A theoretical study of the first transition for the non-linear Stokes problem in a horizontal annulus.

A. PASSERINI, C. FERRARIO

Abstract: For any aspect ratio R_o/R_i of the cylinder radii, the non-linear stability of the steady 2D-solutions of the Non-Linear Stokes system, which is an approximation of the Oberbeck-Boussinesq system, is theoretically studied. The sufficient condition for the stability shows a critical Ra which is function of the aspect ratio. It is the same of the associated homogeneous linear problem and it can be found by looking for the largest eigenvalue of a suitable symmetric operator. The critical Ra so defined proves to be uniformly bounded from below in the space of dimensionless parameters, while it is non uniformly bounded from above for the aspect ratio going to infinity. A scheme to evaluate it as function of the aspect ratio is given. The results do not depend on the Prandtl number Pr.

Keywords: steady convective flows, critical Rayleigh number.

1 Introduction

Natural convective flows arising in horizontal coaxial cylinders are involved in different applications such as energy storage systems, thermal insulation and cooling. A large amount of experimental data and numerical results, concerning both the flow fields and the heat transfer, can be found in the literature, see for instance [1, 2, 3, 4, 5, 6].

In Figure 1, we show as sketch of the problem a graphical outcome with streamlines and isothermal lines for the steady flows. Here, the temperatures $T_i > T_o$ are respectively fixed at the inner R_i and the outer R_o radii. This geometrical setting gives as immediate result, both experimental and numerical, that natural convective motions are always present for any value, no matter how small, of the Rayleigh number

$$Ra := \frac{\alpha g}{\nu k}(T_i - T_o)(R_o - R_i)^3$$

(α is the volumetric expansion coefficient, g the gravity acceleration, ν the kinematic viscosity, k the thermal diffusivity).

Actually, all the authors agree on the assertion that for sufficiently small Rayleigh numbers Ra, independently of the Prandtl number $Pr := \frac{\nu}{\kappa}$ and of the inverse relative gap width

$$\mathcal{A} = 2R_i/(R_o - R_i)\,,$$

a steady flow with unicellular crescent-shaped eddies occurs.

Although in the papers by Yoo, see Figure 2, the critical Rayleigh for the first transition seems to be a strongly decreasing function of $\mathcal{A}$, only the basic steady flow is always observed close to the $\mathcal{A}$-axis.

Therefore, in such region stable steady solutions should exist for the system of partial differential equations which models natural convection. This is actually confirmed by the theoretical papers on the subject [7, 8, 9, 10, 11]

All mathematical models in fluid-dynamics derive from the basic conservation laws, while the Newtonian fluid is the most common model of material [12]. Further, the equations are simplified by means of the Oberbeck-Boussinesq approximation[13, 14], whose rigorous derivation under proper hypotheses on the materials is shown in [15] and which is widely studied in several versions depending on the applications (see for instance [16, 17, 18, 19, 20]). The four classical assumptions to write the O-B system are:

- isochoric motion: $\nabla \cdot \boldsymbol{v} = 0$,

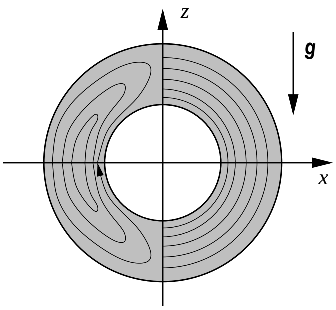

Figure 1.1: basic flow, streamlines (left) and isotherms (right)

- thermal expansion of the material in the weight force: $\rho = \rho_0(1 - \alpha(T - T_0))$,
- uniform density in all the other terms: $\rho = \rho_0$,
- negligible self dissipation: $\mathbb{D} : \mathbb{D} \approx 0$,

(ρ_0 is the reference value for the density and $\mathbb{D}$ is the symmetric part of the gradient of $\boldsymbol{v}$).

The resulting system is

$$\begin{aligned} &\nabla \cdot \mathbf{v} = 0 \\ &\rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \Delta \mathbf{v} + \rho_0(1 - \alpha(T - T_0))\mathbf{g} \\ &\rho_0 C_V \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \Delta T. \end{aligned} \tag{1.1}$$

For (approximately) $2.5 < \mathcal{A} < 8.5$, , see Figure 2, it happens that 3D flows are observable, say *spiral motions*, while elsewhere in the space of dimensionless parameters only 2D solutions are observable, so that a 2D description makes sense. Our 2D-domain is endowed with a reference frame with horizontal x- and vertical z-axis, we set $\boldsymbol{x} = (x, z)$, and the cylindrical axis coincides with the (hidden) y-direction. Of course, polar coordinates (r, φ) with $r \in [R_i, R_o]$ and $\varphi \in [-\pi, \pi)$ come in handy for this geometrical configuration ($\varphi = 0$ corresponds to the positive part of the x-axis). and polar coordinates are chosen. Accordingly, $\mathbf{e}_3 = \mathbf{k} = \nabla(r \sin \varphi) = \sin \varphi \mathbf{e}_r + \cos \varphi \mathbf{e}_\varphi$.

All functions and vector fields are periodic in φ and depend on the dimensionless variable r, in agreement with the following choice:

$$r = \frac{r'}{R_o - R_i}, \qquad z = \frac{z'}{R_o - R_i}, \qquad t = \frac{\kappa}{(R_o - R_i)^2} t', \qquad T = \frac{T'}{T_i - T_o}. \tag{1.2}$$

Next, we write the dimensionless definition of the annulus: for $\mathcal{A} > 0$, the domains in consideration are

$$\Omega_{\mathcal{A}} := \{(r, \varphi) \in \mathbb{R}^2 : r \in (\mathcal{A}/2, 1 + \mathcal{A}/2)\}.$$

If all the variables in the Oberbeck-Boussinesq system are renamed with primes, by expressing all terms as functions of the dimensionless ones, and redefining the pressure by putting together all the gradient-like terms, it follows the system [1]

[1]though this is seldom specified, in polar coordinates the symbol $\mathbf{v} \cdot \nabla \mathbf{v}$ is a strong abuse of notation, since the sum

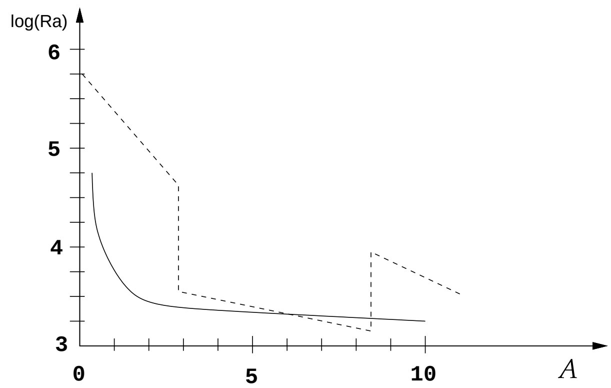

Figure 1.2: The line represents the critical Rayleigh number above which dual steady solutions exist. Below the dashed line experiments [2] show prevailing 2d steady flow. The results are for $Pr = 0.7$

$$\begin{gathered}\nabla\cdot\mathbf{v}=0\\ \frac{1}{Pr}\left(\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)-\Delta\mathbf{v}+\nabla\Pi=\frac{Ra}{b}\sin\varphi\,\boldsymbol{e}_r+Ra\,\tau\boldsymbol{e}_3\\ \frac{\partial\tau}{\partial t}+\mathbf{v}\cdot\nabla\tau-\Delta\tau=\frac{v^r}{rb}\,,\end{gathered}\tag{1.3}$$

with the boundary conditions

$$\mathbf{v}|_{\partial\Omega_{\mathcal{A}}}=0\qquad\qquad\tau|_{\partial\Omega_{\mathcal{A}}}=0\,,$$

where the third equation follows from the definition of the *excess temperature* τ:

$$\tau:=T-T^*=T-\frac{T_i}{T_i-T_o}+\frac{1}{b}\left(\ln r-\ln\frac{R_i}{R_i-R_o}\right)=T-\frac{T_i}{T_i-T_o}+\frac{1}{b}\ln\frac{2r}{\mathcal{A}}\,.\tag{1.4}$$

