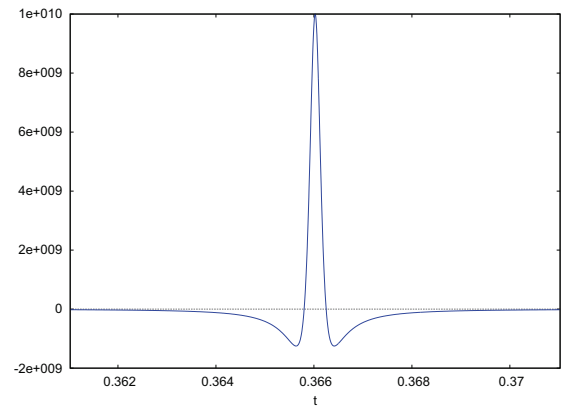
Figure(d) : $H(0.0001, t)$



Figure(e) : $G(0.00001, t)$



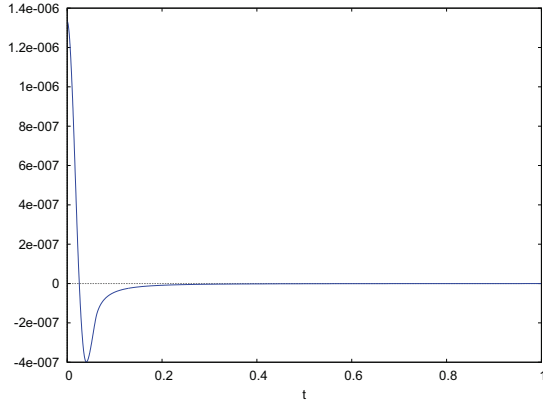
Figure(f) : $H(0.00001, t)$



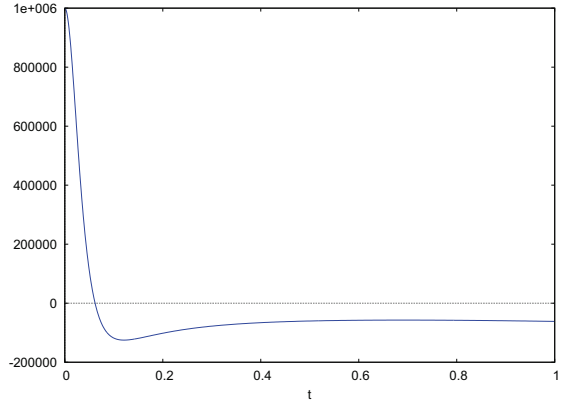
Figures (a),(c) and (e) : The function $G_\epsilon(t, 0.25, 0, 0.5, 1)$ as ϵ gets smaller.

Figures (b),(d) and (f) : The function $H_\epsilon(t, 0.25, 0, 0.5, 1)$ as ϵ gets smaller.

Figure(I) : $G_{0.001}(t, 0.2, 0.3, 0.6, 0.45)$

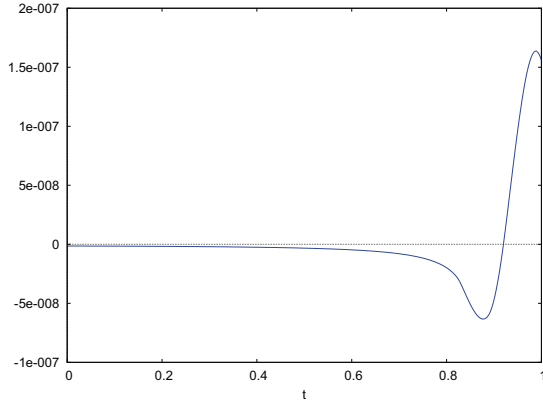


Figure(II) : $H_{0.001}(t, 0.2, 0.3, 0.6, 0.45)$

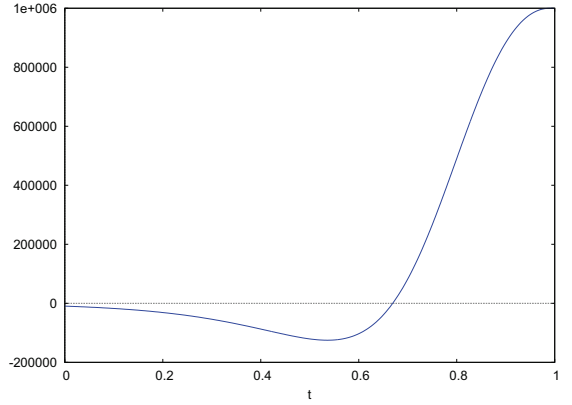


$$t_*(s) = 0.0001$$

Figure(I) : $G_{0.001}(t, 0.2, 0.3, 0.6, 0.47)$



Figure(II) : $H_{0.001}(t, 0.2, 0.3, 0.6, 0.47)$



$$t_*(s) = 0.99$$

Since we no longer have a uniform estimate for $H_\epsilon(t, s_1, s_2, s_3, s_4)$, we must integrate $H_\epsilon(t, s_1, s_2, s_3, s_4)$ in time to estimate $K_\epsilon(s_1, s_2, s_3, s_4)$. Integrating $H_\epsilon(t, s_1, s_2, s_3, s_4)$ by parts, we encounter other difficulties. One difficulty is that we need to estimate the derivatives $\partial_t^j A(t, s)$, $j = 1, 2, 3$, on the time interval where the integral is performed. We obtain the following estimate for $K_\epsilon(t, s)$.

Lemma 5.5.1. *If $\alpha < 0$, $-\alpha \approx \mu\nu$, so that $|A(t, s)| \approx \mu\nu|t-t_*|$ and if, in addition, $\mu\nu \gg \epsilon$, $s_1 - s_2 \approx s_4 - s_3$, $s_3 - s_2 \approx s_4 - s_1$, then*

$$|k_\epsilon(s)| \lesssim \frac{1}{\mu^2\nu^2} \left(\max \left\{ \frac{1}{|t_*(s)|}, \frac{1}{|1-t_*(s)|} \right\} + |\log \epsilon| \right).$$

What we shall do is the following. We begin with assuming that $t_* < \frac{1}{2}$. We will see in the proof of Lemma 5.5.1 above that there is no loss of generality in doing this. We shall divide proving the estimate (5.7) on the subregion $R_{III2iii}$ into three cases

$$(C1). |t_*(s)| \lesssim \epsilon$$

$$(C2). \epsilon \ll |t_*(s)| \lesssim \frac{\epsilon}{\mu\nu}$$

$$(C3). \frac{\epsilon}{\mu\nu} \ll |t_*(s)| \lesssim 1$$

Symmetrically, if $t_* > \frac{1}{2}$, we could have one of the following

$$(SC1). |1 - t_*(s)| \lesssim \epsilon$$

$$(SC2). \epsilon \ll |1 - t_*(s)| \lesssim \frac{\epsilon}{\mu\nu}$$

$$(SC3). \frac{\epsilon}{\mu\nu} \ll |1 - t_*(s)| \lesssim 1.$$

We shall see that, thanks to the estimates given in Lemma 5.5.2 below, the argument for the cases (SC1), (SC2) and (SC3) is totally similar to that for the cases (C1), (C2) and (C3), respectively. Before we go on outlining how to prove (5.7) in each of these cases, we summarize all the restrictions on s belonging to the region $R_{III2iii}$ taking into account Remark 5.2.1 and excluding the subcase when $-\alpha \approx \mu\nu \lesssim \epsilon$ which has already been studied in Sections 5.3.1 and 5.4.1 where we obtained the estimates (5.23) and (5.31), respectively.

