ON CERTAIN ANISOTROPIC ELLIPTIC EQUATIONS ARISING IN CONGESTED OPTIMAL TRANSPORT: LOCAL GRADIENT BOUNDS

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Abstract. Motivated by applications to congested optimal transport problems, we prove higher integrability results for the gradient of solutions to some anisotropic elliptic equations, exhibiting a wide range of degeneracy. The model case we have in mind is the following:

$$\partial_x\left[(|u_x|-\delta_1)_+^{q-1}\,\frac{u_x}{|u_x|}\right]+\partial_y\left[(|u_y|-\delta_2)_+^{q-1}\,\frac{u_y}{|u_y|}\right]=f,$$

for $2\le q<\infty$ and some non negative parameters δ_1,δ_2. Here $(\,\cdot\,)_+$ stands for the positive part. We prove that if $f\in L^\infty_{loc}$, then $\nabla u\in L^r_{loc}$ for every $r\ge 1$.

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1. Introduction

1.1. **Background and motivations.** Let $\Omega\subset\mathbb{R}^N$ be an open bounded (smooth) set and let us consider a variational integral of the type

$$\mathcal{F}(u)=\int_\Omega F(\nabla u(x))\,dx+\int_\Omega f(x)\,u(x)\,dx, \tag{1.1}$$

2010 *Mathematics Subject Classification.* 35J70, 35B65, 49K20.
Key words and phrases. Degenerate elliptic equations, Anisotropic problems, Traffic congestion.

with $z \mapsto F(z)$ being a convex energy with $q-$growth at infinity (here $q > 1$), uniformly convex for $|z| \gg 1$, but not necessarily strictly convex in the whole. The prototypical case of such an energy is given by

$$F(z) = \frac{1}{q}\,(|z| - \delta)_+^q, \qquad z \in \mathbb{R}^N, \tag{1.2}$$

for some $\delta > 0$, which identically vanishes on the ball $\{z \,:\, |z| \le \delta\}$. Some pioneering works on the regularity of minimizers of variational integrals of this type are [11, 19] and [21]. Then more general functionals exhibiting such a lack of uniform convexity have been considered in other papers, like for example in [10, 12, 17] and [18].

As pointed out in [7], regularity results for minimizers of such a kind of functionals are tightly connected with *optimal transport problems with congestion effects*. In order to neatly motivate the purposes of this paper, we want to spend some words about this point. Suppose that the positive f^+ and negative parts f^- of f stand for the densities of centers of production and consumption of a given commodity in the region $\Omega \subset \mathbb{R}^N$ (the physical case clearly corresponds to $N = 2$). The transportation programs are represented by vector fields ϕ satisfying the balance laws

$$\operatorname{div}\phi = f^+ - f^- \qquad + \qquad \text{Neumann boundary conditions},$$

where these boundary conditions are zero if $\int_\Omega f^+ = \int_\Omega f^-$, i.e. if the region is economically balanced, so there is no need for import/export activities. Observe that the constraint on the divergence simply states that in each point the incoming/outcoming transportation flow is ruled by the difference between the demand and the supply. Then a function $G : \mathbb{R}^N \to \mathbb{R}^+$ is given, such that for every transportation program ϕ, the quantity

$$\mathcal{G}(\phi) = \int_\Omega G(\phi(x))\,dx,$$

gives the total transportation cost. In order to capture the effects of congestion, the function G is typically taken to be strictly convex and superlinear. In economical terms, this comes from the fact that in a congested situation, the marginal cost ∇G is strictly monotone and divergent at infinity. In other words, the transport problem we are facing is the so called *Beckmann's problem*, introduced in [2] and defined by

$$\min\left\{\mathcal{G}(\phi) \,:\, \operatorname{div}\phi = f, \text{ in } \Omega, \langle \phi, \nu_\Omega\rangle = 0 \text{ on } \partial\Omega\right\}. \tag{1.3}$$

Here and in what follows for simplicity we are considering the balanced case, i.e. we assume $\int_\Omega f\,dx = 0$. The link between (1.3) and our original problem (1.1) is given by

$$\min_\phi \mathcal{G}(\phi) = \max_u -\mathcal{F}(u),$$

provided that F is the Legendre-Fenchel transform of G. Then optimizers of the two problems are linked by the primal-dual optimality conditions

$$\nabla u_0 \in \partial G(\phi_0) \qquad \text{or equivalently} \qquad \phi_0 \in \partial F(\nabla u_0),$$

which hold pointwisely almost everywhere in Ω. In other words (1.1) is the dual (in the sense of convex analysis) of Beckmann's problem. As always in Optimal Transport, the dual variables u of problem (1.1) have to be thought as *price systems* for a company handling the transport in a congested situation. An optimizer u_0 then gives the price system which maximizes the profit of the company.

Observe that the function (1.2) considered in [6, 7] corresponds to Beckmann's problem with cost

$$G(\xi) = \frac{1}{p}\,|\xi|^p + \delta\,|\xi|, \qquad \xi \in \mathbb{R}^N,$$

where $p = q/(q-1)$. In this case, a Lipschitz estimate and the higher differentiability of an optimal price u_0 can be proved, by looking at the corresponding Euler-Lagrange equation

$$\operatorname{div}\left((|\nabla u_0| - \delta)_+^{q-1}\,\frac{\nabla u_0}{|\nabla u_0|}\right) = f. \tag{1.4}$$

Appealing to the primal-duality optimality conditions, these in turn give that the optimal ϕ_0 in (1.3) is a bounded Sobolev vector field, provided f is smooth enough (see [6, Theorem 2.1] and [7, Theorem 3.4]). We also mention the papers [13] and [25] where the continuity of ϕ_0 is proved. Of course the latter *does not imply* that ∇u_0 as well is continuous.

It is crucial to observe that for such a cost G, the linear part $|\xi|$ prevails on the strictly convex one $|\xi|^p$ as $|\xi| \ll 1$. This means that

$(\star)$ "*congestion effects are negligible, in the small mass regime*"

an hypothesis which is very reasonable. The lack of strict convexity for the Lagrangian (1.2) is precisely a consequence of this assumption. This implies that *elliptic equations that are relevant for these transport problems typically exhibit a severe degeneracy*, like in (1.4).

1.2. **More degeneracy: technical issues.**

However, the hypothesis of an isotropic cost function G is not always well-motivated. For example, as shown in [1], the analysis of discrete congested transport problems settled on a network grid of small size ε, naturally leads to (1.3) with a transportation cost of the form

$$G(\xi) = \sum_{i=1}^{N} h_i(|\xi_i|), \qquad \xi \in \mathbb{R}^N,$$

as the parameter ε goes to 0. Roughly speaking, this anisotropic cost keeps memory of the rigid geometry (i.e. the network grid) of the approximating discrete problems.

Here again, the functions $h_1, \dots, h_N : \mathbb{R}^+ \to \mathbb{R}^+$ are strictly convex and superlinear, such that $h_i(0) = 0$ and $h_i'(0) = \delta_i > 0$. As before, this last hypothesis is motivated by the realistic assumption $(\star)$ above, i.e. G should behave linearly for small masses. Back to our original problem (1.1), it is then natural to ask which kind of Lagrangians we are lead to study, with such a choice. It is easily seen that in this case we have

$$F(z) = \sum_{i=1}^{N} h_i^*(|z_i|), \qquad z \in \mathbb{R}^N,$$

where h_i^* are C^1 functions of one variable, constantly equal to 0 on the interval $[0, \delta_i]$, due to the assumption $h_i'(0) = \delta_i > 0$. A significant instance of such a Lagrangian is given by

$$F(z) = \sum_{i=1}^{N} \frac{(|z_i| - \delta_i)_+^q}{q}, \qquad z \in \mathbb{R}^N. \tag{1.5}$$

This function considerably differs from (1.2), in that this time the Hessian matrix D^2F is given by a diagonal matrix, whose $i-$th entry $(h_i^*)''(|z_i|)$ is constantly zero as $0 \le |z_i| \le \delta_i$. In terms of the corresponding Euler-Lagrange equation

$$\operatorname{div} \nabla F(\nabla u) = f, \tag{1.6}$$

this implies that *ellipticity breaks down at every point where a single component of the gradient is small.* Also observe that due to the particular structure of D^2F, the function F not only lacks strict convexity, but it is not even uniformly convex "at infinity", i.e. outside a ball, contrary to the case of (1.2). This fact is the main source of difficulties. Typically, in order to derive higher integrability results for the gradient of minimizers of $\mathcal{F}$, one considers the differentiated equations

$$\mathrm{div}\left(D^2F(\nabla u)\nabla u_{x_j}\right) = f_{x_j}, \qquad j=1,\dots,N, \tag{1.7}$$

which are solved by the (components of the) gradient of a minimizer u_0. Then it is sufficient to know that this equation is uniformly elliptic at least "at infinity" to conclude that ∇u_0 is in L^∞. A typical assumption is the following[1]:

- there exists $M_0 \ge 0$, such that $\quad (1+|z|^2)^{\frac{q-2}{2}} \lesssim \min\limits_{|\vartheta|=1}\langle D^2F(z)\,\vartheta,\vartheta\rangle, \quad$ for every $|z|\ge M_0$;
- $|D^2F(z)| \lesssim (1+|z|^2)^{\frac{q-2}{2}}, \quad$ for every $z\in\mathbb{R}^N$;

In this case the natural idea, which is somehow common to [6, 10, 17, 18], is that of cutting away the degeneracy region, by localizing equation (1.7) "in a neighborhood of infinity". In a nutshell, this is done by selecting suitable test functions in the weak formulation of (1.7), for examples quantities like $(|\nabla u_0|^\beta - M_0^\beta)_+$ would do the job. Thanks to the hypotheses on D^2F, one can derive Caccioppoli inequalities for these quantities, which combined with the Sobolev inequality give a recursive scheme of reverse Hölder inequalities, leading to the L^∞ estimate for ∇u_0.