Here, the scalar field T^* is the conductive solution [2]. Moreover, the pressure Π is not the thermodynamical pressure but is redefined by adding gradient-like terms arising from the right hand side of the second equation in (1.1). Of course, $\boldsymbol{e}_r$ is the unit vector in direction r, $\boldsymbol{e}_3=\sin\varphi\boldsymbol{e}_r+\cos\varphi\boldsymbol{e}_\varphi$ the one in direction z, while

$$b:=\ln\frac{R_o}{R_i}=\ln\left(1+\frac{2}{\mathcal{A}}\right)\,,\tag{1.5}$$

is a purely geometric parameter, unbounded as $\mathcal{A}$ tends to zero.

Further, by considering the region in which stable basic steady flows occur, one can see how the linear Stokes-like system, studied in [3], works as approximation of the complete model which is

over repeated indices $v^i\partial_i\mathbf{v}$ is understood in Cartesian coordinates. In polar coordinates, $\mathbf{v}\cdot\nabla\mathbf{v}$ actually stands for

$$\mathbf{v}\cdot\nabla\mathbf{v}=\begin{pmatrix}\partial_r v^r & \frac{1}{r}(\partial_\varphi v^r-v^\varphi)\\ \partial_r v^\varphi & \frac{1}{r}(v^r+\partial_\varphi v^\varphi)\end{pmatrix}\begin{pmatrix}v^r\\ v^\varphi\end{pmatrix}=\begin{pmatrix}v^r\partial_r v^r+\frac{v^\varphi}{r}(\partial_\varphi v^r-v^\varphi)\\ v^r\partial_r v^\varphi+\frac{v^\varphi}{r}(v^r+\partial_\varphi v^\varphi)\end{pmatrix}.$$

In what follows, this involved expression is understood unless its manipulation was strictly necessary.

[2]In particular, T^* solves $\Delta T^*=0$ with boundary conditions $T^*(\frac{\mathcal{A}}{2},\varphi)=\frac{T_i}{T_i-T_o}$ and $T^*(\frac{\mathcal{A}}{2}+1,\varphi)=\frac{T_o}{T_i-T_o}$. So that, with the same boundary conditions, T solves $-\Delta T+\mathbf{v}\cdot\nabla T=0$.

most commonly used: it gives a stable flow for any Ra close to zero. However, in [8] it is proved that by using such simplified model, although the fluid flows for any Ra, the heat transfer is the same as for conduction: the Nusselt number is not increased by the motion if one neglects the transport term in the heat equation. This feature makes the model very weak in describing how the thermal energy can be transported by convection.

The present paper deals with the non-linear stability of the non-linear Stokes problem [21], a less simplified and more realistic approximation of the full system of equations. This simplified version of (1.3) is got by erasing only the non-linear term $\mathbf{v} \cdot \nabla \mathbf{v}$ lying in the linear momentum balance.

The mathematical tools herein used are the classical ones of the functional analysis applied to fluid dynamics and can be found, for instance in [22, 23, 24].

In particular, in the next section by using the Straughan scheme [25], we define the critical Rayleigh number Ra_{cr} of the non-linear Stokes problem in the annulus. It depends on the curvature of the domain but it is uniformly bounded from below, and we prove that for $Ra < Ra_{cr}(\mathcal{A})$ the steady solutions, which can easily be found with the same techniques as in [11], are asymptotically stable and then unique. Actually, a further bound $Ra < Ra*(\mathcal{A})$ must hold, confining Ra to a region of the space of parameters which is however unbounded.

In Section 3, we furnish the computational scheme to identify $Ra_{cr}(\mathcal{A})$, hoping that such function, once plotted by numerical methods, will be compared with Yoo's first transition line.

2 The non linear stability

As announced in the Introduction, the model in consideration is

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0 \\ \frac{1}{Pr}\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} + \nabla \Pi &= \frac{Ra}{b} \sin\varphi\, \boldsymbol{e}_r + Ra\, \tau \boldsymbol{e}_3 \\ \frac{\partial \tau}{\partial t} + \mathbf{v} \cdot \nabla \tau - \Delta \tau &= \frac{v^r}{rb}\,, \end{aligned} \tag{2.1}$$

with the boundary conditions

$$\mathbf{v}|_{\partial\Omega_{\mathcal{A}}} = 0 \qquad\qquad \tau|_{\partial\Omega_{\mathcal{A}}} = 0\,,$$

The steady solutions obey to the system:

$$\begin{aligned} \nabla \cdot \mathbf{v}_0 &= 0 \\ -\Delta \mathbf{v}_0 + \nabla \Pi_0 &= \frac{Ra}{b} \sin\varphi\, \boldsymbol{e}_r + Ra\, \tau_0 \boldsymbol{e}_3 \\ -\Delta \tau_0 + \mathbf{v}_0 \cdot \nabla \tau_0 &= \frac{v_0^r}{rb}\,, \end{aligned} \tag{2.2}$$

Both for (2.1) and (2.2), some existence and regularity theorems can easily be found by particularizing the corresponding proofs given in [9] and [11] for the full O-B system (1.3).

For system (2.1), everything is simplified by the 2D setting. For system (2.2) -whose basic solution is not the rest state as in the Bénard problem, since (2.2) is not homogeneous- one has global existence in the space of the dimensionless parameters. This is essentially due to the pointwise boundedness of the temperature field, consequence of a maximum principle holding for the heat equation.

Thus, giving as understood the existence of regular solutions, the aim of the present paper is to present a clear result about non linear stability and suggest a procedure to find numerically a theoretically well defined critical Ra. In fact, it should be compared with Yoo's one, which was numerically found for the full system, and the reason is that the non linear stability implies uniqueness by the Poincaré inequality, so that the asserted (in [**?**]) existence of dual steady solutions has to be excluded for the Non-Linear Stokes Problem in the region where stability is proved. In [**?**], it looks like the basic solution of the full system looses its stability while another steady solution becomes stable, on the opposite in [11] the non linear stability result proved for the full system (although only in the "small $\mathcal{A}$" region) shows a unique steady solution, which is an attractor as in the present paper. only.

