

ON THE CESÀRO AVERAGE OF THE “LINNIK NUMBERS”

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ABSTRACT. Let Λ be the von Mangoldt function and $r_Q(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)$ be the counting function for the numbers that can be written as sum of a prime and two squares (that we will call “Linnik numbers”, for brevity). Let N a sufficiently large integer and let

$$\begin{aligned}
 M_1(N, k) &= \frac{\pi N^{k+2}}{4\Gamma(k+3)} + \frac{N^{k+1}}{4\Gamma(k+2)} - \frac{\pi^{1/2} N^{k+3/2}}{2\Gamma(k+5/2)} \\
 M_2(N, k) &= -\frac{\pi}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} N^{k+1+\rho} - \frac{1}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho} \\
 &\quad + \frac{\pi^{1/2}}{2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho} \\
 M_3(N, k) &= \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{J_{k+2} \left(2\pi (l_1^2 + l_2^2)^{1/2} N^{1/2} \right)}{(l_1^2 + l_2^2)^{k/2+1}} \\
 &\quad - \pi^{-k} N^{k/2+1/2} \sum_{\rho} \frac{\Gamma(\rho)}{\pi^{\rho}} N^{\rho/2} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{J_{k+1+\rho} \left(2\pi (l_1^2 + l_2^2)^{1/2} N^{1/2} \right)}{(l_1^2 + l_2^2)^{(k+1+\rho)/2}} \\
 M_4(N, k) &= \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{m \geq 1} \frac{J_{k+2} (2m\pi N^{1/2})}{m^{k+2}} - \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{m \geq 1} \frac{J_{k+3/2} (2m\pi N^{1/2})}{m^{k+3/2}} \\
 &\quad - \pi^{-k} N^{(k+1)/2} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m \geq 1} \frac{J_{k+1+\rho} (2m\pi\sqrt{N})}{m^{k+1+\rho}} \\
 &\quad + \pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m \geq 1} \frac{J_{k+1/2+\rho} (2m\pi\sqrt{N})}{m^{k+1/2+\rho}},
 \end{aligned}$$

where $J_v(u)$ denotes the Bessel function of complex order v and real argument u . We prove that

$$\sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = M_1(N, k) + M_2(N, k) + M_3(N, k) + M_4(N, k) + O(N^{k+1})$$

for $k > 3/2$, where ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. We also prove that with this technique the bound $k > 3/2$ is optimal.

1. INTRODUCTION

We continue the recent work of Languasco and Zaccagnini on additive problems with prime summands. In [9] and [10] they study the Cesàro weighted explicit formula for the Goldbach numbers (the integers that can be written as sum of two primes) and for the Hardy-Littlewood numbers (the integers that can be written as sum of a prime and a square). In a similar manner, we will study a Cesàro weighted explicit formula for the integers that can be written as sum of a prime and two squares. We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function.

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Letting

$$r_Q(n) = \sum_{m_1+m_2^2+m_3^2=n} \Lambda(m_1)$$

and

$$(1) \quad M_1(N, k) = \frac{\pi N^{k+2}}{4\Gamma(k+3)} + \frac{N^{k+1}}{4\Gamma(k+2)} - \frac{\pi^{1/2} N^{k+3/2}}{2\Gamma(k+5/2)}$$

$$(2) \quad M_2(N, k) = -\frac{\pi}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} N^{k+1+\rho} - \frac{1}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho}$$

$$+ \frac{\pi^{1/2}}{2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho}$$

$$(3) \quad M_3(N, k) = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{J_{k+2} \left(2\pi (l_1^2 + l_2^2)^{1/2} N^{1/2} \right)}{(l_1^2 + l_2^2)^{k/2+1}}$$

$$- \pi^{-k} N^{k/2+1/2} \sum_{\rho} \frac{\Gamma(\rho)}{\pi^{\rho}} N^{\rho/2} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{J_{k+1+\rho} \left(2\pi (l_1^2 + l_2^2)^{1/2} N^{1/2} \right)}{(l_1^2 + l_2^2)^{(k+1+\rho)/2}}$$

$$(4) \quad M_4(N, k) = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{m \geq 1} \frac{J_{k+2} \left(2m\pi N^{1/2} \right)}{m^{k+2}} - \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{m \geq 1} \frac{J_{k+3/2} \left(2m\pi N^{1/2} \right)}{m^{k+3/2}}$$

$$- \pi^{-k} N^{(k+1)/2} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m \geq 1} \frac{J_{k+1+\rho} \left(2m\pi \sqrt{N} \right)}{m^{k+1+\rho}}$$

$$+ \pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m \geq 1} \frac{J_{k+1/2+\rho} \left(2m\pi \sqrt{N} \right)}{m^{k+1/2+\rho}},$$

where $J_v(u)$ is the Bessel function of complex order v and real argument u (see below). The main result of this paper is the following

Theorem 1. *Let N be a sufficient large integer. We have*

$$\sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = M_1(N, k) + M_2(N, k) + M_3(N, k) + M_4(N, k) + O(N^{k+1})$$

for $k > 3/2$, where ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$ and $J_v(u)$ is the Bessel function of complex order v and real argument u . Furthermore the bound $k > 3/2$ is optimal using this technique.