$\begin{aligned} \alpha &< 0, \\ -\alpha &\approx \mu\nu \gg \epsilon, \\ A(t, s) &\approx \mu\nu t - t_*(s) , \\ s_1 - s_2 &\approx s_4 - s_3 \gg \epsilon, \\ s_3 - s_2 &\approx s_4 - s_1 \gg \epsilon \end{aligned}$

In the first two cases, the cases (C1) and (C2), we will consider a division of the interval $[0, 1]$ into two intervals I_i where

$$I_1(s) = \{t \in [0, 1] : |t - t_*(s)| \lesssim \frac{\epsilon}{\mu\nu}\},$$

$$I_2(s) = [0, 1] - I_1(s) = \{t \in [0, 1] : |t - t_*(s)| \gg \frac{\epsilon}{\mu\nu}\}.$$

In the case (C1), to estimate

$$K_\epsilon(s_1, s_2, s_3, s_4) = \int_0^1 H_\epsilon(t, s_1, s_2, s_3, s_4) dt.$$

we clearly have to integrate $H_\epsilon(t, s_1, s_2, s_3, s_4)$ on both intervals $I_1(s)$ and $I_2(s)$. On the interval $I_1(s)$, we merely use the trivial estimate

$$H_\epsilon(t, s_1, s_2, s_3, s_4) \lesssim \frac{1}{\epsilon^2}$$

and integrate it there. On the interval $I_2(s)$, we actually integrate $H_\epsilon(t, s_1, s_2, s_3, s_4)$. The integration on $I_2(s)$ won't be as difficult as it may seem though. This is because on $I_2(s)$, we have that

$$|t - t_*(s)| \gg \frac{\epsilon}{\mu\nu}.$$

Hence we have the following estimate for $A(t, s)$.

$$|A(t, s)| \gg \epsilon.$$

This in turn implies that

$$|H_\epsilon(t, s_1, s_2, s_3, s_4)| \lesssim \frac{1}{A^2(t, s)} \approx \frac{1}{\mu\nu} \frac{1}{(t - t_*(s))^2}$$

which we integrate on $I_2(s)$.

In the case (C2), since, on the interval $I_1(s)$, we have that

$$|t - t_*(s)| \lesssim \frac{\epsilon}{\mu\nu} \ll 1,$$

then we find from Lemma 5.5.2 that the derivative $\partial_t A(t, s)$ sustains the estimate $-\partial_t A(t, s) \approx \mu\nu$ on the whole interval. This enables us to integrate $H_\epsilon(t, s_1, s_2, s_3, s_4)$ by parts using the formula given below in Lemma 5.5.3.

On the interval $I_2(s)$, we, similarly to what we did in case (C1), integrate

$$|H_\epsilon(s_1, s_2, s_3, s_4)| \approx \frac{1}{A^2(t, s)}.$$

This estimate follows obviously from the fact that for all $t \in I_2(s)$, we have that

$$t - t_*(s) \gg \frac{\epsilon}{\mu\nu}$$

and therefore the function $A(t, s)$ is uniformly bounded from below by

$$A(t, s) \gg \epsilon.$$

In the case (C3) we basically repeat what we have done on the intervals I_i , in the previous two cases, on the intervals $\tilde{I}_i(s)$ where

$$\tilde{I}_1(s) = \{t \in [0, 1] : |t - t_*(s)| \ll 1\},$$

$$\tilde{I}_2(s) = [0, 1] - \tilde{I}_1(s) = \{t \in [0, 1] : |t - t_*(s)| \gtrsim 1\}.$$

After estimating $K_\epsilon(s_1, s_2, s_3, s_4)$, in each one of the cases (C1) - (C3), we go on to the next step and verify the condition (5.7).

Now, we go through the details. We start by proving the Lemmas 5.5.3 and 5.5.4 below that we shall need later. First, we recall the following Lemma (Lemma 3.4.5, Section 3.4.5, Chapter 3).

Lemma 5.5.2. *If*

$$s_2 < s_1 < s_3 < s_4, \quad \alpha < 0, \quad -\alpha \approx \mu\nu, \quad s_3 - s_2 \approx s_4 - s_1,$$

then the derivatives $\partial_t A(t, s)$, $\partial_{tt} A(t, s)$ and $\partial_{ttt} A(t, s)$ satisfy the following uniform estimates whenever $|t - t_(s)| \ll 1$ where $t_*(s)$ is such that $A(t_*(s), s) = 0$.*

$$\boxed{\begin{array}{l} -\partial_t A(t, s) \approx \mu\nu \\ \partial_{tt} A(t, s) \approx \mu\nu \\ -\partial_{ttt} A(t, s) \approx \mu\nu \end{array}} \quad (5.38)$$

In particular, when $t_(s) > \mu\nu$, we have*

$$\boxed{\begin{array}{l} -\partial_t A(t, s) \approx \mu\nu \text{ whenever } t \in (t_*(s) - \mu\nu, t_*(s) + \mu\nu), \\ -\partial_t A(1, s) \approx \mu\nu \text{ whenever } |1 - t_*(s)| \ll 1, \\ -\partial_t A(0, s) \approx \mu\nu. \end{array}} \quad (5.39)$$

In Lemma 5.5.3 below, we estimate the integral

$$\int_a^b H_\epsilon(t, s_1, s_2, s_3, s_4) dt$$

where the interval $[a, b]$ is such that the unique zero $t_*(s)$ of $A(t, s)$ is inside $[a, b]$ so that $A(t, s)$ changes its sign on it. We also require that the interval $[a, b]$ is small enough so that the derivative $\partial_t^j A(t, s)$, that satisfy the estimates $(-1)^j \partial_t^j A(t_*(s), s) \approx \mu\nu$, $j = 1, 2, 3$, by Lemma 5.5.2, sustain these uniform estimates on the whole interval $[a, b]$. Lemma 5.5.2 tells us that it is enough for this purpose to assume that $|b - a| \ll 1$.

Lemma 5.5.3. *Let $[a, b] \subset [0, 1]$ be such that $t_*(s) \in (a, b)$ and*

$$\max \{b - t_*(s), t_*(s) - a\} \ll 1.$$

Then, we have that

$$\left| \int_a^b H_\epsilon(t, s_1, s_2, s_3, s_4) dt \right| \lesssim \frac{1}{\mu^2 \nu^2} \left(|\log \epsilon| + \max \left\{ \frac{1}{|a - t_*(s)|}, \frac{1}{|b - t_*(s)|} \right\} \right).$$

Proof. Notice that $H_\epsilon(t, s) = f_\epsilon(A(t, s))$ where $f_c(x) = \frac{c^2 - x^2}{(c^2 + x^2)^2}$. The function $f_c(x)$ has the following property

$$f_c(x) = \frac{d}{dx} \frac{1}{c^2 + x^2} = \frac{1}{2} \frac{d^2}{dx^2} \log(c^2 + x^2).$$

Exploiting this and integrating by parts we get

$$\begin{aligned} & \int_a^b H_\epsilon(t, s) dt \\ &= \int_a^b \frac{\epsilon^2 - A^2(t, s)}{[\epsilon^2 + A^2(t, s)]^2} B(t, s) dt \\ &= \int_a^b B(t, s) \frac{d}{dA} \frac{A(t, s)}{\epsilon^2 + A^2(t, s)} dt \\ &= \int_a^b \frac{B(t, s)}{\partial_t A(t, s)} \partial_t \left(\frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \right) dt \\ &= \frac{B(b, s)}{\partial_t A(b, s)} \frac{A(b, s)}{\epsilon^2 + A^2(b, s)} - \frac{B(a, s)}{\partial_t A(a, s)} \frac{A(a, s)}{\epsilon^2 + A^2(a, s)} - \int_a^b \frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) dt, \end{aligned}$$

and integrating by parts again we have

$$\begin{aligned}
& \int_a^b \frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) dt \\
&= \frac{1}{2} \int_a^b \frac{d}{dA} \log(\epsilon^2 + A^2(t, s)) \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) dt \\
&= \frac{1}{2} \int_a^b \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] \partial_t \log(\epsilon^2 + A^2(t, s)) dt \\
&= \frac{1}{2} \left[\frac{\log(\epsilon^2 + A^2(b, s))}{\partial_t A(b, s)} \partial_t \left(\frac{B(b, s)}{\partial_t A(b, s)} \right) \right] - \frac{1}{2} \left[\frac{\log(\epsilon^2 + A^2(a, s))}{\partial_t A(a, s)} \partial_t \left(\frac{B(a, s)}{\partial_t A(a, s)} \right) \right] + \\
&\quad - \frac{1}{2} \int_a^b \log(\epsilon^2 + A^2(t, s)) \partial_t \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_a^b H_\epsilon(t, s) dt \\
&= \frac{B(b, s)}{\partial_t A(b, s)} \frac{A(b, s)}{\epsilon^2 + A^2(b, s)} - \frac{B(a, s)}{\partial_t A(a, s)} \frac{A(a, s)}{\epsilon^2 + A^2(a, s)} + \\
&\quad - \frac{1}{2} \left[\frac{\log(\epsilon^2 + A^2(b, s))}{\partial_t A(b, s)} \partial_t \left(\frac{B(b, s)}{\partial_t A(b, s)} \right) \right] + \frac{1}{2} \left[\frac{\log(\epsilon^2 + A^2(a, s))}{\partial_t A(a, s)} \partial_t \left(\frac{B(a, s)}{\partial_t A(a, s)} \right) \right] + \\
&\quad + \frac{1}{2} \int_a^b \log(\epsilon^2 + A^2(t, s)) \partial_t \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] dt. \tag{5.40}
\end{aligned}$$

Now, we estimate the terms on the right hand side of (5.40) using the estimates obtained in Lemma 5.5.2.