Let us write down the equation for the perturbation of the solutions of (2.2):

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \frac{1}{Pr}\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla P &= +Ra\,\sigma \boldsymbol{e}_3 \\ \frac{\partial \sigma}{\partial t} + \mathbf{u}\cdot\nabla\sigma + \mathbf{v}_0\cdot\nabla\sigma + \mathbf{u}\cdot\nabla\tau_0 - \Delta\sigma &= \frac{u^r}{rb}\,, \end{aligned} \tag{2.3}$$

By performing the scalar product in the Hilbert space, precisely $L^(\Omega_{\mathcal{A}})$ for scalar functions and $H(\Omega_{\mathcal{A}})$ for vectorial divergence free functions, one immediately finds the energy equalities:

$$\begin{aligned} \frac{1}{Pr}\frac{d}{dt}\frac{1}{2}\int_{\Omega_{\mathcal{A}}} \mathbf{u}^2 + \int_{\Omega_{\mathcal{A}}} |\nabla \mathbf{u}|^2 &= Ra\int_{\Omega_{\mathcal{A}}} \sigma u_3 \\ \frac{d}{dt}\frac{1}{2}\int_{\Omega_{\mathcal{A}}} \sigma^2 + \int_{\Omega_{\mathcal{A}}} |\nabla\sigma|^2 + \int_{\Omega_{\mathcal{A}}} \mathbf{u}\cdot\nabla\tau_0\sigma &= \frac{1}{b}\int_{\Omega_{\mathcal{A}}} \frac{u^r}{r}\sigma\,, \end{aligned} \tag{2.4}$$

In [11], some bounds for τ_0 can be found and, in particular, we are going to use:

$$\|\nabla\tau_0\|_2 \le \frac{RaC(Ra,\mathcal{A})}{\sqrt{b^3(\mathcal{A})}}\,, \tag{2.5}$$

where $C(Ra,\mathcal{A})$ is a uniformly bounded function of both the variables, whose form we are going to specify, since it's important to get the optimal critical value of Ra. To this end we increase the details of the proof in [11].

Theorem 2.1 *For any Ra and $\mathcal{A} > 0$ any solution of system (2.2) verifies*

$$\|\nabla \mathbf{v}_0\|_2 \le \ Ra\frac{c(Ra,\mathcal{A})}{b} \qquad \|\nabla\tau_0\|_2 \le Ra\frac{C(Ra,\mathcal{A})}{b^{\frac{3}{2}}}\,, \tag{2.6}$$

where

$$\begin{aligned} c(Ra,\mathcal{A}) &= c_p(\mathcal{A})\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\left(1+Ra\,c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{\sqrt{b(\mathcal{A})}}+Rac_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right)\right) \\ C(Ra,\mathcal{A}) &= c_p^2(\mathcal{A})\sqrt{\pi}\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\left(1+Ra\,c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{\sqrt{b(\mathcal{A})}}+Rac_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right)\right). \end{aligned} \tag{2.7}$$

are continuous function on $[0,\infty)\times(0,\infty)$ and, in particular, uniformly continuous with respect to Ra, while $c_p(\mathcal{A})$ is the Poincaré constant.

Proof: Since $|\tau_0(\boldsymbol{x})| < 1$, we can compute

$$\|\tau_0\|_2 < \sqrt{2\pi}\left(\int_{\mathcal{A}/2}^{1+\mathcal{A}/2} r\,dr\right)^{1/2} = \sqrt{\pi(1+\mathcal{A})}\,. \tag{2.8}$$

We put $\mathbf{v}_0$ and τ_0 as test functions in (2.2), and consider the first of the two *energy equalities*. We first apply the Schwarz inequality and calculate

$$\|\sin\varphi\|_2 = \sqrt{\frac{\pi}{2}(1+\mathcal{A})}\,,$$

then we insert estimate (2.8) and, finally, with the Poincaré inequality, one arrives at

$$\|\nabla\mathbf{v}_0\|_2^2 \le \frac{Ra}{b(\mathcal{A})}\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\|\mathbf{v}_0\|_2 + Ra\,\|\tau_0\|_2\|\mathbf{v}_0\|_2 \le Ra\,c_p(\mathcal{A})\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\|\nabla\mathbf{v}_0\|_2$$

where we can insert the estimate (see [9])

$$c_p(\mathcal{A}) \le \max\left\{\frac{1}{2}\sqrt{1+\frac{2}{\mathcal{A}}};1+\frac{\mathcal{A}}{2}\right\}$$

of the Poincaré constant, and the first expression is the best one for $\mathcal{A} > \sqrt{2}-1$.

This implies

$$\|\nabla\mathbf{v}_0\|_2 \le Ra\,c_p(\mathcal{A})\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\sqrt{\frac{\pi}{2}(1+\mathcal{A})} =: f_1(Ra,\mathcal{A}). \tag{2.9}$$

For the second energy equality we get by Hölder's inequality

$$\|\nabla\tau_0\|_2^2 \le \frac{1}{b(\mathcal{A})}\left(\frac{v_0^r}{r},\tau_0\right) \le \frac{1}{b(\mathcal{A})}\|\mathbf{v}_0\|_4\|\tau_0\|_4\left\|\frac{1}{r}\right\|_2, \tag{2.10}$$

where

$$\left\|\frac{1}{r}\right\|_2 = \sqrt{2\pi b(\mathcal{A})}.$$

Since, by taking into account Ladyzenskaya's and Poincaré's inequalities, we see

$$\|\mathbf{v}_0\|_4 \le \frac{1}{\sqrt[4]{2}}\sqrt{\|\mathbf{v}_0\|_2\|\nabla\mathbf{v}_0\|_2} \le \frac{1}{\sqrt[4]{2}}\sqrt{c_p(\mathcal{A})}\|\nabla\mathbf{v}_0\|_2$$

and, analogously, also

$$\|\tau_0\|_4 \le \frac{1}{\sqrt[4]{2}}\sqrt{c_p(\mathcal{A})}\|\nabla\tau_0\|_2\,,$$

then (2.10) becomes

$$\|\nabla\tau_0\|_2^2 \le \frac{1}{b(\mathcal{A})}c_p(\mathcal{A})\|\nabla\mathbf{v}_0\|_2\|\nabla\tau_0\|_2\sqrt{\pi b(\mathcal{A})} \le \frac{1}{\sqrt{b(\mathcal{A})}}c_p(\mathcal{A})\sqrt{\pi}f_1(Ra,\mathcal{A})\|\nabla\tau_0\|_2. \tag{2.11}$$

This implies

$$\|\nabla\tau_0\|_2 \le \sqrt{\frac{\pi}{b(\mathcal{A})}}c_p(\mathcal{A})f_1(Ra,\mathcal{A}).$$

The former estimate can be inserted in the chain of inequalities leading to (2.9), replacing the use of (2.8). Thus, we obtain

$$\begin{aligned}\|\nabla\mathbf{v}_0\|_2^2 &\le \frac{Ra}{b(\mathcal{A})}\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\|\mathbf{v}_0\|_2 + Ra\|\tau_0\|_2\|\mathbf{v}_0\|_2 \le Ra\left(\frac{1}{b(\mathcal{A})}\sqrt{\frac{\pi}{2}(1+\mathcal{A})}+c_p(\mathcal{A})\,\|\nabla\tau_0\|_2\right)\|\mathbf{v}_0\|_2\\ &\le \frac{Ra}{\sqrt{b(\mathcal{A})}}c_p(\mathcal{A})\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\left[\frac{1}{\sqrt{b(\mathcal{A})}}+Ra\,c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right]\|\nabla\mathbf{v}_0\|_2,\end{aligned} \tag{2.12}$$

from which:

$$\|\nabla \mathbf{v}_0\|_2 \leq \frac{Ra}{\sqrt{b(\mathcal{A})}} f_2(Ra, \mathcal{A}) \,, \tag{2.13}$$

where

$$f_2(Ra, \mathcal{A}) = c_p(\mathcal{A})\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\left[\frac{1}{\sqrt{b(\mathcal{A})}} + Rac_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right]$$

is a continuous function of both the variables. Now, we go back to the estimate derived by the momentum balance equation, to improve the decay as $\mathcal{A}$ tends to 0 (read: as b tends to ∞) up to the optimal one. If we substitute (2.13) in (2.11), we obtain

$$\|\nabla \tau_0\|_2^2 \leq \sqrt{\frac{\pi}{b(\mathcal{A})}} c_p(\mathcal{A})\|\nabla \mathbf{v}_0\|_2\|\nabla \tau_0\|_2 \leq \frac{Ra}{b(\mathcal{A})} c_p(\mathcal{A})\sqrt{\pi} f_2(Ra, \mathcal{A})\|\nabla \tau_0\|_2. \tag{2.14}$$