The study of these numbers is classical. For example Hardy and Littlewood in [7] studied the number of solutions of the equation

$$n = p + a^2 + b^2$$

and Linnik in [13] derived an asymptotic formula for the number of representations of these numbers. Similar averages of arithmetical functions are common in literature, see, e.g., Chandrasekharan - Narasimhan [2] and Berndt [1] who built on earlier classical work. For our work we will need the Bessel functions $J_v(u)$ of complex order v and real argument u . For their definition and main properties we refer to Watson [15], but we recall that they were introduced by Daniel Bernoulli and they are the canonical solution of the differential equation

$$u^2 \frac{d^2 J}{du^2} + u \frac{dJ}{du} + (u^2 - v^2) J = 0$$

for any complex number v . In particular, equation (8) on page 177 of [15] gives the Sonine representation

$$(5) \quad J_\nu(u) = \frac{(u/2)^\nu}{2\pi i} \int_{(a)} e^s s^{-\nu-1} e^{-u^2/(4s)} ds$$

where the notation $\int_{(a)}$ means $\int_{a-i\infty}^{a+i\infty}$. As noted by Languasco and Zaccagnini in [10] the estimates of such Bessel functions are harder to perform than the ones already present in the Number Theory literature (as far as we know, Bessel functions of complex order arise in a similar problem for the first time in [10]) since the real argument and the complex order are both unbounded while, in the previous papers, either the real order or the complex argument is bounded. The method we will use in this additive problem is based on a formula due to Laplace [11], namely

$$(6) \quad \frac{1}{2\pi i} \int_{(a)} v^{-s} e^v dv = \frac{1}{\Gamma(s)}$$

with $\operatorname{Re}(s) > 0$ and $a > 0$ (see, e.g., formula 5.4 (1) on page 238 of [4]). As in [10], we combine this approach with line integrals with the classical methods dealing with infinite sum over primes and integers. Similarly as [10] the problem naturally involves the modular relation for the complex Jacobi θ_3 function; the presence of the Bessel functions in our statement strictly depends on such modularity relation.

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2. PRELIMINARY DEFINITIONS AND LEMMAS

Let $z = a + iy$, $a > 0$, and

$$(7) \quad \theta_3(z) = \sum_{m \in \mathbb{Z}} e^{-m^2 z}$$

$$(8) \quad \tilde{S}(z) = \sum_{m \geq 1} \Lambda(m) e^{-mz}$$

$$(9) \quad \omega_2(z) = \sum_{m \geq 1} e^{-m^2 z}$$

and we can see that

$$(10) \quad \theta_3(z) = 1 + 2\omega_2(z).$$

Furthermore we have the functional equation (see, for example, the proposition VI.4.3 of Freitag-Busam [5] page 340)

$$(11) \quad \theta_3(z) = \left(\frac{\pi}{z}\right)^{1/2} \theta_3\left(\frac{\pi^2}{z}\right), \operatorname{Re}(z) > 0$$

and so

$$(12) \quad \omega_2^2(z) = \left(\frac{1}{2} \left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2}\right)^2 + \frac{\pi}{z} \omega_2^2\left(\frac{\pi^2}{z}\right) + \left(\left(\frac{\pi}{z}\right)^{1/2} - 1\right) \left(\left(\frac{\pi}{z}\right)^{1/2} \omega_2\left(\frac{\pi^2}{z}\right)\right).$$

A trivial but important estimate is

$$(13) \quad |\omega_2(z)| \leq \omega_2(a) \leq \int_0^\infty e^{-at^2} dt = \frac{\sqrt{\pi}}{2\sqrt{a}} \ll a^{-1/2}.$$

Let us introduce the following

Lemma 2. *Let $z = a + iy$, $a > 0$ and $y \in \mathbb{R}$. Then*

$$(14) \quad \tilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(a, y)$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$ and

$$E(a, y) \ll |z|^{1/2} \begin{cases} 1, & |y| \leq a \\ 1 + \log^2(|y|/a), & |y| > a. \end{cases}$$

(For a proof see Lemma 1 of [9]. The bound for $E(a, y)$ has been corrected in [8]). So in particular, taking $z = \frac{1}{N} + iy$ we have

$$(15) \quad \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| = \left| \frac{1}{z} - \tilde{S}(z) + E\left(\frac{1}{N}, y\right) \right| \ll N + \frac{1}{|z|} + \left| E\left(\frac{1}{N}, y\right) \right|$$

$$\ll \begin{cases} N, & |y| \leq 1/N \\ N + |z|^{1/2} \log^2(2N|y|), & |y| > 1/N. \end{cases}$$

Now we have to recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 2, to the statement

$$\tilde{S}(a) \sim a^{-1}, \text{ when } a \rightarrow 0^+$$

(see Lemma 9 of [7]). For our purposes it is important to introduce the Stirling approximation

$$(16) \quad |\Gamma(x + iy)| \sim \sqrt{2\pi} e^{-\pi|y|/2} |y|^{x-1/2}$$

(see for example §4.42 of [14]) uniformly for $x \in [x_1, x_2]$, x_1 and x_2 fixed, and the identity

$$(17) \quad |z^{-w}| = |z|^{-\operatorname{Re}(w)} \exp(\operatorname{Im}(w) \arctan(y/a)).$$

We now quote Lemmas 2 and 3 from [9]:

Lemma 3. *Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function and let $\alpha > 1$ be a parameter. The series*

$$\sum_{\rho, \gamma > 0} \gamma^{\beta-1/2} \int_1^{\infty} \exp(-\gamma \arctan(1/u)) \frac{dy}{u^{\alpha+\beta}}$$

converges provided that $\alpha > 3/2$. For $\alpha \leq 3/2$ the series does not converge. The result remains true if we insert in the integral a factor $\log^c(u)$, for any fixed $c \geq 0$.

Lemma 4. *Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function, let $z = a + iy$, $a \in (0, 1)$, $y \in \mathbb{R}$ and $\alpha > 1$. We have*

$$\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{Y}_1 \cup \mathbb{Y}_2} \exp\left(\gamma \arctan\left(\frac{y}{a}\right) - \frac{\pi}{2} |\gamma|\right) \frac{dy}{|z|^{\alpha+\beta}} \ll_{\alpha} a^{-\alpha}$$

where $\mathbb{Y}_1 = \{y \in \mathbb{R} : \gamma y \leq 0\}$ and $\mathbb{Y}_2 = \{y \in [-a, a] : y\gamma > 0\}$. The result remains true if we insert in the integral a factor $\log^c(|y|/a)$, for any fixed $c \geq 0$.