Here on the contrary, with the choice (1.5) we have

$$\min_{|z|\ge M}\left(\min_{|\xi|=1}\langle D^2F(z)\,\xi,\xi\rangle\right) = 0, \qquad \text{for every } M>0,$$

so that we have an obstruction in deriving a true Caccioppoli inequality for positive subsolutions of the linearized equation (1.7). This is the main technical difficulty one faces with this type of degeneracy.

1.3. **Strategy of the proof.**

First of all, a weak form of the Caccioppoli inequality can still be derived (see Lemma 2.7 below). This time, rather than having an integral control on ∇u_{x_j} in terms of u_{x_j} itself, we have a control on a "weighted" norm of ∇u_{x_j}, where the weights depend on *all the other components* u_{x_i} *of the gradient*, through the nonlinear functions $(h_i^*)''$. Once we have this, we avoid the use of Sobolev inequality for these weighted integrals. Rather, we apply a sort of *very weak weighted Gagliardo-Nirenberg inequality* (see Lemma 2.6 below), valid for solutions of equation (1.6). This can be derived by means of a weird choice for the test function to be inserted in the weak formulation of (1.6). Such a test function is given by a mixture of the solution and its gradient. We learnt this trick from Di Benedetto's celebrated paper on $C^{1,\alpha}$ regularity for solutions of $p-$Laplacian type equations (see [15, Lemma 2.4 and Proposition 3.1]). We point out that a similar idea can be found in the papers [4, 5] (dealing with variational integrals similar to those considered here, in less degenerate situations) and [24].

Once we have these two surrogates of the classical tools, we can join them to get the desired recursive scheme of reverse Hölder inequalities (Lemma 2.10), like in the standard Moser's technique. Here there is a drawback, since at each step the gain of integrability for the gradient is quite poor (it is of additive type) and does not permit to implement a real Moser iteration so to obtain an L^∞ estimate. However, integrability of any finite exponent can be obtained and this is the main achievement of this paper.

The whole proof is carried on by approximation. As it is classical, we approximate our equation by less degenerate ones and aim to prove a priori estimates on the gradient which are independent of the approximating equation. Due to its degeneracy, our equation does not permit to infer uniqueness of the solution, then some care is needed in this argument as well. Here we take advantage of the variational nature of our equation and use some penalization arguments, somehow inspired to [10].

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[1] As shown in [18], in general the upper bound on D^2F is not really necessary and can be replaced by a growth assumption on ∇F. Here for ease of exposition, we stick to a more classical hypothesis.

As for the growth conditions we are to going to consider, for technical reasons we will confine ourselves to the case of *superquadratic growth*, i.e. we will restrict to the case $q \geq 2$. We point out that for Lagrangians displaying a structure similar to (1.5), the restriction $q \geq 2$ gives some slight simplifications in the choice of the regularized approximating problems. Moreover, it is quite typical (see [5, Remark 1.2] or [20, Theorem 4.1], for example). Further investigations for the singular case $1 < q < 2$ are left for future research.

Last but not least, some final words concerning the hypothesis on the datum f, which is required to be in L^∞. One may wonder whether this hypothesis is optimal or not, since usually $f \in L^{N+\varepsilon}$ is the sharp assumption (in the scale of Lebesgue spaces) that could guarantee a Lipschitz estimate for the solution. The key point is that the usual proof of this result uses the Sobolev inequality, to treat the term f as a lower order perturbation. For the reason described before, such a strategy seems not to work in the present situation. We leave as an (interesting) open question to know if the L^∞ hypothesis on f can be weakened.

1.4. **The result of this paper.** Motivated by the previous discussion, the main aim of this paper is to investigate regularity properties of local minimizers of the following convex energy

$$\mathfrak{F}_q(u;\Omega) = \sum_{i=1}^{N} \int_\Omega g_i(u_{x_i})\,dx + \int_\Omega f\,u\,dx, \qquad u \in W^{1,q}(\Omega), \tag{1.8}$$

where $q \geq 2$, the functions g_i are defined by

$$g_i(t) = \frac{1}{q}\,(|t| - \delta_i)_+^q, \qquad t \in \mathbb{R},$$

for some $\delta_1, \dots, \delta_N \geq 0$ and $f \in L^\infty(\Omega)$ is given. For the reader's convenience, we recall that $u \in W^{1,q}_{loc}(\Omega)$ is said to be a *local minimizer* of $\mathfrak{F}_q$ if for every $\Omega' \Subset \Omega$ we have

$$\mathfrak{F}_q(u;\Omega') \leq \mathfrak{F}_q(u+\varphi;\Omega'), \qquad \text{for every } \ \varphi \in W^{1,q}_0(\Omega').$$

Observe that a local minimizer does not necessarily belong to $W^{1,q}(\Omega)$. In this paper, we are going to prove the following higher integrability result for the gradient.

Main Theorem. *Let $f \in L^\infty_{loc}(\Omega)$ and $q \geq 2$. If $u \in W^{1,q}_{loc}(\Omega)$ is a local minimizer of $\mathfrak{F}_q$, then $u \in W^{1,r}_{loc}(\Omega)$ for every $r \geq 1$.*

Remark 1.1. In the particular case $\delta_1 = \dots = \delta_N = 0$, the gradient term of $\mathfrak{F}_q$ coincides with the anisotropic Dirichlet energy, i.e.

$$\frac{1}{q}\int_\Omega \|\nabla u\|^q_{\ell^q}\,dx + \int_\Omega f\,u\,dx, \qquad u \in W^{1,q}(\Omega), \tag{1.9}$$

where for every $z \in \mathbb{R}^N$, we set $\|z\|_{\ell^q} = \left(\sum_{i=1}^N |z_i|^q\right)^{1/q}$. Observe that a local minimizer of this anisotropic energy is a local weak solution of the equation

$$\widetilde{\Delta}_q u := \sum_{i=1}^{N} \left(|u_{x_i}|^{q-2}\,u_{x_i}\right)_{x_i} = f,$$

where the differential operator on the left-hand side is sometimes called *pseudo $q-$Laplacian*. We cite the papers [3, 8], where some spectral properties of this nonlinear operator are investigated. The paper [3] also proves a local Lipschitz result for positive viscosity solutions of $-\widetilde{\Delta}_q u \geq 0$ when $q \geq 2$, see [3, Theorem 2.9].

In order to put into the right framework the previous result, a remark on related existing regularity results is in order.

Remark 1.2. We already mentioned the papers [4, 5, 20] as a (non exhaustive) list of works considering functionals similar to ours. As gradient regularity of local minimizers is concerned, the model case there studied is given by

$$\int_\Omega \widetilde{F}(\nabla u)\,dx, \qquad \text{with} \quad \widetilde{F}(z)=\sum_{i=1}^N (\mu+|z_i|^2)^{\frac{q_i}{2}}, \tag{1.10}$$

where $\mu > 0$ and $1 < q_1 \le q_2 \le \dots \le q_N$ are given exponents, possibly different. We observe that such a functional belong to the class of *problems with non standard growth conditions*, whose systematic study started with the paper [23] by Marcellini. However, already in the standard growth case, i.e. when $q_1 = \dots = q_N$, the type of degeneracy is quite different from that of our functional $\mathfrak{F}_q$ in (1.8). For example, when $2 \le q_1 = q_2 = \dots = q_N$ the corresponding Euler-Lagrange equation is not even degenerate, in the sense that

$$0 < \min_{|\vartheta|=1} \langle D^2\widetilde{F}(z)\,\vartheta, \vartheta\rangle.$$

Then the result of our Main Theorem is not directly comparable to those in the above mentioned references. Also observe that due to the peculiar structure, it is not even true that our $\mathfrak{F}_q$ has the same behaviour of a functional like (1.10) "asymptotically at infinity".

We also mention that the degenerate case $\mu = 0$ has been considered in the pioneering paper [26]. There the Lipschitz character of minimizers has been shown under some restrictions on the exponents $q_1, \dots, q_N$, by using the so-called *Bernstein method.* For example, their result applies to minimizers of (1.9) for $q > 3$. Again, though the growth conditions considered are more general than ours, the type of degeneracy is weaker than that admitted in $\mathfrak{F}_q$.

1.5. **Plan of the paper.** The rest of the paper is devoted to prove the Main Theorem. In Section 2 we will derive local uniform estimates for the gradients of minimizers of some regularized problems. The crucial estimate is contained in Proposition 2.1, whose proof occupies the whole section. Then in Section 3 we will show how to take these estimates to the limit, in order to prove the desired result. Since the functional $\mathfrak{F}_q$ is not strictly convex, a further penalization argument will be needed, so to select the desired local minimizer in the limit. Finally, the concluding Section 4 gives an application of the Main Theorem to the relevant optimal transport problem.