Therefore:

$$\|\nabla \tau_0\|_2 \leq \frac{Ra}{b(\mathcal{A})}\sqrt{\pi} c_p^2(\mathcal{A})\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\left[\frac{1}{\sqrt{b(\mathcal{A})}} + Rac_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right] := f_3(Ra, \mathcal{A}) \,. \tag{2.15}$$

By inserting (2.15) in (2.12):

$$\|\nabla \mathbf{v}_0\|_2 \leq Ra\, c_p(\mathcal{A})\left(\frac{1}{b(\mathcal{A})}\sqrt{\frac{\pi}{2}(1+\mathcal{A})} + c_p(\mathcal{A})\,\|\nabla \tau_0\|_2\right)$$

one obtains

$$\|\nabla \mathbf{v}_0\|_2 \leq Ra\,\frac{c_p(\mathcal{A})}{b(\mathcal{A})}\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\left(1 + Ra\, c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{\sqrt{b(\mathcal{A})}} + Rac_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right)\right) :=$$

$$:= \frac{Ra}{b(\mathcal{A})} c(Ra, \mathcal{A}). \tag{2.16}$$

From the structure of (2.12), it can easily be deduced that the exponent of the decay with respect to b can not further be improved for the velocity field. On the opposite, by inserting (2.16) in (2.14) one finally obtains the optimal estimate for the temperature gradient:

$$\|\nabla \tau_0\|_2 \leq Ra\sqrt{\frac{\pi}{b^3(\mathcal{A})}} c_p(\mathcal{A}) c(Ra, \mathcal{A}) =$$

$$= Ra\sqrt{\frac{\pi}{b^3(\mathcal{A})}} c_p^2(\mathcal{A})\sqrt{\frac{\pi}{2}(1+\mathcal{A})}\left(1 + Ra\, c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{\sqrt{b(\mathcal{A})}} + Rac_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right)\right). \tag{2.17}$$

Thus (2.6) is proved. ∎

In fact, by using Hölder's, Ladyzenskaya's, Poincaré's and Young's inequalities (as done in [11]), one can write

$$|(\mathbf{u}\cdot\nabla \tau_0,\, \sigma)| \leq \|\mathbf{u}\|_4\|\sigma\|_4\|\nabla \tau_0\|_2 \leq \frac{1}{\sqrt{2}} c_p(\mathcal{A})\|\nabla \tau_0\|_2\|\nabla \mathbf{u}\|_2\|\nabla \sigma\|_2 \leq \tag{2.18}$$

$$\frac{\mu^2 c_p^2(\mathcal{A})}{4}\,\frac{Ra^2 C^2(Ra, \mathcal{A})}{b^3(\mathcal{A})}\|\nabla \mathbf{u}\|_2^2 + \frac{1}{2\mu^2}\|\nabla \sigma\|_2^2 \,, \tag{2.19}$$

for all μ.

Then, we multiply by Ra the second of (2.4), next we sum the inequalities and finally we increase the trilinear form by using (2.18). We see that the total derivative of the Lyapunov function

$$E := \frac{1}{2}\left(\frac{1}{Pr}\int_{\Omega_{\mathcal{A}}} \mathbf{u}^2 + Ra\int_{\Omega_{\mathcal{A}}} \sigma^2\right)$$

can be positive and we can write:

$$\frac{d}{dt}\frac{1}{2}\left(\frac{1}{Pr}\int_{\Omega_{\mathcal{A}}} \mathbf{u}^2 + Ra\int_{\Omega_{\mathcal{A}}} \sigma^2\right) + (1-\epsilon^2)\left(\int_{\Omega_{\mathcal{A}}} |\nabla \mathbf{u}|^2 + Ra\int_{\Omega_{\mathcal{A}}} |\nabla\sigma|^2\right) \leq \\ \leq Ra\left(\int_{\Omega_{\mathcal{A}}} \sigma u_3 + \frac{1}{b}\int_{\Omega_{\mathcal{A}}} \frac{u^r}{r}\sigma\right), \tag{2.20}$$

where $\epsilon^2 > 0$ can be arbitrarily small, since we can ask

$$1-\epsilon^2 = 1 - \frac{\mu^2 c_p^2(\mathcal{A})}{4}\frac{Ra^3 C^2(Ra,\mathcal{A})}{b^3(\mathcal{A})} \leq \frac{1}{2\mu^2}, \tag{2.21}$$

(where $C(Ra,\mathcal{A})$ is as defined in (2.7)) and this can be verified by arbitrarily small μ^2 for fixed Ra and $\mathcal{A}$. Actually, we see that (2.21) is a bi-quadratic inequality for μ and, provided that for any $\mathcal{A}$ we keep $Ra(\mathcal{A})$ bounded by Ra_* such that

$$1 > \frac{c_p^2(\mathcal{A})}{2}\frac{Ra_*^3 C^2(Ra_*,\mathcal{A})}{b^3(\mathcal{A})},$$

it allows for positive and arbitrarily small real solutions. The explicit form of the bound is given by

$$1 > \frac{c_p^6(\mathcal{A})}{2}\frac{Ra_*^3\pi^2}{b^3(\mathcal{A})}(1+\mathcal{A})\left(1+Ra_* c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{\sqrt{b(\mathcal{A})}}+Ra_* c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right)\right)^2. \tag{2.22}$$

Therefore, in the region whose boundary is defined by (2.22), which is an unbounded region (as one can easily deduce from the behaviour of $b(\mathcal{A})$), a standard study of the non linear stability can be performed.

To this end, we are now going to define the critical Rayleigh number $Ra_{cr}(\mathcal{A})$ of the non linear Stokes problem and prove that it is also bounded from below independently of $\mathcal{A}$.

$$D(\boldsymbol{w},\theta) := \|\nabla \boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2,$$
$$\mathcal{F}(\boldsymbol{w},\theta,\mathcal{A}) := (\tau, w^z) + \frac{1}{b}\left(\theta, \frac{w^r}{r}\right), \qquad \text{where } w^z := \boldsymbol{w}\cdot\boldsymbol{e}_3, \tag{2.23}$$

$$\frac{1}{Ra_{cr}(\mathcal{A})} := \sup \frac{\mathcal{F}(\boldsymbol{w},\theta,\mathcal{A})}{D(\boldsymbol{w},\theta)}, \tag{2.24}$$

where the supremum is taken over all the couples of functions $(\boldsymbol{w},\theta)$ both vanishing on the boundary, with the gradients in the Hilbert space (the definition of the scalar product is of course different for scalar, vector and tensor fields), and with divergence free $\boldsymbol{w}$: we indicate it as $V(\Omega_{\mathcal{A}}) \times \overset{\circ}{W}{}^{1,2}(\Omega_{\mathcal{A}})$, where W is the usual symbol of the Sobolev spaces.