Since

$$B(t, s) = \prod_{l=1}^4 \frac{1}{(1+t+s_l)^2} \approx 1,$$

for all $t \in [0, 1]$, then

$$B(a, s) \approx 1, \quad B(b, s) \approx 1. \tag{5.41}$$

We also have that

$$|A(a, s)| \approx \mu\nu|a - t_*(s)|, \quad |A(b, s)| \approx \mu\nu|b - t_*(s)|. \tag{5.42}$$

It always holds true that

$$|\log(\epsilon^2 + A^2(t, s))| \lesssim |\log \epsilon|. \tag{5.43}$$

Now, because of the assumptions that $t_*(s) \in (a, b)$ and that $|b - a| \ll 1$, we have from the estimate (5.38) proven in Lemma 5.5.2 that

$$(-1)^j \partial_t^j A(t, s) \approx \mu\nu.$$

Moreover, we have that

$$|\partial_t B(t, s)| \lesssim 1, \quad |\partial_{tt} B(t, s)| \lesssim 1.$$

Hence

$$\left| \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right| = \left| \frac{\partial_t B(t, s)}{\partial_t A(t, s)} - \frac{B(t, s) \partial_{tt} A(t, s)}{(\partial_t A(t, s))^2} \right| \lesssim \frac{1}{\mu^2(s) \nu^2(s)}. \quad (5.44)$$

Furthermore, we have

$$\begin{aligned} & \left| \partial_t \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] \right| \\ &= \left| \frac{\partial_{tt} B(t, s)}{(\partial_t A(t, s))^2} - \frac{3 \partial_t B(t, s) \partial_{tt} A(t, s)}{(\partial_t A(t, s))^3} - \frac{B(t, s) \partial_{ttt} A(t, s)}{(\partial_t A(t, s))^3} - \frac{3 B(t, s) (\partial_{tt} A(t, s))^2}{(\partial_t A(t, s))^4} \right| \\ &\lesssim \frac{1}{\mu^2(s) \nu^2(s)}. \end{aligned} \quad (5.45)$$

Finally, it follows from (5.40) and the estimates (5.41)-(5.45) that

$$\left| \int_a^b H_\epsilon(t, s) dt \right| \lesssim \frac{1}{\mu^2 \nu^2} \left(\max \left\{ \frac{1}{|a - t_*(s)|}, \frac{1}{|b - t_*(s)|} \right\} + |\log \epsilon| \right).$$

This completes the proof of Lemma 5.5.3. \square

Lemma 5.5.4 below is concerned with the following question. Let $i \in \{1, 2, 3, 4\}$. If we fix s_j , $j \neq i$, can we get an estimate of the measure of the largest interval in which s_i lives while $|t_*(s)| \lesssim \lambda$? The answer can easily be obtained using the mean value theorem and an estimate for $\partial_{s_i} t_*(s)$.

Lemma 5.5.4. *Let i, j, k, l , be any four distinct integers in $\{1, 2, 3, 4\}$. Fix s_i, s_j, s_k and Δ_λ be the variation of the fourth variable s_l when $t_* \approx \lambda$. Obviously Δ_λ depends on s_i, s_j and s_k . And we have*

$$\Delta_\lambda \lesssim \lambda \mu \nu. \quad (5.46)$$

Proof. By the mean value theorem and the estimate (4.56), we have

$$\frac{\lambda}{\Delta_\lambda} = \partial_{s_l} t_*(s) |_{s_i, s_j, s_k \text{ are fixed}} \approx \frac{1}{\mu \nu} \quad (5.47)$$

which implies the estimate (5.46). \square

5.5.1 (C1): $|t_*(s)| \lesssim \epsilon$

Let

$$I_1(s) = \{t \in [0, 1] : |t - t_*(s)| \lesssim \frac{\epsilon}{\mu\nu}\},$$

$$I_2(s) = [0, 1] - I_1(s) = \{t \in [0, 1] : |t - t_*(s)| \gg \frac{\epsilon}{\mu\nu}\}.$$

Then we have

$$\begin{aligned} |K_\epsilon(s_1, s_2, s_3, s_4)| &= \left| \int_0^1 H_\epsilon(t, s) dt \right| \\ &= \int_{I_1(s)} H_\epsilon(t, s) dt + \int_{I_2(s)} H_\epsilon(t, s) dt \\ &\lesssim \int_{I_1(s)} \frac{dt}{\epsilon^2} + \int_{I_2(s)} \frac{dt}{A^2(2)} \\ &\lesssim \frac{|I_1(s)|}{\epsilon^2} + \frac{1}{\mu^2 \nu^2} \int_{I_2(s)} \frac{dt}{(t - t_*(s))^2} \lesssim \frac{1}{\epsilon \mu \nu}, \end{aligned} \tag{5.48}$$

where we used the global uniform estimate

$$|H_\epsilon(t, s_1, s_2, s_3, s_4)| \lesssim \frac{1}{\epsilon^2}$$

on the interval $I_1(s)$ and used the estimate

$$|H_\epsilon(t, s_1, s_2, s_3, s_4)| \lesssim \frac{1}{A^2(t, s)}$$

on the interval $I_2(s)$. The latter estimate is implied by the fact that on $I_2(s)$, we have that

$$|t - t_*(s)| \gg \frac{\epsilon}{\mu\nu}$$

which makes

$$|A(t, s)| \approx \mu\nu |t - t_*(s)| \gg \epsilon.$$

Now we show the estimate (5.7) for this case using the estimate (5.48) we have just obtained for $K_\epsilon(s)$. We have

$$\begin{aligned} &\sup_{s_i \in [0, 1]} \int \int \int_{\substack{s_{i_2}, s_{i_3}, s_{i_4} \\ |s_1 - s_2 + s_3 - s_4| \approx \mu\nu \\ s_1 - s_2 \approx s_4 - s_3 \gg \epsilon \\ s_3 - s_2 \approx s_4 - s_1 \gg \epsilon \\ |t_*(s)| \lesssim \epsilon}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \\ &\lesssim \sup_{s_i \in [0, 1]} \int \int \int_{\substack{|\Delta_\epsilon| \lesssim \epsilon\mu\nu, \\ m \in \{j, k, l\}}} \frac{1}{\epsilon \mu \nu} \lesssim 1, \end{aligned} \tag{5.49}$$

where we interpreted the restriction $|t_*(s)| \lesssim \epsilon$ using Lemma 5.5.4 into the condition $|\Delta_\epsilon| \lesssim \epsilon\mu\nu$.