2. REGULARITY ESTIMATES FOR APPROXIMATING PROBLEMS

Let us fix an open bounded set $\mathcal{O} \subset \mathbb{R}^N$. For every $\varepsilon \ll 1$, we consider the following functional

$$\mathfrak{F}_q^\varepsilon(u) = \sum_{i=1}^N \int_{\mathcal{O}} g_i^\varepsilon(u_{x_i})\,dx + \varepsilon \int_{\mathcal{O}} H(\nabla u)\,dx + \int_{\mathcal{O}} b_\varepsilon(x,u)\,dx, \qquad u \in W^{1,q}(\mathcal{O}), \tag{2.1}$$

where:

- for every $i = 1, \dots, N$, we simply set $g_i^\varepsilon(t) = g_i(t)$ if $q > 2$, while if $q = 2$ this is given by

$$g_i^\varepsilon(t) = \begin{cases} 0, & \text{if } |t| \le \delta_i - \varepsilon, \\[2mm] \dfrac{1}{12\,\varepsilon}\,(|t|-\delta_i+\varepsilon)^3, & \text{if } \delta_i - \varepsilon \le |t| \le \delta_i + \varepsilon, \\[2mm] \dfrac{1}{6}\varepsilon^2 + \dfrac{1}{2}\,(|t|-\delta_i)^2, & \text{if } |t| \ge \delta_i + \varepsilon, \end{cases}$$

  which converges in C^1 to $1/2\,(|t|-\delta_i)_+^2$ as ε goes to 0;

- $H:\mathbb{R}^N\to\mathbb{R}$ is the C^∞ strictly convex function given by

$$H(z)=\frac{1}{q}\,(1+|z|^2)^{\frac{q}{2}},\qquad z\in\mathbb{R}^N;$$

- $b_\varepsilon:\mathcal{O}\times\mathbb{R}\to\mathbb{R}$ are of class C^∞ and such that

$$|b_\varepsilon(x,u)|\le C_1\,(|u|+1)\qquad\text{and}\qquad\left|b'_\varepsilon(x,u):=\frac{\partial}{\partial u}b_\varepsilon(x,u)\right|\le C_2,\qquad (x,u)\in\mathcal{O}\times\mathbb{R},\tag{2.2}$$

  for some positive constants C_1,C_2 independent of ε.

In this section, we will prove the following result.

Proposition 2.1. *Let $q\ge 2$ and $\zeta\in W^{1,q}(\mathcal{O})$. If $u^\varepsilon\in W^{1,q}(\mathcal{O})$ is a solution of*

$$\min\left\{\mathfrak{F}^\varepsilon_q(v)\,:\,v-\zeta\in W^{1,q}_0(\mathcal{O})\right\},\tag{2.3}$$

with $\mathfrak{F}^\varepsilon_q$ defined in (2.1), then $u^\varepsilon\in W^{1,r}_{loc}(\mathcal{O})$ for every $r\ge 1$. Moreover, for every $\Sigma\Subset\mathcal{O}$ and $r\ge 1$, we have the following estimate

$$\|u^\varepsilon\|_{W^{1,r}(\Sigma)}\le C,\tag{2.4}$$

for some positive constant C depending on q, r, $N,\max\{\delta_1,\ldots,\delta_N\}$, $\|u^\varepsilon\|_{W^{1,q}}$, $\mathrm{dist}(\Sigma,\partial\mathcal{O})$ and the constants C_1,C_2 in (2.2).

The rest of this section is devoted to prove Proposition 2.1. For the sake of readability, we divide the proof in five main steps, each corresponding to a subsection.

2.1. **Step 1: machinery and preliminary results.** Let us first collect some basic properties of the convex functions g^ε_i. The proof being elementary, it is left to the reader. From now on, we will always drop the superscript ε on the functions g^ε_i and we will simply denote them by g_i.

Lemma 2.2. *For every $i=1,\ldots,N$ and every $q\ge 2$, the function g_i is $C^{2,\alpha}$, with $\alpha=\min\{q-2,1\}$ for $q>2$ and $\alpha=1$ for $q=2$ (regularized case). Moreover, we have the following estimates*

$$g''_i(t)\le (q-1)\,|t|^{q-2}\qquad\text{and}\qquad\frac{g''_i(t)\,t^2}{q-1}\ge 2^{-q}\,|t|^q-C,\ (C=C(\delta_i,q)),\qquad\text{for every } t\in\mathbb{R},\tag{2.5}$$

and also

$$g'_i(t)\,t\ge\frac{1}{2(q-1)}\,g''_i(t)\,t^2-\frac{\delta_i^2}{2\,(q-1)}\,g''_i(t)\qquad\text{and}\qquad\frac{|g'_i(t)|}{|t|}\le\frac{g''_i(t)}{q-1},\qquad\text{for every } t\in\mathbb{R}.\tag{2.6}$$

We also need the following classical L^∞ result, for local minimizers of integral having $q-$growth conditions in the gradient variable. The important point is the dependence of the constant of the L^∞ estimate. For a proof of this standard result, the reader can consult [22, Theorem 7.5]. The statement has been adapted to suit our simplified hypotheses.

Lemma 2.3. *Let $F:\mathcal{O}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ be a Caratheodory function satisfying the growth conditions*

$$|z|^q-M\,(|u|+1)\le F(x,u,z)\le L\,|z|^q+M\,(|u|+1),\qquad (x,u,z)\in\mathcal{O}\times\mathbb{R}\times\mathbb{R}^N,\tag{2.7}$$

for some positive constants L and M. Then every local minimizer $u\in W^{1,q}(\mathcal{O})$ of the functional

$$\int F(x,u,\nabla u)\,dx,$$

belongs to $L^\infty_{loc}(\mathcal{O})$*. Moreover, there exists a constant* C *depending on* $q, N, \|u\|_{W^{1,q}}, L$ *and* M*, such that for every pair of concentric balls* $B_\varrho(x_0) \subset B_R(x_0) \Subset \mathcal{O}$*, we have*

$$\|u\|_{L^\infty(B_\varrho(x_0))} \le C \left[\frac{1}{(R-\varrho)^{\frac{N}{q}}} \|u\|_{L^q(B_R(x_0))} + 1\right].$$

In what follows, we will drop the superscript ε and we will simply use u to denote the solution of (2.3). We will use the same convention for the functions b_ε. Observe that the Euler-Lagrange equation of problem (2.3) is given by

$$\begin{aligned}\sum_{i=1}^N \int_{\mathcal{O}} g_i'(u_{x_i})\,\varphi_{x_i}\,dx + \varepsilon \int_{\mathcal{O}} \langle \nabla H(\nabla u), \nabla\varphi\rangle\,dx \\ + \int_{\mathcal{O}} b'(x,u)\,\varphi\,dx = 0, \qquad \text{for every } \varphi \in W^{1,q}_0(\mathcal{O}).\end{aligned} \tag{2.8}$$

Particularizing the result of Lemma 2.3 to our problem (2.3), we have the following.

Corollary 2.4. *Let* $u \in W^{1,q}(\mathcal{O})$ *be a solution of problem* (2.3)*. Then for every* $\Sigma \Subset \mathcal{O}$ *we have*

$$\|u\|_{L^\infty(\Sigma)} \le C,$$

for some constant C *depending on* $q, N, \|u^\varepsilon\|_{W^{1,q}}, \mathrm{dist}(\Sigma, \partial\mathcal{O}), \max\{\delta_1, \dots, \delta_N\}$ *and the constant* C_1 *in* (2.2)*.*

Proof. It is sufficient to check that $\mathfrak{F}^\varepsilon_q$ verifies hypothesis (2.7), then we can apply the estimate of Lemma 2.3. To this aim, we simply use the first hypothesis (2.2) on b, the definition of H and the estimates of Lemma 2.2 for the functions g_i. □

Remark 2.5. We observe that the integrand of $\mathfrak{F}^\varepsilon_q$ is a $C^{2,\alpha}$ function, whose Hessian with respect to the gradient variable is bounded from below and such that the ratio between its minimal and maximal eigenvalue is bounded. Then we can infer the $C^{2,\alpha}$ local regularity for the solutions u^ε (see [22, Theorem 10.18]). This implies that quantities of the type $h(u_{x_i})$ are admissible test functions, for every $h : \mathbb{R} \to \mathbb{R}$ locally Lipschitz.

The following notation will be used throughout the rest of the paper:

$$w(x) = 1 + |\nabla u(x)|^2, \quad k_j = \delta_j + 1, \qquad \text{and} \qquad v_j = (u_{x_j} - k_j)^2_+ + 1, \quad j = 1, \dots, N. \tag{2.9}$$

Also, from now on, we will omit to indicate the domain of integration of our integrals each time these are performed on the whole $\mathcal{O}$.

2.2. **Step 2: a Sobolev-type inequality.**

As already remarked in the Introduction, we need a sort of Sobolev inequality for solutions of (2.8). In this sense, the most important term in the right-hand side of (2.10) below is the gradient term. It is not difficult to see that the sum of the powers of the right-hand side is smaller than that on the left-hand one. Heuristically, this means that we are facing a a Gagliardo-Nirenberg inequality. However, things are more complicated, since the partial derivatives u_{x_j} and u_{x_i} are mixed.