We recall a result given in [11]

Theorem 2.2 *For all $\mathcal{A}$ it holds*

$$Ra_{cr}(\mathcal{A}) > 1.$$

Proof:. Let us apply the inequality in (2.10) to increase the functional

$$\begin{aligned}\mathcal{F} &= (\theta, w^z) + \frac{1}{b}\left(\theta, \frac{w^r}{r}\right) \\ &\le \frac{c_p^2(\mathcal{A})}{2}\left(\|\nabla \boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2\right) + \sqrt{\frac{\pi}{b}}\sqrt{\|\boldsymbol{w}\|_2\|\nabla\boldsymbol{w}\|_2\|\theta\|_2\|\nabla\theta\|_2} \\ &\le \left(\frac{c_p^2(\mathcal{A})}{2} + \frac{c_p(\mathcal{A})}{2}\sqrt{\frac{\pi}{b}}\right)\left(\|\nabla\boldsymbol{w}\|_2^2 + \|\nabla\theta\|_2^2\right) .\end{aligned}$$

However, for large $\mathcal{A}$ we can use a better estimate for the second term in $\mathcal{F}(\boldsymbol{w},\theta,\mathcal{A})$: due to $\sup_{\Omega_{\mathcal{A}}}\frac{1}{r} = \frac{2}{\mathcal{A}}$, we obtain by Hölder's and Poincaré's inequalities:

$$\frac{1}{b(\mathcal{A})}\left(\vartheta, \frac{w^r}{r}\right) \le \frac{2c_p^2(\mathcal{A})}{b(\mathcal{A})\mathcal{A}}\|\nabla\vartheta\|_2\|\nabla\mathbf{w}\|_2 .$$

We can choose the best estimate depending on the region by defining

$$f(\mathcal{A}) := \min\left\{\frac{2c_p^2(\mathcal{A})}{b(\mathcal{A})\mathcal{A}}, c_p(\mathcal{A})\sqrt{\frac{\pi}{b(\mathcal{A})}}\right\} ,$$

so that the bound for $\mathcal{F}$ can be summarized as

$$\mathcal{F} \le \frac{1}{2}\left(\max_{\mathcal{A}} c_p^2(\mathcal{A}) + \max_{\mathcal{A}} f(\mathcal{A})\right) D .$$

By replacing the estimate of $c_p(\mathcal{A})$ and plotting the bound of $\frac{\mathcal{F}}{D}$ so obtained, one sees that the supremum of the functional is always less than 1. In particular, for $\mathcal{A}$ close to zero one has

$$\mathcal{F} \le \frac{1}{2}\left(\frac{\mathcal{A}}{2}+1\right)\left(\left(\frac{\mathcal{A}}{2}+1\right) + \sqrt{\frac{\pi}{b}}\right) D ,$$

therefore the limit of the expression in front of D for $\mathcal{A} \to 0$ is $\frac{1}{2}$ and so is the limit of the supremum. ∎

Now, we rewrite (2.20) as

$$\frac{dE}{dt} = -(1-\epsilon^2)\widehat{D} + Ra\mathcal{F} , \tag{2.25}$$

where

$$\widehat{D} = \widehat{D}(\mathbf{u},\sigma) := \|\nabla\mathbf{u}\|_2^2 + Ra\|\nabla\sigma\|_2^2 . \tag{2.26}$$

In order to estimate the right hand side of (2.25) we define

$$d_0 := \min\{1, Ra\} .$$

Then, $d_0 D \le \widehat{D}$. Therefore, we can write [3]

$$\frac{dE}{dt} = -(1-\epsilon^2)\widehat{D} + Ra\mathcal{F} \le -(1-\epsilon^2)d_0 D + RaD\frac{\mathcal{F}}{D} \le -RaD\left(\frac{(1-\epsilon^2)d_0}{Ra} - \frac{1}{Ra_{cr}}\right) ,$$

where we increased $\frac{\mathcal{F}}{D}$ by choosing the worse possible initial condition: $\frac{1}{Ra_{cr}}$.

Then, the stability condition is $Ra_{cr}d_0(1-\epsilon^2) > Ra$, and:

[3]Note that, if $D = 0$ for some t, then the solution is necessarily the rest state, since it holds uniqueness once the initial conditions are given.

- if $d_0 = Ra$ (and, consequently $Ra \leq 1$) then the stability conditions reads $Ra_{cr}(1-\epsilon^2) > 1$;
- if $d_0 = 1$ then the stability conditions reads $Ra_{cr}(1-\epsilon^2) > Ra$, and we have also $Ra \geq 1$.

Thus, in both cases one has to require that

$$Ra_{cr}(1-\epsilon^2) > 1\,. \tag{2.27}$$

However, by the theorem just proved condition (2.27) is automatically satisfied. Finally,

$$Ra < Ra_{cr}(1-\epsilon^2) \tag{2.28}$$

is the stability condition. Since ϵ^2 is arbitrarily small, (2.28) can be replaced by $Ra < Ra_{cr}$.

Moreover, since the Poincaré inequality holds true, it follows that whenever (2.28) is verified then (for some positive κ)

$$\frac{dE}{dt} \leq -\kappa E\,,$$

the exponential decay of the perturbation of the steady solution is inferred independently of the initial condition and, consequently, the uniqueness of the steady solution follows too.

Theorem 2.3 *For all $\mathcal{A}$, if Ra is bounded by Ra_* defined by*

$$1=\frac{c_p^6(\mathcal{A})}{2}\frac{Ra_*^3\pi^2}{b^3(\mathcal{A})}(1+\mathcal{A})\left(1+Ra_*c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{\sqrt{b(\mathcal{A})}}+Ra_*c_p^3(\mathcal{A})\sqrt{\pi}\left(\frac{1}{b(\mathcal{A})}+\sqrt{2}\right)\right)\right)^2, \tag{2.29}$$

where $c_p(\mathcal{A})$ is the Poincaré constant, and if moreover

$$Ra < Ra_{cr}(\mathcal{A}) = \left(\sup_{V(\Omega_{\mathcal{A}})\times \overset{o}{W}{}^{1,2}(\Omega_{\mathcal{A}})} \frac{\mathcal{F}(\boldsymbol{w},\theta,\mathcal{A})}{D(\boldsymbol{w},\theta)}\right)^{-1}, \tag{2.30}$$

then the corresponding steady solution of the non linear Stokes system (2.1) is asymptotically stable and, consequently, unique.

The unbounded region whose bound is defined in (2.29) could contain or not the graph given by the points $(\mathcal{A}, Ra_{cr}(\mathcal{A}))$. Thus, that bound could be a stronger condition than $Ra < Ra_{cr}$. This last is anyway a non trivial condition, because Ra_{cr} is bounded from below. The next section is devoted to offer mathematical tools for computing it numerically.

We conclude by remarking that $Ra < Ra_{cr}(\mathcal{A})$ is also the sufficient condition for the stability of the rest state solution of the linear homogeneous system:

$$\begin{gathered}\nabla\cdot\mathbf{w}=0\\ \frac{1}{Pr}\frac{\partial\mathbf{w}}{\partial t}-\Delta\mathbf{w}+\nabla p = Ra\theta\mathbf{e}_3\\ \frac{\partial\theta}{\partial t}-\Delta\theta=\frac{w^r}{r\,b},\end{gathered} \tag{2.31}$$

We recall a result given in [11], which can be proved by means of the standard Galerkin method and *a priori* estimates of the same kind as ones of Section 2, but simpler. Here, $H(\Omega_{\mathcal{A}})$ denotes the usual Hilbert space but with the divergence free condition

Theorem 2.4 *Let us arbitrarily fix $\mathcal{A}$ and Pr. If $Ra < Ra_c(\mathcal{A})$, then*

(i) *for any $(\mathbf{w}_0, \vartheta_0) \in H \times L^2$ there exists a unique weak solution*

$$(\mathbf{w}, \vartheta) \in \big(L_\infty(0,\infty; H) \cap L^2(0,\infty; V)\big) \times \big(L_\infty(0,\infty; L^2) \cap L_2(0,\infty; \overset{o}{W}{}^{1,2})\big)$$

of (2.31) which is globally exponentially stable, i.e. there exists $\beta = \beta(Ra, \mathcal{A}, Pr) > 0$ such that for a.e. $t \in (0,\infty)$

$$d_1\big(\|\mathbf{w}(t)\|_2^2 + \|\vartheta(t)\|_2^2\big) \le d_2\big(\|\mathbf{w}_0\|_2^2 + \|\vartheta_0\|_2^2\big)\, e^{-\beta t}\,, \tag{2.32}$$

where $d_1 = \min\{Pr^{-1}, Ra\}$ and $d_2 = \max\{Pr^{-1}, Ra\}$.