We now establish an important Lemma. We will use it to prove that there is a limitation in our technique. Essentially the lower bound of k is linked to the number of squares in the problem. We have

Lemma 5. *Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function, let N, d be positive integers, $\|\cdot\|$ the euclidean norm in \mathbb{R}^d and $k > 0$ be a real number. Then the series*

$$\sum_{\bar{l} \in (0, \infty)^d} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^{\gamma} e^{-N\|\bar{l}\|^2 v^2/\gamma^2} e^{-v} v^{k+\beta} dv,$$

where

$$\sum_{\bar{l} \in (0, \infty)^d} = \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \cdots \sum_{l_d \geq 1},$$

converges if $k > d - 1/2$ and this result is optimal.

Proof. From (10) we have that

$$\omega_2^d(z) = \frac{1}{2^d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \theta_3^m(z).$$

Hence

$$\begin{aligned}
I &= \sum_{\bar{l} \in (0, \infty)^d} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^\gamma e^{-N \|\bar{l}\|^2 v^2 / \gamma^2} e^{-v} v^{k+\beta} dv \\
&= \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^\gamma \omega_2^d \left(\frac{Nv^2}{\gamma^2} \right) e^{-v} v^{k+\beta} dv \\
&= \frac{1}{2^d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^\gamma \theta_3^m \left(\frac{Nv^2}{\gamma^2} \right) e^{-v} v^{k+\beta} dv.
\end{aligned}$$

Now, using the functional equation (11) we have that

$$\begin{aligned}
I &= \frac{1}{2^d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \frac{\pi^{m/2}}{N^{m/2}} \sum_{\gamma > 0} \gamma^{m-k-3/2} \int_0^\gamma \theta_3^m \left(\frac{\pi^2 \gamma^2}{Nv^2} \right) e^{-v} v^{k+\beta-m} dv \\
&= \frac{1}{2^d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \frac{\pi^{m/2}}{N^{m/2}} \sum_{\gamma > 0} \gamma^{m-k-3/2} I_{\gamma, m},
\end{aligned}$$

say. Now we claim that

$$\theta_3 \left(\frac{\pi^2 \gamma^2}{Nv^2} \right) \asymp 1,$$

where the notation $f(x) \asymp g(x)$ means $g(x) \ll f(x) \ll g(x)$, since $\theta_3(x)$ is a continuous function in the interval $\left[\frac{\pi^2}{N}, \infty \right)$ (i.e. the range of $1/v^2$) and

$$\lim_{x \rightarrow \infty} \theta_3(x) = 1$$

so we have

$$I_{\gamma, m} \asymp \sum_{\gamma > 0} \gamma^{m-k-3/2} \int_0^\gamma e^{-v} v^{k+\beta-m} dv$$

and now assuming $k + \beta - m + 1 > 0$ we have

$$\int_0^\gamma e^{-v} v^{k+\beta-m} dv \asymp 1$$

and so

$$I_{\gamma, m} \asymp_k \sum_{\gamma > 0} \gamma^{m-k-3/2}$$

and the last series converges if $k > m - 1/2$. Since $m = 0, \dots, d$ for a global convergence we must have $k > d - 1/2$ and this result is optimal. \square

Let us introduce another lemma

Lemma 6. *Let $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function, let $z = \frac{1}{N} + iy$, $N > 1$ natural number, $y \in \mathbb{R}$ and $\alpha > 3/2$. We have*

$$\sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}| |z^{-\rho}| |z|^{-\alpha} |dz| \ll_{\alpha} N^{\alpha}.$$

Proof. Put $a = \frac{1}{N}$. Using the identity (17) and (16) we get that the left hand side in the statement above is

$$(18) \quad \sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{R}} \exp \left(\gamma \arctan \left(\frac{y}{a} \right) - \frac{\pi}{2} |\gamma| \right) \frac{dy}{|z|^{\alpha+\beta}}.$$

and so by Lemma 4 (18) is $\ll_{\alpha} a^{-\alpha}$ in $\mathbb{Y}_1 \cup \mathbb{Y}_2$. For the other part we can see that

$$\begin{aligned} & \sum_{\rho} \gamma^{\beta-1/2} \int_a^{\infty} \exp\left(-\gamma \arctan\left(\frac{a}{y}\right)\right) \frac{dy}{|z|^{\alpha+\beta}} \\ &= a^{-\alpha-\beta+1} \sum_{\rho} \gamma^{\beta-1/2} \int_1^{\infty} \exp\left(-\gamma \arctan\left(\frac{1}{u}\right)\right) \frac{dy}{u^{\alpha+\beta}} \end{aligned}$$

since

$$(19) \quad |z|^{-1} \asymp \begin{cases} a^{-1} & |y| \leq a, \\ |y|^{-1} & |y| \geq a, \end{cases}$$

and so by Lemma 3 we have the convergence if $\alpha > 3/2$. \square

3. SETTINGS

Using (7), (8) and (9) it is not hard to see that

$$\tilde{S}(z) \omega_2^2(z) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \sum_{m_3 \geq 1} \Lambda(m_1) e^{-(m_1+m_2+m_3)z} = \sum_{n \geq 1} r_Q(n) e^{-nz}$$

so let $z = a + iy$, $a > 0$ and let us consider

$$\frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}(z) \omega_2^2(z) dz = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \sum_{n \geq 1} r_Q(n) e^{-nz} dz.$$