Lemma 2.6. *Let* α, β *be two positive exponents such that*

$$0 \le \alpha < \beta,$$

then using the notation introduced in (2.9)*, for every* $\xi \in C^1_0(\mathcal{O})$ *and every* $j = 1, \dots, N$*, we have*

$$
\begin{aligned}
\sum_{i=1}^N \int g_i''(u_{x_i})\, |u_{x_i}|^2\, v_j^\beta\, \xi^2 + \varepsilon \int w^{\frac{q}{2}}\, v_j^\beta\, \xi^2 &\le C \sum_{i=1}^N \int |g_i''(u_{x_i})| \left|\partial_{x_i}\left(v_j^{\beta-\frac{\alpha}{2}}\right)\right|^2 \xi^2 \\
&\quad + C \int w^{\frac{q}{2}}\, v_j^\alpha\, \xi^2 + C \int w^{\frac{q-2}{2}}\, v_j^\beta\, \left(|\nabla \xi|^2 + \xi^2\right) \\
&\quad + \varepsilon\, C \int w^{\frac{q-2}{2}} \left|\nabla\left(v_j^{\beta-\frac{\alpha}{2}}\right)\right|^2 \xi^2,
\end{aligned}
\tag{2.10}
$$

for some constant C *depending on* $q, N, \max\{\delta_1, \dots, \delta_N\}, \operatorname{dist}(\operatorname{support}(\xi), \partial\mathcal{O}), \|u\|_{W^{1,q}}$ *and the constant* C_1 *in* (2.2)*.*

Proof. We take the following test function

$$
\varphi^+_{j,\beta} = u\, v_j^\beta\, \xi^2, \qquad j = 1, \dots, N, \quad \beta > 0,
$$

which is admissible thanks to Remark 2.5. Inserting it into (2.8), we get

$$
\begin{aligned}
\sum_{i=1}^N \int_\Omega g_i'(u_{x_i})\, u_{x_i}\, v_j^\beta\, \xi^2 + \sum_{i=1}^N \int g_i'(u_{x_i})\ \partial_{x_i}\left(v_j^\beta\right) u\, \xi^2 + 2 \sum_{i=1}^N \int_\Omega g_i'(u_{x_i})\, \xi_{x_i}\, \xi\, u\, v_j^\beta & \\
+ \varepsilon \int_\Omega \langle \nabla H(\nabla u), \nabla u\rangle\, v_j^\beta\, \xi^2 & \\
+ \varepsilon \int \langle \nabla H(\nabla u), \nabla (v_j^\beta)\rangle\, \xi^2\, u & \\
+ 2\, \varepsilon \int_\Omega \langle \nabla H(\nabla u), \nabla \xi\rangle\, \xi\, u\, v_j^\beta & \\
= - \int b'\, u\, v_j^\beta\, \xi^2, \qquad j = 1, \dots, N.&
\end{aligned}
\tag{2.11}
$$

We start estimating the second term in (2.11): observe that by using Young inequality we have[2]

$$
\begin{aligned}
g_i'(u_{x_i}) \left|\partial_{x_i}\left(v_j^\beta\right)\right| &= \beta\, g'(u_{x_i})\, v_j^{\beta-1}\ |\partial_{x_i} v_j| \\
&\le \frac{1}{2} \beta^2\, v_j^{2\beta-\alpha-2}\, |\partial_{x_i} v_j|^2\, \frac{|g'(u_{x_i})|}{|u_{x_i}|} + \frac{1}{2}\, v_j^\alpha\, |g'(u_{x_i})|\, |u_{x_i}|\, 1_{\{u_{x_j} > k\}} \\
&= \frac{2\, \beta^2}{(2\,\beta - \alpha)^2} \left|\partial_{x_i}\left(v_j^{\beta-\frac{\alpha}{2}}\right)\right|^2 + \frac{1}{2}\, v_j^\alpha\, |g'(u_{x_i})|\, |u_{x_i}|\, 1_{\{u_{x_j} > k_j\}},
\end{aligned}
$$

where we used that $\partial_{x_i} v_j = 0$ on the set $\{u_{x_j} \le k_j\}$. Also observe that thanks to the fact that we are assuming $\alpha < \beta$, in the previous we can further estimate

$$
\frac{\beta^2}{(2\,\beta - \alpha)^2} \le 1,
\tag{2.12}
$$

---

[2] Observe that the therm $\frac{|g_i'(u_{x_i})|}{|u_{x_i}|}$ is well-defined even when $u_{x_i} = 0$.

so that the constant C that will appear in (2.10) will not depend on α and β. Using the previous estimate and the fact that $u \in L^\infty_{loc}$ by Corollary 2.4, the second term can be estimated by

$$\left|\sum_{i=1}^N \int g_i'(u_{x_i})\, \partial_{x_i} (v_j^\beta)\, u\, \xi^2\right| \le C \sum_{i=1}^N \int \frac{|g_i'(u_{x_i})|}{|u_{x_i}|} \left|\partial_{x_i}\left(v_j^{\beta-\frac{\alpha}{2}}\right)\right|^2 \xi^2 + C \sum_{i=1}^N \int_{\{u_{x_j}>k_j\}} |g_i'(u_{x_i})|\, |u_{x_i}|\, v_j^\alpha\, \xi^2, \tag{2.13}$$

for some constant $C > 0$, clearly depending on the L^∞ norm of u on the support of ξ. Observe that the second integral can be easily estimated by

$$\sum_{i=1}^N \int_{\{u_{x_j}>k_j\}} |g_i'(u_{x_i})|\, |u_{x_i}|\, v_j^\alpha\, \xi^2 \le C \int_{\{u_{x_j}>k_j\}} w^{\frac{q}{2}}\, v_j^\alpha\, \xi^2,$$

using the growth of g_i and the definition of w. The third term in (2.11) is estimated by

$$\begin{aligned}\left|\sum_{i=1}^N \int g_i'(u_{x_i})\, \xi_{x_i}\, \xi u\, v_j^\beta\right| &\le C\,\tau \sum_{i=1}^N \int |g_i'(u_{x_i})|\, |u_{x_i}|\, \xi^2\, v_j^\beta \\ &\quad + \frac{C}{\tau} \sum_{i=1}^N \int \frac{|g_i'(u_{x_i})|}{|u_{x_i}|}\, |\nabla \xi|^2\, v_j^\beta \\ &\le \frac{C}{\tau} \int w^{\frac{q-2}{2}}\, v_j^\beta\, |\nabla\xi|^2 + C\,\tau \sum_{i=1}^N \int |g_i'(u_{x_i})|\, |u_{x_i}|\, \xi^2\, v_j^\beta,\end{aligned}$$

and the second term can be absorbed in the left-hand side, by taking $\tau > 0$ small enough and observing that $g_i'(t)\, t \ge 0$.

We now come to the estimates of the $\varepsilon-$terms: for the fourth term in (2.11), we first observe that

$$\int \langle \nabla H(\nabla u), \nabla u\rangle\, v_j^\beta\, \xi^2 = \int w^{\frac{q}{2}}\, v_j^\beta\, \xi^2 - \int w^{\frac{q-2}{2}}\, v_j^\beta\, \xi^2,$$

so that we will collect the first integral in left-hand side and put the second one in the right-hand side, since this is a lower-order term with respect to the first (just check the sum of the powers). As for the second $\varepsilon-$term, we use:

$$\begin{aligned}|\langle \nabla H(\nabla u), \nabla(v_j^\beta)\rangle| &\le C\,\beta\, |\nabla v_j|\, v_j^{\beta-1}\, |\nabla u|\, (1+|\nabla u|^2)^{\frac{q-2}{2}} \\ &\le C\,\beta\, |\nabla v_j|\, v_j^{\beta-\frac{\alpha}{2}-1}\, v_j^{\frac{\alpha}{2}}\, w^{\frac{q-1}{2}} \\ &\le C\, \frac{\beta^2}{(2\,\beta-\alpha)^2} \left|\nabla\left(v_j^{\beta-\frac{\alpha}{2}}\right)\right|^2 w^{\frac{q-2}{2}} + C\, w^{\frac{q}{2}}\, v_j^\alpha,\end{aligned}$$

then by further using (2.12) we obtain

$$\left|\int \langle \nabla H(\nabla u), \nabla(v_j^\beta)\rangle\, \xi^2\, u\right| \le C \int_{\{u_{x_j}>k_j\}} w^{\frac{q-2}{2}} \left|\nabla\left(v_j^{\beta-\frac{\alpha}{2}}\right)\right|^2 \xi^2 + C \int_{\{u_{x_j}>k_j\}} w^{\frac{q}{2}}\, v_j^\alpha\, \xi^2,$$

with some constant C not depending on α and β. Finally, still in the same way as before we get

$$\left|\int_\Omega \langle \nabla H(\nabla u), \nabla \xi\rangle\, \xi\, u\, v_j^\beta\right| \le C\,\tau \int_\Omega w^{\frac{q}{2}}\, \xi^2\, v_j^\beta + \frac{C}{\tau} \int_\Omega w^{\frac{q-2}{2}}\, |\nabla\xi|^2\, v_j^\beta,$$

so that the first integral can be once again absorbed in the left-hand side.