(ii) *If $(\mathbf{w}_0, \vartheta_0) \in V \times \overset{o}{W}{}^{1,2}$, then the weak solution belongs to*

$$(\mathbf{w}, \vartheta) \in \big(L_\infty(0,\infty; V) \cap L^2(0,\infty; W^{2,2})\big) \times \big(L_\infty(0,\infty; W^{1,2}) \cap L^2(0,\infty; W^{2,2})\big).$$

In particular, for $t \in (0,\infty)$

$$\|\nabla \mathbf{w}(t)\|_2 + \|\nabla \vartheta(t)\|_2 < d_3\,, \tag{2.33}$$

where the constant d_3 depends continuously on of $Ra, \mathcal{A}, Pr$, and $\|\mathbf{w}_0\|_{1,2}, \|\vartheta_0\|_{1,2}$.

Since of the geometry of the domains, a distinction is often made in literature between convection *shear driven* and convection *buoyancy driven*, depending on how large is the relative contribution to the flow of the body force term with respect to the contribution of the two right hand sides in (2.31). For *body force* we mean the first term on the right hand side of the momentum balance equation (1.3), which is identically zero in the Bénard problem. One is naturally lead to think that the shear contribution is weaker in the region of domains with $\mathcal{A}$ close to zero, since of the coefficient $\frac{Ra}{b(\mathcal{A})}$ going to zero.

Then, one could ask whether, in the limit of vanishing shear contribution the dynamical system (2.31) would behave as in the Bénard problem where the uniqueness of the steady solutions is not verified. But the rest state solution of (2.31) is clearly unique as steady solution for $Ra < Ra_{cr}(\mathcal{A})$, which is a truly different situation from that of the corresponding linear system in the geometry of the Bénard problem (with free surface conditions). Therefore, it has to be underlined that the multiplicity of steady solutions exchanging their stabilities at the bifurcation point is not a general feature of the linear systems derived from the O-B system. On the opposite, it looks related to the symmetries of the Bénard domain.

3 Computational scheme for the critical constant

In this section we give a procedure to compute $Ra_{cr}(\mathcal{A})$ with standard methods from [25].

If we come back to the definition of $Ra_{cr}(\mathcal{A})$, we see that its value, provided it is finite, corresponds to a stationary point $(\widetilde{\mathbf{w}}, \widetilde{\theta})$ for the functional $\mathcal{F}/D$. With the help of a parameter $\eta \in \mathbb{R}$, let us write a suitable expression of $\mathcal{F}$ and D around the maximal point [4]:

$$\mathcal{F} = \mathcal{F}(\widetilde{\mathbf{w}} + \eta\widetilde{\mathbf{u}}, \widetilde{\theta} + \eta\widetilde{\sigma})\,, \qquad D = D(\widetilde{\mathbf{w}} + \eta\widetilde{\mathbf{u}}, \widetilde{\theta} + \eta\widetilde{\sigma})\,,$$

with arbitrary *variation direction* $(\widetilde{\mathbf{u}}, \widetilde{\sigma}) \in V \times W_2^1$. We will use the symbol $\frac{d}{d\eta}|_{\eta=0}$ to perform a variational derivative around $(\widetilde{\mathbf{w}}, \widetilde{\theta})$, which is identified as a stationary point if we impose that the

[4] The notation with tilde means that the stationary point has nothing to do with the solutions of (2.31); as we will see, it is solution of a different system of equations.

derivative vanishes. It must vanish for any variation direction when applied to $\mathcal{F}/D$ (and evaluated in $\eta = 0$). So

$$0 = \frac{d}{d\eta}\left(\frac{\mathcal{F}}{D}\right)\Big|_{\eta=0} = \left[\frac{1}{D^2}\left(D\frac{d}{d\eta}\mathcal{F} - \mathcal{F}\frac{d}{d\eta}D\right)\right]_{\eta=0} = \frac{\frac{d\mathcal{F}}{d\eta}}{D}\Big|_{\eta=0} - \frac{1}{Ra_c}\frac{\frac{dD}{d\eta}}{D}\Big|_{\eta=0}.$$

Notice that $\frac{1}{Ra_c}$ is inserted since it is by definition the value of $\frac{\mathcal{F}}{D}$ in the stationary point.

On the other hand,

$$\frac{d\mathcal{F}}{d\eta}\Big|_{\eta=0} = (\widetilde{\sigma}, \widetilde{w}^z) + (\widetilde{\theta}, \widetilde{u}^z) + \frac{1}{b}\Big((\widetilde{\theta}, \frac{\widetilde{u}^r}{r}) + (\widetilde{\sigma}, \frac{\widetilde{w}^r}{r})\Big)$$
$$\frac{dD}{d\eta}\Big|_{\eta=0} = 2(\nabla\widetilde{\mathbf{u}}, \nabla\widetilde{\mathbf{w}}) + 2(\nabla\widetilde{\theta}, \nabla\widetilde{\sigma})\,.$$

Therefore, from the maximum condition it follows for all variations

$$(\widetilde{\sigma}, \widetilde{w}^z) + (\widetilde{\theta}, \widetilde{u}^z) + \frac{1}{b}\Big((\widetilde{\theta}, \frac{\widetilde{u}^r}{r}) + (\widetilde{\sigma}, \frac{\widetilde{w}^r}{r})\Big) = \frac{2}{Ra_c}\Big((\nabla\widetilde{\mathbf{u}}, \nabla\widetilde{\mathbf{w}}) + (\nabla\widetilde{\theta}, \nabla\widetilde{\sigma})\Big)$$

and, integrating by parts

$$(\widetilde{\sigma}, \widetilde{w}^z) + (\widetilde{\theta}, \widetilde{u}^z) + \frac{1}{b}\Big((\widetilde{\theta}, \frac{\widetilde{u}^r}{r}) + (\widetilde{\sigma}, \frac{\widetilde{w}^r}{r})\Big) = -\frac{2}{Ra_c}\Big((\Delta\widetilde{\mathbf{w}}, \widetilde{\mathbf{u}}) + (\Delta\widetilde{\theta}, \widetilde{\sigma})\Big)\,.$$

In order to find a maximum for the functional, among all the variations we have the freedom to choose

1. arbitrary $\widetilde{\mathbf{u}}$ and $\widetilde{\sigma} = 0$;
2. $\widetilde{\mathbf{u}} = 0$ and arbitrary $\widetilde{\sigma}$.