Now we prove that we can exchange the integral with the series. From (13) and the Prime Number Theorem in the form quoted above we have

$$\sum_{n \geq 1} |r_Q(n) e^{-nz}| = \tilde{S}(a) \omega_2^2(a) \ll a^{-2}$$

hence

$$\begin{aligned} \int_{(a)} |e^{Nz} z^{-k-1}| |\tilde{S}(z) \omega_2^2(z)| |dz| &\ll a^{-2} e^{Na} \left(\int_{-a}^a a^{-k-1} dy + 2 \int_a^{\infty} y^{-k-1} dy \right) \\ &\ll_k a^{-2-k} e^{Na} \end{aligned}$$

assuming $k > 0$. So finally we have

$$(20) \quad \sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}(z) \omega_2^2(z) dz.$$

Now, using (14), we can write (20) as

$$\begin{aligned} \sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} &= \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \omega_2^2(z) dz + \\ (21) \quad &+ O\left(\int_{(a)} |e^{Nz}| |z|^{-k-1} |\omega_2^2(z)| |E(a, y)| |dz| \right) \end{aligned}$$

and the error term can be estimated, using Lemma 2, (13) and (19) as

$$a^{-1} e^{Na} \left(\int_{-a}^a a^{-k-1} dy + \int_a^{\infty} y^{-k-1/2} (1 + \log^2(y/a)) dy \right) \ll_k e^{Na} a^{-k-1}$$

assuming $k > 1/2$. Hereafter we will consider $a = 1/N$. We have

$$\sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \omega_2^2(z) dz + O(N^{k+1})$$

and now, using the functional equation (12), we get

$$\begin{aligned}
\sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} &= \frac{1}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \left(\left(\frac{\pi}{z} \right)^{1/2} - 1 \right)^2 dz \\
&+ \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \frac{\pi \omega_2^2 \left(\frac{\pi^2}{z} \right)}{z} dz \\
&+ \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \left(\left(\frac{\pi}{z} \right)^{1/2} - 1 \right) \left(\left(\frac{\pi}{z} \right)^{1/2} \omega_2 \left(\frac{\pi^2}{z} \right) \right) dz \\
&+ O(N^{k+1}) \\
&= I_1 + I_2 + I_3 + O(N^{k+1}),
\end{aligned}$$

say.

4. EVALUATION OF I_1

From I_1 we will find the main terms $M_1(N, k)$ and $M_2(N, k)$ of our asymptotic formulae. We have

$$\begin{aligned}
I_1 &= \frac{1}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \left(\left(\frac{\pi}{z} \right)^{1/2} - 1 \right)^2 dz \\
&- \frac{1}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \sum_{\rho} z^{-\rho} \Gamma(\rho) \left(\left(\frac{\pi}{z} \right)^{1/2} - 1 \right)^2 dz \\
&= I_{1,1} - I_{1,2},
\end{aligned}$$

say. From $I_{1,1}$ we have

$$I_{1,1} = \frac{\pi}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} dz + \frac{1}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} dz - \frac{\pi^{1/2}}{4\pi i} \int_{(1/N)} e^{Nz} z^{-k-5/2} dz$$

so, if we put $Nz = s$, $ds = Ndz$ and use (6) we have immediately

$$\begin{aligned}
I_{1,1} &= \frac{\pi}{4} \frac{N^{k+2}}{2\pi i} \int_{(1)} e^s s^{-k-3} ds + \frac{N^{k+1}}{4} \frac{1}{2\pi i} \int_{(1)} e^s s^{-k-2} ds - \frac{\pi}{2} \frac{N^{k+3/2}}{2\pi i} \int_{(1)} e^s s^{-k-5/2} ds \\
&= M_1(N, k).
\end{aligned}$$

From $I_{1,2}$ we have

$$\begin{aligned}
I_{1,2} &= \frac{\pi}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma(\rho) dz \\
&+ \frac{1}{8\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \sum_{\rho} z^{-\rho} \Gamma(\rho) dz \\
&- \frac{\pi^{1/2}}{4\pi i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_{\rho} z^{-\rho} \Gamma(\rho) dz \\
&= \mathcal{I}_1 + \mathcal{I}_2 - \mathcal{I}_3,
\end{aligned}$$

say. We observe that by Lemma 6 we have the absolute convergence of these integrals if, respectively, we have $k > -1/2$, $k > 1/2$ and $k > 0$. Hence for $k > 1/2$ we have

$$\mathcal{I}_1 = \frac{\pi}{4} \sum_{\rho} \Gamma(\rho) \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-2-\rho} dz = \frac{\pi}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} N^{k+1+\rho}$$

$$\begin{aligned}\mathcal{I}_2 &= \frac{1}{4} \sum_{\rho} \Gamma(\rho) \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1-\rho} dz = \frac{1}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho} \\ \mathcal{I}_3 &= \frac{\pi^{1/2}}{2} \sum_{\rho} \Gamma(\rho) \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3/2-\rho} dz = \frac{\pi^{1/2}}{2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho}.\end{aligned}$$

5. EVALUATION OF I_2

We have

$$\begin{aligned}I_2 &= \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2^2\left(\frac{\pi^2}{z}\right) dz \\ &\quad - \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma(\rho) \omega_2^2\left(\frac{\pi^2}{z}\right) dz \\ &= I_{2,1} - I_{2,2},\end{aligned}$$

say.