Finally, we estimate the term containing b'. By using (2.2) and Corollary 2.4, it is readily seen that we have

$$\left|\int b'\,u\,v_j^\beta\,\xi^2\right| \le C\,\int v_j^\beta\,\xi^2 \le C\,\int w^{\frac{q-2}{2}}\,v_j^\beta\,\xi^2,$$

where the constant C depends again on the L^∞ norm of u on the support of ξ, which is in turn estimated as in Corollary 2.4. Collecting all the estimates and using once again (2.6), we arrive at(2.10). □

2.3. **Step 3: a Caccioppoli-type inequality.** In order to derive a Caccioppoli-type inequality for the gradient, we have to differentiate the equation (2.8) with respect to x_j. More precisely, let us take the test function $\varphi = \eta_{x_j}$ in (2.8), for some $\eta \in C_0^\infty(\mathcal{O})$. By recalling that $u \in C^{2,\alpha}$ and integrating by parts, the resulting equation takes the form

$$-\sum_{i=1}^N \int g_i''(u_{x_i})\,u_{x_i\,x_j}\,\eta_{x_i}\,dx - \varepsilon\,\int \langle D^2H(\nabla u)\,D_j^2 u, \nabla\eta\rangle\,dx + \int b'(x,u)\,\eta_{x_j}\,dx = 0, \tag{2.14}$$

for every $\eta \in C_0^\infty(\mathcal{O})$. Here $D_j^2 u$ stands for the $j-$th column of the Hessian matrix, i.e.

$$D_j^2 u = \begin{bmatrix} u_{x_1\,x_j} \\ \vdots \\ u_{x_N\,x_j}. \end{bmatrix}, \qquad j = 1,\dots,N.$$

By a density argument, we then get that (2.14) holds for every $\eta \in W_0^{1,q}(\mathcal{O})$.

Lemma 2.7. *Using the notation* (2.9)*, for every $s > 0$, $\xi \in C_0^1(\mathcal{O})$ and $j = 1,\dots,N$, we have*

$$\begin{aligned}\frac{s}{(s+1)^2}\sum_{i=1}^N \int g_i''(u_{x_i})\left|\partial_{x_i}\left(v_j^{\frac{s+1}{2}}\right)\right|^2 \xi^2 &+ \sum_{i=1}^N \int_{\{u_{x_j}>k_j\}} g_i''(u_{x_i})\,u_{x_i\,x_j}^2\,v_j^s\,\xi^2 \\ &+ \varepsilon\,\frac{s}{(s+1)^2}\int w^{\frac{q-2}{2}}\left|\nabla\left(v_j^{\frac{s+1}{2}}\right)\right|^2 \xi^2 \\ &+ \varepsilon\int_{\{u_{x_j}>k_j\}} \langle D^2H(\nabla u)\,D_j^2 u, D_j^2 u\rangle\,v_j^s\,\xi^2 \\ &\le C\,\frac{1+s}{s}\int_{\{u_{x_j}>k_j\}} w^{\frac{q-2}{2}}\,v_j^{s+1}\,(|\nabla\xi|^2+\xi^2),\end{aligned} \tag{2.15}$$

for a constant C depending on q, N and the constant C_2 in (2.2).

Proof. We insert the test function

$$\psi_{j,s}^+ = (u_{x_j} - k_j)_+\,v_j^s\,\xi^2,$$

in equation (2.14), with $s > 0$. Then we obtain the following 3 groups of terms that have to be estimated: the terms containing the functions g_i

$$\begin{aligned}\sum_{i=1}^N \int_{\{u_{x_j}>k_j\}} g_i''(u_{x_i})\,u_{x_i\,x_j}^2\,v_j^s\,\xi^2 &+ 2\,s\sum_{i=1}^N \int g_i''(u_{x_i})\,u_{x_i\,x_j}^2\,v_j^{s-1}\,(u_{x_j}-k_j)_+^2\,\xi^2 \\ &+ 2\sum_{i=1}^N \int g_i''(u_{x_i})\,u_{x_i\,x_j}\,(u_{x_j}-k_j)_+\,v_j^s\,\xi_{x_i}\,\xi \\ &=: G_1 + 2\,s\,G_2 + 2\,G_3,\end{aligned}$$

the terms containing H

$$\begin{aligned}\int_{\{u_{x_j}>k_j\}}\langle D^2H(\nabla u)\,D^2_ju,D^2_ju\rangle\,v_j^s\,\xi^2+2\,s\,\int\langle D^2H(\nabla u)\,D^2_ju,D^2_ju\rangle\,v_j^{s-1}\,(u_{x_j}-k_j)_+^2\,\xi^2\,dx\\ +2\int\langle D^2H(\nabla u)\,D^2_ju,\nabla\xi\rangle\,\xi\,(u_{x_j}-k_j)_+\,v_j^s\,dx\\ =:H_1+2\,s\,H_2+2\,H_3,\end{aligned}$$

and the terms containing b', i.e.

$$\begin{aligned}-\int_{\{u_{x_j}>k_j\}}b'\,u_{x_j\,x_j}\,v_j^s\,\xi^2dx+2\,s\,\int b'\,(u_{x_j}-k_j)_+^2\,v_j^{s-1}\,u_{x_j\,x_j}\,\xi^2\\ -2\int b'\,(u_{x_j}-k_j)_+\,v_j^s\,\xi_{x_j}\,\xi=:B_1+2\,s\,B_2+2\,B_3.\end{aligned}$$

Terms G_i. Let us start with the term G_2: by noticing that

$$u^2_{x_i\,x_j}\,v_j^{s-1}\,(u_{x_j}-k_j)_+^2=\left|v_j^{\frac{s-1}{2}}\,u_{x_i\,x_j}\,(u_{x_j}-k_j)_+\right|^2=\frac{1}{(s+1)^2}\,\left|\partial_{x_i}\left(v_j^{\frac{s+1}{2}}\right)\right|^2,$$

we get

$$G_2=\frac{1}{(s+1)^2}\,\sum_{i=1}^N\int g_i''(u_{x_i})\,\left|\partial_{x_i}\left(v_j^{\frac{s+1}{2}}\right)\right|^2\,\xi^2.$$

For the term G_3, we estimate it from above: we use Young inequality, so to get

$$\begin{aligned}|G_3|&\le\sum_{i=1}^N\int g_i''(u_{x_i})\,|u_{x_i\,x_j}|\,(u_{x_j}-k_j)_+\,v_j^s\,|\xi_{x_i}|\,\xi\\ &\le\frac{1}{\tau}\,\sum_{i=1}^N\int_{\{u_{x_j}>k_j\}}g_i''(u_{x_i})\,v_j^{s+1}\,|\nabla\xi|^2\\ &+\tau\,\sum_{i=1}^N\int g_i''(u_{x_i})\,|u_{x_i\,x_j}|^2\,(u_{x_j}-k_j)_+^2\,\,v_j^{s-1}\,\xi^2,\end{aligned}$$

and the last integral is exactly the same as in G_2.

Terms H_i. We keep the term H_1, which is positive, and we estimate H_2 from below by

$$H_2\ge\frac{C}{(s+1)^2}\,\int w^{\frac{q-2}{2}}\,\left|\nabla\left(v_j^{\frac{s+1}{2}}\right)\right|^2\,\xi^2.$$

For H_3 we proceed similarly to G_3, then getting

$$|H_3|\le\frac{1}{\tau}\,\int_{\{u_{x_j}>k_j\}}w^{\frac{q-2}{2}}\,v_j^{s+1}\,|\nabla\xi|^2+\tau\,\int\langle D^2H(\nabla u)\,D^2_ju,D^2_ju\rangle\,\,v_j^{s-1}\,(u_{x_j}-k_j)_+^2\,\xi^2,$$

having used Cauchy-Schwarz inequality in the following form

$$\big|\langle D^2H(\nabla u)\,D^2_ju,\nabla\xi\rangle\big|\le\sqrt{\langle D^2H(\nabla u)\,D^2_ju,D^2_ju\rangle}\,\sqrt{\langle D^2H(\nabla u)\,\nabla\xi,\nabla\xi\rangle},$$

and the growth of $|D^2H|\simeq w^{\frac{q-2}{2}}$.

Terms B_i. We estimate from above each of these terms, replacing $|b'|$ by the constant C_2, thanks to our assumption (2.2). Then we have

$$|B_1| \le C \int_{\{u_{x_j}>k_j\}} |u_{x_j\,x_j}|\, v_j^s\, \xi^2 \le C\,\tau \int_{\{u_{x_j}>k_j\}} |u_{x_j\,x_j}|^2\, v_j^s\, \xi^2 + \frac{C}{\tau} \int_{\{u_{x_j}>k_j\}} v_j^s\, \xi^2$$

and then we observe that we have

$$1_{\{u_{x_j}>k_j\}} \le \frac{1}{q-1}\, g_j''(u_{x_j})\, 1_{\{u_{x_j}>k_j\}}, \tag{2.16}$$

thanks to the fact that $k_j = \delta_j + 1$. Inserting this information in the previous estimate, we finally get

$$|B_1| \le C\,\tau \int_{\{u_{x_j}>k_j\}} g_j''(u_{x_j})\, |u_{x_j\,x_j}|^2\, v_j^s\, \xi^2 + \frac{C}{\tau} \int_{\{u_{x_j}>k_j\}} g_j''(u_{x_j})\, v_j^{s+1}\, \xi^2,$$

possibly with a different constant C. Notice that we further estimated $v_j^s \le v_j^{s+1}$, thanks to the fact that $v_j \ge 1$.