In the first case, one can deduce

$$\int_{\Omega_{\mathcal{A}}} \big(\widetilde{\theta}\boldsymbol{e}_3 + \frac{\widetilde{\theta}}{br}\boldsymbol{e}_r + \frac{2}{Ra_c}\Delta\widetilde{\mathbf{w}}\big)\cdot\widetilde{\mathbf{u}}d\boldsymbol{x} = 0 \qquad \Rightarrow \qquad \widetilde{\theta}\boldsymbol{e}_3 + \frac{\widetilde{\theta}}{br}\boldsymbol{e}_r + \frac{2}{Ra_c}\Delta\widetilde{\mathbf{w}} + \nabla\widetilde{p} = 0\,.$$

Since $\widetilde{\mathbf{u}}$ is arbitrary but divergence free, then the term in brackets belongs to the orthogonal complement with respect to Helmholtz's decomposition. That's why $\nabla\widetilde{p}$ is added. In the second case:

$$\int_{\Omega_{\mathcal{A}}} \left(\widetilde{w}^z + \frac{\widetilde{w}^r}{br} + \frac{2}{Ra_c}\Delta\widetilde{\theta}\right)\widetilde{\sigma}d\boldsymbol{x} = 0 \qquad \Rightarrow \qquad \widetilde{w}^z + \frac{\widetilde{w}^r}{br} + \frac{2}{Ra_c}\Delta\widetilde{\theta} = 0\,.$$

Moreover, we set

$$\boldsymbol{e}_3 + \frac{\boldsymbol{e}_r}{br} = \nabla S, \qquad \text{where} \quad S := r\sin\varphi + \frac{1}{b}\ln r,$$

so that the Euler-Lagrange equations derived from the maximum condition become

$$\widetilde{\theta}\nabla S + \lambda\Delta\widetilde{\mathbf{w}} = -\nabla\widetilde{p} \tag{3.1}$$
$$\widetilde{\mathbf{w}}\cdot\nabla S + \lambda\Delta\widetilde{\theta} = 0 \qquad \text{with} \qquad \lambda = \frac{2}{Ra_c}\,. \tag{3.2}$$

To be more precise, the maximum of the functional corresponds to the largest positive eigenvalue λ of problem (3.1)-(3.2).

Theorem 3.1 *For all $\mathcal{A}$, system (3.1)-(3.2) has non trivial solutions $(\widetilde{\mathbf{w}}, \widetilde{\theta}) \in V \times \overset{o}{W}{}^{1,2}$ and all the eigenvalues are real.*

Proof: Let us consider the representation of two-dimensional solenoidal vector fields by means of the streamfunction $\Psi = \Psi(r, \varphi)$, or, by considering the reference frame here chosen and the (hidden) y-direction,

$$\widetilde{\mathbf{w}} = \nabla\Psi \wedge \boldsymbol{e}_2.$$

By using the vectorial identity $\nabla \times (F\mathbf{V}) = F\nabla \times \mathbf{V} + \nabla F \wedge \mathbf{V}$, one can calculate the curl of all terms in (3.1)

$$\begin{aligned} \nabla \times (\widetilde{\theta}\nabla S) &= \nabla\widetilde{\theta} \wedge \nabla S \\ \nabla \times (\lambda\Delta\widetilde{\mathbf{w}}) &= -\lambda\Delta^2\Psi\boldsymbol{e}_2 \\ \nabla \times \nabla\widetilde{p} &= 0\,. \end{aligned}$$

Hence, problem (3.1)-(3.2) becomes

$$\boldsymbol{e}_2 \cdot \nabla\widetilde{\theta} \wedge \nabla S - \lambda\Delta^2\Psi = 0 \tag{3.3}$$

$$\boldsymbol{e}_2 \cdot \nabla\Psi \wedge \nabla S - \lambda\Delta\widetilde{\theta} = 0\,. \tag{3.4}$$

Now, we are going to show that solving the eigenvalue problem is equivalent to diagonalise a symmetric bilinear form. To this end, one has first to prove that

$$\int_{\Omega_{\mathcal{A}}} (\boldsymbol{e}_2 \cdot \nabla\Psi \wedge \nabla S)\widetilde{\theta} d\boldsymbol{x} = -\int_{\Omega_{\mathcal{A}}} (\boldsymbol{e}_2 \cdot \nabla\widetilde{\theta} \wedge \nabla S)\Psi d\boldsymbol{x}\,. \tag{3.5}$$

Actually, since $\nabla \times \nabla S = 0$, the left-hand side is equal to

$$\int_{\Omega_{\mathcal{A}}} \boldsymbol{e}_2 \cdot (\widetilde{\theta}\nabla \times (\Psi\nabla S))d\boldsymbol{x} = \int_{\Omega_{\mathcal{A}}} \boldsymbol{e}_2 \cdot \nabla \times (\widetilde{\theta}\Psi\nabla S)d\boldsymbol{x} - \int_{\Omega_{\mathcal{A}}} \boldsymbol{e}_2 \cdot (\nabla\widetilde{\theta} \wedge (\Psi\nabla S))d\boldsymbol{x}$$

and, on the other hand, here the first integral on the right-hand side vanishes, because of the Gauss theorem and of the boundary condition for $\widetilde{\theta}$:

$$\int_{\Omega_{\mathcal{A}}} \boldsymbol{e}_2 \cdot \nabla \times (\widetilde{\theta}\Psi\nabla S)d\boldsymbol{x} = \int_{\partial\Omega_{\mathcal{A}}} \widetilde{\theta}\Psi\nabla S d\boldsymbol{x}\,.$$

Identity (3.5), so proved, is useful in projecting (3.3)-(3.4) on the eigenfunctions of the Bilaplacian and of the Laplacian [24], respectively (the first one with boundary condition of vanishing gradient, as follows from the definition of the streamfunction). For better understanding the completeness of the bases, look at the Appendix.

In this way, the problem is reduced to an algebraic one of finding normal modes. Thus, let us indicate the eigenfunctions as

$$\Delta^2\psi^k = \mu^2_{(k)}\psi^k\,, \qquad \Delta\chi^k = -\omega^2_{(k)}\chi^k\,,$$

where the definite sign of the eigenvalues can be proved by multiplying the first equation by ψ^k, the second by χ^k and integrating by parts. Then, the streamfunction and the temperature can be expressed as linear combinations

$$\Psi = c_k\psi^k\,, \qquad \widetilde{\theta} = d_k\chi^k$$

and substituted in (3.3)-(3.4), which in turn can be projected on each basis function:

$$\begin{aligned} d_j(\boldsymbol{e}_2 \cdot \nabla\chi^j \wedge \nabla S, \psi^k) - \lambda c_j(\Delta^2\psi^j, \psi^k) &= 0 \\ c_j(\boldsymbol{e}_2 \cdot \nabla\psi^j \wedge \nabla S, \chi^k) - \lambda d_j(\Delta\chi^j, \chi^k) &= 0\,. \end{aligned}$$

Thus, the algebraic system

$$d_j(\boldsymbol{e}_2 \cdot \nabla\chi^j \wedge \nabla S, \psi^k) - \lambda\mu^2_{(k)} c_k = 0 \tag{3.6}$$

$$c_j(\boldsymbol{e}_2 \cdot \nabla\psi^j \wedge \nabla S, \chi^k) + \lambda\omega^2_{(k)} d_k = 0 \tag{3.7}$$

arises. Finally identity (3.5) can be used to show an important feature of the block matrix

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

whose blocks are defined and characterised as follows

$$B^j_k := (\boldsymbol{e}_2 \cdot \nabla\chi^j \wedge \nabla S, \psi^k) = -(\boldsymbol{e}_2 \cdot \nabla\psi^k \wedge \nabla S, \chi^j) = -B^k_j = -\left(B^j_k\right)^T . \tag{3.8}$$

The algebraic system of equations associated with any finite projection of equations (3.3) and (3.4) is

$$\left[\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} - \begin{pmatrix} \lambda\,\mathrm{diag}(\mu^2_{(k)}) & 0 \\ 0 & \lambda\,\mathrm{diag}(\omega^2_{(k)}) \end{pmatrix}\right] \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} = 0 . \tag{3.9}$$

Its eigenvalue equation is

$$\det\left[\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} - \lambda \begin{pmatrix} \mathrm{diag}(\mu^2_{(k)}) & 0 \\ 0 & \mathrm{diag}(\omega^2_{(k)}) \end{pmatrix}\right] = 0 , \tag{3.10}$$

and the solutions are all real numbers, because the first matrix is symmetric and the second is diagonal and positive defined.