Evaluation of $I_{2,1}$. We have that

$$I_{2,1} := \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2^2\left(\frac{\pi^2}{z}\right) dz = \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-3} \left(\sum_{l_1 \geq 1} e^{-l_1^2 \pi^2 / z} \right) \left(\sum_{l_2 \geq 1} e^{-l_2^2 \pi^2 / z} \right) dz$$

so let us prove that we can exchange the integral with the series. Let us consider

$$A_1 := \sum_{l_1 \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-3} e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} \left| \omega_2\left(\frac{\pi^2}{z}\right) \right| |dz|.$$

From

$$(22) \quad \operatorname{Re}(1/z) = \frac{N}{1+N^2 y^2} \gg \begin{cases} N & |y| \leq 1/N \\ 1/(Ny^2) & |y| > 1/N \end{cases}$$

we have

$$A_1 \ll \sum_{l_1 \geq 1} \int_0^{1/N} \frac{e^{-l_1^2 N}}{|z|^{k+3}} \omega_2(N) dy + N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{y e^{-l_1^2 / (Ny^2)}}{|z|^{k+3}} dy = U_1 + U_2$$

hence, recalling (13) and (19),

$$U_1 \ll N^{k+2} \omega_2^2(N) \ll N^{k+1}$$

and from (19) (with $a = 1/N$) we get

$$\begin{aligned}U_2 &\ll N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{e^{-l_1^2 / (Ny^2)}}{y^{k+2}} dy \ll N^{k/2+1} \sum_{l_1 \geq 1} \frac{1}{l_1^{k+1}} \int_0^{l_1^2 N} u^{k/2-1/2} e^{-u} du \leq \\ &\leq \Gamma\left(\frac{k+1}{2}\right) N^{k/2+1} \sum_{l_1 \geq 1} \frac{1}{l_1^{k+1}} \ll_k N^{k/2+1}\end{aligned}$$

assuming $k > 0$. Now we have to study the convergence of

$$A_2 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-3} e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} e^{-l_2^2 \pi^2 \operatorname{Re}(1/z)} |dz|$$

and again from (19) we have

$$\begin{aligned}A_2 &\ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_0^{1/N} \frac{e^{-(l_1^2 + l_2^2)N}}{|z|^{k+3}} dy + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \frac{e^{-(l_1^2 + l_2^2)/(Ny^2)}}{|z|^{k+3}} dy \\ &= V_1 + V_2.\end{aligned}$$

For V_1 we can repeat the same reasoning of U_1

$$V_1 \ll N^{k+2} \omega_2^2(N) \ll N^{k+1}$$

and for V_2 , assuming $k > 1$, we have

$$V_2 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \frac{e^{-(l_1^2 + l_2^2)/(Ny^2)}}{y^{k+3}} dy \ll_k N^{k/2+1/2}.$$

Then finally we have

$$I_{2,1} = \frac{\pi}{2\pi i} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{(1/N)} e^{Nz} z^{-k-3} e^{-(l_1^2 + l_2^2)\pi^2/z} dz = N^{k+2} \pi \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2\pi i} \int_{(1)} e^s s^{-k-3} e^{-(l_1^2 + l_2^2)\pi^2 N/s} ds$$

from which, recalling the definition of the Bessel functions (5) we have, taking $u = 2\pi(l_1^2 + l_2^2)^{1/2} N^{1/2}$ and assuming $k > 1$,

$$J_{2,1} = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{J_{k+2} \left(2\pi(l_1^2 + l_2^2)^{1/2} N^{1/2} \right)}{(l_1^2 + l_2^2)^{k/2+1}}.$$

Evaluation of $I_{2,2}$. We have to calculate

$$I_{2,2} := \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma(\rho) \left(\sum_{l_1 \geq 1} e^{-l_1^2 \pi^2/z} \right) \left(\sum_{l_2 \geq 1} e^{-l_2^2 \pi^2/z} \right) dz$$

and again we have to prove that is possible to exchange the integral with the series. So let us consider

$$A_3 := \sum_{l_1 \geq 1} \int_{(1/N)} |e^{Nz}| |z^{-k-2}| \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} \left| \omega_2 \left(\frac{\pi^2}{z} \right) \right| |dz|.$$

Now using (15) and (13) we have

$$\begin{aligned} A_3 &\ll N^{1/2} \sum_{l_1 \geq 1} \int_0^{1/N} \frac{e^{-l_1^2 N}}{|z|^{k+2}} dy + N^{3/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{y e^{-l_1^2/(Ny^2)}}{|z|^{k+2}} dy + N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} y \log^2(2Ny) \frac{e^{-l_1^2/(Ny^2)}}{|z|^{k+3/2}} dy \\ &= W_1 + W_2 + W_3. \end{aligned}$$

For W_1 and W_2 we can easily see that

$$W_1 \ll N^{k+3/2} \omega_2(N) \ll N^{k+1}$$

and taking $u = l_1^2/(Ny^2)$

$$\begin{aligned} W_2 &\ll N^{3/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{e^{-l_1^2/(Ny^2)}}{y^{k+1}} dy \\ &\ll N^{k/2+3/2} \sum_{l_1 \geq 1} \frac{1}{l_1^k} \int_0^{l_1^2 N} e^{-u} u^{k/2-1} du \ll_k N^{k/2+3/2} \end{aligned}$$

assuming $k > 1$. We have now to check W_3 . Taking again $u = l_1^2/(Ny^2)$ we have, assuming $k > 3/2$,

$$\begin{aligned} W_3 &\ll N^{k/2-1/4} \sum_{l_1 \geq 1} \frac{1}{l_1^{k-1/2}} \int_0^{l_1^2 N} \log^2 \left(\frac{4Nl_1^2}{u} \right) e^{-u} u^{k/2-5/4} du \\ &\ll N^{k/2-1/4} \sum_{l_1 \geq 1} \frac{1}{l_1^{k-1/2}} \ll_k N^{k/2}. \end{aligned}$$