The term B_2 is readily estimated in a similar manner: we have

$$\begin{aligned}|B_2| \le C \int (u_{x_j}-k_j)_+^2\, v_j^{s-1}\, |u_{x_j\,x_j}|\, \xi^2 &\le C\,\tau \int g_j''(u_{x_j})\, (u_{x_j}-k_j)_+^2\, v_j^{s-1}\, |u_{x_j\,x_j}|^2\, \xi^2 \\ &\quad + \frac{C}{\tau} \int_{\{u_{x_j}>k_j\}} g_j''(u_{x_j})\, v_j^{s+1}\, \xi^2,\end{aligned}$$

where we used again (2.16) and $(u_{x_j}-k_j)_+^2 \le v_j \le v_j^2$. Finally, we come to the term B_3: we obtain

$$\begin{aligned}|B_3| \le C \int (u_{x_j}-k_j)_+\, v_j^s\, |\nabla\xi|\, |\xi| &\le C \int_{\{u_{x_j}>k_j\}} v_j^{s+1}\, |\nabla\xi|\, |\xi| \\ &\le C \int_{\{u_{x_j}>k_j\}} g_j''(u_{x_j})\, v_j^{s+1}\, \left(\xi^2 + |\nabla\xi|^2\right),\end{aligned}$$

still using (2.16) and $(u_{x_j}-k_j)_+ \le v_j$.

We are now ready to put all these estimates together. We keep the lower estimates on G_2 and H_2 on the left, while we put all the other terms on the right. By taking $\tau > 0$ small enough, in order to absorb all the terms appearing on the right and containing the Hessian of u, we finally get

$$\begin{aligned}\frac{s}{(s+1)^2} \sum_{i=1}^N \int g_i''(u_{x_i}) \left|\partial_{x_i}\left(v_j^{\frac{s+1}{2}}\right)\right|^2 \xi^2 &+ \sum_{i=1}^N \int_{\{u_{x_j}>k_j\}} g_i''(u_{x_i})\, u_{x_i\,x_j}^2\, v_j^s\, \xi^2 \\ &+ \varepsilon\, \frac{s}{(s+1)^2} \int w^{\frac{q-2}{2}} \left|\nabla\left(v_j^{\frac{s+1}{2}}\right)\right|^2 \xi^2 \\ &+ \varepsilon \int_{\{u_{x_j}>k_j\}} \langle D^2H(\nabla u)\, D_j^2 u, D_j^2 u\rangle\, v_j^s\, \xi^2 \\ &\le C \left(1+\frac{1}{s}\right) \sum_{i=1}^N \int_{\{u_{x_j}>k_j\}} g_i''(u_{x_i})\, v_j^{s+1}\, |\nabla\xi|^2 \\ &+ \varepsilon\, \frac{C}{s} \int_{\{u_{x_j}>k_j\}} w^{\frac{q-2}{2}}\, v_j^{s+1}\, \left(\xi^2 + |\nabla\xi|^2\right), \qquad j = 1, \dots, N,\end{aligned}$$

i.e. we showed the validity of (2.15). We only have to remark that

$$g_i''(u_{x_i}) \le w^{\frac{q-2}{2}}, \qquad i = 1, \dots, N.$$

Observe that the first integral on the left-hand side is equally performed on the set $\{u_{x_j} > k_j\}$, since otherwise v_j is constant. □

Let us now pay special attention to the case $s = 0$. Computations are very much the same.

Lemma 2.8. *For every $\xi \in C^1_0(\mathcal{O})$, we have*

$$\begin{aligned}\sum_{i=1}^{N} \int g_i''(u_{x_i}) \left|\partial_{x_i}\left(v_j^{\frac{1}{2}}\right)\right|^2 \xi^2 + \varepsilon \int w^{\frac{q-2}{2}} \left|\nabla\left(v_j^{\frac{1}{2}}\right)\right|^2 \xi^2 \\ \le C \int_{\{u_{x_j}>k_j\}} w^{\frac{q-2}{2}}\, v_j\, (|\nabla\xi|^2 + \xi^2),\end{aligned} \tag{2.17}$$

for $j = 1, \dots, N$, for some constant C independent of ε.

Proof. We repeat the previous computations, using the test function $\psi^+_{j,0} = (u_{x_j} - k_j)_+\, \xi^2$. This gives

$$\begin{aligned}\sum_{i=1}^{N} \int_{\{u_{x_j}>k_j\}} g_i''(u_{x_i})\, u^2_{x_i x_j}\, \xi^2 + \varepsilon \int_{\{u_{x_j}>k_j\}} \langle D^2H(\nabla u)\, D^2_j u\, D^2_j u\rangle\, \xi^2 \\ \le C \sum_{i=1}^{N} \int_{\{u_{x_j}>k_j\}} g_i''(u_{x_i})\, v_j\, |\nabla\xi|^2 \\ + \varepsilon\, C \int_{\{u_{x_j}>k_j\}} w^{\frac{q-2}{2}}\, v_j\, \left(\xi^2 + |\nabla\xi|^2\right), \qquad j = 1, \dots, N.\end{aligned}$$

By combining this together with

$$u^2_{x_i x_j} \ge \left|\partial_{x_i}\left(v_j^{\frac{1}{2}}\right)\right|^2 \qquad \text{and} \qquad \int_{\{u_{x_j}>k_j\}} \langle D^2H(\nabla u)\, D^2_j u, D^2_j u\rangle\, \xi^2 \ge \int w^{\frac{q-2}{2}} \left|\nabla\left(v_j^{\frac{1}{2}}\right)\right|^2 \xi^2,$$

and using again $g_i''(u_{x_i}) \le w^{\frac{q-2}{2}}$, we readily get the thesis. □

Remark 2.9. The previous estimates are valid for the functions v_j, which are different from 0 if u_{x_j} is large and positive. If on the contrary u_{x_j} is large in absolute value but negative, we can repeat the same estimates of Lemma 2.6 and Lemma 2.7, this time using as test functions

$$\varphi^-_{j,\beta} = u\, z_j^{\beta}\, \xi^2 \qquad \text{and} \qquad \psi^-_{j,s} = (-u_{x_j} - k_j)_+\, z_j^s\, \xi^2,$$

where z_j is given by

$$z_j = (-u_{x_j} - k_j)^2_+ + 1, \qquad j = 1, \dots, N.$$

Then we derive inequalities (2.10), (2.15) and (2.17), with z_j in place of v_j.

2.4. **Step 4: an iterative scheme of reverse Hölder inequalities.** Gluing together the estimates of Lemmas 2.6, 2.7 and 2.8 and tuning the exponents β and s, we obtain the following intermediate estimate, that we enunciate as a separate result for the sake of readability.

Lemma 2.10. *For every $\xi \in C^1_0(\mathcal{O})$, we have*

$$\int w^{\frac{q}{2}}\, v_j^{\frac{1}{2}}\, \xi^2 \le C \int w^{\frac{q}{2}} \left(\xi^2 + |\nabla\xi|^2\right), \qquad j = 1, \dots, N, \tag{2.18}$$

and for $\alpha \ge 1/2$

$$\int w^{\frac{q}{2}}\, v_j^{\frac{1}{2}+\alpha}\, \xi^2 \le C\, \alpha \int w^{\frac{q}{2}}\, v_j^{\alpha} \left(\xi^2 + |\nabla\xi|^2\right), \qquad j = 1, \dots, N, \tag{2.19}$$

for some constant C depending on $q, N, \max\{\delta_1, \dots, \delta_N\}, \mathrm{dist}(\mathrm{support}(\xi), \partial\mathcal{O}), \|u\|_{W^{1,q}}$ and the constants C_1, C_2 in (2.2).

Proof. First of all, we consider the case $\alpha \geq 1/2$ and make the choices

$$\beta = \frac{1}{2} + \alpha \qquad \text{and} \qquad s = 2\,\beta - \alpha - 1 = \alpha,$$

in (2.10) and (2.15). Then we drop the $\varepsilon-$term in the left-hand side of (2.10). In this way, by using $v_j \leq w$, we obtain

$$\begin{split}\sum_{i=1}^{N} \int g_i''(u_{x_i})\,|u_{x_i}|^2\,v_j^{\frac{1}{2}+\alpha}\,\xi^2 \leq C\,\frac{(\alpha+1)^3}{\alpha^2}\,\int w^{\frac{q-2}{2}}\,v_j^{\alpha+1}\,\left(\xi^2 + |\nabla\xi|^2\right)\\ + C \int w^{\frac{q}{2}}\,v_j^{\alpha}\,(|\nabla\xi|^2 + \xi^2) \leq C\,\alpha \int w^{\frac{q}{2}}\,v_j^{\alpha}\,\left(\xi^2 + |\nabla\xi|^2\right),\end{split} \tag{2.20}$$

for some constant C depending on the relevant data of the problem. We now estimate from below the left-hand side of (2.20): thanks to (2.5), we have

$$\sum_{i=1}^{N} g_i''(u_{x_i})\,|u_{x_i}|^2 \geq C_q \sum_{i=1}^{N} |u_{x_i}|^q - C_q' \geq C_{q,N}\,w^{\frac{q}{2}} - C_{q,N}'', \tag{2.21}$$

by using that in $\mathbb{R}^N$ all norms are equivalent and the simple convexity estimate

$$(t-1)^{\frac{q}{2}} \geq 2^{\frac{2-q}{2}}\,t^{\frac{q}{2}} - 1, \qquad t \geq 1.$$

Using this into (2.20) and using as always C as a generic constant depending on the data of the problem, we can thus infer

$$\int w^{\frac{q}{2}}\,v_j^{\frac{1}{2}+\alpha}\,\xi^2 \leq C\,\alpha \int w^{\frac{q}{2}}\,v_j^{\alpha}\,\left(\xi^2 + |\nabla\xi|^2\right) + C\,\alpha \int v_j^{\frac{1}{2}+\alpha}\,\xi^2,$$

which finally yields the thesis, by exploiting again that $1 \leq v_j \leq w$ and $q \geq 2$.