5.5.2 (C2): $\epsilon \ll |t_*(s)| \lesssim \frac{\epsilon}{\mu\nu}$

Let

$$\begin{aligned} I_1(s) &= \{t \in [0, 1] : 0 \leq t \lesssim t_*(s) + \frac{\epsilon}{\mu\nu}\}, \\ I_2(s) &= [0, 1] - I_1(s) = \{t \in [0, 1] : t_*(s) + \frac{\epsilon}{\mu\nu} \ll t \leq 1\}. \end{aligned}$$

Since

$$|t_*(s)| \lesssim \frac{\epsilon}{\mu\nu}$$

then

$$t_*(s) + \frac{\epsilon}{\mu\nu} \approx \frac{\epsilon}{\mu\nu} \ll 1.$$

That is $|I_1(s)| \approx \frac{\epsilon}{\mu\nu} \ll 1$. Hence, by Lemma 5.5.3, we have that

$$\begin{aligned} \left| \int_{I_1(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt \right| &\lesssim \frac{1}{\mu^2\nu^2} \left(\max \left\{ \frac{1}{|t_*(s)|}, \frac{\mu\nu}{\epsilon} \right\} + |\log \epsilon| \right) \\ &\lesssim \frac{1}{\mu^2\nu^2} \left(|\log \epsilon| + \frac{1}{|t_*(s)|} \right). \end{aligned} \quad (5.50)$$

On the interval $I_2(s)$, the function $A(t, s)$ enjoys the estimate $|A(t, s)| \gg \epsilon$. Therefore we have

$$\begin{aligned} \left| \int_{I_2(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt \right| &\lesssim \int_{I_2(s)} \frac{dt}{A^2(t, s)} \\ &\lesssim \frac{1}{\mu^2(s)\nu^2(s)} \int_{t_*(s) + \frac{\epsilon}{\mu\nu}}^1 \frac{dt}{(t - t_*(s))^2} \\ &\lesssim \frac{1}{\mu^2(s)\nu^2(s)} \left[\frac{\mu\nu}{\epsilon} + \frac{1}{1 - t_*(s)} \right] \lesssim \frac{1}{\mu\nu} \frac{1}{\epsilon}. \end{aligned} \quad (5.51)$$

From the estimates (5.50) and (5.51), we deduce that

$$\begin{aligned}
|K_\epsilon(s_1, s_2, s_3, s_4)| &= \left| \int_0^1 H_\epsilon(t, s_1, s_2, s_3, s_4) dt \right| \\
&= \int_{I_1(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt + \int_{I_2(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt \\
&\lesssim \frac{1}{\mu^2 \nu^2} \left(\frac{1}{|t_*(s)|} + \frac{1}{\epsilon \mu \nu} + |\log \epsilon| \right) \lesssim \frac{1}{\mu^2 \nu^2} \left(\frac{1}{|t_*(s)|} + |\log \epsilon| \right)
\end{aligned} \tag{5.52}$$

because

$$\frac{1}{t_*(s)} \gtrsim \frac{\mu \nu}{\epsilon}.$$

In the light of the estimate (5.7) we need to show, we realize that we have to prove

$$\sup_{s_i \in [0,1]} \int \int \int_{\substack{|s_1 - s_2 + s_3 - s_4| \approx \mu \nu, \\ |s_1 - s_2| \approx |s_4 - s_3| \gg \epsilon, \\ |s_3 - s_2| \approx |s_4 - s_1| \gg \epsilon, \\ \epsilon \ll |t_*(s)| \lesssim \frac{\epsilon}{\mu \nu}}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim (|\log \epsilon|)^2. \tag{5.53}$$

Now, by

- a dyadic decomposition of the operator in (5.53),
- the estimate (5.52) we have proved above for this case in s and $t_*(s)$,
- noticing how Lemma 5.5.4 implies that whenever we fix any three of the four variables s_i then, for $t_*(s)$ to lie in an interval of length comparable to the dyadic number λ , where $\epsilon \ll \lambda \lesssim \frac{\epsilon}{\mu \nu} \ll 1$, the variation in the value of the fourth variable $\lesssim \lambda \mu \nu$,
- the fact that the integral

$$\int \int \int_{\substack{|s_1 - s_2 + s_3 - s_4| \approx \mu \nu, \\ |s_1 - s_2| \approx |s_4 - s_3| \approx \mu, \\ |s_3 - s_2| \approx |s_4 - s_1| \approx \nu, \\ |\Delta \lambda| \lesssim \lambda \mu \nu}} ds_j ds_k ds_l \approx \mu^2 \nu^2 \lambda,$$

we see that

$$\begin{aligned}
& \sup_{s_i \in [0,1]} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \approx \mu\nu, \\ s_1-s_2 \approx s_4-s_3 \gg \epsilon, \\ s_3-s_2 \approx s_4-s_1 \gg \epsilon, \\ \epsilon \ll |t_*(s)| \lesssim \frac{\epsilon}{\mu\nu}}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \\
& \lesssim \sum_{\substack{\epsilon \ll \mu \lesssim 1, \\ \mu \text{ dyadic}}} \frac{1}{\mu^2} \sum_{\substack{\epsilon \ll \nu \lesssim 1, \\ \nu \text{ dyadic}}} \frac{1}{\nu^2} \sum_{\substack{\epsilon \ll \lambda \lesssim 1, \\ \lambda \text{ dyadic}}} \left(\frac{1}{\lambda} + |\log \epsilon| \right) \\
& \sup_{s_i \in [0,1]} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \approx \mu\nu, \\ s_1-s_2 \approx s_4-s_3 \approx \mu, \\ s_3-s_2 \approx s_4-s_1 \approx \nu, \\ |\Delta\lambda| \lesssim \lambda\mu\nu}} ds_j ds_k ds_l \\
& \gtrsim \sum_{\substack{\epsilon \ll \mu \lesssim 1, \\ \mu \text{ dyadic}}} 1 \sum_{\substack{\epsilon \ll \nu \lesssim 1, \\ \nu \text{ dyadic}}} \nu \sum_{\substack{\epsilon \ll \lambda \lesssim 1, \\ \lambda \text{ dyadic}}} (1 + |\log \epsilon|) \lesssim (|\log \epsilon|)^2. \tag{5.54}
\end{aligned}$$

5.5.3 (C3): $\frac{\epsilon}{\mu\nu} \ll |t_*(s)| \lesssim 1$

Assume that

$$\begin{aligned}
\tilde{I}_1(s) &= \{t \in [0, 1] : |t - t_*(s)| \ll 1\}, \\
\tilde{I}_2(s) &= [0, 1] - \tilde{I}_1(s) = \{t \in [0, 1] : |t - t_*(s)| \gtrsim 1\}.
\end{aligned}$$

On the interval $\tilde{I}_1(s)$ we have that

$$|t - t_*(s)| \ll 1.$$

In other words $|\tilde{I}_1(s)| \ll 1$. We can then use Lemma 5.5.3 and get the estimate

$$\left| \int_{\tilde{I}_1(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt \right| \lesssim \frac{1}{\mu^2\nu^2} \left(\frac{1}{|t_*(s)|} + |\log \epsilon| \right). \tag{5.55}$$

While for all $t \in \tilde{I}_2(s)$, we have

$$|A(t, s)| \gg \mu\nu |t - t_*(s)| \gtrsim \mu\nu \gg \epsilon.$$

Hence

$$\begin{aligned}
& \left| \int_{\tilde{I}_2(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt \right| \lesssim \int_{\tilde{I}_2(s)} \frac{dt}{A^2(t, s)} \\
& \lesssim \frac{1}{\mu^2(s)\nu^2(s)} \int_{|t-t_*(s)| \gtrsim 1} \frac{dt}{(t - t_*(s))^2} \lesssim \frac{1}{\mu^2\nu^2}. \tag{5.56}
\end{aligned}$$

Using the estimates (5.55) and (5.56), we obtain that

$$\begin{aligned}
|K_\epsilon(s_1, s_2, s_3, s_4)| &= \left| \int_0^1 H_\epsilon(t, s_1, s_2, s_3, s_4) dt \right| \\
&= \int_{\tilde{I}_1(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt + \int_{\tilde{I}_2(s)} H_\epsilon(t, s_1, s_2, s_3, s_4) dt \lesssim \frac{1}{\mu^2 \nu^2} \left(|\log \epsilon| + \frac{1}{t_*(s)} \right).
\end{aligned} \tag{5.57}$$

Now we proceed in a way similar to that of section 5.5.3. That is, we employ

- a dyadic decomposition of the operator we want to estimate,
- the estimate (5.57) we have proved for $k_\epsilon(s_1, s_2, s_3, s_4)$ in this case in s and $t_*(s)$,
- Lemma 5.5.4 that implies the restriction $|\Delta_\lambda| \lesssim \lambda \mu \nu$ whenever $t_*(s)$ lies inside an interval of length about λ where $\epsilon/\mu\nu \ll \lambda \lesssim 1$ is a dyadic number.