Let us consider

$$A_4 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 2} \int_{(1/N)} |e^{Nz}| |z^{-k-2}| \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} e^{-l_2^2 \pi^2 \operatorname{Re}(1/z)} |dz|$$

and again for the estimation of $\left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right|$ we have, using (15),

$$\begin{aligned} A_4 &\ll N \sum_{l_1 \geq 1} \sum_{l_2 \geq 2} \int_0^{1/N} \frac{e^{-(l_1^2 + l_2^2)N}}{|z|^{k+2}} dy + \sum_{l_1 \geq 1} \sum_{l_2 \geq 2} \int_{1/N}^{\infty} \frac{e^{-(l_1^2 + l_2^2)/(Ny^2)}}{|z|^{k+2}} dy \\ &\quad + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \log^2(2Ny) \frac{e^{-(l_1^2 + l_2^2)/(Ny^2)}}{|z|^{k+3/2}} dy \\ &= R_1 + R_2 + R_3, \end{aligned}$$

say. So we have immediately

$$R_1 \ll N^{k+2} \omega^2(N) \ll N^{k+1}$$

and, if we take $u = (l_1^2 + l_2^2) / (Ny^2)$, we obtain

$$R_2 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \frac{e^{-(l_1^2 + l_2^2)/(Ny^2)}}{y^{k+2}} dy \ll_k N^{(k+1)/2}$$

for $k > 1$. So it remains to evaluate R_3 . Again we take $u = (l_1^2 + l_2^2) / (Ny^2)$ and we have

$$\begin{aligned} R_3 &\ll N^{k/2+1/4} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{\log^2(4N(l_1^2 + l_2^2))}{(l_1^2 + l_2^2)^{k/2+1/4}} \int_0^{(l_1^2 + l_2^2)^{1/2}N} e^{-u} u^{k/2-3/4} du \\ &\quad - N^{k/2+1/4} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{(l_1^2 + l_2^2)^{k/2+1/4}} \int_0^{(l_1^2 + l_2^2)^{1/2}N} \log^2(u) e^{-u} u^{k/2-3/4} du \end{aligned}$$

and the convergence follows if $k > 3/2$. Note that the estimation of R_3 is optimal. For proving it, take $c = (l_1^2 + l_2^2) / N$, assume $k \leq 3/2$ and $y > 1$. We have

$$S := \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} \log^2(2Ny) \frac{e^{-c/y^2}}{y^{k+3/2}} dy \geq \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_1^{\infty} \log^2(2Ny) \frac{e^{-c/y^2}}{y^{k+3/2}} dy.$$

Now, since $y \geq 1$ we have $\log^2(2Ny) \geq \log^2(2N)$ and since $k \leq 3/2$ we have

$$\begin{aligned} S &\geq \log(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_1^{\infty} \frac{e^{-c/y^2}}{y^{k+3/2}} dy \geq \log(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_1^{\infty} \frac{e^{-c/y^2}}{y^3} dy \\ &= \log(2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2c} (1 - e^{-c}) \geq \frac{N \log(2N) (1 - e^{-2/N})}{2} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{l_1^2 + l_2^2} \end{aligned}$$

and the last double series diverges since

$$\sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{l_1^2 + l_2^2} \geq \sum_{l_1 \geq 1} \sum_{1 \leq l_2 \leq l_1} \frac{1}{l_1^2 + l_2^2} \geq \frac{1}{2} \sum_{l_1 \geq 1} \frac{1}{l_1}.$$

Now we have to estimate

$$A_5 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}| |z^{-k-2}| |z^{-\rho}| e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} e^{-l_2^2 \pi^2 \operatorname{Re}(1/z)} |dz|.$$

Using (16) and (17) we have

$$A_5 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} e^{-\pi \gamma / 2} \gamma^{\beta-1/2} \int_{1/N}^{\infty} |z|^{-k-2} |z|^{-\beta} \exp(\gamma \arctan(Ny)) e^{-l_1^2 \pi^2 \operatorname{Re}(1/z)} e^{-l_2^2 \pi^2 \operatorname{Re}(1/z)} |dz|.$$

Let $Q_k = \sup_{\beta} \left\{ \Gamma \left(\frac{k}{2} + \frac{\beta}{2} + \frac{1}{2} \right) \right\}$ and assume $y < 0$. Using the obvious bound $\gamma \arctan(Ny) - \gamma \frac{\pi}{2} \leq -\gamma \frac{\pi}{2}$ we have

$$(23) \quad \begin{aligned} A_5 &\ll N^{k+1} \sum_{l_1 \geq 1} e^{-l_1^2 N} \sum_{l_2 \geq 1} e^{-l_2^2 N} \sum_{\rho, \gamma > 0} N^{\beta} e^{-\pi \gamma / 2} \gamma^{\beta-1/2} \\ &+ N^{(k+1)/2} Q_k \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{(l_1^2 + l_2^2)^{(k+1)/2}} \sum_{\rho, \gamma > 0} N^{\beta} \frac{e^{-\pi \gamma / 2} \gamma^{\beta-1/2}}{(l_1^2 + l_2^2)^{\beta}} \ll_k N^k \end{aligned}$$

for $k > 1$, where (23) follows from the density estimate $\gamma_m \sim \frac{2\pi m}{\log(m)}$ where γ_m is the imaginary part of the m -th non trivial zeros of the Riemann zeta function. If $y > 0$ we have

$$\begin{aligned} A_5 &\ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} e^{-\pi \gamma / 2} \gamma^{\beta-1/2} \int_0^{1/N} N^{k+2+\beta} e^{-(l_1^2 + l_2^2)N} dy \\ &+ \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp \left(\gamma \left(\arctan(Ny) - \frac{\pi}{2} \right) \right) \frac{e^{-(l_1^2 + l_2^2)/(Ny^2)}}{y^{k+2+\beta}} dy \end{aligned}$$