To treat the case $\alpha = 0$, which gives the first gain of integrability, we proceed similarly. We combine (2.17) and (2.10) with $\beta = 1/2$ and $\alpha = 0$. Then we use $v_j^{1/2} \leq v_j$, thus arriving at

$$\sum_{i=1}^{N} \int g_i''(u_{x_i})\,|u_{x_i}|^2\,v_j^{\frac{1}{2}}\,\xi^2 \leq C\,\int w^{\frac{q-2}{2}}\,v_j\,\left(\xi^2 + |\nabla\xi|^2\right) + C \int w^{\frac{q}{2}}\,\xi^2 \leq 2\,C\,\int w^{\frac{q}{2}}\,\left(\xi^2 + |\nabla\xi|^2\right).$$

Again using (2.21), we immediately deduce the thesis. □

2.5. **Step 5: proof of Proposition 2.1.** Keeping in mind Remark 2.9, the same estimate (2.19) holds with z_j in place of v_j, so that summing up we get

$$\int w^{\frac{q}{2}}\,\left(v_j^{\frac{1}{2}+\alpha} + z_j^{\frac{1}{2}+\alpha}\right)\,\xi^2\,dx \leq C\,(\alpha+1) \int w^{\frac{q}{2}}\,\left(v_j^{\alpha} + z_j^{\alpha}\right)\,\left(\xi^2 + |\nabla\xi|^2\right)\,dx, \quad j = 1, \dots, N.$$

If we set

$$T_j = \max\{v_j, z_j\}, \qquad j = 1, \dots, N,$$

from the previous we can easily infer

$$\int w^{\frac{q}{2}}\,T_j^{\frac{1}{2}+\alpha}\,\xi^2\,dx \leq 2\,C\,(\alpha+1)\,\int w^{\frac{q}{2}}\,T_j^{\alpha}\,\left(\xi^2 + |\nabla\xi|^2\right)\,dx, \qquad j = 1, \dots, N.$$

Summing up all these inequalities and setting $T = \max\{T_1, \dots, T_N\}$, we get

$$\int w^{\frac{q}{2}}\,T^{\frac{1}{2}+\alpha}\,\xi^2\,dx \leq 2\,N\,C\,(\alpha+1)\,\int w^{\frac{q}{2}}\,T^{\alpha}\,\left(\xi^2 + |\nabla\xi|^2\right)\,dx.$$

Finally, we observe that there exist two constants $\gamma_1, \gamma_2 > 0$ depending only on the dimension N and $\max\{\delta_1, \dots, \delta_N\}$, such that

$$\gamma_1\, T \le w \le \gamma_2\, T.$$

Inserting this information in the previous inequality, we get

$$\int T^{\frac{q}{2}+\alpha+\frac{1}{2}}\, \xi^2\, dx \le C\,(\alpha+1) \int T^{\frac{q}{2}+\alpha}\, \left(\xi^2 + |\nabla \xi|^2\right)\, dx.$$

By fixing two balls $B_\varrho(x_0) \subset B_R(x_0)$ and suitably choosing a sequence of cut-off functions $\{\xi_k\}_{k\in\mathbb{N}} \subset C^1_0(\mathcal{O})$ supported on an infinite family of shrinking balls $B_\varrho(x_0) \Subset B_{r_{k+1}}(x_0) \Subset B_{r_k}(x_0) \Subset B_R(x_0)$, we can iterate the previous estimate, taking

$$\alpha_k = \frac{k}{2}, \qquad \text{with } k \in \mathbb{N}.$$

Then a standard covering argument concludes the proof of Proposition 2.1.

3. Proof of the Main Theorem

We now come to the proof of the Main Theorem. Let $f \in L^\infty_{loc}(\Omega)$ and call u a local minimizer of (1.8). We have to show that $u \in W^{1,r}_{loc}(\Omega)$ for every $r \ge 1$. At this aim, it is sufficient to show that for every ball $B \Subset \Omega$ there holds $u \in W^{1,r}(B)$ for every $r \ge 1$.

We then fix a ball $B \Subset \Omega$ and consider a slightly larger ball $B' \Subset \Omega$. For $\delta > 0$ sufficiently small, we take a standard mollification kernel η_δ with compact support and set

$$u^\delta = (u * \eta_\delta) \cdot 1_{B'} \in C^\infty(\overline{B'}).$$

The localization on B' is needed since by definition our local minimizer u just belongs to $W^{1,q}_{loc}(\Omega)$. Now we fix $\varepsilon \ll 1$ and take $f^\varepsilon \in C^\infty(\Omega)$, such that f^ε $*-$weak converges in L^∞_{loc} to f. In particular, we can assume that

$$\|f^\varepsilon\|_{L^\infty(B')} \le C\, \|f\|_{L^\infty(B')},$$

with C independent of ε. Then, let $u^{\varepsilon,\delta}$ be a solution of

$$\min\left\{ J^\varepsilon_q(v; B') + P(v - u^\delta; B') \,:\, v - u^\delta \in W^{1,q}_0(B') \right\}, \tag{3.1}$$

where J^ε_q is given by

$$J^\varepsilon_q(v; B') = \sum_{i=1}^N \int_{B'} g^\varepsilon_i(v_{x_i})\, dx + \int_{B'} f^\varepsilon\, v\, dx + \varepsilon \int_{B'} H(\nabla v)\, dx, \qquad v \in W^{1,q}(B'),$$

and P is a penalization term, defined by

$$P(v; B') = \int_{B'} \Big[1 - \exp(-|v(x)|^2)\Big]\, dx, \qquad v \in W^{1,q}(B').$$

Lemma 3.1 (Uniform estimates)**.** *The following estimate holds*

$$\|u^{\varepsilon,\delta}\|_{W^{1,q}(B')} \le C\, \left(\|u\|_{W^{1,q}(B')} + 1\right), \tag{3.2}$$

for some constant $C > 0$ not depending on δ and ε. Moreover, $u^{\varepsilon,\delta} \in W^{1,r}(B)$ for every $r \ge 1$ and we have the estimate

$$\|u^{\varepsilon,\delta}\|_{W^{1,r}(B)} \le C,$$

for a constant $C > 0$ depending on $N, q, r, \mathrm{dist}(B, \partial\Omega), \|u\|_{W^{1,q}(B')}$ and $\|f\|_{L^\infty(B')}$, but not on ε and δ.

Proof. We consider the Euler-Lagrange equation of problem (3.1), tested with $\varphi = u^{\varepsilon,\delta} - u^\delta \in W^{1,q}_0(B')$. This yields

$$\begin{aligned}\sum_{i=1}^N \int_{B'} (g_i^\varepsilon)'(u^{\varepsilon,\delta}_{x_i})\, u^{\varepsilon,\delta}_{x_i} + \varepsilon \int_{B'} \langle \nabla H(\nabla u^{\varepsilon,\delta}), \nabla u^{\varepsilon,\delta}\rangle \le \sum_{i=1}^N \int_{B'} &|(g_i^\varepsilon)'(u^{\varepsilon,\delta}_{x_i})|\, |u^\delta_{x_i}| \\ &+ \varepsilon \int_{B'} |\nabla H(\nabla u^{\varepsilon,\delta})|\, |\nabla u^\delta| \\ &+ C\, \|u^{\varepsilon,\delta} - u^\delta\|_{L^q(B')},\end{aligned}$$

for a constant depending on q, $|B'|$ and $\|f\|_{L^\infty(B')}$, but not on ε and δ. Using the growth conditions on ∇H and g_i' and Young inequality, from the previous we can infer in a standard way

$$\int_{B'} |\nabla u^{\varepsilon,\delta}(x)|^q\, dx \le C \int_{B'} |\nabla u^\delta(x)|^q\, dx + C\, \|u^{\varepsilon,\delta} - u^\delta\|_{L^q(B')}\, dx + C.$$

Finally using Poincaré inequality for the function $u^{\varepsilon,\delta} - u^\delta \in W^{1,q}_0(B')$, we obtain

$$\|u^{\varepsilon,\delta}\|_{W^{1,q}(B')} \le C\, \left(\|u^\delta\|_{W^{1,q}(B')} + 1 \right),$$

for a different constant C, yet still independent of ε and δ. Recalling the definition of u^δ, this finally gives (3.2).