We deduce then from Remark ?? and the estimate (5.54) that

$$\begin{aligned}
&\sup_{s_i \in [0,1]} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \approx \mu\nu, \\ s_1-s_2 \approx s_4-s_3 >> \epsilon, \\ s_3-s_2 \approx s_4-s_1 >> \epsilon, \\ \frac{\epsilon}{\mu\nu} \ll |t_*(s)| \lesssim 1}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \\
&\lesssim \sum_{\substack{\epsilon \ll \mu \lesssim 1, \\ \mu \text{ dyadic}}} \frac{1}{\mu^2} \sum_{\substack{\epsilon \ll \nu \lesssim 1, \\ \nu \text{ dyadic}}} \frac{1}{\nu^2} \sum_{\substack{\epsilon \ll \lambda \lesssim 1, \\ \lambda \text{ dyadic}}} \left(|\log \epsilon| + \frac{1}{\lambda} \right) \\
&\sup_{s_i \in [0,1]} \int \int \int_{\substack{|s_1-s_2+s_3-s_4| \approx \mu\nu, \\ s_1-s_2 \approx s_4-s_3 \approx \mu, \\ s_3-s_2 \approx s_4-s_1 \approx \nu, \\ |\Delta_\lambda| \lesssim \lambda \mu \nu}} ds_j ds_k ds_l \lesssim (|\log \epsilon|)^2.
\end{aligned} \tag{5.58}$$

Finally by the estimates (5.49), (5.53) and (5.58) that are proven in sections 5.5.1, 5.5.2 and 5.5.3 respectively, we obtain

$$\sup_{s_i \in [0,1]} \int \int \int_{s \in R_{III2iii}} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim (|\log \epsilon|)^2. \tag{5.59}$$

5.6 In R_{IIISYM}

Recall from the study in Chapter 3 Section 3.4.2 of the function $A(t, s)$ that the region R_{III} is the region in $[0, 1]^4$ where s satisfies one of the following eight conditions

	$s_1 - s_2$	$s_3 - s_4$	$s_3 - s_2$	$s_1 - s_4$
$s_2 < s_1 < s_3 < s_4$	+	-	+	-
$s_2 < s_3 < s_1 < s_4$	+	-	+	-
$s_4 < s_1 < s_3 < s_2$	-	+	-	+
$s_4 < s_3 < s_1 < s_2$	-	+	-	+
$s_1 < s_2 < s_4 < s_3$	-	+	+	-
$s_1 < s_4 < s_2 < s_3$	-	+	+	-
$s_3 < s_2 < s_4 < s_1$	+	-	-	+
$s_3 < s_4 < s_2 < s_1$	+	-	-	+

And we have seen that thanks to the symmetry properties satisfied by the function $A(t, s)$, namely

$$\begin{aligned} A(t, s_1, s_2, s_3, s_4) &= A(t, s_3, s_2, s_1, s_4) = A(t, s_1, s_4, s_3, s_2) = A(t, s_3, s_4, s_1, s_2) \\ &= -A(t, s_2, s_1, s_4, s_3) = -A(t, s_2, s_3, s_4, s_1) = -A(t, s_4, s_1, s_2, s_3) = -A(t, s_4, s_3, s_2, s_1), \end{aligned}$$

we were able to choose any of the arrangements above for the components of (s_1, s_2, s_3, s_4) and prove the estimate for it. The estimate for any of the remaining cases would follow in the same way. Here we try to make this clear and precise.

Assume that we choose the following arrangement

$$\boxed{s_i < s_j < s_k < s_l}$$

where obviously

$$\begin{aligned} (i, j, k, l) \in \{ &(2, 1, 3, 4), (2, 3, 1, 4), (4, 1, 3, 2), (4, 3, 1, 2), \\ &(1, 2, 4, 3), (1, 4, 2, 3), (3, 2, 4, 1), (3, 4, 2, 1) \}. \end{aligned}$$

We let

$$\mu = s_j - s_i,$$

$$\nu = s_k - s_i,$$

$$\tau = 1 + t + s_i,$$

$$\alpha = s_i - s_j + s_k - s_l.$$

We have then that

$$A(t, s) = \frac{g(\tau)}{\tau(\tau + \mu)(\tau + \nu)(\tau + \mu + \nu - \alpha)},$$

where

$$g(\tau) = \alpha\tau^2 + 2\mu\nu\tau + \mu\nu(\mu + \nu - \alpha).$$

Hence the argument can be continued exactly as in Sections 5.2, 5.3, 5.4 and 5.5 simply by replacing

$$s_1 \rightarrow s_j,$$

$$s_2 \rightarrow s_i,$$

$$s_3 \rightarrow s_k,$$

$$s_4 \rightarrow s_l.$$

5.7 An inhomogeneous Strichartz estimate for the special inhomogeneity (Proof of Theorem 5.0.4)

From the estimates (5.17), (5.18), (5.29), (5.37), (5.59) and the symmetry observation given in Section 5.6, we get the estimate (5.7) that reads

$$\sup_{s_i \in [0,1]} \int \int \int_{[0,1]^3} |K_\epsilon(s_1, s_2, s_3, s_4)| ds_j ds_k ds_l \lesssim (|\log \epsilon|)^2$$

where (i, j, k, l) is a permutation of the integers $\{1, 2, 3, 4\}$.

Using this estimate and the interpolation result given in Theorem 3.3.3 implies the estimate of Theorem 5.0.4. The following estimate is a direct consequence of Theorem 5.0.4.

Theorem 5.7.1. *Consider the fundamental solution*

$$u(t, x) = \int_0^1 \frac{e^{\frac{i|x|^2}{1+t+\sigma}}}{(1+t+\sigma)^2} f(\sigma) d\sigma$$

of the Cauchy problem

$$i\partial_t u(t, x) + \Delta_x u(t, x) = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad u(0, x) = 0,$$

where the forcing term F is given by

$$F(t, x) = f(t) \delta_0(x),$$

and f is supported on $[0, 1]$. Then we have the following estimate

$$\frac{\| u e^{-\epsilon|x|^2} \|_{L^4([2,3] \times \mathbb{R}^4)}}{\| f \|_{L^4([0,1])}} \lesssim |\log \epsilon|^{\frac{1}{2}}. \quad (5.60)$$

5.8 Remarks on the results

5.8.1 A remark on the singularities of the kernel $K_\epsilon(s)$

In the following Lemma we calculate the limit $\lim_{\epsilon \rightarrow 0^+} K_\epsilon(s)$. This helps us understand the nature of the singularities the kernel involves.

Lemma 5.8.1. *Let*

$$A(t, s) = \sum_{l=1}^4 \frac{(-1)^l}{1+t+s_l}, \quad B(t, s) = \prod_{l=1}^4 \partial_{s_l} A(t, s), \quad K_\epsilon(s) = \int_0^1 \frac{\epsilon^2 - A^2(t, s)}{[\epsilon^2 + A^2(t, s)]^2} B(t, s) dt.$$

Let s be such that $t_*(s) \notin \{0, 1\}$. Then

$$\lim_{\epsilon \rightarrow 0^+} K_\epsilon(s) = K_0(s)$$

where

$$K_0(s) = \begin{cases} \frac{1}{4\Delta(s)} \left[\frac{1}{t_*(s)} + \frac{1}{t_-(s)} + \frac{1}{1-t_*(s)} + \frac{1}{1-t_-(s)} \right] - \frac{\alpha(s)}{4\Delta^{\frac{3}{2}}(s)} \\ \quad (\log |1-t_*(s)| - \log |t_*(s)| - \log |1-t_-(s)| + \log |t_-(s)|) & \text{when } \Delta(s) > 0 \\ \frac{1}{4|\Delta(s)|} \left[\frac{1}{t_*(s)} + \frac{1}{t_-(s)} + \frac{1}{1-t_*(s)} + \frac{1}{1-t_-(s)} \right] - \frac{i\alpha(s)}{4|\Delta(s)|^{\frac{3}{2}}} \\ \quad [\log_{\mathbb{C}}(1-t_*(s)) - \log_{\mathbb{C}}(-t_*(s)) - \log_{\mathbb{C}}(1-t_-(s)) + \log_{\mathbb{C}}(-t_-(s))] & \text{when } \Delta(s) < 0. \end{cases}$$