and by a well-known trigonometric identity follows that

$$\begin{aligned} A_5 &\ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp \left(-\gamma \arctan \left(\frac{1}{Ny} \right) \right) \frac{e^{-(l_1^2 + l_2^2)/(Ny^2)}}{y^{k+2+\beta}} dy \\ &\ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp \left(-\frac{\gamma}{Ny} - \frac{l_1^2 + l_2^2}{Ny^2} \right) y^{-k-2-\beta} dy \end{aligned}$$

and if we put $\frac{\gamma}{Ny} = v$ we get

$$(24) \quad \begin{aligned} A_5 &\ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{\beta-1/2} \int_0^{\gamma} e^{-v} e^{-(Nv^2(l_1^2 + l_2^2)/\gamma^2)} \left(\frac{\gamma}{Nv} \right)^{-k-2-\beta} \frac{\gamma}{Nv^2} dv \\ &\ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho: \gamma > 0} \gamma^{-k-3/2} \int_0^{\infty} e^{-v} e^{-(Nv^2(l_1^2 + l_2^2)/\gamma^2)} v^{k+\beta} dv. \end{aligned}$$

Now we can observe that we are in the situation of Lemma 5 with $d = 2$ and so we can conclude immediately that we have the convergence for $k > 3/2$ and this result is optimal.

We studied the convergence, so we finally have, using again the identity (5), that

$$I_{2,2} = \pi^{-k} N^{k/2+1/2} \sum_{\rho} \frac{\Gamma(\rho)}{\pi^{\rho}} N^{\rho/2} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{J_{k+1+\rho} \left(2\pi (l_1^2 + l_2^2)^{1/2} N^{1/2} \right)}{(l_1^2 + l_2^2)^{(k+1+\rho)/2}}.$$

6. EVALUATION OF I_3

We have

$$\begin{aligned} I_3 &= \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left(\frac{\pi^{1/2}}{z^{3/2}} - \left(\frac{\pi}{z} \right)^{1/2} \sum_{\rho} z^{-\rho} \Gamma(\rho) - \frac{1}{z} + \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \left(\left(\frac{\pi}{z} \right)^{1/2} \omega_2 \left(\frac{\pi^2}{z} \right) \right) dz \\ &= \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2 \left(\frac{\pi^2}{z} \right) dz - \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma(\rho) \omega_2 \left(\frac{\pi^2}{z} \right) dz \\ &\quad - \frac{1}{2\pi^{1/2}i} \int_{(1/N)} e^{Nz} z^{-k-5/2} \omega_2 \left(\frac{\pi^2}{z} \right) + \frac{1}{2\pi^{1/2}i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_{\rho} z^{-\rho} \Gamma(\rho) \omega_2 \left(\frac{\pi^2}{z} \right) dz \\ &= I_{3,1} - I_{3,2} - I_{3,3} + I_{3,4}. \end{aligned}$$

Evaluation of $I_{3,1}$. We have

$$I_{3,1} := \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2 \left(\frac{\pi^2}{z} \right) dz = \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-3} \sum_{m \geq 1} e^{-m^2 \pi^2 / z} dz$$

hence we have to establish the convergence of

$$A_6 := \sum_{m \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-3} e^{-m^2 \operatorname{Re}(1/z)} |dz|.$$

Using (13), (19) and (22) we have

$$(25) \quad A_6 \ll N^{k+3/2} + \sum_{m \geq 1} \int_0^\infty y^{-k-3} e^{-m^2 / (Ny^2)} dy \ll_k N^{k+3/2}$$

for $k > -1$. So we obtain, recalling (5), that

$$J_{3,1} = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{m \geq 1} \frac{J_{k+2}(2m\pi N^{1/2})}{m^{k+2}}.$$

Evaluation of $I_{3,3}$. We have

$$I_{3,3} := \frac{1}{2\pi^{1/2}i} \int_{(1/N)} e^{Nz} z^{-k-5/2} \sum_{m \geq 1} e^{-m^2 \pi^2 / z} dz$$

so we have to establish the convergence of

$$\sum_{m \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-5/2} e^{-m^2 \operatorname{Re}(1/z)} |dz|$$

and so using the same argument that we used for the estimation of $I_{3,1}$ we have the convergence for $k > -1/2$ and so

$$I_{3,3} = \frac{N^{k/2+3/4}}{\pi^{k+1}} \sum_{m \geq 1} \frac{J_{k+3/2}(2m\pi N^{1/2})}{m^{k+3/2}}.$$

Evaluation of $I_{3,2}$. We have to establish the convergence of

$$A_7 := \sum_{m \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-2} \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| \left| e^{-m^2 \pi^2 / z} \right| |dz|$$

so using (13), (19), (22) and (15) we get

$$\begin{aligned} A_7 &\ll N^{k+1/2} + N \sum_{m \geq 1} \int_{1/N}^\infty y^{-k-2} e^{-m^2 / (Ny^2)} dy \\ &\quad + \log^2(2N) \sum_{m \geq 1} \int_{1/N}^\infty y^{-k-3/2} e^{-m^2 / (Ny^2)} dy \\ &\quad + \sum_{m \geq 1} \int_{1/N}^\infty \log^2(y) y^{-k-3/2} e^{-m^2 / (Ny^2)} dy. \end{aligned}$$