Appendix

We want to find a basis for functions in $\overset{\circ}{W}{}^{1,2}(\Omega_{\mathcal{A}})$ which are periodic in φ. Let us start by considering the eigenvalue equation:

$$-\Delta\Theta = \lambda^2\Theta \tag{3.11}$$

or

$$\frac{\partial^2\Theta}{\partial r^2} + \frac{1}{r}\frac{\partial\Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Theta}{\partial\varphi^2} = -\lambda^2\Theta . \tag{3.12}$$

By using Dirichlet boundary conditions, equation (3.11) can be written in the weak form, so defining a symmetric operator in $\overset{\circ}{W}{}^{1,2}$, whose eigenfunctions can have orthogonal gradients. Moreover, one sees that λ^2 is in fact a positive number:

$$\lambda^2 \int_{\Omega_{\mathcal{A}}} \Theta^2 =$$

$$-\int_{\Omega_{\mathcal{A}}} \Theta\Delta\Theta = -\int_{\Omega_{\mathcal{A}}} \nabla\cdot(\Theta\nabla\Theta) + \int_{\Omega_{\mathcal{A}}} |\nabla\Theta|^2 =$$

$$= -\int_{\partial\Omega_{\mathcal{A}}} \Theta\mathbf{n}\cdot\nabla\Theta + \int_{\Omega_{\mathcal{A}}} |\nabla\Theta|^2$$

since the integral on the boundary vanishes. Next, we first look for solutions via separation of variables and next ask ourselves whether such solutions are a complete basis:

$$\Theta(r,\varphi) = F(r)G(\varphi) \Rightarrow \frac{r^2}{F}\frac{d^2F}{dr^2} + \frac{r}{F}\frac{dF}{dr} + \lambda^2 r^2 = -\frac{1}{G}\frac{d^2G}{d\varphi^2} = n^2, \qquad n \in \mathbb{R}.$$

We get the two equations:

$$\frac{d^2G}{d^2\varphi^2} = -n^2 G \tag{3.13}$$

$$s^2 \frac{d^2\hat{F}}{ds^2} + s\frac{d\hat{F}}{ds} + (s^2 - n^2)\hat{F} = 0\,, \tag{3.14}$$

where we set $s = \lambda r$, so that $F(r) = \hat{F}(s) = \hat{F}(\lambda r)$. Equation (3.13) in the space of periodic functions has the two independent solutions

$$G_n(\varphi) = \begin{cases} \cos n\varphi = G_{n1} \\ \sin n\varphi = G_{n2} \end{cases} \tag{3.15}$$

with integer n. For even n one sees that G_{n1} is even and G_{n2} is odd, while for odd n is the opposite. To span the space, we'll write G_{nk} with $k = 1, 2$.

Equation (3.14) is Bessel's equation, whose general solution is

$$\hat{F}(s) = AJ_n(\lambda r) + BY_n(\lambda r), \tag{3.16}$$

with J_n and Y_n first and second kind Bessel functions of order n, respectively. For the moment, A and B are arbitrary constants. Now, their ratio will be fixed by the boundary conditions.

$$\Theta_{nk} = (AJ_n(\lambda r) + BY_n(\lambda r))G_{nk}(\varphi),$$

has to verify

$$\begin{aligned} AJ_n\left(\lambda\mathcal{A}/2\right) &\;+\; BY_n\left(\lambda\mathcal{A}/2\right) = 0 \\ AJ_n\left(\lambda\left(1+\mathcal{A}/2\right)\right) &\;+\; BY_n\left(\lambda\left(1+\mathcal{A}/2\right)\right) = 0\,. \end{aligned}$$

This algebraic homogeneous system allows for non trivial solutions if and only if the matrix of the coefficients is singular: by asking vanishing determinant the eigenvalues are found, by solving the system B/A is found. In particular, elementary numerical analysis shows that the eigenvalues, which are from now on denoted as λ_{nl}, are infinitely many for each n. As a consequence, the radial part of the eigenfunction is denoted by Θ_{nkl}. Finally, the constant A is fixed by normalization.

However, other eigenfunctions different from the ones we found could exist. In order to prove that we have a complete basis, we first notice that any periodic $f \in \overset{\circ}{W}{}^{1,2}$ can be written in a unique way as

$$f(r,\varphi) = \sum_0^\infty a_n(r)\cos n\varphi + b_n(r)\sin n\varphi\,,$$

since the G_{nk}'s are a complete basis in one dimension and, in particular, an orthonormal basis in L^2. Moreover, one can prove that $a_n(r), b_n(r) \in \overset{\circ}{W}{}^{1,2}(\mathcal{A}/2, 1+\mathcal{A}/2)$, where the measure is redefined with the weight r. In fact, one has

$$\int_{\mathcal{A}/2}^{1+\mathcal{A}/2} (\sum_0^\infty |a_n(r)|^2 + |b_n(r)|^2) r\, dr = \int_{\Omega_\mathcal{A}} |f(r,\varphi)|^2 < \infty\,,$$

$$\int_{\mathcal{A}/2}^{1+\mathcal{A}/2} (\sum_0^\infty |a_n'(r)|^2 + |b_n'(r)|^2) r\, dr = \int_{\Omega_\mathcal{A}} |\partial_r f(r,\varphi)|^2 < \infty\,.$$

At this point, one can rewrite (3.14) in the weak form

$$\int_{\mathcal{A}/2}^{1+\mathcal{A}/2} F'G'r\,dr = \lambda_{nl}^2 \int_{\mathcal{A}/2}^{1+\mathcal{A}/2} FGr\,dr - n^2 \int_{\mathcal{A}/2}^{1+\mathcal{A}/2} \frac{1}{r^2} FGr\,dr \tag{3.17}$$

with the testfunction $G \in \overset{\circ}{W}^{1,2}(\mathcal{A}/2, 1+\mathcal{A}/2)$.

In fact this is a variational formulation for problem (3.14) from which we easily see that the corresponding operator S is self-adjoint and linear. Moreover, due to the Poincare inequality solutions exist in $\overset{\circ}{W}^{1,2}(\mathcal{A}/2, 1+\mathcal{A}/2)$ and are unique for any $F \in (\overset{\circ}{W}^{1,2}(\mathcal{A}/2, 1+\mathcal{A}/2))^*$ (and for any $F \in L_2(A/2, 1+\mathcal{A}/2)$). Due to the compact imbedding of $\overset{\circ}{W}^{1,2}(\mathcal{A}/2, 1+\mathcal{A}/2)$ into $L_2(\mathcal{A}/2, 1+\mathcal{A}/2)$ this means that S is compact in L_2 and has a orthonormal sequence of eigenfunctions, which is a basis of $\overset{\circ}{W}^{1,2}(\mathcal{A}/2, 1+\mathcal{A}/2)$ as we wanted to show.

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