We may also write

$$\begin{aligned} \log_{\mathbb{C}} \frac{1-t_*(s)}{-t_*(s)} - \log_{\mathbb{C}} \frac{1-t_-(s)}{-t_-(s)} &= i \left[\arg \frac{1-t_*(s)}{-t_*(s)} - \arg \frac{1-t_-(s)}{-t_-(s)} \right] \\ &= 2i \arctan \frac{\sqrt{|\Delta(s)|}}{\alpha(s)(t_*(s)+t_-(s))(2-t_*(s)-t_-(s))+4\Delta(s)} \end{aligned}$$

so that

$$K_0(s) = \begin{cases} \frac{1}{4\Delta(s)} \left[\frac{1}{t_*(s)} + \frac{1}{t_-(s)} + \frac{1}{1-t_*(s)} + \frac{1}{1-t_-(s)} \right] - \frac{\alpha(s)}{4\Delta^{\frac{3}{2}}(s)} \\ \quad (\log |1-t_*(s)| - \log |t_*(s)| - \log |1-t_-(s)| + \log |t_-(s)|) & \text{when } \Delta(s) > 0 \\ \frac{1}{|\Delta(s)|} \frac{1}{r(s)(1-r(s))} + \frac{\alpha(s)}{2|\Delta(s)|^{\frac{3}{2}}} \arctan \frac{\sqrt{|\Delta(s)|}}{4\alpha(s)r(s)(1-r(s))+4\Delta(s)}, & \text{when } \Delta(s) < 0. \\ r(s) = \Re t_*(s) = \Re t_-(s), \end{cases}$$

Proof. We have only three possibilities as follows:

1. $\Delta(s) > 0$ but the two real roots $t_*(s)$ and $t_-(s)$ of $A(t, s)$ are outside $[0, 1]$. In this case $A(t, s)$ does not vanish on $[0, 1]$ and consequently the integral

$$- \int_0^1 \frac{B(t, s)}{A^2(t, s)} dt = \lim_{\epsilon \rightarrow 0^+} K_{\epsilon}(s)$$

exists and defines $K_0(s)$ for this case.

2. $\Delta(s) > 0$ but one of the real roots, namely $t_*(s)$, lies inside $]0, 1[$. (Recall from lemma 3.4.2 that one real root at most can satisfy this). In this case the integral $\int_0^1 \frac{B(t, s)}{A^2(t, s)} dt$ does not exist because of the singularity near $t = t_*(s)$. We treat this difficulty by isolating the singularity in a "small" interval on which we integrate by parts and discover that this singularity is ruined by what comes out of the integration outside the previously described interval where the integral exists.
3. $\Delta(s) < 0$. This case is treated as in (1) with some technical differences coming from the fact that the roots are conjugate complex numbers.

1. We first consider the case when $\Delta(s) > 0$ and $t_*(s), t_-(s) \notin [0, 1]$.

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0^+} K_\epsilon(s) &= \lim_{\epsilon \rightarrow 0^+} \int_0^1 \frac{\epsilon^2 - A^2(t, s)}{[\epsilon^2 + A^2(t, s)]^2} B(t, s) dt = - \int_0^1 \frac{B(t, s)}{A^2(t, s)} dt \\
&= - \int_0^1 \frac{\partial_{s_1} A(t, s) \partial_{s_2} A(t, s) \partial_{s_3} A(t, s) \partial_{s_4} A(t, s)}{A^2(t, s)} dt \\
&= - \int_0^1 \frac{dt}{\alpha^2(s) (t - t_*(s))^2 (t - t_-(s))^2} \\
&= \frac{-1}{\alpha^2(s)} \left[\frac{-2}{[t_*(s) - t_-(s)]^3} \int_0^1 \frac{dt}{t - t_*(s)} + \frac{2}{[t_*(s) - t_-(s)]^3} \int_0^1 \frac{dt}{t - t_-(s)} + \right. \\
&\quad \left. + \frac{1}{[t_*(s) - t_-(s)]^2} \int_0^1 \frac{dt}{[t - t_*(s)]^2} + \frac{1}{[t_*(s) - t_-(s)]^2} \int_0^1 \frac{dt}{[t - t_-(s)]^2} \right] \\
&= - \frac{\alpha(s)}{4 \Delta^{\frac{3}{2}}(s)} \log \frac{|1 - t_*(s)|}{|t_*(s)|} + \frac{\alpha(s)}{4 \Delta^{\frac{3}{2}}(s)} \log \frac{|1 - t_-(s)|}{|t_-(s)|} + \\
&\quad + \frac{1}{4 \Delta(s)} \frac{1}{t_*(s) [1 - t_*(s)]} + \frac{1}{4 \Delta(s)} \frac{1}{t_-(s) [1 - t_-(s)]} = K_0(s).
\end{aligned}$$

2. Let $\delta > 0$ be an arbitrary small number and write

$$K_\epsilon(s) = \left(\int_0^{t_*(s)-\delta} + \int_{t_*(s)-\delta}^{t_*(s)+\delta} + \int_{t_*(s)+\delta}^1 \right) \frac{\epsilon^2 - A^2(t, s)}{[\epsilon^2 + A^2(t, s)]^2} B(t, s) dt.$$

Obviously

$$\begin{aligned}
L_1(s, \delta) &= \lim_{\epsilon \rightarrow 0^+} \left(\int_0^{t_*(s)-\delta} + \int_{t_*(s)+\delta}^1 \right) \frac{\epsilon^2 - A^2(t, s)}{[\epsilon^2 + A^2(t, s)]^2} B(t, s) dt \\
&= - \left(\int_0^{t_*(s)-\delta} + \int_{t_*(s)+\delta}^1 \right) \frac{B(t, s)}{A^2(t, s)} dt
\end{aligned}$$

exists for $\frac{B(t, s)}{A^2(t, s)}$ is continuous on $]0, t_*(s) - \delta[\cup]t_*(s) + \delta, 1[$ as $A(t, s)$ does not vanish

there. By the computations done in 1 we have that

$$\begin{aligned}
L_1(s, \delta) &= \frac{1}{4\Delta(s)} \left[\frac{1}{t_*(s)} + \frac{1}{t_-(s)} + \frac{-1}{\delta} + \frac{1}{t_*(s) - t_-(s) - \delta} \right] - \frac{2\alpha(s)}{\Delta^{\frac{3}{2}}(s)} \\
&\quad (\log |-\delta| - \log |t_*(s)| - \log |t_*(s) - t_-(s) - \delta| + \log |t_-(s)|) + \\
&\quad + \frac{1}{4\Delta(s)} \left[\frac{-1}{\delta} + \frac{-1}{t_*(s) - t_-(s) + \delta} + \frac{1}{1 - t_*(s)} + \frac{1}{1 - t_-(s)} \right] - \frac{2\alpha(s)}{\Delta^{\frac{3}{2}}(s)} \\
&\quad (\log |1 - t_*(s)| - \log |\delta| - \log |1 - t_-(s)| + \log |t_*(s) - t_-(s) + \delta|) \\
&= \frac{-1}{2\Delta(s)} \frac{1}{\delta} + \frac{1}{4\Delta(s)} \left[\frac{1}{t_*(s)} + \frac{1}{t_-(s)} + \frac{1}{1 - t_*(s)} + \frac{1}{1 - t_-(s)} \right] - \frac{2\alpha(s)}{\Delta^{\frac{3}{2}}(s)} \\
&\quad (\log |-\delta| - \log |t_*(s)| - \log |t_*(s) - t_-(s) - \delta| + \log |t_-(s)|) + \\
&\quad + \frac{1}{4\Delta(s)} \left[\frac{-1}{t_*(s) - t_-(s) + \delta} + \frac{1}{t_*(s) - t_-(s) - \delta} \right] - \frac{2\alpha(s)}{\Delta^{\frac{3}{2}}(s)} \\
&\quad (\log |1 - t_*(s)| - \log |\delta| - \log |1 - t_-(s)| + \log |t_*(s) - t_-(s) + \delta|)
\end{aligned}$$

Now we look at

$$\begin{aligned}
M_2(s, \epsilon, \delta) &= \int_{t_*(s)-\delta}^{t_*(s)+\delta} \frac{\epsilon^2 - A^2(t, s)}{[\epsilon^2 + A^2(t, s)]^2} B(t, s) dt \\
&= \int_{t_*(s)-\delta}^{t_*(s)+\delta} \partial_t \left(\frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \right) \frac{B(t, s)}{\partial_t A(t, s)} dt \\
&= \left[\frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \frac{B(t, s)}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} - \int_{t_*(s)-\delta}^{t_*(s)+\delta} \frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) dt \\
&= \left[\frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \frac{B(t, s)}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} - \frac{1}{2} \int_{t_*(s)-\delta}^{t_*(s)+\delta} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \frac{\partial_t \log(\epsilon^2 + A^2(t, s))}{\partial_t A(t, s)} dt \\
&= \left[\frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \frac{B(t, s)}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} - \frac{1}{2} \left[\partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \frac{\log(\epsilon^2 + A^2(t, s))}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} + \\
&\quad + \frac{1}{2} \int_{t_*(s)-\delta}^{t_*(s)+\delta} \log(\epsilon^2 + A^2(t, s)) \partial_t \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] dt.
\end{aligned}$$