Now if we put $m^2 / (Ny^2) = u$ we have

$$N \sum_{m \geq 1} \int_{1/N}^\infty y^{-k-2} e^{-m^2 / (Ny^2)} dy \ll N^{k/2+3/2} \Gamma\left(\frac{k+1}{2}\right) \sum_{m \geq 1} m^{-k-1}$$

which converges if $k > 0$. With the same substitution we get

$$\log^2(2N) \sum_{m \geq 1} \int_{1/N}^{\infty} y^{-k-3/2} e^{-m^2/(Ny^2)} dy \ll \log^2(2N) N^{k/2+1/4} \Gamma\left(\frac{k}{2} + \frac{1}{4}\right) \sum_{m \geq 1} m^{-k-1/2}$$

and so the convergence for $k > 1/2$. For the estimation of the last integral in the bound of A_7 we observe that if we take $\epsilon > 0$ we have

$$\sum_{m \geq 1} \int_{1/N}^{\infty} \log^2(y) y^{-k-3/2} e^{-m^2/(Ny^2)} dy \ll \sum_{m \geq 1} \int_{1/N}^{\infty} y^{-k-3/2+\epsilon} e^{-m^2/(Ny^2)} dy$$

and so, arguing analogously as we did for (25), we get

$$\ll N^{k/2+1/4-\epsilon/2} \Gamma\left(\frac{k}{2} + \frac{1}{4} - \frac{\epsilon}{2}\right) \sum_{m \geq 1} m^{-k-1/2+\epsilon}$$

and for the arbitrariness of ϵ we have the convergence for $k > 1/2$. We have now to study

$$A_8 := \sum_{m \geq 1} \sum_{\rho} |\Gamma(\rho)| \int_{(1/N)} |e^{Nz}| |z^{-k-2}| |z^{-\rho}| \left| e^{-m^2 \pi^2 / z} \right| |dz|.$$

By symmetry we may assume that $\gamma > 0$. If $y \leq 0$ we have $\gamma \arctan(y/a) - \frac{\pi}{2} \gamma \leq -\frac{\pi}{2} \gamma$ and so using (16) and (17) we get

$$\begin{aligned} A_8 &\ll \sum_{m \geq 1} \sum_{\gamma > 0} \gamma^{\beta-1/2} \exp\left(-\frac{\pi}{2} \gamma\right) \left(\int_{-1/N}^0 N^{k+2+\beta} e^{-m^2 N} dy + \int_{-\infty}^{-1/N} \frac{e^{-m^2/(Ny^2)}}{|y|^{k+2+\beta}} dy \right) \\ &\ll_k N^{k+3/2} + N^{k/2+1/2} Q_k \sum_{m \geq 1} \frac{1}{m^{k+1}} \sum_{\gamma > 0} N^{\beta/2} \frac{\gamma^{\beta-1/2}}{m^{\beta}} \exp\left(-\frac{\pi}{2} \gamma\right) \ll_k N^{k+3/2} \end{aligned}$$

provided that $k > 0$ and $Q_k = \sup_{\beta} \left\{ \Gamma\left(\frac{k}{2} + \frac{1}{2} + \frac{\beta}{2}\right) \right\}$. Let $y > 0$. We have

$$\begin{aligned} A_8 &\ll \sum_{m \geq 1} \sum_{\gamma > 0} \gamma^{\beta-1/2} \exp\left(-\frac{\pi}{4} \gamma\right) \int_0^{1/N} N^{k+2+\beta} e^{-m^2 N} dy \\ &\quad + \sum_{m \geq 1} \sum_{\gamma > 0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\left(\gamma \arctan(Ny) - \frac{\pi}{2} \gamma\right) \frac{e^{-m^2/(Ny^2)}}{y^{k+2+\beta}} dy \\ &= L_1 + L_2, \end{aligned}$$

say. From (13) and (19) we have

$$L_1 \ll N^{k+1} \sum_{m \geq 1} e^{-m^2 N} \sum_{\gamma > 0} N^{\beta} \gamma^{\beta-1/2} \exp\left(-\frac{\pi}{4} \gamma\right) \ll_k N^{k+3/2}$$

and again by a well-known trigonometric identity and taking $v = m/(N^{1/2}y)$ we have

$$\begin{aligned} L_2 &\ll \sum_{m \geq 1} \sum_{\gamma > 0} \gamma^{\beta-1/2} \int_{1/N}^{\infty} \exp\left(-\frac{\gamma}{Ny} - \frac{m^2}{Ny^2}\right) \frac{dy}{y^{k+2+\beta}} \\ &= N^{(k+1)/2} \sum_{m \geq 1} \frac{1}{m^{k+1}} \sum_{\gamma > 0} \frac{N^{\beta/2}}{m^{\beta}} \gamma^{\beta-1/2} \int_0^{m\sqrt{N}} \exp\left(-\frac{\gamma v}{N^{1/2}m} - v^2\right) v^{k+\beta} dv \end{aligned}$$

and now since $e^{-v^2} v^k = O_k(1)$ if $k > 0$ we have, taking $s = \gamma v/(N^{1/2}m)$,

$$\ll N^{k/2+1} \sum_{m \geq 1} \frac{1}{m^k} \sum_{\gamma > 0} N^{\beta} \gamma^{-3/2} \int_0^{\infty} \exp(-s) s^{\beta} ds \ll_k N^{k/2+2}$$

for $k > 1$. Now we can exchange the series with the integral and so we have

$$I_{3,2} = \pi^{-k} N^{(k+1)/2} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma(\rho) \sum_{m \geq 1} \frac{J_{k+1+\rho}(2m\pi\sqrt{N})}{m^{k+1+\rho}}$$

Evaluation of $I_{3,4}$. We have to establish the convergence of

$$I_{3,4} := \frac{1}{2\pi^{1/2}i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_{\rho} z^{-\rho} \Gamma(\rho) \omega_2\left(\frac{\pi^2}{z}\right) dz$$

and so we have the similar situation of $I_{3,2}$. Then arguing analogously as we did for estimating $I_{3,2}$ we obtain the condition $k > 1$. We can exchange the series with the integral and obtain

$$I_{3,4} = \pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho} \Gamma(\rho) \sum_{m \geq 1} \frac{J_{k+1/2+\rho}(2m\pi\sqrt{N})}{m^{k+1/2+\rho}}.$$

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