The $W^{1,r}$ estimates are a straightforward consequence of Proposition 2.1 applied with

$$\mathcal{O} = B', \qquad \zeta = u^\delta \qquad \text{and} \qquad b_\varepsilon(x,t) = f^\varepsilon(x)\, t + \left[1 - \exp(-|t - u^\delta(x)|^2)\right], \quad x \in B',\ t \in \mathbb{R},$$

in conjunction with the uniform estimate (3.2) on $\|u^{\varepsilon,\delta}\|_{W^{1,q}(B')}$ and the fact that the C^∞ functions b_ε verify (2.2) with constants C_1 and C_2 depending only on $\|f\|_{L^\infty(B')}$. □

We also need the $\Gamma-$convergence result below: the proof is standard and we just skecth it, referring the reader to [14] for more details. The interesting point here is the convergence of the minimizers. We recall that $\mathfrak{F}_q$ is the original functional defined in (1.8).

Lemma 3.2. *Let $\delta > 0$ be given and $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive reals converging to 0, then the functionals*

$$u \mapsto J_q^{\varepsilon_k}(u) + P(u - u^\delta), \qquad u \in W^{1,q}(B'),$$

are $\Gamma-$converging to $\mathfrak{F}_q + P(\cdot - u^\delta)$ with respect to the $W^{1,q}(B')$ weak topology. Moreover, a sequence of minimizers $\{u^{\varepsilon_k,\delta}\}_{k\in\mathbb{N}}$ weakly converges (up to a subsequence) in $W^{1,q}(B')$ to a minimizer $u^{0,\delta}$ of

$$\min\left\{ \mathfrak{F}_q(v; B') + P(v - u^\delta; B') \,:\, v - u^\delta \in W^{1,q}_0(B') \right\}. \tag{3.3}$$

Proof. First of all, we observe that the additive term $u \mapsto P(u - u^\delta)$ is not dependent on ε and it is continuous with respect to the $W^{1,q}(B')$ weak convergence, then it is sufficient to prove the $\Gamma-$convergence of the functionals $J_q^{\varepsilon_k}$, thanks to [14, Proposition 6.21]. Also, we observe that for every sequence $\{u^{\varepsilon_k}\}_{k\in\mathbb{N}}$ strongly converging in $L^q(B')$ to a function u, we have

$$\lim_{k\to\infty} \int_{B'} f^{\varepsilon_k}\, u^{\varepsilon_k}\, dx = \lim_{k\to\infty} \int_{B'} f^{\varepsilon_k}\, u\, dx = \int_{B'} f\, u\, dx,$$

then by [14, Proposition 6.20] we only need to prove that the functional

$$v \mapsto \sum_{i=1}^N \int_{B'} g_i^\varepsilon(v_{x_i})\, dx + \varepsilon \int_{B'} H(\nabla v)\, dx,$$

is $\Gamma-$converging to $\sum_{i=1}^N \int_{B'} g_i(v_{x_i})\,dx$. By using the convexity of g_i and H and the uniform convergence of $g_i^{\varepsilon_k}$, we can conclude by appealing to [14, Theorem 5.14].

For the last part of the statement, we can use [14, Corollary 7.20], once it is observed that the minimizers $\{u^{\varepsilon_k,\delta}\}_{k\in\mathbb{N}}$ satisfy the equi-coercivity condition (3.2). □

Proof of the Main Theorem. We first pass to the limit as ε goes to 0. Thanks to (3.2), there exists a subsequence $\{u^{\varepsilon_k,\delta}\}_{k\geq 0}$ weakly converging in $W^{1,q}(B')$ to a limit function $u^{0,\delta}$. Moreover, thanks to Lemma 3.2, this $u^{0,\delta}$ is a minimizer of (3.3). We now take the limit as δ goes to 0. Still from (3.2) we have

$$\|u^{0,\delta}\|_{W^{1,q}(B')} \leq C\,\left(\|u\|_{W^{1,q}(B')}+1\right),$$

then there exists a subsequence $\{u^{0,\delta_k}\}_{k\geq 0} \subset W^{1,q}(B')$ which weakly converges in $W^{1,q}(B')$ to a function u^0. Using the minimality of u^{0,δ_k}, the semicontinuity of the penalized functional and the continuity of the functional $\mathfrak{F}_q(\cdot\,;B')$ with respect to the strong convergence, we get

$$\begin{aligned}\mathfrak{F}_q(u^0;B')+P(u^0-u;B') &\leq \liminf_{k\to\infty}\left[\mathfrak{F}_q(u^{0,\delta_k};B')+P(u^{0,\delta_k}-u^{\delta_k};B')\right]\\ &\leq \liminf_{k\to\infty}\mathfrak{F}_q(u^{\delta_k};B')=\mathfrak{F}_q(u;B').\end{aligned}$$

Finally, we use the fact that u is a local minimum and that $u^0-u\in W^{1,q}_0(B')$, then

$$\mathfrak{F}_q(u^0;B')+P(u^0-u;B')\leq \mathfrak{F}_q(u;B')\leq \mathfrak{F}_q(u^0;B'),$$

which implies that $P(u^0-u;B')=0$. By recalling the definition of P, the latter implies $u=u^0$ almost everywhere in B'.

Let us now observe that thanks to Lemma 3.1, we have that $u^{\varepsilon_k,\delta}\in W^{1,r}_{loc}(B')$, for every $r\geq 1$. In particular, $u^{\varepsilon_k,\delta}\in W^{1,r}(B)$ for every $r\geq 1$ and we have a uniform estimate of the type

$$\|u^{\varepsilon_k,\delta}\|_{W^{1,r}(B)}\leq C,$$

with C independent of ε_k and δ. Using the fact that $u^{\varepsilon_k,\delta}$ converges to $u^{0,\delta}$, we get that $u^{0,\delta}\in W^{1,r}(B)$ for every $r\geq 1$ as well, with an estimate uniform in δ. Finally, taking the limit as δ goes to 0, from the previous discussion we get that $u\in W^{1,r}(B)$ as well, for every $r\geq 1$. This finally concludes the proof of the Main Theorem. □

4. Applications to Beckmann's problem

Going back to our original purpose, it is mandatory to conclude the paper with some applications to Beckmann's problem (1.3).

Corollary 4.1. *Let $q\geq 2$ and $f\in L^\infty(\Omega)$ be such that $\int_\Omega f\,dx=0$. Every solution $u\in W^{1,q}(\Omega)$ of the following variational problem*

$$\min\left\{\sum_{i=1}^N\int_\Omega \frac{1}{q}\,(|v_{x_i}|-\delta_i)_+^q\,dx+\int_\Omega f\,v\,dx\;:\;v\in W^{1,q}(\Omega)\right\}, \tag{4.1}$$

satisfies $u\in W^{1,r}_{loc}(\Omega)$, for every $r\geq 1$.

Proof. At first, we observe that (4.1) is equivalent to

$$\min\left\{\sum_{i=1}^N\int_\Omega \frac{1}{q}\,(|v_{x_i}|-\delta_i)_+^q\,dx+\int_\Omega f\,v\,dx\;:\;v\in W^{1,q}(\Omega)\ \text{ and }\ \int_\Omega v\,dx=0\right\},$$

since the functional is unchanged if we replace v with $v + c$, for every constant c. Then the existence of a minimizer u follows from the Direct Methods in a standard way. A global minimizer u is of course a local minimizer of the functional $\mathfrak{F}_q$, thus the thesis follows by applying the Main Theorem to u. □

An "almost" L^∞ estimate for the optimal transportation program is now an easy consequence of the previous result and the primal-dual optimality condition.

Corollary 4.2. *Let $f \in L^\infty(\Omega)$ be such that $\int_\Omega f(x)\,dx = 0$ and $1 < p \le 2$. Then the (unique) vector field $\widetilde{\phi} \in L^p(\Omega;\mathbb{R}^N)$ which solves*

$$\min_{\phi\in L^p(\Omega;\mathbb{R}^N)} \left\{ \sum_{i=1}^N \int_\Omega \left[\frac{|\phi_i|^p}{p} + \delta_i\,|\phi_i| \right] dx \,:\, \operatorname{div}\phi = f \ \textit{in } \Omega,\ \langle \phi, \nu_\Omega\rangle = 0 \ \textit{on } \partial\Omega \right\},$$

is in $L^r_{loc}(\Omega;\mathbb{R}^N)$, for every $r \ge 1$.

Proof. Existence and uniqueness of $\widetilde{\phi} \in L^p(\Omega;\mathbb{R}^N)$ is straightforward, since we are minimizing a strictly convex energy with $p-$growth, under a linear and closed constraint. By standard convex duality (see [16, Proposition 5, page 89]), we get the primal-dual optimality condition

$$\widetilde{\phi}_i = (|u_{x_i}| - \delta_i)_+^{q-1}\,\frac{u_{x_i}}{|u_{x_i}|}, \qquad i = 1,\dots,N,$$

with $q = p/(p-1) \ge 2$ and $u \in W^{1,q}(\Omega)$ solution of (4.1). Then the result follows from Corollary 4.1. □

Acknowledgements. We would like to thank Filippo Santambrogio for some useful discussions. Nina Ural'tseva is gratefully acknowledged for having pointed out to us the reference [26]. This work has been supported by the ANR through the projects ANR-09-JCJC-0096-01 EVAMEF and ANR-07-BLAN- 0235 OTARIE, as well as by the ERC Advanced Grant n. 226234.

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