Now, since $A(t_*(s) \pm \delta, s) \neq 0$, $\partial_t A(t, s) \neq 0$ on $]t_*(s) - \delta, t_*(s) + \delta[$ and by the local integrability of the logarithmic function, we have

$$\begin{aligned}
L_2(s, \delta) &= \lim_{\epsilon \rightarrow 0^+} M_2(s, \epsilon, \delta) \\
&= \left[\frac{1}{A(t, s)} \frac{B(t, s)}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} - \frac{1}{2} \left[\partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \frac{\log |A(t, s)|}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} + \\
&\quad + \frac{1}{2} \int_{t_*(s)-\delta}^{t_*(s)+\delta} \log(|A(t, s)|) \partial_t \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] dt.
\end{aligned}$$

Writing

$$\begin{aligned}
A(t, s) &= \frac{\alpha(s)(t - t_*(s))(t - t_-(s))}{\prod_{l=1}^4 p_l(t, s)}, & p_l(t, s) &= 1 + t + s_l, \\
B(t, s) &= \prod_{l=1}^4 \partial_{s_l} A(t, s) = \frac{1}{\prod_{l=1}^4 p_l^2(t, s)} \\
\partial_t A(t, s) &= \partial_t \left[\frac{\alpha(s)(t - t_*(s))(t - t_-(s))}{\prod_{l=1}^4 p_l(t, s)} \right] \\
&= A(t, s) \left[\frac{1}{t - t_*(s)} + \frac{1}{t - t_-(s)} - q(t, s) \right], & q(t, s) &= \sum_{l=1}^4 \frac{1}{p_l(t, s)}
\end{aligned}$$

and further simplifying of the functions that define $L_2(s, \delta)$ gives that

$$\begin{aligned}
\frac{1}{A(t, s)} \frac{B(t, s)}{\partial_t A(t, s)} &= \frac{\frac{1}{\prod_{l=1}^4 p_l^2(t, s)}}{\frac{\alpha^2(s)(t - t_*(s))^2(t - t_-(s))^2}{\prod_{l=1}^4 p_l^2(t, s)} \left[\frac{1}{t - t_*(s)} + \frac{1}{t - t_-(s)} - q(t, s) \right]} \\
&= \frac{1}{\alpha^2(s)(t - t_*(s))(t - t_-(s)) \left[(2t - t_*(s) - t_-(s)) - q(t, s)(t - t_*(s))(t - t_-(s)) \right]}
\end{aligned}$$

so

$$\begin{aligned}
\left[\frac{1}{A(t, s)} \frac{B(t, s)}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} &= \frac{2}{\alpha^2(s)(t_*(s) - t_-(s))^2} + o(\delta) \\
&= \frac{1}{2\Delta(s)} \frac{1}{\delta} + o(\delta), \quad \text{as } \delta \rightarrow 0^+.
\end{aligned} \tag{5.61}$$

By lemma 3.4.5, we have that the function

$$C(t, s) = \frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \log \frac{|A(t, s)|}{|t - t_*(s)|}$$

is a smooth function on $]t_*(s) - \delta, t_*(s) + \delta[$. Hence

$$\begin{aligned}
C(t_*(s) + \delta, s) &= C(t_*(s), s) + \delta \partial_t C(\tilde{t}(s), s), \\
C(t_*(s) - \delta, s) &= C(t_*(s), s) - \delta \partial_t C(\tilde{\tilde{t}}(s), s),
\end{aligned}$$

for some points $\tilde{t}(s), \tilde{\tilde{t}}(s) \in (t_*(s) \pm \delta)$.

$$\begin{aligned}
&\left[\partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \frac{\log |A(t, s)|}{\partial_t A(t, s)} \right]_{t_*(s)-\delta}^{t_*(s)+\delta} \\
&= \log |\delta| [C(t_*(s), s) + \delta \partial_t C(\tilde{t}(s), s)] - \log |-\delta| [C(t_*(s), s) - \delta \partial_t C(\tilde{\tilde{t}}(s), s)] \\
&= \delta \log |\delta| [\partial_t C(\tilde{t}(s), s) - \partial_t C(\tilde{\tilde{t}}(s), s)].
\end{aligned} \tag{5.62}$$

The local integrability of the logarithmic function and the smoothness of the function $\partial_t \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] dt$ that is implied by lemma 3.4.5 yield that

$$\lim_{\delta \rightarrow 0^+} \int_{t_*(s) - \delta}^{t_*(s) + \delta} \log(\epsilon^2 + A^2(t, s)) \partial_t \left[\frac{1}{\partial_t A(t, s)} \partial_t \left(\frac{B(t, s)}{\partial_t A(t, s)} \right) \right] dt = 0. \quad (5.63)$$

It follows from (5.61), (5.62) and (5.63) that

$$\lim_{\epsilon \rightarrow 0^+} K_\epsilon(s) = \lim_{\delta \rightarrow 0^+} (L_1(s, \delta) + L_2(s, \delta)) = K_0(s).$$

Finally, when $\Delta(s) < 0$, we follow the same steps as in 1 noticing that the roots $t_*(s)$ and $t_-(s)$ in this case are complex so the integrals give complex logarithmic functions. \square

Now that we compute precisely $K_0(s)$, we can study its behavior. We notice the following

Lemma 5.8.2. *Let $\lim_{\epsilon \rightarrow 0^+} K_\epsilon(s)$ be the function calculated in Lemma 5.8.1. Then*

$$K_0(s) = O\left(\frac{1}{\alpha^2(s)}\right) \quad \text{when} \quad \Delta(s) \rightarrow 0.$$

Proof. Recall from Chapter 3 Section 3.4.2 that

$$\boxed{t_*(s) = t_-(s) - \frac{2\sqrt{\Delta(s)}}{\alpha}}.$$

We lose no generality if we assume the worst scenario when $|t_*(s)| < 1$. In that case we have both the estimates $|t_-(s)| \approx 1$ and $1 - t_-(s) \approx 1$. Therefore and when $\Delta(s) \rightarrow 0$ while α stays bounded away from zero we have

$$\frac{1}{t_*(s)} = \frac{1}{t_-(s) - \frac{2\sqrt{\Delta(s)}}{\alpha}} = \frac{1}{t_-(s)} + \frac{2\sqrt{\Delta(s)}}{\alpha t_-^2(s)} + \frac{4\Delta(s)}{\alpha^2(s) t_-^3(s)} + o\left(\frac{\Delta(s)}{\alpha^2(s)}\right). \quad (5.64)$$

Similarly

$$\begin{aligned} \frac{1}{1 - t_*(s)} &= \frac{1}{1 - t_-(s) + \frac{2\sqrt{\Delta(s)}}{\alpha}} = \frac{1}{1 - t_-(s)} - \frac{2\sqrt{\Delta(s)}}{\alpha(1 - t_-(s))^2} + \\ &\quad + \frac{4\Delta(s)}{\alpha^2(s)(1 - t_-(s))^3} + o\left(\frac{\Delta(s)}{\alpha^2(s)}\right). \end{aligned} \quad (5.65)$$

Also

$$\begin{aligned} \log |t_*(s)| - \log |t_-(s)| &= \log \left| \frac{t_-(s) - \frac{2\sqrt{\Delta(s)}}{\alpha}}{t_-(s)} \right| = \log \left(1 - \frac{2\sqrt{\Delta(s)}}{\alpha t_-(s)} \right) \\ &= -\frac{2\sqrt{\Delta(s)}}{\alpha t_-(s)} - \frac{2\Delta(s)}{\alpha^2(s) t_-^2(s)} - \frac{8}{3} \frac{\Delta^{\frac{3}{2}}(s)}{\alpha^3(s) t_-^3(s)} + o\left(\frac{\Delta^{\frac{3}{2}}(s)}{\alpha^3(s)}\right). \end{aligned} \quad (5.66)$$

And

$$\begin{aligned}
& \log |1 - t_*(s)| - \log |1 - t_-(s)| \\
&= \log \left| \frac{1 - t_-(s) + \frac{2\sqrt{\Delta(s)}}{\alpha}}{1 - t_-(s)} \right| = \log \left(1 + \frac{2\sqrt{\Delta(s)}}{\alpha(1 - t_-(s))} \right) \\
&= \frac{2\sqrt{\Delta(s)}}{\alpha(1 - t_-(s))} - \frac{2\Delta(s)}{\alpha^2(s)(1 - t_-(s))^2} + \frac{8}{3} \frac{\Delta^{\frac{3}{2}}(s)}{\alpha^3(s)(1 - t_-(s))^3} + o\left(\frac{\Delta^{\frac{3}{2}}(s)}{\alpha^3(s)}\right). \tag{5.67}
\end{aligned}$$

Substituting from the Taylor expansions (5.64)-(5.67) we deduce that

$$\begin{aligned}
& \frac{1}{\Delta(s)} \left[\frac{1}{t_*(s)} + \frac{1}{t_-(s)} + \frac{1}{1 - t_*(s)} + \frac{1}{1 - t_-(s)} \right] - \frac{\alpha}{\Delta^{\frac{3}{2}}(s)} \\
& \quad (\log |1 - t_*(s)| - \log |t_*(s)| - \log |1 - t_-(s)| + \log |t_-(s)|) \\
&= \frac{4}{3} \left[\frac{1}{t_-^3(s)} + \frac{1}{(1 - t_-(s))^3} \right] \frac{1}{\alpha^2(s)} + O\left(\frac{1}{\alpha^2(s)}\right) = O\left(\frac{1}{\alpha^2(s)}\right).
\end{aligned}$$

This shows that when $|t_*(s)| < 1$ and $\Delta(s) \rightarrow 0^+$ while α stays bounded away from zero we have that $K_0(s) = O\left(\frac{1}{\alpha^2(s)}\right)$. Proving the same result for when $\Delta(s) \rightarrow 0^-$ is analogous. \square

5.8.2 Remarks on the logarithmic divergence

In both chapters 4 and 5, we obtained a logarithmic divergence. The following discussion indicates the possibility that this is not the best estimate that can be obtained.

Consider the data $f(s) = c$, where c is a real constant. Clearly $\|f\|_{L^p([0,1])} \approx 1$ for all $1 \leq p \leq \infty$. According to (3.17), the corresponding solution in the dimension $n = 4$ is given from

$$u(t, x) = \int_0^1 \frac{e^{i\frac{|x|^2}{t-s}}}{(t-s)^2} ds = \frac{1}{i|x|^2} [e^{i\frac{|x|^2}{t}} - e^{i\frac{|x|^2}{t-1}}].$$

Thus we have

$$|u(t, x)| = \frac{1}{|x|^2} |e^{i\frac{|x|^2}{t}} - e^{i\frac{|x|^2}{t-1}}| = \frac{2}{|x|^2} \left| \sin \frac{|x|^2}{2t(t-1)} \right|.$$

Consequently, and since

$$\lim_{z \rightarrow 0} \frac{|\sin \frac{z}{2t(t-1)}|^r}{z^{r-1}} = 0$$

we have

$$\begin{aligned}
& \int_{\mathbb{R}^4} |u(t, x)|^r dx \\
& \approx \int_{\mathbb{R}^4} \frac{1}{|x|^{2r}} \left| \sin \frac{|x|^2}{2t(t-1)} \right|^r dx \approx \int_0^\infty \frac{1}{\rho^{2r-3}} \left| \sin \frac{\rho^2}{2t(t-1)} \right|^r d\rho \\
& \approx \int_0^\infty \frac{1}{\rho^{r-1}} \left| \sin \frac{\rho}{2t(t-1)} \right|^r d\rho \\
& \approx \int_0^1 \frac{1}{\rho^{r-1}} \left| \sin \frac{\rho}{2t(t-1)} \right|^r d\rho + \int_1^\infty \frac{1}{\rho^{r-1}} \left| \sin \frac{\rho}{2t(t-1)} \right|^r d\rho \approx 1
\end{aligned}$$

whenever $r > 2$. This implies that

$$\|u\|_{L^q([2,3], L_x^r(\mathbb{R}^4))} \approx 1,$$

whenever $r > 2$. This shows that, for a constant data, the corresponding mixed L^4 norm is finite. The same result follows if we use the quadrilinear form involving the kernel $K_\epsilon(s)$.

Indeed, since

$$|K_\epsilon(s)| \lesssim \frac{1}{\epsilon^2},$$

then we can change the order of integration and obtain that when $f(s) = 1$,

$$\begin{aligned}
& \int_0^1 \int_0^1 \int_0^1 \int_0^1 K_\epsilon(s) ds_1 ds_2 ds_3 ds_4 = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{\epsilon^2 - A^2(t, s)}{(\epsilon^2 + A^2(t, s))^2} B(t, s) dt ds_1 ds_2 ds_3 ds_4 \\
& = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{\epsilon^2 - A^2(t, s)}{(\epsilon^2 + A^2(t, s))^2} \partial_{s_1} A(t, s) \partial_{s_2} A(t, s) \partial_{s_3} A(t, s) \partial_{s_4} A(t, s) ds_1 ds_2 ds_3 ds_4 dt \\
& = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left[\frac{A(t, s)}{\epsilon^2 + A^2(t, s)} \right]_{s_1=0}^{s_1=1} \partial_{s_2} A(t, s) \partial_{s_3} A(t, s) \partial_{s_4} A(t, s) ds_2 ds_3 ds_4 dt \\
& = \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \left[\left[\log(\epsilon^2 + A^2(t, s)) \right]_{s_1=0}^{s_1=1} \right]_{s_2=0}^{s_2=1} \partial_{s_3} A(t, s) \partial_{s_4} A(t, s) ds_3 ds_4 dt \\
& = \int_0^1 \int_0^1 \left[\left[\left[\frac{1}{2} A(t, s) \log(\epsilon^2 + A^2(t, s)) - A(t, s) + \right. \right. \right. \\
& \quad \left. \left. \left. + \epsilon \arctan\left(\frac{1}{\epsilon} A(t, s)\right) \right]_{s_1=0}^{s_1=1} \right]_{s_2=0}^{s_2=1} \right]_{s_3=0}^{s_3=1} \partial_{s_4} A(t, s) ds_4 dt \\
& = \int_0^1 \left[\left[\left[\left[\epsilon A(t, s) \arctan \frac{A(t, s)}{\epsilon} - \frac{1}{4} (\epsilon^2 - A^2(t, s)) \log(\epsilon^2 + A^2(t, s)) - \right. \right. \right. \right. \\
& \quad \left. \left. \left. + \frac{3}{4} A^2(t, s) \right]_{s_1=0}^{s_1=1} \right]_{s_2=0}^{s_2=1} \right]_{s_3=0}^{s_3=1} \right]_{s_4=0}^{s_4=1} dt \lesssim 1
\end{aligned}$$

by continuity of the integrand. Even though, if we try to estimate the quadrilinear form T_ϵ that approximates $\|u\|_{L^4([2,3],L^4(\mathbb{R}^4))}$ on some of the subregions on which we estimated the kernel $K_\epsilon(s)$, we still get a bound that diverges like a log. Take for instance the region where $s_1 - s_2 \gg s_3 - s_4 > 0$ and $s_1 - s_2 \gg \epsilon$. when $f_j \approx 1$. Recall from 5.1.1 that on this subregion $|K_\epsilon(s)| \approx \frac{1}{(s_1 - s_2)^2}$. Therefore, when $f \approx 1$, we have

$$\begin{aligned}
 T_\epsilon(f_1, f_2, f_3, f_4) &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 K_\epsilon(s) f_1(s_1) f_2(s_2) f_3(s_3) f_4(s_4) ds_1 ds_2 ds_3 ds_4 \\
 &\approx \int \int \int \int_{\substack{(s_1, s_2, s_3, s_4) \in [0,1]^4, \\ s_1 - s_2 \gg s_3 - s_4 > 0, \\ s_1 - s_2 \gg \epsilon}} \frac{ds_1 ds_2 ds_3 ds_4}{(s_1 - s_2)^2} \approx \log \epsilon
 \end{aligned}$$

by computations similar to those done in 5.1.2. This confirms the fact that there is some kind of cancellations lost in the decomposing process.

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